STUDENTS' MENTAL CONSTRUCTIONS OF CONCEPTS IN VECTOR CALCULUS: INSIGHTS FROM TWO UNIVERSITIES

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ABSTRACT

Student success in Mathematics is a global priority. Mathematics is a fundamental part of engineering programs in higher education, necessary for application in further engineering studies and yet often becomes a stumbling block for engineering students. Concerningly, even successful students frequently exhibit weak understanding of key mathematical concepts. The vector calculus course is known to be particularly challenging for students.

While much research has been done into students' constructions of core concepts at school level, less has been done on advanced mathematical topics such as vector calculus, yet this important insight has the potential to impact curriculum and pedagogy and to inform relevant support. This research conducted at the University of Twente, the Netherlands and the University of Cape Town, South Africa will use the constructivist APOS (Action-Process-Object-Schema) Theory to explore how students mentally construct concepts such as partial derivatives, directional derivatives and double integrals. APOS theory is based on the hypotheses that individuals construct mental actions, processes, and objects and organise them in schemas to solve mathematical problems.

In this exploratory case study we attempt to explore and understand how our students understand the limits of integration of double integrals, informing the design of our teaching of vector calculus to improve students' understanding and ultimately increase success. Students will participate in an assessment, complete a survey and participate in individual interviews. APOS Theory can be used directly in the analysis of data. Although initially intended we were unable to use this opportunity to compare the mental constructions of the cohorts from the two different universities and it will be assigned to future research.

1 INTRODUCTION

Mathematics is fundamental to the study of engineering courses and has an important bearing on the success of engineering students. There are increasing concerns about the transition from studying mathematics at high school to university and many research studies focus on “students' preparedness” to study higher education mathematics. Engineering programs are known to be very selective and even for those students who meet the mathematics requirements for engineering enrolment, mathematics problems still persist and the lack of mathematical preparedness and mathematical proficiency remain a barrier to the study of engineering [1].

Although there are various problems associated with this transition from high school to university it is said that the procedural approach to learning mathematics in school particularly aggravates that transition. Conceptual understanding is commonly known as deep level understanding of underlying concepts in mathematics and their relationships with each other. Recent research focus has been on students' understanding of mathematics at undergraduate level and a call for learning approaches in mathematics to change from procedural to conceptual and for teaching for conceptual understanding in mathematics. It is asserted that mathematics courses whilst providing necessary skills for the study of other courses should also foster cognitive and metacognitive abilities allowing students to be lifelong learners and creative and critical problem solvers.

Studies have shown that in some cases there is a disconnect in the teaching and learning dynamic between mathematics taught and what students learn. This occurs for various reasons, some of which from the teaching end involve an underestimation of the difficulty of the concept for the student, an assumption that students have the prerequisite knowledge, and an unintentional omission of knowledge vital to students understanding the concept. Research into how students learn and understand mathematics allows for a better articulation of teaching practice and an alignment between what is taught and what students learn. In this research we recognise that in accordance with the philosophy of constructivism a better facilitation of student meaning making in mathematics is central to their learning.

1.1 Background

Calculus is central to engineering. Calculus, as the mathematics of change and motion, is indispensable in any form of mathematical modelling. Ordinary and partial differential equations, multiple integrals, curl of a vector field, Stokes’ and Gauss’s Theorems are all found in any introductory textbook of engineering mathematics.

Research suggests that vector calculus is one of the important and difficult courses in undergraduate mathematics studies, challenging for any student. Certain issues contribute to this challenge such as difficulty with imaging and sketching in three dimensions, a lack of problem-solving skills, students' beliefs and students' learning styles. Visualisation and the many conceptual challenges around continuity and differentiability in these contexts challenge all students. Another suggestion for students' difficulty is that the course demands of students that they absorb complex and new ideas in a limited time. It is our experience that students are already contending with an overload of other courses in their respective engineering qualifications. [2:23] point out that "the shift from single variable to multivariate calculus is more than simply a matter of the symbolic demand of calculus with more variables".

The study of functions of two variables forms an important foundation to the study of engineering. The double integral is frequently used in engineering from finding areas, volumes, areas of surfaces to computing mass, electric charge, work, centre of mass and moments of inertia to name some of the many applications.

Our focus in this research on double integrals is driven by the fact that research in this area is not extensively explored. Research shows that students experience difficulties with double integrals especially with determining the limits of integration and changing of order of integration [3]. In the research reported on here we analyse students' mental constructions of the limits of integration of double integrals with the main objective to provide evidence to inform the teaching and learning of double integrals. The authors acknowledge that whilst there are various factors which impact student understanding this research focuses only on the cognitive facet of student understanding.

1.2 **Theoretical Framework**

This study uses APOS (Action-Process-Object-Schema) theory to analyse the mental constructions of students taking a test on double integrals. Learning takes place in four stages when students construct mathematical concepts. The four stages are: action, process, object and schema. Actions are a transformation of mathematical objects when students follow some explicit algorithm to perform the operation and this is perceived by the student as externally driven, for example something they would have been taught or from memory. This is a step by step procedure where one step cues the next. At this point students are not able to anticipate or skip any steps. Upon repeating the action and reflection on the action the student internalises the action as a process. At this stage the student has gained control over the actions. Upon performing actions on processes, it is said that students have encapsulated the process and constructed a cognitive object. "In many mathematical objects it may be necessary to de-encapsulate an object and work with the process from which it came" [4]. A schema for a mathematical concept is a student's response when presented with a mathematical problem situation and is based on their framework which is built from a collection of the students' actions, processes, objects and other schema linking the mathematical concept to general principles.

Genetic Decomposition (GD) or model of cognition of the mathematical concept describes a possible and not necessarily unique way in which a student constructs a mathematical concept in terms of the mental constructions in the framework of APOS theory. This description of specific mental constructions made by the student to develop an understanding of the concept is for design and analysis of teaching and learning.

2 METHODOLOGY

In this section we give the context, research design, research questions and propose a genetic decomposition for finding the limits of double integrals.

2.1 Context

Two cohorts of students were intended to contribute data for this project. The University of Twente participants were the electrical engineering and advanced technology students in their first year of study in 2019/2020 and for the University of Cape Town, the second-year engineering students in 2020. The project will investigate students' understanding of a range of concepts in multivariable and vector calculus, including but not limited to directional derivatives, double integrals, divergence and curl.

For the purposes of this paper we report on a pilot study that will inform the project going forward. Unfortunately, the global coronavirus crisis and ensuing constraints on our institutions resulted in data being gathered from 97 students from the University of Twente only. In essence this paper can be seen as a first step towards realising our vision of the project, that of understanding students' mental constructions of vector calculus concepts to inform our teaching. In this concept paper we will focus on students' understanding of the limits of integration for double integrals. We shall present the analysis of that data as an indication of the expectation from the project.

2.2 Research Design

For each of the multivariable and vector calculus concepts of interest in the study we shall postulate a genetic decomposition (GD). A GD is a detailed description of a set of mental constructions a student will use in developing an understanding of the concept under study. These mental constructions are called actions, processes, objects and schemas and play a role in the development of an understanding of the concept. Our GD for determining the limits of integration for double integrals is informed by past research, literature and the researchers' mathematical knowledge and teaching experience. Arriving at a genetic decomposition which describes the students' actual mental constructions and informs the teaching of the mathematical concept requires many cycles of research involving GD posing or refining, classroom activities and data gathering [4].

This research describes only the first cycle in this process. Students' understanding of the concepts will be analysed through the lens of APOS theory using data from assessment and, where possible, interviews. The data presented in this paper were drawn from a test taken by the participants from the University of Twente. The tests were graded, and the responses analysed using APOS Theory by one of the authors. The grading and the analysis were also undertaken by the other author for consensus.

The two authors are the lecturers and graders for the applicable mathematics courses and hence have access to students' test responses. To include students' written work in a publication ethics approval will be applied for. For this report on the pilot study no direct examples of student work will be presented.

The limitations of this research study as a consequence of the global pandemic and lockdown was that we were unable to further probe students' mental constructions of the vector calculus concept during interviews and to compare the mental constructions of the cohorts from the two different universities.

2.3 **Research Questions**

"What are Vector Calculus students' visual and analytic understanding of vector calculus concepts?
This paper contributes to answering the sub question: "What are Vector Calculus students' visual and analytic understanding of the limits of integration for double integrals?"

2.4 **Genetic Decomposition for Double Integrals**

In this section we start with prerequisite knowledge that students will need before performing the mathematical task and propose a genetic decomposition of finding the limits of integration of double integrals.

Prerequisite knowledge:

- Recall of and understanding of notation encountered previously in differential calculus- e.g dx - with respect to the variable x, dy - with respect to y.
- Techniques previously used in single integration
- The integrals of polynomial, trigonometric, inverse trigonometric, exponential and natural logarithmic functions;
- Determining limits for the definite integral and basic algebraic operations
- Double integrals over rectangular regions

Actions: These are mechanical procedures which lack meaningful internal relations to other mathematical ideas. At this stage there is a transformation of a mathematical object by applying a rigid step by step algorithm which is perceived as externally driven. A student evaluates a double integral by computing an iterated integral over a region which is either a:
Type 1 region: integration is first with respect to y, in the vertical direction (bottom to top) and then with respect to x by seeing the region between two functions of x and two vertical lines respectively, or a
Type 2 region: integration is first with respect to x, in the horizontal direction (left to right) and then with respect to y by seeing the region between two functions of y and two horizontal lines respectively
At this stage the student has an action understanding of setting up a double integral with limits of integration.

Process: Students when repeating actions and reflecting upon actions will internalize them. Specifically, the student can imagine performing the transformation without having to execute each step explicitly, seeing a step by step algorithm as no longer necessary. When faced with the iterated integral where the integration order is $dydx$, integration with respect to y is required first and then followed by integration with respect to x, students will no longer need to identify the region as type 1. Students will realise that integration will take place vertically first followed by horizontally. Now students will identify the function of x which lies above the region i.e the region is bounded above by that function and the function of x which lies below the region i.e the region is bounded below by the function for values of x in an interval (left to right). Similarly, students will proceed when the iterated integral requires integrating $dxdy$, first with respect to x and then with respect to y.

Object: When a student applies and can imagine applying such transformations then it is said that the process is encapsulated into a cognitive object. The student with an object conception of the mathematical operation may, unprompted and needing no further instructions, recognise the applicability of the mathematical operation in a given problem situation. For example, the student understands that for double integrals we are sweeping the area under volume. Realising that taking slices of the volume into two dimensional slices of area, computing that area of each slice and summing over all areas of slices will give the volume as a whole.
In our particular test example, a student would when confronted with setting up an iterated integral in the order $dydx$ will then “de-encapsulate” the object to the process that it came from and apply it to a particular situation. For example, here a student will note that when looking at the order of integration firstly with respect to y and then with respect to x. Taking slices of the volume yield the same lower boundary function for the region in question but not the same upper boundary function. That would require thinking of the region as a sum of two regions over which integration will take place and hence the double integral in question will be

reflected as a sum of two double integrals each of which is bounded by the same function above and below.

Schema: This is a coherent collection of actions, processes and objects and other previously constructed schemas.

3 RESULTS AND DISCUSSION

Two items from a mathematics test taken by 97 electrical engineering students are analysed here. Focus is on the students who interpreted the task incorrectly. It must be noted that this is an initial analysis as further probing by interviewing students to obtain a clearer understanding of students' mental constructions could not take place.

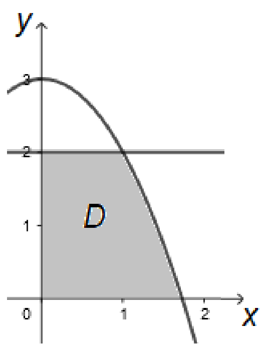

Fig. 1. Test item on double integral

The density of lamina *D* at point (x, y) is twice the distance from the point to the *y*-axis. Write down (<u>but do not evaluate</u>) an iterated double integral to represent the mass of lamina *D* shown below. Do this first in the order $dxdy$ and then in the order $dydx$. *D* is bounded by $x = 0$, $y = 0$, $y = 2$ and $y = 3 - x^2$.

3.1 **Double integral APOS analysis, UT electrical engineering**

Given the context of the test question item 1 required students to set up an iterated integral for integration in the direction $dxdy$. The correct interpretation of item 1 would result in integral 1 as shown below:

Integral 1: $\int_0^2 \int_0^{\sqrt{3-y}} \sigma dx\, dy$ where σ is the density of the lamina.

Student Responses:

Four students responded with: $\int_0^2 \int_0^{3-x^2} \sigma dx\, dy$

Students displayed the action stage and were able to determine the curves that make left and right boundaries of the region. They recognised that integration firstly was with respect to x and this formed the 'nested' integral and that the outer integral limits go from $y = 0$ to $y = 2$. However they failed to recognise that the boundary curves are functions of y and not x. It may be correct to suggest that the students are in the action stage and have not made the transformation to the process stage yet.

Two students responded with: $\int_0^2 \int_0^{\sqrt{y-3}} \sigma dx\, dy$

Here students having successfully recognised the order of integration have proceeded to find $x = f(y)$ for boundary curves, thus showing that a process stage has been reached. However, an algebraic error leads the student to an upper limit of $x = \sqrt{y-3}$ and the student is unable to reflect that $y \leq 3$ for the region under consideration. The object stage has not been reached since a real understanding of limits as a description of the region should have flagged the domain of the upper bound function for x.

One student responded with: $\int_0^2 \int_{\sqrt{3-y}}^1 \sigma dx\, dy$

Here the student changes the region of integration and integrates from right to left on the inner integral. There is a recognition that the inner limits of integration need to be functions of y, yet the visual interpretation is not present. It may be said that the student is not following an algorithmic approach here and cannot be said to have reached the action stage.

One student responded with: $\int_0^3 \int_0^2 \sigma dx\, dy$

This student proceeds as if the region is a rectangular one. Therefore, the student does not show knowledge of an algorithm in terms of how to proceed. It can be said that this student has not reached the action stage. In fact, prerequisite knowledge of integration is lacking.

Given the context of the test question, item 2 required students to set up an iterated integral for integration in the direction $dydx$. The correct interpretation of item 2 would result in integral 2 as shown below:

Integral 2 : $\int_0^2 \int_0^{\sqrt{3-y}} \sigma dx\, dy = \int_0^1 \int_0^2 \sigma dy\, dx + \int_1^{\sqrt{3}} \int_0^{3-x^2} \sigma dy\, dx$ where σ is the density of the lamina.

Student Responses:

Eighteen students responded with: $\int_0^{\sqrt{3}} \int_0^{3-x^2} \sigma dy\, dx$

These students are considering the entire area as if it were bounded by the same function above, not realising that $0 \leq y \leq 3 - x^2$ is not true for the entire interval, i.e $x \in [0,3]$. Students have merely followed steps here without acknowledging visually the shape of the area which in this case warrants that the area be represented as a sum of two integrals. There appears to be an action stage reached but certainly no process stage can be observed.

Three students responded with: $\int_0^1 \int_0^2 \sigma dy\, dx + \int_1^2 \int_0^{3-x^2} \sigma dy\, dx$

Students are at the process stage where the shape of the region is recognised to give rise to the sum of two integrals. There is however a failure to recognise that $x =$

2 is clearly outside the region under consideration. It is clear from the diagram given in the test question that x does not extend to 2.

One student responded with: $\int_0^1 \int_0^2 \sigma dy\, dx + \int_1^{\sqrt{3}} \int_0^{\sqrt{y}} \sigma dy\, dx$
This student has recognised that the region should be split in two regions and integrals evaluated over each of these regions should be summed. The student seems to be at the process stage as the student reads bounds bottom to top, however does not show an understanding that the limits of the inner integral are functions of x, which is an indication that the student is actually at the action stage.

One student responded with: $\int_0^1 \int_0^2 \sigma dy\, dx + \int_1^{\sqrt{3}} \int_0^{\sqrt{3-y^2}} \sigma dy\, dx$
The actions of "read region bottom to top" and "find bounding curves" and "bounding curves of inner integrals must be $y = f(x)$" when the nested integral calls for integration with respect to y become internalised as a process that results in an object. Recognising the context here where the given integral when the order is $dydx$ is the sum of two integrals is applicable in the given problem situation calls for decapsulating the object from the process it came from. However here the student has given the inner function of the second integral as a function of y whereas it should be given as a function of x.

One student responded with: $\int_0^1 \int_0^2 \sigma dy\, dx + \int_1^{\sqrt{3}} \int_{3-x^2}^{0} \sigma dy\, dx$
This student is in the action stage as shows an understanding of the limits of boundary curves however there is a swapping around of upper and lower limits.

One student responded with: $\int_0^2 \int_0^3 \sigma dy\, dx$
As with the similar $dxdy$ construction this represents a failure of prerequisite knowledge.

One student responded with: $\int_0^{\sqrt{3}} \int_{3-x^2}^{2} \sigma dy\, dx$
This student has a vague grasp of how to set up integrals but seems to lack geometric understanding of what the limits represent. This represents a failure of prerequisite knowledge.

One student responded with: $\int_0^{\sqrt{3-y}} \int_0^2 \sigma dy\, dx$
Here there is merely a swapping of the limits from the $dxdy$ form. Fundamental knowledge when setting up double integrals requires firstly to focus on the inner integral and to consider the limits of that integral that are boundary curves defined as functions of x. It is worrisome that the limits on the integral on the outer integral are functions of y and therefore not constant making the integral meaningless.

One student responded with: $\int_{-\sqrt{3}}^{\sqrt{3}} \int_0^2 \sigma dy\, dx$
This is a reflection of student thinking that intersections and roots of functions are often limits of integration. Here student has solved for roots of $y = 3 - x^2$ and used as limits. There is a failure to relate these roots to the diagram and hence to the limits of integration.

4. SUMMARY AND FUTURE RESEARCH

Although this is a pilot study and the findings cannot be generalised the APOS theory provided a valuable exploration of the learning of the limits of integration of double integrals in a vector calculus class. We note that the part of the question that required the action level of understanding, setting up a double integral with limits of integration over a type 1 region or a type 2 region, was well within the capabilities of the majority of the students. However, the responses to the part of the question requiring an object level of understanding of limits of a double integral were problematic for a number of students. Although a graphical representation was given students found difficulty in identifying the region of integration. The majority of those students who had difficulty merely followed steps without acknowledging visually the shape of the region which warranted that it be represented as a sum of two double integrals. This suggests that the students' engagement of the concept of a double integral and how it refers to the region is not well grounded.

In partially answering our research question at this stage of analysis in the research project, we observe that some students find difficulty with recognising the region of integration if it is other than rectangular, some find difficulty when dealing with integration first with respect to y and there is no understanding that the limits of integration of the inner integral are functions of x and similarly for the integration with respect to x first, and others lack a geometric understanding of what the limits of integration represent.

Important insights have been gained from students' mental constructions of limits of integration for double integrals which the next cycle of teaching will focus on. Reflecting on teaching this concept, more consideration will be given to graphical representation of regions and how the choice of integration in one direction first and then the other depends on the region of integration and hence implies the choice of limits of integration.

This pilot study has not only provided valuable insights on students' mental constructions of double integrals but has also illustrated the potential of the APOS framework to be used in future research and to influence teaching and learning of vector calculus concepts. This aligns with our intention to present implications for the teaching and learning of Vector Calculus concepts that promote deeper conceptual understanding.

The next stage of this research would involve exploring student understanding of the limits of integration of double integrals with a larger population and, as initially intended, the comparison with the second research cohort including probing of this understanding using interviews. We suggest that future research could study the impact of such teaching of double integrals on students' learning.

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