

A Polyhedral Study for the Cubic Formulation of the Unconstrained Traveling Tournament Problem

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Abstract

We consider the unconstrained traveling tournament problem, a sports timetabling problem that minimizes traveling of teams. Since its introduction about 20 years ago, most research was devoted to modeling and reformulation approaches. In this paper we carry out a polyhedral study for the cubic integer programming formulation by establishing the dimension of the integer hull as well as of faces induced by model inequalities. Moreover, we introduce a new class of inequalities and show that they are facet-defining. Finally, we evaluate the impact of these inequalities on the linear programming bounds.

1 Introduction

The *traveling tournament problem* is an optimization problem that involves aspects from tournament timetabling as well as from tour problems such as the traveling salesman problem. It was introduced by Easton, Nemhauser and Trick in 2001 [4]. To formally state the problem, we consider an even number $n \geq 4$ of sports teams, each playing at its own venue, and the problem of designing a *double round-robin tournament*. Such a tournament consists of *slots* $S := \{1, 2, \dots, 2n - 2\}$ and in each slot, each team $i \in V := \{1, 2, \dots, n\}$ plays against another team $j \in V$, either at its home venue or away, i.e., at j 's home venue. Moreover, every two teams $i, j \in V$ play each other exactly twice, once at i and once at j . Finally, distances $d_{i,j}$ between the venues $i, j \in V$ (with $i \neq j$) are given and the goal is to find a tournament with the minimum total traveling distance. Between two consecutive slots in which a team plays at different venues j and k , it travels $d_{j,k}$ units. In particular, if both matches are played away, then it directly travels from venue j to venue k . Before slot 1 and after slot $2n - 2$ each team shall reside at its home venue, i.e., if the first or last match is played away, then the team has to travel between this venue and its home venue. This problem is known as the *unconstrained traveling tournament problem (TTP)*, which is known to be NP-hard [2].

There exist several variants, including the *classic TTP*. Here, the unconstrained TTP is further restricted by requiring that the two matches of teams i and j shall not be in consecutive slots. Moreover, no team shall play more than 3 consecutive home matches and no more than 3 consecutive away matches. Also this variant is NP-hard [14].

The first solution approaches were developed in [5], where a column generation framework was combined with constraint programming techniques. The authors of [12] discuss several integer programming formulations in their paper on a single-round-robin variant of the TTP. In particular,

they describe a cubic formulation (with $\mathcal{O}(n^3)$ variables) that naturally generalizes to one for the unconstrained TTP.

Already the tournament construction without a traveling aspect is nontrivial. While there exist several efficient methods to construct a feasible solution (see [10, 3, 13, 8]), the addition of more constraints or an objective function often makes the problem intractable. For instance, the optimization version, called the *planar 3-index assignment problem*, is NP-hard [7]. However, there exist several polyhedral studies in which the integer hull of the natural integer programming formulation of the planar 3-index assignment problem was investigated [6, 1, 11].

Outline. In Section 2 we introduce the cubic integer programming formulation in order to define the unconstrained traveling tournament polytope as its integer hull. In Section 3 we deal with equations valid for the polytope and establish its dimension. Moreover, in Section 4 we show that some of the model inequalities are facet-defining while others are lifted to have this property. Finally, in Section 5 we introduce a new class of inequalities and show that they are facet-defining. For the proofs in Sections 3 to 5 we need to construct tournaments with a variety of properties. These constructions can be found in Appendix A. The paper is concluded in Section 6 where we evaluate the impact of our findings on the dual linear programming bounds.

2 The unconstrained traveling tournament polytope

We denote the set of teams and venues by $V := \{1, 2, \dots, n\}$. For the set of traveling arcs $A := \{(i, j) \in V \times V : i \neq j\}$, we are given distances $d_{i,j} \in \mathbb{R}_{\geq 0}$. In a double round-robin tournament with n teams, each team plays $n - 1$ matches at home and $n - 1$ matches away, and hence we consider *slots* $S := \{1, 2, \dots, 2n - 2\}$. A *match* between i and j at venue i that is played in slot $k \in S$ is denoted by the triple (k, i, j) and by \mathcal{M} we denote the set of all possible matches. The formulation has *play variables* $x_m \in \{0, 1\}$ for each match $m \in \mathcal{M}$ and *travel variables* $y_{t,i,j} \in \{0, 1\}$ for all $t \in V$ and all $(i, j) \in A$. The interpretation is that $x_{k,i,j} = 1$ if and only if match (k, i, j) is played, and $y_{t,i,j} = 1$ if (but not only if) team t travels from venue i to venue j . Note that in a tournament each team travels along such an arc at most once. The formulation reads

$$\min \sum_{(i,j) \in A} d_{i,j} \sum_{t \in V} y_{t,i,j} \quad (1a)$$

$$\text{s.t.} \quad \sum_{j \in V \setminus \{i\}} (x_{k,i,j} + x_{k,j,i}) = 1 \quad \forall k \in S : k \geq 2, \forall i \in V, \quad (1b)$$

$$\sum_{k \in S} x_{k,i,j} = 1 \quad \forall (i, j) \in A, \quad (1c)$$

$$x_{k,i,t} + x_{k+1,j,t} - 1 \leq y_{t,i,j} \quad \forall k \in S \setminus \{2n - 2\}, \forall (i, j) \in A, \forall t \in V \setminus \{i, j\}, \quad (1d)$$

$$\sum_{i \in V \setminus \{t\}} x_{k,t,i} + x_{k+1,j,t} - 1 \leq y_{t,t,j} \quad \forall k \in S \setminus \{2n - 2\}, \forall (t, j) \in A, \quad (1e)$$

$$x_{k-1,i,t} + \sum_{j \in V \setminus \{t\}} x_{k,t,j} - 1 \leq y_{t,i,t} \quad \forall k \in S \setminus \{1\}, \forall (i, t) \in A, \quad (1f)$$

$$x_{1,j,t} \leq y_{t,t,j} \quad \forall (t, j) \in A, \quad (1g)$$

$$x_{2n-2,i,t} \leq y_{t,i,t} \quad \forall (i, t) \in A, \quad (1h)$$

$$x_{k,i,j} \in \{0, 1\} \quad \forall (k, i, j) \in \mathcal{M}, \quad (1i)$$

$$y_{t,i,j} \in \{0, 1\} \quad \forall (t, i, j) \in V \times A. \quad (1j)$$

The objective (1a) minimizes the total traveled distance. Constraints (1b) ensure that each team plays exactly once (either home or away) in each slot $k \geq 2$. For $k = 1$, the same equations are implied (see Proposition 1). Constraints (1c) ensure that each home-away pair occurs exactly once. This constitutes a correct model for a double round-robin schedule with binary variables x . Note that the *classic* traveling tournament instances also require custom constraints such as a no-repeater constraint (requiring that the two matches of two teams are not scheduled in a row) and upper bounds on the number of consecutive home/away games. However, for our polyhedral study we omit these constraints to keep the model simple. The remaining constraints (1d)–(1h) force the travel variables to be 1 if the corresponding travel occurs.

To carry out a polyhedral study, it is worth to define the integer hull of IP (1). To this end, we define a *tournament* as a subset $T \subseteq \mathcal{M}$ of matches whose *play vector* $\chi(T) \in \{0, 1\}^{\mathcal{M}}$, defined via $\chi(T)_{k,i,j} = 1 \iff (k, i, j) \in T$, satisfies (1b) and (1c). Its *travel vector* is the vector $\psi(T) \in \{0, 1\}^{V \times A}$ with $\psi(T)_{t,i,j} = 1$ if and only if team t travels from venue i to venue j . In the IP, a travel variable $y_{t,i,j}$ can be set to 1 although team t does not travel from i to j . If the distances $d_{i,j}$ are positive, this will however never happen in an optimal solution. The integer hull of the IP, which we call the *unconstrained traveling tournament polytope*, is thus equal to

$$P_{\text{utt}}(n) := \text{conv}\{(\chi(T), y) \in \{0, 1\}^{\mathcal{M}} \times \{0, 1\}^{V \times A} : T \text{ tournament and } y \geq \psi(T)\}.$$

Finally, by \odot we denote the zero vector, where its length can be derived from the context.

3 Equations and dimension

3.1 Known equations

Proposition 1. *For each team $t \in V$, equations (1b) for $(k, i) = (1, t)$ follow from equations (1b) for all $k \in S \setminus \{1\}$ and $i = t$ together with equations (1c) for all $(i, j) \in A$ with $t \in \{i, j\}$.*

Proof. Let $t \in V$. The sum of equations (1c) for all $(i, j) \in A$ with $j = t$ plus the sum of equations (1c) for all $(i, j) \in A$ with $i = t$ minus the sum of equations (1b) for all $k \in S \setminus \{1\}$ and $i = t$ yields

$$\begin{aligned} \sum_{i \in V \setminus \{t\}} \sum_{k \in S} x_{k,i,t} + \sum_{j \in V \setminus \{t\}} \sum_{k \in S} x_{k,t,j} - \sum_{k \in S \setminus \{1\}} \sum_{j \in V \setminus \{t\}} (x_{k,t,j} + x_{k,j,t}) \\ = (n-1) + (n-1) - (2n-3) \iff \sum_{j \in V \setminus \{t\}} (x_{1,t,j} + x_{1,j,t}) = 1, \end{aligned}$$

which is equation (1b) for $(k, i) = (1, t)$. □

We define the following *column basis* $\mathcal{B}_{\bar{k}} \subseteq \mathcal{M}$ via

$$\mathcal{B}_{\bar{k}} := \{(k, i, j) \in \mathcal{M} : k = \bar{k} \text{ or } i = 1 \text{ or } (i, j) = (2, 3)\}. \quad (2)$$

We will often use the following lemma which states that the play variables indexed by $\mathcal{B}_{\bar{k}}$ induce an invertible submatrix of the equation system of interest.

Lemma 2. *Let $\bar{k} \in S$ and let $Cx = d$ be the system defined by equations (1b) and (1c). Then the submatrix of C induced by variables x_m for $m \in \mathcal{B}_{\bar{k}}$ is invertible. In particular, these $|\mathcal{B}_{\bar{k}}| = 3n^2 - 4n$ equations are irredundant.*

Proof. Observe that variables $x_{\bar{k},i,j}$ only appear in equation (1c) for $(i,j) \in A$. Thus, by Laplace expansion it remains to prove invertibility of the coefficient submatrix C' of C whose rows correspond to equations (1b) and whose columns correspond to variables $x_{k,i,j}$ for $(k,i,j) \in \mathcal{M}$ with $k \neq \bar{k}$ and $i = 1$ or $(i,j) = (2,3)$.

The matrix C' is a block diagonal matrix. The blocks are the submatrices C^k whose rows and columns are the same as those of C' but for fixed k . For the remainder of the proof we fix $k \in S \setminus \{\bar{k}\}$ and prove that C^k is invertible. For $\ell \in \{3,4,\dots,n\}$, consider the submatrices $C^{k,\ell} \in \mathbb{R}^{\ell \times \ell}$ of C induced by equations (1b) for k and for $i = 1,2,\dots,\ell$ and by variables $x_{k,2,3}, x_{k,1,2}, x_{k,1,3}, x_{k,1,4}, \dots, x_{k,1,\ell}$. One easily verifies that $C^{k,3}$ is invertible and that for $\ell \geq 4$, $C^{k,\ell}$ is obtained from $C^{k,\ell-1}$ by adding a unit row with the one in the added column. By induction on ℓ , Laplace expansion shows that $C^{k,n}$ is invertible. The fact that $C^{k,n} = C^k$ holds, concludes the proof. \square

A consequence of Lemma 2 is that every equation that is valid for $P_{\text{utt}}(n)$ or some of its faces can be turned into an equivalent one that involves no x_m for $m \in \mathcal{B}_{\bar{k}}$. Hence, in many subsequent proofs we will assume that such an equation $a^\top x + b^\top y = \gamma$ satisfies

$$a_m = 0 \text{ for each } m \in \mathcal{B} := \mathcal{B}_1. \quad (\mathcal{B})$$

3.2 Tournaments from 1-factors

We consider the tournament construction based on perfect matchings (also called 1-factors) of the complete graphs on n nodes (see [3]). In each tournament T , for each $k \in S$, the matches $(k,i,j) \in T$ in slot k , interpreted as edges $\{i,j\}$, form a perfect matching. Thus, each tournament is characterized by $|S|$ such perfect matchings whose edges are oriented so that no oriented edge $(i,j) \in A$ appears twice. Since the latter is the only restriction, we can first determine the $|S|$ perfect matchings M_k for all $k \in S$ and afterwards orient their edges in a complementary fashion, that is,

each edge $\{i,j\}$ is oriented differently in the two perfect matchings in which it is contained. (3)

We call such an orientation *complementary*. The following *canonical factorization* is one specific set $\{M_1, M_2, \dots, M_{2n-2}\}$ of perfect matchings [3], where M_k for $k < n$ is determined by

$$M_k := \{\{k,n\}\} \cup \{\{k+i, k-i\} : i = 1, 2, \dots, n/2 - 1\},$$

where $k+i$ and $k-i$ are taken modulo $n-1$ as one of the numbers $1, 2, \dots, n-1$. The remaining perfect matchings are $M_k := M_{k-n+1}$ for all $k \in \{n, n+1, \dots, 2n-2\}$. Hence,

for each edge $\{i,j\}$ there is a unique $k \in \{1, 2, \dots, n-1\}$ with $\{i,j\} \in M_k$ and $\{i,j\} \in M_{k+n-1}$ a unique $k' \in \{n, n+1, \dots, 2n-2\}$ with $\{i,j\} \in M_{k'}$ (which satisfies $k' = k + n - 1$). (4)

We will often construct tournaments obtained from the canonical factorizations by permuting slots or teams. In many cases, it is easy to see that corresponding permutations exist. Hence, we typically state that a tournament is constructed from a canonical factorization such that certain requirements are satisfied, e.g., by specifying certain matches that shall be played.

Operations on tournaments. Three fundamental operations to modify a given tournament are the *cyclic shift*, the *home-away swap* and the *partial slot swap*, defined as follows.

Let $s \in \mathbb{Z}$ and let T be a tournament. We say that tournament T' is obtained by a *cyclic shift by s* if T' arises from T by mapping each slot $k \in S$ to slot $k+s$, where slots are considered modulo $2n-2$ in the range $1, 2, \dots, 2n-2$.

Proposition 3 (Home-away swap). *Let T be a tournament with matches $(k_1, i, j), (k_2, j, i) \in T$. Then*

$$T' := T \setminus \{(k_1, i, j), (k_2, j, i)\} \cup \{(k_1, j, i), (k_2, i, j)\} \quad (\text{HA}_{k_1, k_2, i, j})$$

is also a tournament.

Proposition 4 (Partial slot swap). *Let T be a tournament with matches $(k_1, i, j), (k_1, i', j'), (k_2, i, j'), (k_2, i', j) \in T$. Then*

$$T' := T \setminus \{(k_1, i, j), (k_1, i', j'), (k_2, i, j'), (k_2, i', j)\} \\ \cup \{(k_1, i, j'), (k_1, i', j), (k_2, i, j), (k_2, i', j')\} \quad (\text{PS}_{k_1, k_2, i, j, i', j'})$$

is also a tournament.

3.3 Dimension of the unconstrained traveling tournament polytope

Theorem 5. *The affine hull of $P_{\text{utt}}(n)$ is described completely by the irredundant equations (1b) and (1c).*

Proof. We first prove that the equations are valid for $P_{\text{utt}}(n)$. To this end, consider a tournament T . Since in each round each team plays exactly once, equation (1b) is satisfied by $(\chi(T), \psi(T))$. The vector also satisfies equation (1c) since each pair of teams plays against each other once at each of the two venues. Irredundancy of the equations follows from Lemma 2.

We now prove that every valid equation is a linear combination of these equations. To this end, we show that for any equation $a^\top x + b^\top y = \gamma$ valid for $P_{\text{utt}}(n)$ and satisfying (B) that $(a, b) = \mathbb{O}$ holds.

Claim 5.1. *For each $(t, i, j) \in V \times A$ there exists a tournament in which team t never travels from venue i to venue j .*

A tournament T from Claim 5.1 satisfies $\psi(T)_{t, i, j} = 0$. Let $y := \psi(T)$ and let y' be equal to y except for $y'_{t, i, j} = 1$. Hence, $(\chi(T), y), (\chi(T), y') \in P_{\text{utt}}(n)$ and thus $a^\top \chi(T) + b^\top y = \gamma = a^\top \chi(T) + b^\top y'$ holds. We obtain

$$b = \mathbb{O}. \quad (\S 5.1)$$

Claim 5.2. *For each $k \in S \setminus \{1\}$ and for distinct $i, j \in V$ there exist tournaments T and T' satisfying $(\text{HA}_{1, k, i, j})$.*

For the tournaments T and T' from Claim 5.2 we have $b^\top \psi(T) = b^\top \psi(T')$ due to (§5.1). Using (B), $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to

$$a_{k, i, j} = a_{k, j, i} \text{ for each } k \in S \setminus \{1\} \text{ and for all distinct } i, j \in V. \quad (\S 5.2)$$

Claim 5.3. *For each $k \in S \setminus \{1\}$ and for distinct $i, j, i', j' \in V$ there exist tournaments T and T' satisfying $(\text{PS}_{1, k, i, j, i', j'})$.*

For the tournaments T and T' from Claim 5.3 we have $b^\top \psi(T) = b^\top \psi(T')$ due to (§5.1). Using (B), $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to

$$a_{k, i, j} + a_{k, i', j'} = a_{k, i, j'} + a_{k, i', j} \text{ for each } k \in S \setminus \{1\} \text{ and for all distinct } i, j, i', j' \in V. \quad (\S 5.3)$$

Consider a slot $k \in S \setminus \{1\}$. For each $\ell \in \{4, 5, \dots, n\}$, (§5.3) implies $a_{k, 1, \ell} + a_{k, 2, 3} = a_{k, 1, 3} + a_{k, 2, \ell}$ which together with (B) yields $a_{k, 2, \ell} = 0$. Combined with (§5.2) we also obtain $a_{k, \ell, 2} = 0$. For all distinct $\ell, \ell' \in \{3, 4, \dots, n\}$, (§5.3) implies $a_{k, 1, \ell'} + a_{k, \ell, 2} = a_{k, 1, 2} + a_{k, \ell, \ell'}$. Together with (B), this shows $a_{k, \ell, \ell'} = 0$. Hence, $a = \mathbb{O}$ holds, which concludes the proof. \square

Corollary 6. *The dimension of $P_{\text{utt}}(n)$ is equal to $3n^3 - 8n^2 + 6n$.*

Proof. The ambient space of $P_{\text{utt}}(n)$ has dimension $|\mathcal{M}| + n \cdot |A|$. By Theorem 5, the affine hull is described by the $3n^2 - 4n$ equations (1b) and (1c), which are irredundant by Lemma 2. Hence,

$$\dim(P_{\text{utt}}(n)) = (2n - 2) \cdot n \cdot (n - 1) + n \cdot n \cdot (n - 1) - (3n^2 - 4n) = 3n^3 - 8n^2 + 6n.$$

This concludes the proof. \square

4 Model inequalities

We consider the inequalities from (1) and determine when they are facet-defining. Within the proofs we will sometimes argue about symmetry of the formulation, for which we state the following lemma.

Lemma 7. *$P_{\text{utt}}(n)$ and formulation (1) are symmetric with respect to permuting teams and with respect to mirroring all slots, i.e., exchanging roles of slots k and $2n - 1 - k$ for all $k \in \{1, 2, \dots, n - 1\}$.*

Proof. Symmetry with respect to team permutations is clear for $P_{\text{utt}}(n)$ and for the formulation.

Moreover, symmetry with respect to mirroring slots is easy to see for $P_{\text{utt}}(n)$: when slots are exchanged, all traveled arcs are simply reversed. For the formulation, the roles of (1e) and (1f) as well as (1g) and (1h) are exchanged. \square

We start with the nonnegativity constraints for the play variables.

Theorem 8. *Inequalities $x_{k,i,j} \geq 0$ are facet-defining for $P_{\text{utt}}(n)$ for all $(k, i, j) \in \mathcal{M}$.*

Proof. Consider the inequality $x_{k^*, i^*, j^*} \geq 0$ for some match $m^* = (k^*, i^*, j^*) \in \mathcal{M}$. By Lemma 7, we can assume $k^* \geq n$ and $i^* = 3$ and $j^* = 4$. This implies $m^* \notin \mathcal{B}$. Let $a^\top x + b^\top y \geq \gamma$ define any facet F that contains the face induced by this inequality. Without loss of generality, a satisfies (B). It remains to prove that $b = \mathbb{0}$ and $\gamma = 0$ hold and that a is a multiple of $\chi(\{(k^*, i^*, j^*)\})$.

Claim 8.1. *For each $(t, i, j) \in V \times A$ there exists a tournament T with $m^* \notin T$ and in which team t never travels from venue i to venue j .*

A tournament T from Claim 8.1 satisfies $\psi(T)_{t,i,j} = 0$. Let $y := \psi(T)$ and let y' be equal to y except for $y'_{t,i,j} = 1$. Since $\chi(T)_{m^*} = 0$ holds, we have $(\chi(T), y), (\chi(T), y') \in F$. The equation $a^\top \chi(T) + b^\top y = \gamma = a^\top \chi(T) + b^\top y'$ simplifies to $b_{t,i,j} = 0$. We obtain

$$b = \mathbb{0}. \tag{\S 8.1}$$

Claim 8.2. *For each $(k, i, j) \in \mathcal{M}$ with $k \geq 2$ and $(k, i, j) \neq (k^*, i^*, j^*), (k^*, j^*, i^*)$ there exist tournaments T and T' satisfying (HA_{1,k,i,j}) and $(k^*, i^*, j^*), (k^*, j^*, i^*) \notin T \cup T'$.*

The tournaments T and T' from Claim 8.2 satisfy $\chi(T)_{m^*} = \chi(T')_{m^*} = 0$ and thus we have $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$. Using (B) and (8.1), $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to

$$a_{k,i,j} = a_{k,j,i} \text{ for each } (k, i, j) \in \mathcal{M} \text{ with } k \geq 2 \text{ and } (k, i, j) \notin \{(k^*, i^*, j^*), (k^*, j^*, i^*)\}. \tag{\S 8.2}$$

Claim 8.3. *For each slot $k \in S \setminus \{1\}$ and for distinct $i, j, i', j' \in V$ with $k \neq k^*$ or $(i^*, j^*) \notin \{(i, j), (i', j'), (i', j), (i, j')\}$ there exist tournaments T and T' satisfying (PS_{1,k,i,j,i',j'}) and $m^* \notin T \cup T'$.*

The tournaments T and T' from Claim 8.3 satisfy $\chi(T)_{m^*} = \chi(T')_{m^*} = 0$ and thus we have $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$. Using (B) and (8.1), $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to

$$a_{k,i,j} + a_{k,i',j'} = a_{k,i,j'} + a_{k,i',j} \text{ for each } k \in S \setminus \{1\} \text{ and for all distinct } (i, j, i', j') \in V \\ \text{with } k \neq k^* \text{ or } (i^*, j^*) \notin \{(i, j), (i', j'), (i, j'), (i', j)\}. \quad (\S 8.3)$$

Consider a slot $k \in S \setminus \{1\}$. For each $\ell \in \{4, 5, \dots, n\}$, (§8.3) implies $a_{k,1,\ell} + a_{k,2,3} = a_{k,1,3} + a_{k,2,\ell}$ which together with (B) yields $a_{k,2,\ell} = 0$. Combined with (§8.2) we also obtain $a_{k,\ell,2} = 0$. For all distinct $\ell, \ell' \in \{3, 4, \dots, n\}$ except for $(\ell, \ell') = (4, 3)$, (§8.3) implies $a_{k,1,\ell'} + a_{k,\ell,2} = a_{k,1,2} + a_{k,\ell,\ell'}$. Together with (B), this shows $a_{k,\ell,\ell'} = 0$ for all but the entry corresponding to match (k^*, i^*, j^*) .

Hence, the inequality reads $a_{k^*, i^*, j^*} \cdot x_{k^*, i^*, j^*} \geq \gamma$. Since $\chi(T)_{k^*, i^*, j^*} = 0$ holds for each of the considered tournaments T , we obtain $\gamma = 0$. Finally, since there exist tournaments T for which $\chi(T)_{k^*, i^*, j^*} = 1$ holds, a_{k^*, i^*, j^*} must be positive, which concludes the proof. \square

We continue with inequalities (1d) which are not facet-defining. However, they can be lifted to these two stronger ones.

$$x_{k,j,t} + x_{k,i,t} + x_{k+1,j,t} - 1 \leq y_{t,i,j} \quad \forall k \in S \setminus \{2n-2\}, \forall (i, j) \in A, \forall t \in V \setminus \{i, j\} \quad (5a)$$

$$x_{k+1,i,t} + x_{k,i,t} + x_{k+1,j,t} - 1 \leq y_{t,i,j} \quad \forall k \in S \setminus \{2n-2\}, \forall (i, j) \in A, \forall t \in V \setminus \{i, j\} \quad (5b)$$

Indeed, in order to obtain (1d) they only need to be combined with nonnegativity constraints for x . These inequalities turn out to be facet-defining.

Theorem 9. *Inequalities (5) are facet-defining for $P_{\text{utt}}(n)$ for each slot $k \in S \setminus \{2n-2\}$ and all distinct teams $i, j, t \in V$.*

Proof. We only prove the statement for inequalities (5a) since the proof for (5b) is similar. Moreover, we assume $n \geq 6$ since we verified the statement for $n = 4$ computationally. For this, we used the software package IPO [15], which can exactly compute dimensions of polyhedra that are defined implicitly via an optimization oracle, in this case a MIP solver.

Consider the inequality $x_{k^*, j^*, t^*} + x_{k^*, i^*, t^*} + x_{k^*+1, j^*, t^*} - y_{t^*, i^*, j^*} \leq 1$ for some slot $k^* \in S \setminus \{2n-2\}$, and distinct teams $i^*, j^*, t^* \in V$. By Lemma 7, we can assume $k^* \geq n$, $i^* = 4$, $j^* = 5$ and $t^* = 6$. The inequality is valid for $P_{\text{utt}}(n)$ since the only possibility of scheduling more than one of the three matches (k^*, j^*, t^*) , (k^*, i^*, t^*) and (k^*+1, j^*, t^*) consists of the latter two which implies that team t^* travels from venue i^* to venue j^* . The following claim is used several times throughout the proof.

Claim 9.1. *Let T be a tournament that contains*

- (a) *match (k^*, i^*, t^*) and in which team t^* plays away in slot $k^* + 1$, or*
- (b) *one of the matches (k^*, j^*, t^*) , (k^*, i^*, t^*) or (k^*+1, j^*, t^*) , and in which team t^* never travels from venue i^* to venue j^* .*

Then $(\chi(T), \psi(T))$ satisfies (5a) with equality.

In order to prove that the inequality is facet-defining, let $a^\top x + b^\top y \leq \gamma$ define any facet F that contains the face induced by this inequality. We will prove that it is a multiple of inequality (5a). Without loss of generality, we assume that a satisfies (B).

Claim 9.2. For all $(t, i, j) \in V \times A$ with $(t, i, j) \neq (t^*, i^*, j^*)$ there exists a tournament T in which team t never travels from venue i to venue j and which satisfies condition (a) of Claim 9.1.

A tournament T from Claim 9.2 satisfies $\psi(T)_{t,i,j} = 0$. Let $y := \psi(T)$ and let y' be equal to y except for $y'_{t,i,j} = 1$. By Claim 9.1 we have $(\chi(T), y), (\chi(T), y') \in F$. In this case, $a^\top \chi(T) + b^\top y = \gamma = a^\top \chi(T) + b^\top y'$ simplifies to

$$b_{t,i,j} = 0 \text{ for all } (t, i, j) \in V \times A \text{ with } (t, i, j) \neq (t^*, i^*, j^*). \quad (\S 9.2)$$

Claim 9.3. For each $(k, i, j) \in \mathcal{M} \setminus \{(k^*, i^*, t^*), (k^*, t^*, i^*), (k^*, j^*, t^*), (k^*, t^*, j^*), (k^* + 1, j^*, t^*), (k^* + 1, t^*, j^*)\}$ with $k \geq 2$ there exist tournaments T and T' satisfying (HA_{1,k,i,j}) and condition (b) of Claim 9.1.

The tournaments T and T' from Claim 9.3 satisfy $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$ by Claim 9.1. Using (B) and (9.2), $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to

$$a_{k,i,j} = a_{k,j,i} \text{ for each } (k, i, j) \in \mathcal{M} \setminus \{(k^*, i^*, t^*), (k^*, t^*, i^*), (k^*, j^*, t^*), (k^*, t^*, j^*), (k^* + 1, j^*, t^*), (k^* + 1, t^*, j^*)\}. \quad (\S 9.3)$$

Claim 9.4. Let $k \in S \setminus \{1\}$, let $i, j, i', j' \in V$ be distinct and let $P := \{(i, j), (i', j'), (i, j'), (i', j)\}$. If

- (i) $(i^*, t^*) \notin P$ and $(j^*, t^*) \notin P$, or
- (ii) $(i^*, t^*) \notin P$, $(j^*, t^*) \in P$ and $k \notin \{k^*, k^* + 1\}$, or
- (iii) $(i^*, t^*) \in P$, $(j^*, t^*) \notin P$ and $k \neq k^*$, or
- (iv) $(i^*, t^*) \in P$, $(j^*, t^*) \in P$ and $k = k^*$

holds, then there exist tournaments T and T' satisfying (PS_{1,k,i,j,i',j'}) and condition (b) of Claim 9.1.

The tournaments T and T' from Claim 9.4 satisfy $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$ by Claim 9.1. Using (B) and (9.2), $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to

$$a_{k,i,j} + a_{k,i',j'} = a_{k,i,j'} + a_{k,i',j} \text{ for all } (k, i, j, i', j') \text{ satisfying the conditions in Claim 9.4.} \quad (9.4a)$$

Consider a slot $k \in S \setminus \{1\}$. For each $\ell \in \{4, 5, \dots, n\}$, (9.4a) for $(i, j, i', j') = (1, \ell, 2, 3)$ is applicable since condition (i) of Claim 9.4 is satisfied due to $\{i^*, j^*\} \cap \{1, 2\} = \emptyset$. This implies $a_{k,1,\ell} + a_{k,2,3} = a_{k,1,3} + a_{k,2,\ell}$ which together with (B) yields $a_{k,2,\ell} = 0$. Moreover, for each $\ell \in \{3, 4, \dots, n\}$, (§9.3) for $(i, j) = (\ell, 2)$ implies $a_{k,\ell,2} = a_{k,2,\ell} = 0$.

For distinct $\ell, \ell' \in \{3, 4, \dots, n\}$ with $(k, \ell, \ell') \notin \{(k^*, t^*, i^*), (k^*, t^*, j^*), (k^* + 1, t^*, j^*)\}$, (9.4a) for $(i, j, i', j') = (1, \ell', \ell, 2)$ is applicable, which implies $a_{k,1,\ell'} + a_{k,\ell,2} = a_{k,1,2} + a_{k,\ell,\ell'}$. Together with (B) this shows

$$a_{k,i,j} = 0 \text{ for all } (k, i, j) \in \mathcal{M} \setminus \{(k^*, i^*, t^*), (k^*, j^*, t^*), (k^* + 1, j^*, t^*)\}. \quad (9.4b)$$

Since for each of the matches $(k^*, i^*, t^*), (k^*, j^*, t^*), (k^* + 1, j^*, t^*)$ there exists a tournament containing exactly this match and in which team t^* never travels from venue i^* to venue j^* , and since there exists a tournament satisfying condition (a) of Claim 9.1, we obtain

$$\gamma = a_{k^*, i^*, t^*} = a_{k^*, j^*, t^*} = a_{k^*+1, j^*, t^*} = \gamma = a_{k^*, i^*, t^*} + a_{k^*, j^*, t^*} - b_{t^*, i^*, j^*}.$$

This shows that $a^\top x + b^\top y \leq \gamma$ is a positive multiple of inequality (5a), which concludes the proof. \square

Similar to (1d), inequalities (1e) are not facet-defining. A lifted inequality reads

$$x_{1,j,t} + x_{k,j,t} + \sum_{i \in V \setminus \{t\}} x_{k,t,i} + x_{k+1,j,t} - 1 \leq y_{t,t,j} \quad \forall k \in S \setminus \{2n-2\}, \forall (t,j) \in A \quad (6)$$

Indeed, in order to obtain (1e) one only needs to combine (6) with nonnegativity constraints for x . The lifted inequalities turn out to be facet-defining.

Theorem 10. *Inequalities (6) are facet-defining for $P_{\text{utt}}(n)$ for all $k \in S \setminus \{2n-2\}$ and $(t,j) \in A$.*

Proof. We assume $n \geq 6$ since we verified the statement for $n = 4$ computationally [15]. Consider the inequality $x_{1,j^*,t^*} + x_{k^*,j^*,t^*} + \sum_{i \in V \setminus \{t^*\}} x_{k^*,t^*,i} + x_{k^*+1,j^*,t^*} - y_{t^*,t^*,j^*} \leq 1$ for some slot $k^* \in S \setminus \{2n-2\}$ and distinct teams $t^*, j^* \in V$. By Lemma 7, we can assume $j^* = 3$ and $t^* = 4$. The inequality is valid for $P_{\text{utt}}(n)$ since the only possibilities in which $x_{1,j^*,t^*} + x_{k^*,j^*,t^*} + \sum_{i \in V \setminus \{t^*\}} x_{k^*,t^*,i} + x_{k^*+1,j^*,t^*}$ exceeds 1 are for $k^* = 1$ (since then $(1, j^*, t^*)$ and (k^*, j^*, t^*) are identical) or if team t^* plays at home in slot k^* and away against team j^* in slot 1 or $k^* + 1$. In either case, team t^* travels from its home venue to j^* , forcing $y_{t^*,t^*,j^*} = 1$.

The following claim is used several times throughout the proof.

Claim 10.1. *Let T be a tournament with*

- (a) $(1, j^*, t^*) \in T$ and $k^* = 1$ holds, or
- (b) $(1, j^*, t^*) \in T$ and team t^* plays at home in slot k^* , or
- (c) $(k^* + 1, j^*, t^*) \in T$ and team t^* plays at home in slot k^* , or
- (d) $(k^* + 1, j^*, t^*) \in T$ and team t^* plays away in slot k^* , or
- (e) $(k^*, j^*, t^*) \in T$, $k^* \geq 2$ and team t^* plays away in slot $k^* - 1$, or
- (f) team t^* plays at home in slot k^* and never travels from its home venue to venue j^* .

Then $(\chi(T), \psi(T))$ satisfies (6) with equality. Moreover, team t^* travels from its home venue to venue j^* if and only if one of conditions (a)–(c) is satisfied.

In order to prove that the inequality is facet-defining, let $a^\top x + b^\top y \leq \gamma$ define any facet F that contains the face induced by this inequality. We will prove that it is a multiple of inequality (6). Let $\bar{k} \in S \setminus \{1, k^*, k^* + 1\}$. By Lemma 2 we can assume that a satisfies

$$a_m = 0 \text{ for each } m \in \mathcal{B}_{\bar{k}}. \quad (\S 10.1)$$

Claim 10.2. *For all $(t, i, j) \in V \times A$ with $(t, i, j) \neq (t^*, t^*, j^*)$ there exists a tournament T satisfying a condition from Claim 10.1.*

A tournament T from Claim 10.2 satisfies $\psi(T)_{t,i,j} = 0$. Let $y := \psi(T)$ and let y' be equal to y except for $y'_{t,i,j} = 1$. By Claim 10.1 we have $(\chi(T), y), (\chi(T), y') \in F$. In this case, $a^\top \chi(T) + b^\top y = \gamma = a^\top \chi(T) + b^\top y'$ simplifies to

$$b_{t,i,j} = 0 \text{ for all } (t, i, j) \in V \times A \text{ with } (t, i, j) \neq (t^*, t^*, j^*). \quad (\S 10.2)$$

Claim 10.3. *For each $(k, i, j) \in \mathcal{M}$ with $k \neq \bar{k}$, $\{i, j\} \neq \{j^*, t^*\}$ and for which $k = k^*$ implies $t^* \notin \{i, j\}$ there exist tournaments T and T' satisfying $(HA_{\bar{k}, k, i, j})$ such that T and T' satisfy the same condition from Claim 10.1.*

The tournaments T and T' from Claim 10.3 satisfy $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$ by Claim 10.1. Using (§10.1) and (10.2), $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to

$$a_{k,i,j} = a_{k,j,i} \text{ for each } (k, i, j) \in \mathcal{M} \text{ with } \{i, j\} \neq \{j^*, t^*\} \text{ for which } k = k^* \text{ implies } t^* \notin \{i, j\}. \quad (\S 10.3)$$

Claim 10.4. *Let $k \in S \setminus \{\bar{k}\}$, let $i, j, i', j' \in V$ be distinct such that $(j^*, t^*) \notin \{(i, j), (i', j'), (i, j'), (i', j)\}$ or $k \notin \{1, k^*, k^* + 1\}$ holds. Then there exist tournaments T and T' satisfying $(PS_{\bar{k}, k, i, j, i', j'})$ such that T and T' satisfy the same condition from Claim 10.1.*

The tournaments T and T' from Claim 10.4 satisfy $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$ by Claim 10.1. Using (§10.1) and (10.2), $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to

$$a_{k,i,j} + a_{k,i',j'} = a_{k,i,j'} + a_{k,i',j} \text{ for all distinct } i, j, i', j' \in V \text{ with } (j^*, t^*) \notin \{(i, j), (i', j'), (i, j'), (i', j)\} \text{ or } k \notin \{1, k^*, k^* + 1\}. \quad (\S 10.4a)$$

Consider a slot $k \in S \setminus \{\bar{k}\}$. For each $\ell \in \{4, 5, \dots, n\}$, (§10.4a) for $(i, j, i', j') = (1, \ell, 2, 3)$ is applicable since $(j^*, t^*) = (3, 4)$ is not among the matches $(i, j), (i', j'), (i, j'), (i', j)$. This implies $a_{k,1,\ell} + a_{k,2,3} = a_{k,1,3} + a_{k,2,\ell}$ which together with (§10.1) yields $a_{k,2,\ell} = 0$. Moreover, for each $\ell \in \{3, 4, \dots, n\}$ with $(k, \ell) \neq (k^*, t^*)$, (§10.3) for $(i, j) = (\ell, 2)$ implies $a_{k,\ell,2} = a_{k,2,\ell} = 0$.

For distinct $\ell, \ell' \in \{3, 4, \dots, n\}$ with $(\ell, \ell') \neq (3, 4)$ or $k \notin \{1, k^*, k^* + 1\}$, (§10.4a) for $(i, j, i', j') = (1, \ell', \ell, 2)$ is applicable, which implies $a_{k,1,\ell'} + a_{k,\ell,2} = a_{k,1,2} + a_{k,\ell,\ell'}$. Together with (§10.1) this shows

$$a_{k,i,j} = 0 \text{ for all } (k, i, j) \in \mathcal{M} \text{ with } (k, i) \neq (k^*, t^*) \text{ and for which } (i, j) = (j^*, t^*) \text{ implies } k \notin \{1, k^*, k^* + 1\}. \quad (\S 10.4b)$$

Together with (§10.2), we obtain that the support of inequality $a^\top x + b^\top y \leq \gamma$ is a subset of the support of inequality (6). It remains to prove that the coefficients agree (up to a positive multiple).

It is easy to see that for each condition of Claim 10.1 there exists a tournament T satisfying it. From (§10.2) and (§10.4b) we obtain the following equations: If $k^* = 1$, then

$$\gamma \stackrel{(a)}{=} a_{1,j^*,t^*} - y_{t^*,t^*,j^*} \stackrel{(c)}{=} a_{k^*+1,j^*,t^*} + a_{k^*,t^*,j} - y_{t^*,t^*,j^*} \stackrel{(d)}{=} a_{k^*+1,j^*,t^*} \stackrel{(f)}{=} a_{k^*,t^*,j}$$

holds, which implies $a_{1,j^*,t^*} = 2$ and $a_{1,t^*,j} = a_{2,j^*,t^*} = b_{t^*,t^*,j^*} = \gamma = 1$ for each $j \in V \setminus \{t^*\}$. Otherwise, i.e., if $k^* \geq 2$, then

$$\gamma \stackrel{(b)}{=} a_{1,j^*,t^*} + a_{k^*,t^*,j} - y_{t^*,t^*,j^*} \stackrel{(c)}{=} a_{k^*+1,j^*,t^*} + a_{k^*,t^*,j} - y_{t^*,t^*,j^*} \stackrel{(d)}{=} a_{k^*+1,j^*,t^*} \stackrel{(e)}{=} a_{k^*,j^*,t^*} \stackrel{(f)}{=} a_{k^*,t^*,j}$$

holds, which implies $a_{1,j^*,t^*} = a_{k^*,j^*,t^*} = a_{k^*,t^*,j} = a_{k^*+1,j^*,t^*} = b_{t^*,t^*,j^*} = \gamma = 1$ for each $j \in V \setminus \{t^*\}$. This shows that $a^\top x + b^\top y \leq \gamma$ is a positive multiple of inequality (6), which concludes the proof. \square

The symmetric lifted version of inequality (1f) reads

$$x_{2n-2,i,t} + x_{k,i,t} + \sum_{j \in V \setminus \{t\}} x_{k,t,j} + x_{k-1,i,t} - 1 \leq y_{t,i,t} \quad \forall k \in S \setminus \{1\}, \forall (i, t) \in A \quad (7)$$

Using Lemma 7, we obtain the following corollary of Theorem 10.

Corollary 11. *Inequalities (7) are facet-defining for $P_{\text{utt}}(n)$ for all $k \in S \setminus \{2n-2\}$ and $(i, t) \in A$.*

Theorem 12. *Inequalities (1g), $x_{1,j,t} \leq y_{t,t,j}$, are facet-defining for $P_{\text{utt}}(n)$ for all $(t, j) \in A$.*

Proof. We assume $n \geq 6$ since we verified the statement for $n = 4$ computationally [15]. Consider the inequality $x_{1,j^*,t^*} \leq y_{t^*,t^*,j^*}$ for distinct teams $t^*, j^* \in V$. By Lemma 7, we can assume $j^* = 3$ and $t^* = 4$. The inequality is valid for $P_{\text{utt}}(n)$ since the team t^* has to travel from its home venue to venue j^* if it plays there in slot 1.

The following claim is used several times throughout the proof.

Claim 12.1. *Let T be a tournament*

- (a) *in which team t^* never travels from its home venue to venue j^* , or*
- (b) *with $(1, j^*, t^*) \in T$.*

Then $(\chi(T), \psi(T))$ satisfies (1g) with equality. Moreover, team t^ travels from its home venue to venue j^* if and only if condition (b) is satisfied.*

In order to prove that the inequality is facet-defining, let $a^\top x + b^\top y \leq \gamma$ define any facet F that contains the face induced by this inequality. We will prove that it is a multiple of inequality (1g). By Lemma 2 we can assume that a satisfies

$$a_m = 0 \text{ for each } m \in \mathcal{B}_n. \quad (\S 12.1)$$

Claim 12.2. *For all $(t, i, j) \in V \times A$ with $(t, i, j) \neq (t^*, t^*, j^*)$ there exists a tournament T satisfying a condition of Claim 12.1.*

A tournament T from Claim 12.2 satisfies $\psi(T)_{t,i,j} = 0$. Let $y := \psi(T)$ and let y' be equal to y except for $y'_{t,i,j} = 1$. By Claim 12.1 we have $(\chi(T), y), (\chi(T), y') \in F$. In this case, $a^\top \chi(T) + b^\top y = \gamma = a^\top \chi(T) + b^\top y'$ simplifies to

$$b_{t,i,j} = 0 \text{ for all } (t, i, j) \in V \times A \text{ with } (t, i, j) \neq (t^*, t^*, j^*). \quad (\S 12.2)$$

Claim 12.3. *For each $(k, i, j) \in \mathcal{M}$ with $k \neq n$ and $\{i, j\} \neq \{j^*, t^*\}$ there exist tournaments T and T' satisfying $(HA_{n,k,i,j})$ such that T and T' satisfy the same condition from Claim 12.1.*

The tournaments T and T' from Theorem 12 satisfy $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$ by Claim 12.1. Using (§12.1) and (12.2), $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to

$$a_{k,i,j} = a_{k,j,i} \text{ for each } (k, i, j) \in \mathcal{M} \text{ with } \{i, j\} \neq \{j^*, t^*\}. \quad (\S 12.3)$$

Claim 12.4. *Let $k \in S \setminus \{n\}$, let $i, j, i', j' \in V$ be distinct such that $(k, j^*, t^*) \notin \{(1, i, j), (1, i', j'), (1, i, j'), (1, i', j)\}$ holds. Then there exist tournaments T and T' satisfying $(PS_{n,k,i,j,i',j'})$ such that T and T' satisfy the same condition from Claim 12.1.*

The tournaments T and T' from Claim 12.4 satisfy $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$ by Claim 12.1. Using (§12.1) and (12.2), $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to

$$a_{k,i,j} + a_{k,i',j'} = a_{k,i,j'} + a_{k,i',j} \text{ for all distinct } i, j, i', j' \in V \text{ with } (j^*, t^*) \notin \{(i, j), (i', j'), (i, j'), (i', j)\}. \quad (\S 12.4a)$$

Consider a slot $k \in S \setminus \{n\}$. For each $\ell \in \{4, 5, \dots, n\}$, (§12.4a) for $(i, j, i', j') = (1, \ell, 2, 3)$ is applicable since $(j^*, t^*) = (3, 4)$ is not among the matches (i, j) , (i', j') , (i, j') , (i', j) . This implies $a_{k,1,\ell} + a_{k,2,3} = a_{k,1,3} + a_{k,2,\ell}$ which together with (§12.1) yields $a_{k,2,\ell} = 0$. Moreover, for each $\ell \in \{3, 4, \dots, n\}$, (§12.3) for $(i, j) = (\ell, 2)$ implies $a_{k,\ell,2} = a_{k,2,\ell} = 0$.

For distinct $\ell, \ell' \in \{3, 4, \dots, n\}$ with $(k, \ell, \ell') \neq (1, 3, 4)$, (§12.4a) for $(i, j, i', j') = (1, \ell', \ell, 2)$ is applicable, which implies $a_{k,1,\ell'} + a_{k,\ell,2} = a_{k,1,2} + a_{k,\ell,\ell'}$. Together with (§12.1) this shows

$$a_{k,i,j} = 0 \text{ for all } (k, i, j) \in \mathcal{M} \setminus \{(1, j^*, t^*)\}. \quad (\S12.4b)$$

Together with (§12.2), we obtain that the support of inequality $a^\top x + b^\top y \leq \gamma$ is a subset of the support of inequality (1g).

It remains to prove that the coefficients agree (up to a positive multiple). From Claim 12.1 it is clear that $a_{1,j^*,t^*} = -b_{t^*,t^*,j^*}$ and that the right-hand side γ must be equal to 0. This concludes the proof. \square

Again, we obtain the following corollary by applying Lemma 7.

Corollary 13. *Inequalities (1h), $x_{2n-2,i,t} \leq y_{t,i,t}$, are facet-defining for $P_{\text{utt}}(n)$ for all $(i, t) \in A$.*

5 New inequality classes

Flow inequalities. Formulation (1) can be strengthened by the following *flow inequalities*.

$$\sum_{j \in V \setminus \{i\}} y_{t,i,j} \geq 1 \quad \forall i, t \in V : i \neq t \quad (8a)$$

$$\sum_{j \in V \setminus \{i\}} y_{t,j,i} \geq 1 \quad \forall i, t \in V : i \neq t \quad (8b)$$

They state that each team t has to leave (resp. enter) each other team's venue at least once. We now prove that all these inequalities define facets of $P_{\text{utt}}(n)$.

Theorem 14. *Inequalities (8) are facet-defining for $P_{\text{utt}}(n)$ for all $i, t \in V$ with $i \neq t$.*

Proof. We only prove the statement for inequalities (8a). For (8b), it then follows from Lemma 7. In addition, we assume $n \geq 8$ since we verified the statement for $n \in \{4, 6\}$ computationally [15].

Let $i^*, t^* \in V$ with $i^* \neq t^*$. The inequality for $i := i^*$ and $t := t^*$ is valid since team t^* has to play an away match against team i^* after which it leaves to some other venue.

To establish that the inequality is facet-defining, let $a^\top x + b^\top y \geq \gamma$ define any facet F that contains the face induced by $\sum_{j \in V \setminus \{i^*\}} y_{t^*,i^*,j} \geq 1$. Without loss of generality, a satisfies (B).

Claim 14.1. *For all $(t, i, j) \in V \times A$ with $(t, i) \neq (t^*, i^*)$ there exists a tournament in which team t never travels from venue i to venue j and in which team t^* leaves venue i^* exactly once.*

A tournament T from Claim 14.1 satisfies $\psi(T)_{t,i,j} = 0$. Let $y := \psi(T)$ and let y' be equal to y except for $y'_{t,i,j} = 1$. We have $(\chi(T), y) \in F$ and if $(t, i) \neq (t^*, i^*)$ holds, also $(\chi(T), y') \in F$. In this case, $a^\top \chi(T) + b^\top y = \gamma = a^\top \chi(T) + b^\top y'$ simplifies to $b_{t,i,j} = 0$. We obtain

$$b_{t,i,j} = 0 \text{ for all } (t, i, j) \in V \times A \text{ with } (t, i) \neq (t^*, i^*). \quad (14.1)$$

Claim 14.2. *For all distinct $i, j \in V$ and for each $k \in S \setminus \{1\}$ there exist tournaments T and T' satisfying $(HA_{1,k,i,j})$ such that in both tournaments team t^* leaves venue i^* exactly once and to the same venue.*

In the tournaments T and T' from Claim 14.2 team t^* leaves venue i^* exactly once and to the same venue. Hence, we have $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$. Moreover, together with (14.1) it implies $b^\top \psi(T) = b^\top \psi(T')$. Combining this with (\mathcal{B}) , $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to $a_{k,i,j} = a_{k,j,i}$. Thus, we have

$$a_{k,i,j} = a_{k,j,i} \text{ for each } (k, i, j) \in \mathcal{M}. \quad (14.2)$$

Claim 14.3. *For each slot $k \in S \setminus \{1\}$ and for distinct teams $i, j, i', j' \in V$ with $(i^*, t^*) \notin \{(i, j), (i', j'), (i, j'), (i', j)\}$ there exist tournaments T and T' satisfying $(PS_{1,k,i,j,i',j'})$ such that in both tournaments team t^* leaves venue i^* exactly once and to the same venue.*

In the tournaments T and T' from Claim 14.3 team t^* leaves venue i^* exactly once and to the same venue. Hence, we have $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$. Moreover, together with (14.1) it implies $b^\top \psi(T) = b^\top \psi(T')$. Combining this with (\mathcal{B}) , $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to

$$a_{k,i,j} + a_{k,i',j'} = a_{k,i,j'} + a_{k,i',j} \text{ for each } k \in S \setminus \{1\} \text{ and for all distinct } i, j, i', j' \in V \\ \text{with } (i^*, t^*) \notin \{(i, j), (i', j'), (i, j'), (i', j)\} \quad (14.3)$$

Since the formulation is symmetric with respect to teams, we can now, by permuting teams, assume $(i^*, t^*) = (4, 3)$. Consider a slot $k \in S \setminus \{1\}$. For each $\ell \in \{4, 5, \dots, n\}$, (14.3) implies $a_{k,1,\ell} + a_{k,2,3} = a_{k,1,3} + a_{k,2,\ell}$ which together with (\mathcal{B}) yields $a_{k,2,\ell} = 0$. Combined with (14.2) we also obtain $a_{k,\ell,2} = 0$. For all $\ell, \ell' \in \{3, 4, \dots, n\}$ except for $(\ell, \ell') = (4, 3)$, (14.3) implies $a_{k,1,\ell'} + a_{k,\ell,2} = a_{k,1,2} + a_{k,\ell,\ell'}$. Together with (\mathcal{B}) , this shows $a_{k,\ell,\ell'} = 0$ for all $(\ell, \ell') \neq (4, 3)$. From (14.2) we also have $a_{k,4,3} = a_{k,3,4} = 0$ and obtain $a = \mathbb{O}$.

Claim 14.4. *For distinct $j, j' \in V \setminus \{i^*\}$ there exist tournaments T and T' such that in both tournaments team t^* leaves venue i^* exactly once, namely to venue j in T and to venue j' in T' .*

In the tournaments T and T' from Claim 14.4 team t^* leaves venue i^* exactly once. Hence, we have $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$. From $a = \mathbb{O}$ and (14.1) we have that $b_{t^*,i^*,j} = a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T') = b_{t^*,i^*,j'}$. This shows that (a^\top, b^\top) is a multiple of the coefficient vector of (8a). The fact that it is a positive multiple follows from the observation that we can take any feasible solution and setting all entries of y to 1 yields another feasible solution (which is not in the face anymore). \square

Home-flow inequalities. Inequalities (8) also hold for t 's home venue, i.e., $i = t$, but in this case they are dominated by the following *home-flow inequalities*.

$$\sum_{j \in V \setminus \{t\}} y_{t,t,j} + \sum_{j \in V \setminus \{t\}} (x_{k,t,j} + x_{k+n-1,t,j}) \geq 2 \quad \forall k \in \{1, 2, \dots, n-1\}, \forall t \in V \quad (9a)$$

$$\sum_{j \in V \setminus \{t\}} y_{t,t,j} + \sum_{j \in V \setminus \{t\}} (x_{k,j,t} + x_{k+n-1,j,t}) \geq 2 \quad \forall k \in \{1, 2, \dots, n-1\}, \forall t \in V \quad (9b)$$

$$\sum_{i \in V \setminus \{t\}} y_{t,i,t} + \sum_{i \in V \setminus \{t\}} (x_{k,t,i} + x_{k+n-1,t,i}) \geq 2 \quad \forall k \in \{1, 2, \dots, n-1\}, \forall t \in V \quad (9c)$$

$$\sum_{i \in V \setminus \{t\}} y_{t,i,t} + \sum_{i \in V \setminus \{t\}} (x_{k,i,t} + x_{k+n-1,i,t}) \geq 2 \quad \forall k \in \{1, 2, \dots, n-1\}, \forall t \in V \quad (9d)$$

They are valid for $P_{\text{utt}}(n)$ since team t leaves (resp. enters) its home venue either at least twice or it leaves (resp. enters) it only once in which case it cannot play at home (resp. away) in slots k and $k+n-1$. The sum of the first two reads

$$\sum_{j \in V \setminus \{t\}} 2y_{t,t,j} + \sum_{j \in V \setminus \{t\}} (x_{k,t,j} + x_{k,j,t} + x_{k+n-1,t,j} + x_{k+n-1,j,t}) \geq 4,$$

for which the subtraction of equation (1b) for team t and slots k and $k+n-1$ yields

$$\sum_{j \in V \setminus \{t\}} 2y_{t,t,j} \geq 4 - 1 - 1,$$

which in turn equals (8a) for $i = t$. The corresponding result is as follows.

Theorem 15. *Inequalities (9) are facet-defining for $P_{\text{utt}}(n)$ for each team $t \in V$ and each slot $k \in \{1, 2, \dots, n-1\}$.*

Proof. We only prove the statement for inequalities (9a). The proof for inequalities (9b) is very similar. Moreover, the result for inequalities (9c) and (9d) follows from Lemma 7. In addition, we assume $n \geq 6$ since we verified the statement for $n = 4$ computationally [15].

Let $t^* \in V$ and $k^* \in \{1, 2, \dots, n-1\}$. To see that the inequalities are valid, first observe that team t^* has to leave its own venue at least once. If it does so at least twice, the inequality is certainly satisfied. The remaining case is settled by the following observation which we will use several times throughout the proof.

Claim 15.1. *Let T be a tournament in which t^* leaves its home venue exactly once. Then all away matches of t^* take place in consecutive slots, and hence t^* plays at home in exactly one of the two slots k^* and $k^* + n - 1$. In particular, $(\chi(T), \psi(T))$ satisfies (9a) and (9b) with equality.*

To prove that inequality (9a) is facet-defining, let $a^\top x + b^\top y \geq \gamma$ define any facet F that contains the face induced by $\sum_{j \in V \setminus \{t^*\}} y_{t^*,t^*,j} + \sum_{j \in V \setminus \{t^*\}} (x_{k^*,t^*,j} + x_{k^*+n-1,t^*,j}) \geq 2$.

Since the formulation is symmetric with respect to teams we can, by permuting teams, assume $t^* = 4$ for the remainder of the proof. Let $\bar{k} \in S$ with $k^* < \bar{k} < k^* + n - 1$. By Lemma 2 we can assume that a satisfies

$$a_m = 0 \text{ for each } m \in \mathcal{B}_{\bar{k}}. \quad (\S 15.1)$$

Claim 15.2. *For all $(t, i, j) \in V \times A$ with $(t, i) \neq (t^*, t^*)$ there exists a tournament in which team t never travels from venue i to venue j and in which team t^* leaves its home venue exactly once.*

A tournament T from Claim 15.2 satisfies $\psi(T)_{t,i,j} = 0$. Let $y := \psi(T)$ and let y' be equal to y except for $y'_{t,i,j} = 1$. By Claim 15.1 we have $(\chi(T), y) \in F$ and if $(t, i) \neq (t^*, t^*)$ holds, also $(\chi(T), y') \in F$. In this case, $a^\top \chi(T) + b^\top y = \gamma = a^\top \chi(T) + b^\top y'$ simplifies to $b_{t,i,j} = 0$. We obtain

$$b_{t,i,j} = 0 \text{ for all } (t, i, j) \in V \times A \text{ with } (t, i) \neq (t^*, t^*). \quad (\S 15.2)$$

Claim 15.3. *For any slot $k \in \{1, 2, \dots, n-1\}$ and distinct $j, j' \in V \setminus \{t^*\}$ there exist tournaments T and T' satisfying $(HA_{k,k+n-1,t^*,j})$ and such that team t^* leaves its home venue exactly once and to the venues j in T and to j' in T' .*

In the tournaments T and T' from Claim 15.3 team t^* leaves its home venue exactly once. Hence, by Claim 15.1 we have $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$. Due to (§15.1) and (§15.2) the equation $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to $a_{k,t^*,j} + a_{k+n-1,j,t^*} + b_{t^*,t^*,j} = a_{k,j,t^*} + a_{k+n-1,t^*,j} + b_{t^*,t^*,j'}$. Since j' only appears in the last term, varying j' yields $b_{t^*,t^*,j_1} = b_{t^*,t^*,j_2}$ for all $j_1, j_2 \in V \setminus \{t^*\}$. Together with (§15.2), this shows

$$b^\top \psi(T) = b^\top \psi(T') \text{ for all tournaments } T, T' \text{ with } (\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F \\ \text{in which } t^* \text{ leaves its home venue as often in } T \text{ as in } T'. \quad (\S 15.3a)$$

This further simplifies the equation to

$$a_{k,t^*,j} + a_{k+n-1,j,t^*} = a_{k,j,t^*} + a_{k+n-1,t^*,j} \text{ for all } k \in \{1, 2, \dots, n-1\} \text{ and all } j \in V \setminus \{t^*\}. \quad (15.3b)$$

Claim 15.4. *For each slot $k \in S \setminus \{\bar{k}\}$ and for all distinct $i, j \in V \setminus \{t^*\}$ there exist tournaments T and T' satisfying $(HA_{\bar{k},k,i,j})$ and such that in both tournaments team t^* leaves its home venue exactly once.*

In the tournaments T and T' from Claim 15.4 team t^* leaves its home venue exactly once. Hence, by Claim 15.1 we have $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$ and by (§15.3a) also $b^\top \psi(T) = b^\top \psi(T')$. Combining this with (§15.1), $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to $a_{k,i,j} = a_{k,j,i}$. Thus, we have

$$a_{k,i,j} = a_{k,j,i} \text{ for each } (k, i, j) \in \mathcal{M} \text{ with } t^* \notin \{i, j\}. \quad (\S 15.4)$$

Claim 15.5. *For distinct slots $k_1, k_2 \in S$ and distinct teams $i, j, i', j' \in V$ with $t^* \notin \{i, i'\}$ and with $k_2 = k_1 + 1$ if $t^* \in \{j, j'\}$ there exist tournaments T and T' satisfying $(PS_{k_1,k_2,i,j,i',j'})$ such that in both tournaments team t^* leaves its home venue exactly once.*

In the tournaments T and T' from Claim 15.5 team t^* leaves its home venue exactly once. Hence, by Claim 15.1 we have $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$ and by (§15.3a) also $b^\top \psi(T) = b^\top \psi(T')$. Combining this with (§15.1), equation $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ yields

$$a_{k_1,i,j} + a_{k_1,i',j'} + a_{k_2,i,j'} + a_{k_2,i',j} = a_{k_1,i,j'} + a_{k_1,i',j} + a_{k_2,i,j} + a_{k_2,i',j'} \text{ for all distinct slots } k_1, k_2 \in S \\ \text{and for all distinct } i, j, i', j' \in V \text{ with } t^* \notin \{i, i'\} \text{ and with } |k_1 - k_2| = 1 \text{ if } t^* \in \{j, j'\}. \quad (\S 15.5a)$$

For each $k \in S \setminus \{\bar{k}\}$ and each $\ell \in \{5, 6, \dots, n\}$ (noting $\ell \neq t^* = 4$), (§15.5a) with $(k_1, k_2, i, j, i', j') = (\bar{k}, k, 1, 3, 2, \ell)$ implies $a_{\bar{k},1,3} + a_{\bar{k},2,\ell} + a_{k,1,\ell} + a_{k,2,3} = a_{\bar{k},1,\ell} + a_{\bar{k},2,3} + a_{k,1,3} + a_{k,2,\ell}$. By (§15.1), this simplifies to $a_{k,2,\ell} = 0$, from which (§15.4) yields $a_{k,\ell,2} = 0$.

For each $k \in S \setminus \{\bar{k}\}$ and all distinct $\ell, \ell' \in \{3, 5, 6, \dots, n\}$, (§15.5a) with $(k_1, k_2, i, j, i', j') = (\bar{k}, k, 1, \ell', \ell, 2)$ implies $a_{\bar{k}, 1, \ell'} + a_{\bar{k}, \ell, 2} + a_{k, 1, 2} + a_{k, \ell, \ell'} = a_{\bar{k}, 1, 2} + a_{\bar{k}, \ell, \ell'} + a_{k, 1, \ell'} + a_{k, \ell, 2}$. By (§15.1) and the previous observation $a_{k, \ell, 2} = 0$, this simplifies to $a_{k, \ell, \ell'} = 0$. Since also $a_{\bar{k}, \star, \star} = \mathbb{O}$, we have

$$a_{k, i, j} = 0 \text{ for all } k \in S \text{ and all } i, j \in V \setminus \{t^*\}. \quad (\S15.5b)$$

Let $\ell \in V \setminus \{t^*\}$. For $k \in S \setminus \{\bar{k}\}$, the tuple $(k_1, k_2, i, j, i', j') = (k - 1, k, \ell, t^*, 1, 2)$ satisfies the conditions of (§15.5a), and thus for $\ell \in \{3, 5, 6, \dots, n\}$ implies $a_{k-1, \ell, t^*} + a_{k-1, 1, 2} + a_{k, \ell, 2} + a_{k, 1, t^*} = a_{k-1, \ell, 2} + a_{k-1, 1, t^*} + a_{k, \ell, t^*} + a_{k, 1, 2}$. By (§15.1) and (§15.5b), this simplifies to $a_{k-1, \ell, t^*} = a_{k, \ell, t^*}$. By induction on k and $a_{\bar{k}, \ell, t^*} = 0$, we obtain

$$a_{k, \ell, t^*} = 0 \text{ for all } k \in S \text{ and all } \ell \in V \setminus \{t^*\}. \quad (\S15.5c)$$

With this, (15.3b) is simplified to

$$a_{k, t^*, j} = a_{k+n-1, t^*, j} \text{ for all } k \in \{1, 2, \dots, n-1\} \text{ and all } j \in V \setminus \{t^*\}. \quad (\S15.5d)$$

Claim 15.6. *For each slot $k \in \{k^* + 1, k^* + 2, \dots, k^* + n - 3\}$ and each team $j \in V \setminus \{t^*\}$ there exist tournaments T and T' satisfying $(HA_{k, k+1, j, t^*})$ such that in both tournaments team t^* leaves its home venue exactly twice and plays away in slots k^* and $k^* + n - 1$.*

In the tournaments T and T' from Claim 15.6 team t^* leaves its home venue exactly twice and does not play home in slots k^* and $k^* + n - 1$. Hence, we have $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$ and by (§15.3a) also $b^\top \psi(T) = b^\top \psi(T')$. Combining this with (§15.1) and (§15.5c), equation $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to

$$a_{k+1, t^*, j} = a_{k, t^*, j} \text{ for each } k \in S \text{ with } k^* < k < k^* + n - 1 \text{ and each } j \in V \setminus \{t^*\}.$$

Induction on k yields that $a_{k, t^*, j}$ is the same for all these k . Moreover, for each slot $k \in S$ with $k < k^*$ or $k > k^* + n - 1$ the slot $k + n - 1$ (resp. $k - n + 1$) lies between k^* and $k^* + n - 1$. Application of (§15.5d) yields that $a_{k, t^*, j}$ is the same for all $k \in S \setminus \{k^*, k^* + n - 1\}$. As \bar{k} is among those, (§15.1) yields

$$a_{k, t^*, j} = 0 \text{ for each } k \in S \setminus \{k^*, k^* + n - 1\} \text{ and each } j \in V \setminus \{t^*\}. \quad (\S15.6)$$

Claim 15.7. *For all distinct teams $j, j' \in V \setminus \{t^*\}$ there exist tournaments T and T' satisfying $(HA_{k^*, k^*+1, t^*, j})$ such that team t^* leaves its home venue exactly once, namely to venue j , in tournament T and exactly twice, namely to venues j and j' , in tournament T' where it plays away in slots k^* and $k^* + n - 1$.*

In the tournaments T and T' from Claim 15.7 team t^* leaves its home venue either once or twice, and in the latter case it does not play home in slots k^* and $k^* + n - 1$. Hence, we have $(\chi(T), \psi(T)), (\chi(T'), \psi(T')) \in F$. Using (§15.1), (§15.2), (§15.5c) and (§15.6), equation $a^\top \chi(T) + b^\top \psi(T) = \gamma = a^\top \chi(T') + b^\top \psi(T')$ simplifies to

$$a_{k^*, t^*, j} + b_{t^*, t^*, j} = \gamma = b_{t^*, t^*, j} + b_{t^*, t^*, j'}. \quad (\S15.7)$$

By varying j and j' and considering (§15.5d), we obtain that $a^\top x + b^\top y \geq \gamma$ is a positive multiple of inequality (9a). This concludes the proof. \square

Translated home-flow inequalities for the constrained polytope. Let us briefly consider the case of home-stand constraints, i.e., that a team may play at most U subsequent matches at home. In this case, the *translated home-flow inequalities*

$$\sum_{j \in V \setminus \{i\}} y_{t,i,j} \geq \left\lceil \frac{n-1}{U} \right\rceil \quad \forall i, t \in V : i \neq t \quad (10a)$$

$$\sum_{j \in V \setminus \{i\}} y_{t,j,i} \geq \left\lceil \frac{n-1}{U} \right\rceil \quad \forall i, t \in V : i \neq t \quad (10b)$$

are valid since the $n-1$ home matches of each team must be divided into at least $(n-1)/U$ home stands. Clearly, these inequalities are invalid for $P_{\text{utt}}(n)$ for $n \geq 6$ and hence we do not study it theoretically.

A face defined by flow inequalities. Recall the definition of the unconstrained traveling tournament polytope:

$$P_{\text{utt}}(n) := \text{conv}\{(\chi(T), y) \in \{0, 1\}^{\mathcal{M}} \times \{0, 1\}^{V \times A} : T \text{ tournament and } y \geq \psi(T)\}.$$

Allowing vectors $y \geq \psi(T)$ augments the set of feasible solutions by suboptimal ones, which is advantageous for finding facet-defining inequalities due to a larger dimension. Now we examine what happens if we set the flow inequalities (8a) and (8b) to equality:

$$\sum_{j \in V \setminus \{i\}} y_{t,i,j} = 1 \quad \forall i, t \in V : i \neq t \quad (11a)$$

$$\sum_{i \in V \setminus \{j\}} y_{t,i,j} = 1 \quad \forall j, t \in V : j \neq t \quad (11b)$$

The following theorem shows how we obtain the convex hull of all pairs of play- and travel-vectors as the corresponding face of $P_{\text{utt}}(n)$.

Theorem 16. *The face of $P_{\text{utt}}(n)$ defined by equations (11) is equal to*

$$\text{conv}\{(\chi(T), \psi(T)) \in \{0, 1\}^{\mathcal{M}} \times \{0, 1\}^{V \times A} : T \text{ tournament}\}.$$

Consequently, formulation (1) together with these equations is an integer programming formulation for this polytope.

Proof. Let Q be the polytope defined in the statement of the theorem.

To see that Q is contained in the mentioned face, let T be a tournament. For each $i^*, t^* \in V$ with $i^* \neq t^*$, equation (11a) is satisfied by $\psi(T)$ since team t^* has to play exactly one away match against team i^* after which it leaves this venue. Moreover, it never visits venue i^* again. Similarly, $\psi(T)$ satisfies all equations (11b).

It remains to prove that every vertex (x, y) of the face lies in Q . Since $P_{\text{utt}}(n)$ is integral, all its faces are integral as well, and thus $(x, y) \in \{0, 1\}^{\mathcal{M}} \times \{0, 1\}^{V \times A}$. The vector x defines a tournament T and we have $y \geq \psi(T)$. We have to show $y = \psi(T)$. Consider an entry $(t, i, j) \in V \times A$. By $i \neq j$, we have $t \neq i$ or $t \neq j$. If $t \neq i$, then $y_{t,i,j}$ appears in equation (11a) for (i, t) and otherwise it appears in equation (11b) for (j, t) . Since y must be equal to $\psi(T)$ on the support of this equation, we have $y = \psi(T)$, which concludes the proof. \square

6 Impact on lower bounds

In this section we discuss the practical effect of adding the facet-defining inequalities to the linear relaxation. To this end, we consider formulation (1) augmented by inequalities

$$x_{k,i,j} + x_{k+1,j,i} \leq 1 \quad \forall (k,i,j) \in \mathcal{M} : k \neq 2n - 2 \quad (12a)$$

$$\sum_{\ell=k}^{k+3} \sum_{j \in V \setminus \{t\}} x_{\ell,t,j} \leq 3 \quad \forall k \in \{1, 2, \dots, 2n - 5\}, \forall t \in V \quad (12b)$$

$$\sum_{\ell=k}^{k+3} \sum_{i \in V \setminus \{j\}} x_{\ell,i,t} \leq 3 \quad \forall k \in \{1, 2, \dots, 2n - 5\}, \forall t \in V. \quad (12c)$$

The *no-repeater constraints* (12a) ensure that the matches among two teams are not consecutive, while the *home-stand constraints* (12b) (resp. *road-trip constraints* (12c)) ensure that team t does not play more than $U = 3$ consecutive home (resp. away) matches.

We consider the *NL instances* from [5] and computed the bounds of the linear programming relaxations using Gurobi [9]. The baseline is the lower bound of formulation (1) with (12).

Table 1: Lower bound improvements for NL instances. Column UB contains the best known upper bounds. The LB columns contain the lower bounds.

Instance	LB				UB
	(1, 12)	(1, 12, 8, 9)	(1, 12, 8, 9, 10)	Best	Best
NL4	2004	8016	8016	8276	8276
NL6	2186	13116	17422	23916	23916
NL8	2686	21488	31916	39721	39721
NL10	2980	29800	40264	59436	59436
NL12	4736	56832	83552	108629	110729
NL14	5652	79128	136821	183354	188728
NL16	6028	96448	146730	249477	261687

Table 1 shows the bound improvements when adding inequalities (8) and (9) and/or inequalities (10) to the formulation. We also computed the lower bounds when adding the lifted model inequalities (5), (6) and (7). However, for none of the instance/formulation combinations this improved the lower bounds.

The bound improvement from the addition of inequalities (8) and (9) is quite impressive. Also the improvement due to the translated home-flow inequalities (10) is significant. However, even for NL4, this alone does not close the integrality gap. Hence, we can only conclude that further classes of inequalities need to be characterized to achieve bounds that can make a difference in computations.

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A Tournaments for facet proofs

A.1 Tournaments for Theorem 5

Claim 5.1. *For each $(t, i, j) \in V \times A$ there exists a tournament in which team t never travels from venue i to venue j .*

Proof. If $t = i$, let $i' \in V \setminus \{i, j, t\}$ and $j' := j$. Otherwise, let $i' := i$ and $j' \in V \setminus \{i, j, t\}$. Note that in either case i', j' and t are distinct. We construct tournament T from a canonical factorization by permuting slots and teams such that $(1, i', t), (2, j', t) \in T$. Hence, team t travels from venue i' to venue j' , which implies that team t never travels from venue i to venue j since exactly one of the teams i', j' is equal to its counterpart i, j . \square

Claim 5.2. *For each $k \in S \setminus \{1\}$ and for distinct $i, j \in V$ there exist tournaments T and T' satisfying $(HA_{1,k,i,j})$.*

Proof. We construct tournament T from a canonical factorization by permuting slots and teams such that $(1, i, j), (k, j, i) \in T$. Tournament T' is obtained from T by $(HA_{1,k,i,j})$. \square

Claim 5.3. *For each $k \in S \setminus \{1\}$ and for distinct $i, j, i', j' \in V$ there exist tournaments T and T' satisfying $(PS_{1,k,i,j,i',j'})$.*

Proof. Let M_ℓ for all $\ell \in S$ be the perfect matchings of the canonical factorization. By permuting teams we can assume $\{i, j\}, \{i', j'\} \in M_1 = M_n$. We now exchange the roles of teams j and j' only in perfect matchings $M_n, M_{n+1}, \dots, M_{2n-2}$, which maintains the property that each edge appears in exactly two perfect matchings. Tournament T is obtained by orienting the edges in a complementary fashion and permuting slots such that $(1, i, j), (1, i', j'), (k, i, j'), (k, i', j) \in T$.

Finally, tournament T' is constructed from T by $(PS_{1,k,i,j,i',j'})$. \square

A.2 Tournaments for Theorem 8

For the claims in the proof of Theorem 8, we are given a particular match $m^* = (k^*, i^*, j^*) = (2, 4, 3) \in \mathcal{M}$.

Claim 8.1. *For each $(t, i, j) \in V \times A$ there exists a tournament T with $m^* \notin T$ and in which team t never travels from venue i to venue j .*

Proof. Let tournament T be as constructed in the proof of Claim 5.1. If $m^* \in T$, we apply a cyclic permutation of the slots, mapping slot k to $k + 1$ for $k \in S \setminus \{2n - 2\}$ and slot $2n - 2$ to 1. This preserves the second requirement and establishes $m^* \notin T$. \square

Claim 8.2. *For each $(k, i, j) \in \mathcal{M}$ with $k \geq 2$ and $(k, i, j) \neq (k^*, i^*, j^*), (k^*, j^*, i^*)$ there exist tournaments T and T' satisfying $(HA_{1,k,i,j})$ and $(k^*, i^*, j^*), (k^*, j^*, i^*) \notin T \cup T'$.*

Proof. We construct tournament T'' from a canonical factorization by permuting slots and teams such that $(1, i, j), (k, j, i) \in T''$. If $m^* \notin T''$, let $T := T''$.

Otherwise, if $k^* \neq k$, then let $k' \in S$ be such that $(k', j^*, i^*) \in T''$. Tournament T is obtained from T'' by (HA_{k^*,k',i^*,j^*}) . Due to $k^* \neq k$ and $k^* \neq 1$, we have $(1, i, j), (k, j, i) \in T''$, but $m^* \notin T$, and hence T satisfies all requirements.

Otherwise, $k^* = k$ and $\{i, j\} \neq \{i^*, j^*\}$ hold. Together with $(k, j, i), (k^*, i^*, j^*) \in T''$ this implies that i, j, i^*, j^* must be distinct. Again, let $k' \in S$ be such that $(k', j^*, i^*) \in T''$ and construct T from T'' by (HA_{k^*,k',i^*,j^*}) . Since i, j, i^*, j^* are distinct, we also have $(1, i, j), (k, j, i) \in T$, but $m^* \notin T$ in this case, and hence T satisfies all requirements.

Finally, tournament T' is obtained from T by $(HA_{1,k,i,j})$. \square

Claim 8.3. *For each slot $k \in S \setminus \{1\}$ and for distinct $i, j, i', j' \in V$ with $k \neq k^*$ or $(i^*, j^*) \notin \{(i, j), (i', j'), (i', j), (i, j')\}$ there exist tournaments T and T' satisfying $(PS_{1,k,i,j,i',j'})$ and $m^* \notin T \cup T'$.*

Proof. Let T be as constructed in the proof of Claim 5.3. If $(k^*, i^*, j^*) \in T$, let $k' \in V$ be such that $(k', j^*, i^*) \in T_1$ and modify T via a home-away swap (HA_{k^*, k', i^*, j^*}). By assumptions on k^* , i^* and j^* , this operation does not affect the matches $(1, i, j)$, $(1, i', j')$, (k, i, j') , (k, i', j) above, i.e., these remain in T . However, after the modification, we have $(k^*, i^*, j^*) \notin T$.

Finally, tournament T' is constructed from T by ($PS_{1, k, i, j, i', j'}$). \square

A.3 Tournaments for Theorem 9

For the claims in the proof of Theorem 9, we are given a slot $k^* \in \{n, n+1, \dots, 2n-3\}$ and three distinct teams t^* , i^* and j^* . Note that we also assume $n \geq 6$. To enhance readability of the proofs we restate the claim that contains sufficient conditions for satisfying (1d) with equality.

Claim 9.1. *Let T be a tournament that contains*

- (a) *match (k^*, i^*, t^*) and in which team t^* plays away in slot $k^* + 1$, or*
- (b) *one of the matches (k^*, j^*, t^*) , (k^*, i^*, t^*) or $(k^* + 1, j^*, t^*)$, and in which team t^* never travels from venue i^* to venue j^* .*

Then $(\chi(T), \psi(T))$ satisfies (5a) with equality.

Claim 9.2. *For all $(t, i, j) \in V \times A$ with $(t, i, j) \neq (t^*, i^*, j^*)$ there exists a tournament T in which team t never travels from venue i to venue j and which satisfies condition (a) of Claim 9.1.*

Proof. If $t = i$, let $i' \in V \setminus \{i, j, t\}$ and $j' := j$. Otherwise, let $i' := i$ and $j' \in V \setminus \{i, j, t\}$. Note that in either case i' , j' and t are distinct. We distinguish three cases.

Case 1: $t^* \neq t$ or $i^* \notin \{i', j'\}$. We construct tournament T from a canonical factorization by permuting slots and teams such that $(1, i', t), (2, j', t), (k^*, i^*, t^*) \in T$ holds and such that t^* plays away in slot $k^* + 1$. Hence, team t travels from venue i' to venue j' , which implies that team t never travels from venue i to venue j since exactly one of the teams i', j' is equal to its counterpart i, j .

Case 2: $t^* = t$ and $i^* = i'$. We construct tournament T from a canonical factorization by permuting slots and teams such that $(k^*, i^*, t^*), (k^* + 1, j', t^*) \in T$ holds.

Case 3: $t^* = t$ and $i^* = j'$. We construct tournament T from a canonical factorization by permuting slots and teams such that $(k^*, i', t^*), (k^*, i^*, t^*) \in T$ holds and such that t^* plays away in slot $k^* + 1$.

In all cases team t travels from venue i' to venue j' , which implies that team t never travels from venue i to venue j since exactly one of the teams i', j' is equal to its counterpart i, j . Moreover, $(k^*, i^*, t^*) \in T$ holds and team t^* plays away in slot $k^* + 1$, which concludes the proof. \square

Claim 9.3. *For each $(k, i, j) \in \mathcal{M} \setminus \{(k^*, i^*, t^*), (k^*, t^*, i^*), (k^*, j^*, t^*), (k^*, t^*, j^*), (k^* + 1, j^*, t^*), (k^* + 1, t^*, j^*)\}$ with $k \geq 2$ there exist tournaments T and T' satisfying ($HA_{1, k, i, j}$) and condition (b) of Claim 9.1.*

Proof. Let M_ℓ for all $\ell \in S$ be the perfect matchings of the canonical factorization. We distinguish ten cases:

Case 1: $\{i, j\} \cap \{j^*, t^*\} = \emptyset$. We permute slots such that $\{i, j\} \in M_1, M_k$ and such that there are edges $\{t', i'\} \in M_{k^*}$ and $\{t', j'\} \in M_{k^*+1}$ with distinct $t', i', j' \in V \setminus \{i, j\}$. Then we permute the teams $V \setminus \{i, j\}$ such that i' is mapped to some team $i^\# \neq i^*$, j' is mapped to j^* and t' is mapped to t^* . Tournament T is obtained by orienting the matching edges in a complementary fashion such that $(1, i, j), (k, j, i), (k^*, i^\#, t^*), (k^* + 1, j^*, t^*) \in T$ hold. Note that team t^* travels from venue $i^\#$ to venue j^* and thus never from venue i^* to venue j^* .

Case 2: $\{i, j\} \cap \{i^*, t^*\} = \emptyset$. We permute slots such that $\{i, j\} \in M_1, M_k$ and such that there are edges $\{t', i'\} \in M_{k^*}$ and $\{t', j'\} \in M_{k^*+1}$ with distinct $t', i', j' \in V \setminus \{i, j\}$. Then we permute the teams $V \setminus \{i, j\}$ such that i' is mapped to i^* , j' is mapped to some team $j^\# \neq j^*$ and t' is mapped to t^* . Tournament T is obtained by orienting the matching edges in a complementary fashion such that $(1, i, j), (k, j, i), (k^*, i^*, t^*), (k^* + 1, j^\#, t^*) \in T$ hold. Note that team t^* travels from venue i^* to venue $j^\#$ and thus never from venue i^* to venue j^* .

Case 3: $\{i, j\} = \{i^*, j^*\}$ and $k = k^*$. We construct tournament T by permuting slots such that $(1, i, j), (k, j, i), (k^* + 1, j^*, t^*) \in T$ holds. Note that team t^* plays against a team in $V \setminus \{i^*, j^*\}$ before traveling to venue j^* and thus never travels from venue i^* to venue j^* .

Case 4: $\{i, j\} = \{i^*, j^*\}$ and $k \neq k^*$. We permute slots such that $\{i, j\} \in M_1, M_k, \{i^*, t^*\} \in M_{k^*}$ and $\{j', t^*\} \in M_{k^*+1}$ holds for some team $j' \neq j^*$. Tournament T is obtained by orienting the matching edges in a complementary fashion such that $(1, i, j), (k, j, i), (k^*, i^*, t^*), (k^*, j', t^*) \in T$ holds. Note that team t^* travels from venue i^* to venue $j' \neq j^*$ and thus never travels from venue i^* to venue j^* .

Case 5: $\{i, j\} \cap \{i^*, j^*, t^*\} = \{t^*\}$ and $k = k^*$. We construct tournament T by permuting slots such that $(1, i, j), (k, j, i), (k^* + 1, j^*, t^*) \in T$ holds. Note that team t^* travels from its home venue or some venue different from i^* to venue j^* and thus never from venue i^* to venue j^* .

Case 6: $\{i, j\} \cap \{i^*, j^*, t^*\} = \{t^*\}$ and $k = k^* + 1$. We construct tournament T by permuting slots such that $(1, i, j), (k, j, i), (k^*, i^*, t^*) \in T$ holds. Note that team t^* travels from i^* to its home venue or to some venue different from j^* and thus never from venue i^* to venue j^* .

Case 7: $\{i, j\} \cap \{i^*, j^*, t^*\} = \{t^*\}$ and $k \notin \{k^*, k^* + 1\}$. We construct tournament T by permuting slots such that $(1, i, j), (k, j, i), (k^*, j^*, t^*), (k^* + 1, i^*, t^*) \in T$ holds. Note that team t^* travels from venue j^* to venue i^* and thus never from venue i^* to venue j^* .

Case 8: $\{i, j\} = \{i^*, t^*\}$ and $k = k^* + 1$. We construct tournament T by permuting slots such that $(1, i, j), (k, j, i), (k^*, j^*, t^*) \in T$ and such that $(k^* - 1, i^*, t^*) \notin T$ holds. Due to the last condition, team t^* never travels from venue i^* to venue j^* .

Case 9: $\{i, j\} = \{i^*, t^*\}$ and $k \notin \{k^*, k^* + 1\}$. We construct tournament T by permuting slots such that $(1, i, j), (k, j, i), (k^* + 1, j^*, t^*) \in T$ holds. Since team t^* plays at venue j^* in slot $k^* + 1$ but does not play against i^* in slot k^* it never travels from venue i^* to venue j^* .

Case 10: $\{i, j\} = \{j^*, t^*\}$ and $k \notin \{k^*, k^* + 1\}$. We construct tournament T by permuting slots such that $(1, i, j), (k, j, i), (k^*, i^*, t^*) \in T$ holds. Since team t^* plays at venue i^* in slot k^* but does not play against j^* in slot $k^* + 1$ it never travels from venue i^* to venue j^* .

It is easy to check that all allowed triples (k, i, j) are covered by the cases. Moreover, in all cases tournament T' is obtained from T by [\(HA_{1,k,i,j}\)](#) and also satisfies the required properties. In particular, also in T' team t^* does not travel from venue i^* to venue j^* since all previous arguments were symmetric in i and j . \square

Claim 9.4. *Let $k \in S \setminus \{1\}$, let $i, j, i', j' \in V$ be distinct and let $P := \{(i, j), (i', j'), (i, j'), (i', j)\}$. If*

- (i) $(i^*, t^*) \notin P$ and $(j^*, t^*) \notin P$, or
- (ii) $(i^*, t^*) \notin P, (j^*, t^*) \in P$ and $k \notin \{k^*, k^* + 1\}$, or
- (iii) $(i^*, t^*) \in P, (j^*, t^*) \notin P$ and $k \neq k^*$, or

(iv) $(i^*, t^*) \in P$, $(j^*, t^*) \in P$ and $k = k^*$

holds, then there exist tournaments T and T' satisfying $(PS_{1,k,i,j,i',j'})$ and condition (b) of Claim 9.1.

Proof. Denote by $I := \{(1, i, j), (1, i', j'), (1, i, j'), (1, i', j), (k, i, j), (k, i', j'), (k, i, j'), (k, i', j)\}$ the set of matches required for tournaments T or T' . Let M_ℓ for all $\ell \in S$ be the perfect matchings of the canonical factorization. By permuting teams we can assume $\{i, j\}, \{i', j'\} \in M_1 = M_n$. We now exchange the roles of teams j and j' only in perfect matchings $M_n, M_{n+1}, \dots, M_{2n-2}$, which maintains the property that each edge appears in exactly two perfect matchings. Then we permute slots such that $\{i, j\}, \{i', j'\} \in M_1$ and $\{i, j'\}, \{i', j\} \in M_k$ hold.

We first describe two constructions of tournament T that are applicable in many cases. Note that via $(PS_{1,k,i,j,i',j'})$, also tournament T' is determined.

Case 1: $(i^*, t^*) \notin P$, $k \neq k^*$ and $(k^* + 1, j^*, t^*) \notin I$. We can assume that $\{j^*, t^*\} \notin M_1 \cup M_k$ holds since otherwise we have $\{j^*, t^*\} \cap \{i, j, i', j'\} = \emptyset$ due to $(i^*, t^*) \notin P$, which allows to permute teams in $V \setminus \{i, j, i', j'\}$ in order to avoid this situation. Hence, we can permute slots such that $\{j^*, t^*\} \in M_{k^*}$ holds, which is possible due to $k^* \neq k$ and $k^* \geq n$. We can now orient the matching edges in a complementary fashion such that $(1, i, j), (1, i', j'), (k, i, j'), (k, i', j), (k^*, i^*, t^*), (k^* + 1, j'', t^*) \in T$ holds for some $j'' \neq j^*$. The latter is possible due to $(k^* + 1, j^*, t^*) \notin I$. Since in both tournaments T and T' , team t^* travels from venue i^* to venue $j'' \neq j^*$, it never travels from venue i^* to venue j^* . Hence, T and T' satisfy the requirements of the claim.

Case 2: $(j^*, t^*) \notin P$, $k \neq k^* + 1$ and $(k^*, i^*, t^*) \notin I$. We can assume that $\{j^*, t^*\} \notin M_1 \cup M_k$ holds since otherwise we have $\{j^*, t^*\} \cap \{i, j, i', j'\} = \emptyset$ due to $(j^*, t^*) \notin P$, which allows to permute teams in $V \setminus \{i, j, i', j'\}$ in order to avoid this situation. Hence, we can permute slots such that $\{j^*, t^*\} \in M_{k^*+1}$ holds, which is possible due to $k^* + 1 \neq k$ and $k^* \geq n$. We can now orient the matching edges in a complementary fashion such that $(1, i, j), (1, i', j'), (k, i, j'), (k, i', j), (k^*, i'', t^*), (k^* + 1, j^*, t^*) \in T$ holds for some $i'' \neq i^*$. The latter is possible due to $(k^*, i^*, t^*) \notin I$. Since in both tournaments T and T' , team t^* travels from venue $i'' \neq i^*$ to venue j^* , it never travels from venue i^* to venue j^* . Hence, T and T' satisfy the requirements of the claim.

If condition (i) of the claim is satisfied, then $(i^*, t^*) \notin P$ implies $(k^*, i^*, t^*) \notin I$ and $(j^*, t^*) \notin P$ implies $(k^* + 1, j^*, t^*) \notin I$. Hence, depending on k , (at least) one of the two cases above is applicable and we are done.

If condition (ii) of the claim is satisfied, then case 1 is applicable unless $(k^* + 1, j^*, t^*) \in I$ holds. However, this implies $k = k^* + 1$, which is excluded by condition (ii).

If condition (iii) of the claim is satisfied, then case 2 is applicable unless $k = k^* + 1$ or $(k^*, i^*, t^*) \in I$ holds. However, the latter would imply $k = k^*$, which is excluded by condition (iii). Hence, $k = k^* + 1$, $i^* \in \{i, i'\}$, $t^* \in \{j, j'\}$ and $j^* \notin \{i, j, i', j'\}$ hold. We permute teams $V \setminus \{i, j, i', j'\}$ and slots $S \setminus \{1, k\}$ such that $\{j^*, t^*\} \in M_{k^*}$ holds. We can now orient the matching edges in a complementary fashion such that $(1, i, j), (1, i', j'), (k, i, j'), (k, i', j), (k^*, j^*, t^*) \in T$ holds. In tournaments T and T' , team t^* plays away at venue i^* in slots 1 or $k^* + 1$. Due to $k^* \geq n$ we have $k^* - 1 \neq 1$, and thus team t^* travels from a venue different from i^* to venue j^* , and thus never travels from venue i^* to venue j^* .

If condition (iv) of the claim is satisfied, then $\{i, i'\} = \{i^*, j^*\}$ and $t^* \in \{j, j'\}$ holds. We orient the matching edges in a complementary fashion such that $(1, i, j), (1, i', j'), (k, i, j'), (k, i', j) \in T$ holds. In tournaments T and T' , team t^* plays away at venues i^* and j^* in slots 1 and $k^* \geq n$, and thus never travels from venue i^* to venue j^* . Moreover, match (k^*, i^*, t^*) is contained in one tournament and match (k^*, j^*, t^*) in the other. \square

A.4 Tournaments for Theorem 10

For the claims in the proof of Theorem 10, we are given a slot $k^* \in S \setminus \{2n - 2\}$ and two distinct teams t^*, j^* . Note that we also assume $n \geq 8$. To enhance readability of the proofs we restate the claim that contains sufficient conditions for satisfying (6) with equality.

Claim 10.1. *Let T be a tournament with*

- (a) $(1, j^*, t^*) \in T$ and $k^* = 1$ holds, or
- (b) $(1, j^*, t^*) \in T$ and team t^* plays at home in slot k^* , or
- (c) $(k^* + 1, j^*, t^*) \in T$ and team t^* plays at home in slot k^* , or
- (d) $(k^* + 1, j^*, t^*) \in T$ and team t^* plays away in slot k^* , or
- (e) $(k^*, j^*, t^*) \in T$, $k^* \geq 2$ and team t^* plays away in slot $k^* - 1$, or
- (f) team t^* plays at home in slot k^* and never travels from its home venue to venue j^* .

Then $(\chi(T), \psi(T))$ satisfies (6) with equality. Moreover, team t^* travels from its home venue to venue j^* if and only if one of conditions (a)–(c) is satisfied.

Claim 10.2. *For all $(t, i, j) \in V \times A$ with $(t, i, j) \neq (t^*, t^*, j^*)$ there exists a tournament T satisfying a condition from Claim 10.1.*

Proof. We distinguish two cases.

Case 1: $(t, i) \neq (t^*, j^*)$ or $k^* \leq 2n - 4$. If $t = i$, let $i' \in V \setminus \{i, j, t, t^*, j^*\}$ and $j' := j$. If $t \neq i$, let $i' := i$ and $j' \in V \setminus \{i, j, t, t^*, j^*\}$. Note that in either case i', j' and t are distinct. Now observe that $(j^*, t^*) \neq (j', t)$ holds since otherwise $j' = j$, and thus $t = i$ would hold, contradicting $(t, i, j) \neq (t^*, t^*, j^*)$. We construct tournament T from a canonical factorization by permuting slots and teams such that $(k^* + 1, j^*, t^*), (k, i', t), (k + 1, j', t) \in T$ holds for $k = k^* + 1$ if $(i', t) = (j^*, t^*)$ and for some $k \in S \setminus \{k^*, k^* + 1, 2n - 2\}$ otherwise. In the former case, since the first two matches $(k^* + 1, j^*, t^*)$ and (k, i', t) are equal and $k + 1 = k^* + 2 \leq 2n - 2$ holds, the existence of T is obvious. In the latter case, the three distinct matches $(j^*, t^*), (i', t)$ and (j', t) have two scheduled in different slots. Since t appears in two of the matches, edges $\{i', t\}$ and $\{j', t\}$ already appear in different matchings of a canonical factorization, and thus only slots must be permuted to construct T . It is easy to see that T satisfies either condition (c) or (d) of Claim 10.1.

Case 2: $(t, i) = (t^*, j^*)$ and $k^* = 2n - 3$. We construct tournament T from a canonical factorization by permuting slots such that $(k^*, j^*, t^*), (k^* + 1, i', t^*) \in T$ holds for some $i' \in V \setminus \{j, t^*, j^*\}$, and such that t^* plays away in slot $k^* - 1$. In this case T satisfies condition (e) of Claim 10.1. \square

Claim 10.3. *For each $(k, i, j) \in \mathcal{M}$ with $k \neq \bar{k}$, $\{i, j\} \neq \{j^*, t^*\}$ and for which $k = k^*$ implies $t^* \notin \{i, j\}$ there exist tournaments T and T' satisfying $(\text{HA}_{\bar{k}, k, i, j})$ such that T and T' satisfy the same condition from Claim 10.1.*

Proof. We distinguish two cases. Note that via $(\text{HA}_{\bar{k}, k, i, j})$, also tournament T' is determined.

Case 1: $k \neq k^* + 1$. We construct tournament T from a canonical factorization by permuting slots and teams such that $(\bar{k}, i, j), (k, j, i), (k^* + 1, j^*, t^*) \in T$ holds. Since $\bar{k} \neq k^*$ holds and $k = k^*$ implies $t^* \notin \{i, j\}$, tournaments T and T' both satisfy condition (c) or both satisfy condition (d) of Claim 10.1.

Case 2: $k = k^* + 1$. We construct tournament T from a canonical factorization by permuting slots and teams such that $(\bar{k}, i, j), (k, j, i), (1, j^*, t^*) \in T$ holds and, if $k^* \neq 1$, team t^* plays at home in slot k^* . Tournaments T and T' satisfy condition (a) (resp. condition (b) if $k^* \neq 1$) of Claim 10.1. \square

Claim 10.4. *Let $k \in S \setminus \{\bar{k}\}$, let $i, j, i', j' \in V$ be distinct such that $(j^*, t^*) \notin \{(i, j), (i', j'), (i, j'), (i', j)\}$ or $k \notin \{1, k^*, k^* + 1\}$ holds. Then there exist tournaments T and T' satisfying $(PS_{\bar{k}, k, i, j, i', j'})$ such that T and T' satisfy the same condition from Claim 10.1.*

Proof. Let M_ℓ for all $\ell \in S$ be the perfect matchings of the canonical factorization. By permuting teams we can assume $\{i, j\}, \{i', j'\} \in M_1 = M_n$. We now exchange the roles of teams j and j' only in perfect matchings $M_n, M_{n+1}, \dots, M_{2n-2}$, which maintains the property that each edge appears in exactly two perfect matchings. Then we permute slots such that $\{i, j\}, \{i', j'\} \in M_{\bar{k}}$ and $\{i, j'\}, \{i', j\} \in M_k$ hold. We describe how to construct tournament T . Note that via $(HA_{\bar{k}, k, i, j})$, also tournament T' is determined. We distinguish three cases.

Case 1: $k \neq k^* + 1$ and $(j^*, t^*) \notin \{(i, j), (i', j'), (i, j'), (i', j)\}$. We can permute teams $V \setminus \{i, j, i', j'\}$ and slots $S \setminus \{\bar{k}, k\}$ such that $\{j^*, t^*\} \in M_{k^*+1}$ holds. Tournament T is obtained by orienting the matching edges in a complementary fashion such that $(\bar{k}, i, j), (\bar{k}, i', j'), (k, i, j'), (k, i', j), (k^* + 1, j^*, t^*) \in T$ holds. Since a partial slot swap does not change the home-away pattern of t^* , tournaments T and T' both satisfy condition (c) or both satisfy condition (d) of Claim 10.1.

Case 2: $k = k^* + 1$ and $(j^*, t^*) \notin \{(i, j), (i', j'), (i, j'), (i', j)\}$. We can permute teams $V \setminus \{i, j, i', j'\}$ and slots $S \setminus \{\bar{k}, k\}$ such that $\{j^*, t^*\} \in M_1$ holds. Tournament T is obtained by orienting the matching edges in a complementary fashion such that $(\bar{k}, i, j), (\bar{k}, i', j'), (k, i, j'), (k, i', j), (1, j^*, t^*) \in T$ holds and, if $k^* \neq 1$, such that team t^* plays at home in slot k^* . Since a partial slot swap does not change the home-away pattern of t^* , tournaments T and T' both satisfy condition (a) or both satisfy condition (b) of Claim 10.1.

Case 3: $k \notin \{1, k^*, k^* + 1\}$ and $(j^*, t^*) \in \{(i, j), (i', j'), (i, j'), (i', j)\}$. Let $k' \in S \setminus \{1, k^* - 1, k^*, k^* + 1, \bar{k} - 1, \bar{k}, k - 1, k\}$ (note that $n \geq 8$ implies $|S| \geq 14$). We can permute teams $V \setminus \{i, j, i', j'\}$ and slots $S \setminus \{\bar{k}, k\}$ such that $\{j'', t^*\} \in M_{k^*}, \{i'', t^*\} \in M_{k'-1}$ and $\{j^*, t^*\} \in M_{k'}$ hold for distinct teams $i'', j'' \in V \setminus \{i, j, i', j'\}$. Tournament T is obtained by orienting the matching edges in a complementary fashion such that $(\bar{k}, i, j), (\bar{k}, i', j'), (k, i, j'), (k, i', j), (k^*, t^*, j''), (k' - 1, i'', t^*), (k', j^*, t^*) \in T$ holds. Both tournaments T and T' satisfy condition (f) of Claim 10.1 since team t^* travels from venue $i'' \neq t^*$ to venue j^* , and thus never travels from its home venue to venue j^* . \square

A.5 Tournaments for Theorem 12

For the claims in the proof of Theorem 12 we are given a team t^* and a venue $j^* \neq t^*$. Note that we also assume $n \geq 6$. To enhance readability of the proofs we restate the claim that contains sufficient conditions for satisfying (1g) with equality.

Claim 12.1. *Let T be a tournament*

- (a) *in which team t^* never travels from its home venue to venue j^* , or*
- (b) *with $(1, j^*, t^*) \in T$.*

Then $(\chi(T), \psi(T))$ satisfies (1g) with equality. Moreover, team t^ travels from its home venue to venue j^* if and only if condition (b) is satisfied.*

Claim 12.2. For all $(t, i, j) \in V \times A$ with $(t, i, j) \neq (t^*, t^*, j^*)$ there exists a tournament T satisfying a condition of Claim 12.1.

Proof. If $t = i$, let $i' \in V \setminus \{i, j, t, t^*, j^*\}$ and $j' := j$. If $t \neq i$, let $i' := i$ and $j' \in V \setminus \{i, j, t, t^*, j^*\}$. Note that in either case i', j' and t are distinct. Moreover, we have $(j', t) \neq (j^*, t^*)$ since otherwise $t = i$ and thus $(t, i, j) = (t^*, t^*, j^*)$ would contradict the assumption of the claim. If $(i', t) = (j^*, t^*)$ holds, then we construct tournament T from a canonical factorization by permuting slots and teams such that $(1, j^*, t^*), (2, j', t) \in T$ holds. Otherwise, we construct T with $(1, j^*, t^*), (2, i', t), (3, j', t) \in T$.

In both cases, tournament T satisfies condition (b) and team t^* travels from venue i' to venue j' , which implies that team t never travels from venue i to venue j since exactly one of the teams i', j' is equal to its counterpart i, j . \square

Claim 12.3. For each $(k, i, j) \in \mathcal{M}$ with $k \neq n$ and $\{i, j\} \neq \{j^*, t^*\}$ there exist tournaments T and T' satisfying $(HA_{n,k,i,j})$ such that T and T' satisfy the same condition from Claim 12.1.

Proof. Note that via $(HA_{n,k,i,j})$, also tournament T' is determined. We distinguish two cases.

Case 1: $k = 1$. Let $i \in V \setminus \{j^*, t^*\}$. We construct tournament T from a canonical factorization by permuting slots and teams such that $(n, i, j), (1, j, i), (2, i, t^*), (3, j^*, t^*) \in T$ holds. In both tournaments, team t^* travels from venue $i \neq t^*$ to venue j^* , which implies that team t^* never travels from its home venue to venue j^* . Hence, T and T' both satisfy condition (a) of Claim 12.1.

Case 2: $k \geq 2$ or $\{j^*, t^*\} \cap \{i, j, i', j'\} = \emptyset$. We construct tournament T from a canonical factorization by permuting slots and teams such that $(n, i, j), (k, j, i), (1, j^*, t^*) \in T$ holds. Tournaments T and T' both satisfy condition (b) of Claim 12.1. \square

Claim 12.4. Let $k \in S \setminus \{n\}$, let $i, j, i', j' \in V$ be distinct such that $(k, j^*, t^*) \notin \{(1, i, j), (1, i', j'), (1, i, j'), (1, i', j)\}$ holds. Then there exist tournaments T and T' satisfying $(PS_{n,k,i,j,i',j'})$ such that T and T' satisfy the same condition from Claim 12.1.

Proof. Let M_ℓ for all $\ell \in S$ be the perfect matchings of the canonical factorization. By permuting teams we can assume $\{i, j\}, \{i', j'\} \in M_1 = M_n$. We now exchange the roles of teams j and j' only in perfect matchings $M_n, M_{n+1}, \dots, M_{2n-2}$, which maintains the property that each edge appears in exactly two perfect matchings. Then we permute slots such that $\{i, j\}, \{i', j'\} \in M_n$ and $\{i, j'\}, \{i', j\} \in M_k$ hold. We describe how to construct tournament T . Note that via $(HA_{n,k,i,j})$, also tournament T' is determined. We distinguish two cases.

Case 1: $k = 1$. We can permute teams $V \setminus \{i, j, i', j'\}$ and slots $S \setminus \{n, k\}$ such that $\{j^*, t^*\} \in M_3$ and $\{i'', t^*\} \in M_2$ for some $i'' \in V \setminus \{j^*, t^*\}$. Tournament T is obtained by orienting the matching edges in a complementary fashion such that $(n, i, j), (n, i', j'), (1, i, j'), (1, i', j), (2, i'', t^*), (3, j^*, t^*) \in T$ holds. Tournaments T and T' both satisfy condition (a) of Claim 12.1.

Case 2: $k \geq 2$ or $\{j^*, t^*\} \cap \{i, j, i', j'\} = \emptyset$. We can permute teams $V \setminus \{i, j, i', j'\}$ and slots $S \setminus \{n, k\}$ such that $\{j^*, t^*\} \in M_1$. Tournament T is obtained by orienting the matching edges in a complementary fashion such that $(n, i, j), (n, i', j'), (k, i, j'), (k, i', j), (1, j^*, t^*) \in T$ holds. Tournaments T and T' both satisfy condition (b) of Claim 12.1.

Case 3: $k \geq 2$ and $(j^*, t^*) \in \{(i, j), (i', j'), (i, j'), (i', j)\}$. We can permute slots such that the edges that match t^* in M_{k-1}, M_k, M_{n-1} and M_n are different (unless $k = n \pm 1$ in which case $M_k = M_{n-1}$ or $M_{k-1} = M_n$ holds). Tournament T is obtained by orienting the matching edges in a complementary fashion such that $(n, i, j), (n, i', j'), (k, i, j'), (k, i', j) \in T$ holds and such that t^*

plays away in slots $k - 1$, k , $n - 1$ and n . Hence, in none of the tournaments T and T' , team t^* travels from its home venue to venue j^* , which shows that T and T' both satisfy condition (a) of Claim 12.1. \square

A.6 Tournaments for Theorem 14

For the claims in the proof of Theorem 14 we are given a team t^* and a venue $i^* \neq t^*$. Note that we also assume $n \geq 8$.

Claim 14.1. *For all $(t, i, j) \in V \times A$ with $(t, i) \neq (t^*, i^*)$ there exists a tournament in which team t never travels from venue i to venue j and in which team t^* leaves venue i^* exactly once.*

Proof. Let T be a tournament from Claim 5.1. We do not need to restrict the schedule of team t^* since it leaves $i^* \neq t^*$ only once, namely after playing away against i^* . \square

Claim 14.2. *For all distinct $i, j \in V$ and for each $k \in S \setminus \{1\}$ there exist tournaments T and T' satisfying $(HA_{1,k,i,j})$ such that in both tournaments team t^* leaves venue i^* exactly once and to the same venue.*

Proof. We distinguish two cases:

Case 1: $\{i, j\} \neq \{i^*, t^*\}$. Since $|S| = 2n - 2 \geq 6$ holds, there exists a slot $k^* \in S \setminus \{1, k - 1, k, 2n - 2\}$. We construct tournament T from a canonical factorization by permuting slots and teams such that $(1, i, j), (k, j, i), (k^*, i^*, t^*), (k^* + 1, i, t^*) \in T$ for some $i \in V$.

Case 2: $\{i, j\} = \{i^*, t^*\}$. We construct tournament T from a canonical factorization by permuting slots and teams such that $(1, i, j), (k, j, i) \in T$. Moreover, t^* shall play at home in slots 2 (unless $k = 2$ and this conflicts with $(k, j, i) \in T$) and $k + 1$ (unless $k + 1 \notin S$).

In both cases, tournament T' is obtained from T by $(HA_{1,k,i,j})$. By construction, in both tournaments team t^* leaves i^* to its home venue t^* , after slot 1 in one tournament and after slot k in the other tournament. \square

Claim 14.3. *For each slot $k \in S \setminus \{1\}$ and for distinct teams $i, j, i', j' \in V$ with $(i^*, t^*) \notin \{(i, j), (i', j'), (i, j'), (i', j)\}$ there exist tournaments T and T' satisfying $(PS_{1,k,i,j,i',j'})$ such that in both tournaments team t^* leaves venue i^* exactly once and to the same venue.*

Proof. Let M_ℓ for all $\ell \in S$ be the perfect matchings of the canonical factorization. By permuting teams we can assume $\{i, j\}, \{i', j'\} \in M_1 = M_n$ and if both i^* and t^* are distinct from i, i', j and j' , then $\{i^*, t^*\} \notin M_1 = M_n$ (this is possible because $n \geq 8$). We now exchange the roles of teams j and j' only in perfect matchings $M_n, M_{n+1}, \dots, M_{2n-2}$, which maintains the property that each edge appears in exactly two perfect matchings. Tournament T is obtained by orienting the edges in a complementary fashion and permuting slots such that $(1, i, j), (1, i', j'), (k, i, j'), (k, i', j) \in T$.

By construction and by the assumptions of the claim, t^* does not play away against i^* in slots 1 or k . Hence, we can permute the slots in $S \setminus \{1, k\}$ such that for some slot $k' \in S \setminus \{1, k - 1, k, 2n - 2\}$, we have $(k', i^*, t^*) \in T$ and such that t^* plays at home in slot $k' + 1$.

Finally, tournament T' is constructed from T by $(PS_{1,k,i,j,i',j'})$. By construction, T and T' satisfy all requirements from the claim. \square

A.7 Tournaments for Theorem 15

For the claims in the proof of Theorem 15, we are given a team t^* and two slots $k^*, \bar{k} \in S$ with $k^* \leq n - 1$ and $k^* < \bar{k} < k^* + n - 1$. Note that we also assume $n \geq 6$.

Claim 15.2. For all $(t, i, j) \in V \times A$ with $(t, i) \neq (t^*, t^*)$ there exists a tournament in which team t never travels from venue i to venue j and in which team t^* leaves its home venue exactly once.

Proof. If $t = i$, let $i' \in V \setminus \{i, j, t, t^*\}$ and $j' := j$. Otherwise, let $i' := i$ and $j' \in V \setminus \{i, j, t, t^*\}$. Note that in either case i', j' and t are distinct. We construct tournament T from a canonical factorization by permuting slots and teams such that $(k, i', t), (k + 1, j', t) \in T$ for some $k \in S \setminus \{2n - 2\}$ and such that all away matches of t^* are in slots $2, 3, \dots, n$.

To see that this is possible, we discuss the cases in which $t^* \in \{i', j', t\}$ holds. If $t^* = i'$, then $t^* = i \neq t, j'$ holds and we can choose $k := 1$ such that in this slot team t^* plays at home against t . If $t^* = j'$, then $t^* = j$ and $t = i$ hold and we can choose $k := n + 1$ such that in this slot team t^* plays at home against t . Finally, if $t^* = t$, then we can choose $k := 2$ such that in slots 2 and 3 team t^* plays away against i' and j' . \square

Claim 15.3. For any slot $k \in \{1, 2, \dots, n - 1\}$ and distinct $j, j' \in V \setminus \{t^*\}$ there exist tournaments T and T' satisfying $(HA_{k, k+n-1, t^*, j})$ and such that team t^* leaves its home venue exactly once and to the venues j in T and to j' in T' .

Proof. We construct tournament T from a canonical factorization by permuting slots such that all away matches of t^* are in slots $k, k + 1, \dots, k + n - 1$ in particular such that $(k, j, t^*), (k + 1, j', t^*) \in T$ holds. Hence, in T , team t^* leaves its home venue exactly once to venue j .

Finally, tournament T' is constructed from T via a home-away swap $(HA_{k, k+n-1, t^*, j'})$, which means that in T' team t^* plays away in slots $k + 1, k + 2, \dots, k + n - 1$, starting at venue j' after playing at home in slot k . \square

Claim 15.4. For each slot $k \in S \setminus \{\bar{k}\}$ and for all distinct $i, j \in V \setminus \{t^*\}$ there exist tournaments T and T' satisfying $(HA_{\bar{k}, k, i, j})$ and such that in both tournaments team t^* leaves its home venue exactly once.

Proof. We construct tournament T from a canonical factorization by permuting slots and teams such that $(\bar{k}, i, j), (k, j, i) \in T$ and such that team t^* plays away in consecutive matches $k', k' + 1, \dots, k' + n - 1$ for some $k' \in \{1, 2, \dots, n\}$. It is easy to see that such a slot k' exists since we only have to make sure that t^* does not play against i or j in slots \bar{k} and k .

Finally, tournament T' is constructed from T via a home-away swap $(HA_{\bar{k}, k, i, j'})$, which does not affect the home-away pattern of team $t^* \notin \{i, j\}$. \square

Claim 15.5. For distinct slots $k_1, k_2 \in S$ and distinct teams $i, j, i', j' \in V$ with $t^* \notin \{i, i'\}$ and with $k_2 = k_1 + 1$ if $t^* \in \{j, j'\}$ there exist tournaments T and T' satisfying $(PS_{k_1, k_2, i, j, i', j'})$ such that in both tournaments team t^* leaves its home venue exactly once.

Proof. Let M_ℓ for all $\ell \in S$ be the perfect matchings of the canonical factorization. By permuting teams we can assume $\{i, j\}, \{i', j'\} \in M_1 = M_n$. We now exchange the roles of teams j and j' only in perfect matchings $M_n, M_{n+1}, \dots, M_{2n-2}$, which maintains the property that each edge appears in exactly two perfect matchings. We distinguish two cases:

Case 1: $t^* \in \{j, j'\}$. By symmetry we can assume $t^* = j$ without loss of generality. We construct tournament T_1 by orienting the edges in a complementary fashion such that T_1 contains the matches $(1, i, j), (1, i', j'), (n, i, j')$ and (n, i', j) and such that t^* plays away in matches $1, 2, \dots, n - 1$, except for match $k' \in \{2, 3, \dots, n - 1\}$ in which t^* plays at home against i' since the corresponding return match is scheduled in slot n . Hence, we have $(k', t^*, i') \in T_1$.

Tournament T_2 is now obtained from T_1 by exchanging slots k' and n , i.e., T_2 contains matches $(1, i, j), (1, i', j'), (k', i, j'), (k', i', j)$ and (n, t^*, i') and team t^* plays away in slots $1, 2, \dots, n - 1$.

We construct tournament T_3 from T_2 via a cyclic shift by $s := \min\{k_1 - 1, n - 1\}$. We obtain $(1 + s, i, j), (1 + s, i', j'), (k' + s, i, j'), (k' + s, i', j) \in T_3$ and team t^* plays away in slots $1 + s, 2 + s, \dots, n - 1 + s$. Note that $k_1, k_2 \in S' := \{1 + s, 2 + s, \dots, n - 1 + s\}$.

Finally, we construct tournament T from T_3 by exchanging slots within S' such that T contains the matches $(k_1, i, j), (k_1, i', j'), (k_2, i, j')$ and (k_2, i', j) while maintaining the property that team t^* plays away in slots S' .

Case 2: $t^* \notin \{j, j'\}$. We construct tournament T_1 by orienting the edges in a complementary fashion such that T_1 contains the matches $(1, i, j), (1, i', j'), (n, i, j')$ and (n, i', j) and such that t^* plays away in matches $1, 2, \dots, n - 1$.

Let $s \in \{0, 1, 2, \dots, n - 1\}$ be such that $S' := \{1 + s, 2 + s, \dots, n - 1 + s\}$ contains exactly one of the two slots k_1, k_2 . By symmetry we can assume $k_1 \in S'$ and $k_2 \in S \setminus S'$ (otherwise exchange k_1 with k_2, i with i' and j with j'). We construct tournament T_2 from T_1 via a cyclic shift by s . We obtain $(1 + s, i, j), (1 + s, i', j'), (n + s, i, j'), (n + s, i', j) \in T_2$ and team t^* plays away in slots S' .

Finally, we construct tournament T from T_2 by exchanging slot $1 + s \in S'$ with $k_1 \in S'$ and slot $n + s \in S \setminus S'$ with $k_2 \in S \setminus S'$, which maintains the property that team t^* plays away in slots S' . Moreover, $(k_1, i, j), (k_1, i', j'), (k_2, i, j'), (k_2, i', j) \in T$ holds.

In both cases we construct T' from T by a partial slot swap ($\text{PS}_{k_1, k_2, i, i', j, j'}$). \square

Claim 15.6. *For each slot $k \in \{k^* + 1, k^* + 2, \dots, k^* + n - 3\}$ and each team $j \in V \setminus \{t^*\}$ there exist tournaments T and T' satisfying $(\text{HA}_{k, k+1, j, t^*})$ such that in both tournaments team t^* leaves its home venue exactly twice and plays away in slots k^* and $k^* + n - 1$.*

Proof. Let M_ℓ for all $\ell \in S$ be the perfect matchings of the canonical factorization. We permute matchings such that $\{t^*, j\} \in M_k, M_{k+1}$ and such that in matchings $M_{k^*}, M_{k^*+1}, \dots, M_{k-1}, M_k, M_{k+2}, M_{k+3}, \dots, M_{k^*+n-2}, M_{k^*+n-1}$ node t^* is matched to every other node exactly once. Tournament T is constructed by orienting edge $\{t^*, j\}$ as (j, t^*) in M_k and as (t^*, j) in M_{k+1} , and such that team t^* plays away in slots $k^*, k^* + 1, \dots, k^* + n - 1$ except for slot $k + 1$. Team t^* leaves its home venue only before slot k^* and after slot $k + 1$. We construct tournament T' by $(\text{HA}_{k, k+1, j, t^*})$, in which t^* plays at home in slot k and away in slot $k + 1$. Moreover, in T' team t^* leaves its home venue only before slot k^* and after slot k . \square

Claim 15.7. *For all distinct teams $j, j' \in V \setminus \{t^*\}$ there exist tournaments T and T' satisfying $(\text{HA}_{k^*, k^*+1, t^*, j})$ such that team t^* leaves its home venue exactly once, namely to venue j , in tournament T and exactly twice, namely to venues j and j' , in tournament T' where it plays away in slots k^* and $k^* + n - 1$.*

Proof. Let M_ℓ for all $\ell \in S$ be the perfect matchings of the canonical factorization. We permute matchings such that $\{t^*, j\} \in M_{k^*}, M_{k^*+1}, \{t^*, j'\} \in M_{k^*+2}$ and such that in matchings $M_{k^*+1}, M_{k^*+2}, \dots, M_{k^*+n-1}$ node t^* is matched to every other node exactly once. Tournament T is constructed by orienting edge $\{t^*, j\}$ as (t^*, j) in M_{k^*} and such that team t^* plays away in slots $k^* + 1, k^* + 2, \dots, k^* + n - 1$. Team t^* leaves its home venue exactly once, namely after slot k^* to venue j . We construct tournament T' by $(\text{HA}_{k^*, k^*+1, t^*, j})$, in which t^* plays away in slot k^* and home in slot $k^* + 1$. Moreover, in T' team t^* leaves its home venue exactly twice, namely before slot k^* to venue j and after slot $k^* + 1$ to venue j' . \square