Editorial

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Recent Advances in Least-Squares and Discontinuous Petrov–Galerkin Finite Element Methods

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Abstract: Least-squares (LS) and discontinuous Petrov–Galerkin (DPG) finite element methods are an emerging methodology in the computational partial differential equations with unconditional stability and built-in a posteriori error control. This special issue represents the state of the art in minimal residual methods in the L^2-norm for the LS schemes and in dual norm with broken test-functions in the DPG schemes.

Keywords: Discontinuous Petrov–Galerkin, Least-Squares, Minimal Residual

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Least-squares (LS) and discontinuous Petrov–Galerkin (DPG) finite element schemes are an attractive class of methods for the numerical treatment of partial differential equations. Among other advantages, they produce hermitian and positive-definite discrete systems of equations and enjoy a built-in *a posteriori* error estimator (based on an appropriate residual norm) on fairly general meshes.

The first-order system least-squares (FOSLS) method minimizes the L^2-residuals for first-order systems of partial differential equations. It has been successfully applied to a wide range of problems arising in computational fluid dynamics, solid mechanics, and electromagnetics. The choice of finite element spaces for the approximation of the different variables is therefore not restricted by compatibility conditions. For an overview of least-squares methods, we refer to [3]. One main ingredient in the analysis of this method is the coercivity of the homogeneous least-squares functional, which in turn, usually implies optimal error estimates in an energy norm for conforming finite element approximations. However, the scaling between the different parts of the LS functional is critical. For instance, weighting the equation corresponding to the symmetry of the stress tensor in solid mechanics gives better results in bending-dominated situations. Motivated by this, the authors of [2] improve the approximation of the momentum balance using a LS finite element method based on the stress-displacement-rotation formulation and the stress-displacement-rotation-pressure formulation.

Using Newton–Raphson iterations, extensions to nonlinear problems are, in general, rather straightforward. However, rigorous results are missing for challenging problems like the Monge–Ampere equation.

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The paper [17] gives some insights into the numerical difficulties arising from the use of outer Newton-like linearizations for this equation. The sequence of first-order div-curl systems converges in a small number of steps, and optimal finite element convergence rates with respect to the mesh size are achieved for problems on convex domains with smooth and appropriately bounded data.

A further application of the least-squares approach consists in the so-called saddle-point least-squares (SPLS) discretization. A discrete test space is paired with a discrete trial space using the operator associated with the bilinear form in such a way that this pair automatically satisfies the discrete inf-sup condition. Extension of this methodology to PDEs with discontinuous coefficients arising in elliptic interface problems, via a non-conforming trial space is performed in [1]. They show that a higher-order approximation of the fluxes is achieved. Combined with the gradient-recovery technique and adaptive refinement, optimal discrete approximation spaces for the flux are constructed. The SPLS method may be put in duality with the DPG method as follows: while the DPG method starts with a trial space and designs an accompanying stable test space [7], the SPLS method does the reverse.

A program for DPG schemes having test function spaces that are automatically computable to guarantee stability was laid out in [8] and a fully automatic adaptive process was presented in [9]. Most DPG schemes are based on ultra-weak formulations. A characterization of interface spaces that connect the broken spaces to their unbroken counterparts is provided in [5]. DPG schemes lead to discretizations that deliver close to optimal L^2-approximations of variables of interest. One implication is that field variables of different regularities are measured in a composite norm where errors in all variables appear coupled together. In [11], the author considers general elliptic problems of second order, and shows that the error in the primal variable may be separately bounded. Thus the primal variable can indeed be approximated optimally up to higher-order terms. Furthermore, superconvergence results are proved for a local postprocessing technique and for a DPG variant based on augmented trial spaces.

The DPG framework is also well suited for space-time discretizations and, in particular, well established for Schrödinger equations and acoustic waves. Avoiding explicit traces of the graph energy spaces, the authors of [10] show that a test space exists which guarantees discrete inf-sup stability for general wave equations. This analysis also transfers to general wave equations in heterogeneous media and provides robust estimates in the energy norm. The relation between the existence of Fortin operators and discrete stability of the practical DPG method is utilized to provide insights on analysis of both ideal and practical DPG methods.

Although the inherent a posteriori error control [4] is a major advantage of the DPG method, the analysis of a standard marking strategy for the stationary linear transport equation is a demanding task. Relationships with DPG error indicators enables a practical reliable and efficient error control in the trial norm for the computed DPG approximations as shown in [6]. A suitable adaptive mesh refinement strategy with Dörfler marking is designed and the contraction property for the errors in adaptive DPG approximations is proven.

The ability of the DPG method to deliver robust convergence for singular perturbation problem relies on its ability for resolving the (Riesz representation of the) residual approximated with the enriched test space technique. The authors of [16] use the convection-dominated diffusion problem with the adjoint graph norm, to study an alternative approximation of the residual using splines.

In order to consider a wider class of irregular solutions driven by possibly irregular sources, and minimize the Gibbs effect inherent in the L^2-approximation, the authors of [14] generalize the DPG method to non-Hilbert Banach spaces, introducing the discrete-dual minimal-residual method. For the approximation of the solution of the advection-reaction equation, they propose to minimize the residual in discrete dual norms in a weak Banach-space setting. The weak formulation allows for the direct approximation of solutions in the Lebesgue L^p-space, $1 < p < \infty$. Several discrete subspace pairs for the discrete stability of the method and quasi-optimality in L^p are studied.

One original motivation of the DPG methodology is the design of stable schemes by the computation of optimal test spaces in the class of Petrov–Galerkin schemes. For non-Hilbert Banach spaces, the idea is generalized in [12] for unsymmetric variational formulations of convection-diffusion-reaction equations in $W^{1,q}$ for $1 < q < 2$. The analogy highlights the duality map as a substitute for the Riesz isometry in the DPG context with a stability proof.

DPG methods are more expensive than standard Galerkin methods, especially at the element level. One of the critical bottlenecks comes with the integration of the Gram matrix corresponding to the test space inner product. The paper [13] presents algorithms for fast integration of Gram matrices corresponding to all exact sequence energy spaces: H^1-, $H(\mathrm{curl})$-, $H(\mathrm{div})$-, and L^2-conforming elements, for the hexahedral element exploiting the tensorization techniques.

Another challenge is concerned with the transformation of a global problem into a set of sub-problems. The paper [18] uses algebraic dual polynomials to set up the Steklov-Poincaré operator for the mixed formulation of the Poisson problem.

The paper [15] presents an update on *Camellia*, a user friendly software for a rapid implementation of the DPG method for a wide class of practical problems in 2D and 3D applications. Recent improvements are extended symbolic manipulations and added support for standard Galerkin formulations.

The papers of this special issue have been carefully selected by the editors on the occasion of the third workshop on minimum residual and least-squares finite element methods held at Portland State University, Portland, Oregon, USA.

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