The Deal-by-Deal Principle for Rational Choice on the *Qui Vive*[*]

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Abstract

Based on the deal-by-deal principle, we propose preference indices as a starting point for the analysis of rational choice. They specify *how* good acts are *as a deal*. Thus, the recognition of reference points in Prospect Theory is interpreted as a crucial step *towards* modeling rationality, to be extended to a degree of goodness level for all acts, in order to reflect context dependency in a more refined way. We show how this leads to an uncomplicated syntax for updating, reconciling familiar concepts in dynamic choice theory. A representation theorem characterizes the weakly decomposable preferences in our framework.

We also describe a relaxation of gain-loss separability in Prospect Theory, by a refinement of comonotonicity, to bring it closer to our

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framework. Furthermore, we show that S-shaped probability weighting can be explained as an artefact under betweenness for binary lotteries, and conclude that rationality comes nearer to behavioral evidence than generally believed.

Keywords: reference point, updating, prospect theory, status quo bias, dynamic consistency, betweenness, consequentialism, probability weighting, rank dependent utility, Choquet integral,
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1 Introduction

Rationality of choice is a contentious issue, for good reasons. Besides the complexity of real world decisions, there is the fundamental philosophical problem of rationalizing the arguments for rationality, spiraling off eventually into questions about free will and the meaning of life, if not the universe. At best, we can formulate conditional statements that take some elementary principles of rationality for granted, for the time being, agreeing with (Gilboa, 2015) that sorting out these principles is essentially a matter of ongoing discussion.

One of these rationality principles is that we should be indifferent to different ways achieving exactly the same prospect on final wealth. It is the cornerstone of Prospect Theory (PT), the best-known behavioral model for decision making, to recognize that we do not choose this way, but take changes as carriers of value, rather than end states (Kahneman and Tversky, 1979). Whereas we should dryly aggregate changes with respect to a fixed status quo reference point, we adjust it time and again. Concretely, we should not discriminate between a low bonus followed by a lottery with gains, and a high bonus followed by a lottery with losses, when their net result is the same.[1] This marks the farewell of behavioral modeling to rationality, confirmed again in Kahneman et al. (1991): "the important notion of a stable preference order must be abandoned in favor of a preference order that depends on the current reference level."

We take the opposite standpoint, and welcome the recognition of reference points as an important step *towards* rationality. Rather than seeking some justification in our *self*, e.g. in how we expect to actually experience the consequences (Tversky and Kahneman (1991)), we emphasize the awareness of the *other*, and argue that it is primarily due to alertness, game-theoretic robustness, decision making *on the qui vive*. Our viewpoint is based on the following three considerations.

1. *A move of nature is not a contract.* We stay close to the mathematical starting point of PT, and model objects of choice as uncertain or risky prospects with monetary outcomes, to which a decision maker (DM) assigns a value that represents her preference ordering. We also stick to the original definition in Kahneman and Tversky (1979) of a prospect as a contract, but we are more keen on keeping it at that: one contract as the natural unit of

[1] For future reference, we summarize this example in Kahneman and Tversky (1979). In an experiment, they compare two situations: in Problem 11, first receiving a bonus of 1000, and then making a choice between (A) a lottery with 50% chance winning 1000, or (B) a gift of 500, and in Problem 12, first receiving a bonus of 2000, and then making a choice between (C) a loss of 1000 with 50% probability, or (D) a sure loss of 500. They find that "the majority of subjects chose B in the first problem and C in the second", while, "when viewed in terms of final states, the two choice problems are identical". They conclude: "The apparent neglect of a bonus that was common to both options in Problems 11 and 12 implies that the carriers of value or utility are changes of wealth".

choice, and a contract is not a windfall gain or loss, not a move of nature.[2]

So we take a starting point in deals as the natural objects of choice - the *deal-by-deal principle* for short. By their nature as agreement, they leave no doubt about the difference between a bonus and a lottery as two seperate, or one aggregated deal.

2. *Deals require alertness.* The role of a counterparty at the human level, not only persons, but also companies, markets, and governments, changes the perspective rigorously. The DM has more urgent questions than what the options would do to her end state, since there are intentions of others in play: who is supporting the choice set, how, why? What would correspond to no agreement? Is that an option? If not, how come? What, in fact, are my current rights and obligations, how exactly does each option on the menu change that, in each state?

Somewhat confusingly, this alertness is the least relevant for gifts. A gift as deal does not change existing rights, it does not make you alert, it improves the status quo, perhaps you ask why but anyhow you gratefully accept it - very much like a windfall gain, in fact it is one.[3]

For losses, it is the opposite. As windfall, a loss of $500, or a loss of $1000

[2]This is not without loss of generality, but a choice set of at least two options for the DM somehow relies on a mutual agreement on rights and obligations with a counterparty, *deals* for short, to effectuate the chosen option, in case it deviates from nature's move in the state that obtains. We assume that the deal aspect in each outcome can be, and has been, isolated from what nature does in each state.

[3]It is perhaps no coincidence that the classic examples on floating reference points all involve gifts before choice: a stranger offering amount X as a gift (Markowitz, 1952), a bonus (cf. footnote 1), a mug (Kahneman et al., 1991) - so that the distinction between deals and moves of nature need not be addressed. Note how Markowitz recognizes the relevance of context by the word 'stranger'.

with 50% probability (cf. Problem 12 in footnote 1), may hardly affect your wealth anyhow, but if 'a stranger' would offer you the choice between both, insisting that the option \$0 indeed does not belong to it, you may call the police. It breaks existing rights, it makes you alert, if not anxious. Obviously, this is not intended in Problem 12, and the options must be perceived as influenced by windfall losses, somehow, but context is missing to identify the deal-aspect in the prospect.

In general, deals require alertness, since there are intentions of others at play. There is the danger of deliberate repitition by others, when our decision rule is flawed (a point emphasized in e.g. Peters and Gell-Mann (2016)). A small step in the wrong direction already counts. We call this loss alertness. At the upside, which we could call gain alertness, it is for instance about recognizing win-win situations, and a feeling for how much a counterparty will give in if you try. Our point is that alertness is not a concession to rationality, but the reflection of rationality at the level of human interaction.

3. *Advanced, not primitive.* So we argue that it is not the bounded rationality of multiple selves that shifts reference points, but the rationality of being aware of the other, and his intentions (*qui vive*, in the literal and historical sense). We view it as a high-level human skill, emerging from social interaction, to enable but also control game theoretic behavior, so as to sublimate competing or even opposite intentions into deals.

This requires an antenna to discern good from bad at a distance, different from the anticipation of what will taste good or bad. But isn't it exactly the cornerstone of PT, that we cannot explain behavior without such a reference point? Is its high level not the reason that it is so elusive, so hard to model as intrinsic reference point (O'Donoghue and Sprenger, 2018)? And that it is problematic to fathom it by deeper introspection on how you will actually perceive your future wealth, since it is directed outward?

We think it should not be suppressed in a normative model, but embraced, as reflection of rationality *beyond* that of maximizing a fixed objective in terms of individual final wealth.

Of course it must have its biases, like our visual perception has - we should not trust it blindly, but it is not primitive. Moreover, it is no longer valid to identify a bias by pointing at a deviation from standard rationality. This strange asymmetry, of being quite diffuse about actual rationality in real life, but at the same time pinning down so many biases so accurately, has gone. In particular, we view loss aversion, the endowment effect, and reluctance to trade, primarily as status quo *wisdom*, rather than bias. For example, when trading at a stock exchange, we intuitively set the reference points at a wide bracket around the small bid-ask spreads that arises as the intersection of the context-dependent wide brackets of all participants. That is not a concession to rationality, it *is* rationality. It is not costly, only hypothetically costly under a rigid objective that we deliberately decline.

To summarize the argument, we draw a parallel with physics. It is as if the alertness for mutual intentions generate attracting and repelling magnetic forces, locally much stronger than the individual gravity of a good prospect on final wealth. It is the achievement of behavioral decision theory, to observe a DM carefully in a narrow frame, rotate the frame in several ways, in order to measure the working of this gravity accurately, but only to discover that there are other forces in play that are surprisingly structured. We argue, however, that it is not twisted gravity, but the magnetism of human interaction, for which we have such a strong compass. It directs to the North, regardless windfalls from the East or West. Time and again one could rightfully elicit the distinct preference for one point at the arctic circle, but only to discover that it moves.

Our reasoning is far from compelling, as announced already in our opening sentence. It is not a matter of syntax and logic, to compare rationality at different levels of reasoning. We think, however, that the arguments for not excluding full rationality outside the standard paradigm are strong. In the discussion of *Heuristics and Biases* versus *Natural Decision Making* (Kahneman and Klein, 2009), they 'fail to disagree' that there is room for skilled heuristics when an environment has a sufficient degree of predictability, and there is sufficient opportunity to learn the regularities. The environment of human interaction we refer to is highly complex, but has structure, and we are naturals. Furthermore, in our framework we allow the compass itself to determine the strenght of its signal, with no signal as limiting case.

The rest of this paper aims to provide a more indirect argument for this viewpoint: it works. Embracing a deal-specific reference point as rational, *simplifies* the logic of rational choice. It is precisely the compass that 'solves' long-standing controversies on updating, isolation, and aggregation, by pulling context-dependent complexity outside the model. We gain further confidence from the fact that in the framework we propose, eventually, we only have to combine some classic, well-established elements at the center of Decision Theory. Rational choice is of all time.

1.1 Outline

First, we refine the reference point in PT to a degree of goodness (dog)-function, more or less as the *utility hill* in the seminal paper of van Praag (1991) on welfare theory, but interpreted differently. We call the combination of an ordering and a compatible dog-function a *preference index*, and take starting point in conventional regularity axioms.

The cornerstone of our framework is a three-fold update principle: Machina's rule (Machina, 1989) for conditional value when there was counterfactual exposure (embedded updates), the fixed point update rule, also known as Pires' rule (Pires, 2002), when there was not (free updates), and intra-deal stability of the degree of goodness.

By a sensitivity axiom we characterize well-definedness of updates, and by a third we ensure that values are in the range of their updates. We claim that updating in the corresponding class is intuitive, respects the stick-to-your-choice principle, and satisfies model-closedness.

Based on an additional cross link between the dog-function and the ordering, namely that emdbedded updates *only* depend on the degree of goodness of a deal, besides of course on the sub-act itself, we arrive at a representation result, mathematically close to the weak-decomposability representation in Grant et al. (2000), and to betweenness under law-invariance.

Finally, we take a closer look at PT from our perspective, and address two issues. Firstly, it is not closed under fixed point updating, but we find a relaxation of gain-loss separability that restores it, corresponding to a two-sided centered Choquet integral. Our third axiom, however, remains problematic for capacities. Secondly, the S-shaped probability weighting in PT is more fundamentally challenged by the observation that precisely this type also arises as an artefact in the betweenness class for binary lotteries. We conclude that normative models get closer to behavioral evidence than generally believed.

2 Preference indices and loss alertness

Objects of choice are acts of the form $f : \Omega \to X$, with Ω a finite outcome space, and X a finite interval $[x_*, x^*] \subset \mathbb{R}$ of monetary outcomes. The set of all acts is denoted by $\mathcal{A}(\Omega, X)$, or simply $\mathcal{A}$. If an act f has $f(\omega) = c \in X$ on Ω, it is called a constant (act), and then we use the symbol c also for f. The interval $[\min f, \max f]$ is denoted as range(f), so range$(c) = \{c\}$ for $c \in X$. For event $E \subset \Omega$, the act $g_E f$ is the act with outcomes $g(\omega)$ on E and $f(\omega)$ in $\bar{E}$, the complement of E.

Acts are interpreted as specification of deals, with $f(\omega)$ the amount to receive (if positive) or pay (if negative) if end state ω obtains. We hence insist on taking 0 as the *zero outcome*. In particular, the *zero deal* 0 is an agreement not to pay or receive.

We consider a DM who has a *preference index* on acts in $\mathcal{A}$, formally defined as follows.

Definition 2.1 *A* preference index *is a pair* (V, Γ) *consisting of a value function* $V : \mathcal{A} \to X$ *and a degree-of-goodness (dog) function* $\Gamma : \mathcal{A} \to [-1, 1]$, *satisfying* $V(c) = c$ *for constant acts* c, *and* $\Gamma(f) = \Gamma(V(f))$. *The function* $\gamma : X \to [-1, 1]$ *defined by* $\gamma(c) = \Gamma(c)$ *is called the corresponding* status quo reference function (sqrf)*, and we also write* (V, γ) *for* (V, Γ).

The value function V represents the purely ordinal part of the preference, by the equivalence $f \succsim g$ if and only if $V(f) \geq V(g)$. The sign of $\Gamma(f)$ defines the *character* attributed to f as a deal: good when positive, ok when zero, bad when negative; absolutely good or bad at the extremes. We sometimes refer to γ, induced by the dog-function Γ, as the *nose* of the preference.

She adopts a preference index in line with her taste, belief, and perception of the inital state. This may include existing rights and obligations, possibly leading to a crisp status quo reference point $f_0 \in \mathcal{A}$, at which Γ crosses zero steeply. Her perception of the status quo can also be much more vague, when the context only provides rough indications for a reference point, reflected in Γ being flat, or even constant.

Also her taste for outcomes may be shaped by the initial state, for instance by loss aversion with respect to f_0, exactly as in PT (in which f_0 would be renamed 0). However, aversion in Γ and V are in principle independent: the DM may value acts according to their expected values, under subjective or objective probabilities, hence exhibit no loss aversion, but still have a strong aversion against any act with mean below that of f_0. To discern the latter type from loss aversion, we call it *loss alertness*, for which, intuitively, one can take as measure the steepness of γ at the left of $V(f_0)$. Of course, loss aversion may also align with loss alertness, to reflect that any outcome below $f_0(\omega)$ must be compensated by a large gain wrt to f_0 in other states.

Preference indices may hence reflect sensitivity with respect to three types of reference points: 0 as the absolute boundary between to receive and to pay, the certainty equivalent $c = V(f)$ of an act marking the balance point between the relatively good and bad side of a deal, and the certainty equivalent $c_0 = \Gamma(f_0)$ of ok-deals for the boundary of good and bad deals.

How and in which order V and Γ amalgamate from the DM's taste for outcomes, belief about states, and compass for the quality of deals, is beyond our scope, and rather a topic of descriptive modeling. We concentrate on the rationality of the resulting preference index, taking a starting point in the following standard axioms for its ordinal part.

Definition 2.2 *$\mathcal{V}$ is the class of value functions that are continuous and monotone. $\mathcal{G}$ is the class of preference indices (V, Γ), with $V \in \mathcal{V}$. $\mathcal{P}$ is the class of preference orderings $\succsim$, represented by $V \in \mathcal{V}$. The classes $\mathcal{V}$, $\mathcal{G}$ and*

$\mathcal{P}$ *are called* regular.

So the class $\mathcal{P}$ consists of preference orderings $\succsim$ on $\mathcal{A}$ that are continuous, complete, transitive, and monotone (strictly on constants c), having certainty equivalent function V in $\mathcal{V}$, i.e., $f \sim c$ if and only if $V(f) = c$. The one-to-one correspondence allows us to freely switch between the level of preference ordering and valuation in our exposition, and also write $(\succsim, \gamma)$. A regular preference index (V, γ) has γ continuous and non-decreasing.

There is some redundancy in V and Γ, for reasons explained later on. When the induced nose γ is strictly increasing, V can be reconstructed from Γ, since they represent the same ordering. In general, however, both elements are needed to specify the ordering and degree of goodness of deals.

We believe our starting point provides a useful stepping stone from standard rationality towards alternative approaches, such as e.g. the aformentioned approach of Klein, and Hausman's view on preference orderings as 'total comparative evaluations', 'more cognitive, more like judgments, than are desires' (Hausman, 2012), perhaps even towards the more radical 'fast and frugal heuristics' approach in Gigerenzer (2006). The conventional regularity properties of V are fully respected by Γ, while Γ still provides an interface to subtle, context dependent considerations per decision. This combination brings an extra dimension to the synthesis proposed in Aumann (2019), relying mainly on the difference between usual and exceptional, or even contrived, contexts.

3 Updating

In this section we address the question, how to update a preference index (V, Γ) when an event $E \subset \Omega$ obtains. We discern two principles as defining property for two different types of update of the ordinal part V. Both principles are well-known, but their distinction and reconciliation is less common, and therefore we present them in parallel, together with a third principle for the degree of goodness Γ.

In fact, we distinguish two conditional states related to an event E. First, the initial state i extended only with the information that E obtains, state $i; E$ for short. Secondly, the state $i; f; E$ she is in, when she agreed on counterfactual exposure f outside E, after i, but before E obtains. Concretely, this applies to cases with initial choice set of the form $\{g_E f, h_E f\}$, and the DM agreeing on this choice set, with the option still to choose in case E obtains - exactly as in the Allais and Ellsberg paradoxes.[4]

Let $\succsim_E$ and $\succsim_E^f$ denote the preferences of the DM in respectively $i; E$ and $i; f; E$, the notation for V and Γ is analogous. We adopt the following defining principles for these updates, to which we refer as resp. the *free* and *embedded* update in E.

Axiom 1 *(Update principle)* For all events $E \subset \Omega$ and acts $f, g, h \in \mathcal{A}$:

a. If $g_E c \succ c$, then $g_E \succ_E c$, and if $g_E c \prec c$, then $g_E \prec_E c$

b. If $g_E f \succ h_E f$, then $g_E \succ_E^f h_E$, and if $g_E f \prec h_E f$, then $g_E \prec_E^f h_E$

c. The degree of goodness is not affected by the event that obtains.

This indeed defines the updates of regular preference orderings, under the following sensitivity conditions, just characterizing their uniqueness.

[4] Our framework accommodates both positions: with acts and lotteries viewed as deals, the Allais and Ellsberg preferences are explained by our model; as moves of nature, no longer, since then Γ is irrelevant. Both positions can be maintained, since the paradoxes concern only gifts, exactly the angle of view where this important distinction becomes invisible, as argued in the introduction.

Axiom 2 *(Sensitivity)* For all events $E \subset \Omega$ and acts $f, g \in \mathcal{A}$:

a. There exists unique $c \in X$ such that $g_E c \sim c$

b. There exists unique $r \in X$ such that $g_E f \sim r_E f$.

Axiom 2b amounts to strict monotonicity in all outcomes. The formulation we use is also valid when updating would be restricted to events in a subset of 2^Ω. We call c the fixed point value of g_E, and r the replacement value of sub-act g_E in the encompassing act $g_E f$. Updating now readily follows from the principles in Axiom 1.

Lemma 3.1 *Consider a preference ordering $\succsim$ in $\mathcal{P}$ that satisfies Axiom 2.*

a. Axiom 1a defines its free update $\succsim_E$ by the fixed point update rule

$$g_E \sim_E c \quad :\Leftrightarrow \quad g_E c \sim c \text{ with } c \in X, \tag{3.1}$$

b. Axiom 1b defines its embedded update $\succsim_E^f$ by the replacement rule

$$g_E \sim_E^f r \quad :\Leftrightarrow \quad g_E f \sim r_E f \text{ with } r \in X \tag{3.2}$$

c. Axiom 1c then defines the free and the embedded update of Γ by the intra-deal stability rule

$$\Gamma_E^f(g_E) = \Gamma(g_E f) \text{ and } \Gamma_E(g_E) = \Gamma(V_E(g_E)). \tag{3.3}$$

The updates are regular.

The proof is straightforward. It is also easily verified that refining an update on E to an update on $E' \subset E$ results in the same as directly updating from Ω to E', both for free and embedded updates.[5]

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[5] This compatibility property is closely related to the commutativity property in Gilboa and Schmeidler (1993). Compatibility of free updates is also addressed in Roorda and Schumacher (2013, Prop. 4.6) (also for a continuous time axis) and Roorda and Schumacher (2016, Prop. 6.7) (for the class of variational preferences).

The fixed point update rule (3.1) is also known as Pires' rule (Pires, 2002), whereas in Eichberger et al. (2007) it is called *conditional certainty equivalent consistency*, but it has never been recognized as central aspect of a universal 'non-consequential' update rule, to our knowledge. The replacement rule (3.2) is Machina's update rule (Machina, 1989), also known as the f-Bayesian update rule in in the context of ambiguous beliefs (Gilboa and Schmeidler, 1993). In Halevy (2004) it is interpreted as *resolute choice*.

The update principle (3.3) is straightforward: don't blame nature. Whatever event obtains, if an initial deal was ok, the remaining sub-act is still ok - it is *part of the deal*, so to speak. More generally, nature does not affect the degree of goodness, only people do. Note here the immediate impact of counterfactual exposure on conditional degree of goodness. The stability rule would collapse if a counterparty or the DM would be able to influence which event obtains - the tacit assumption that *nature* chooses an element of Ω is essential. When acts were not deals, but only moves of nature, the DM could take $\gamma = 0$ as neutral value, since she has no reason to judge moves of nature as a deal.

The controversial non-consequentialist nature of Machina's rule, now arises as a natural property of conditional states. Even if the initial state i has zero status quo reference point, not updating is non-consequentialist, but the conditional state $i; f; E$ itself already is, through the influence of f on the degree of goodness. Moreover, by adhering to model closedness backwardly, we should allow that i itself already may contain information on existing rights and obligations from bygone exposure, before i, and the fixed point update is hence only 'consequentialist' in the sense that it does not rely on *additional* counterfactual exposure, in $\Omega \setminus E$. So, in our framework, non-consequentialism is a less appropriate term, it is rather keeping track of degree of goodness as a relevant aspect of the conditional 'state of the world'.

In this way the threefold update principle further underpins the rationality of resolute choice *within one deal*, and confutes the qualification of non-consequentialism in Wakker (1999) as 'believing in ghosts'.

4 Sequential Consistency

So far the only axiom restricting the class $\mathcal{P}$ of regular preference orderings is Axiom 2, which is just a sensitivity condition to guarantee well-defined updates according to the update principles in Axiom 1. We now add to this a more substantial static restriction, imposing additional structure in terms of updates on mutually exlusive events.

Axiom 3 *(Sequential Consistency)* Values should be in the range of their updates on any partitioning of Ω.[6]

Note that for the extreme partitions, namely the singleton Ω, and the partioning into its separate elements, the axiom already follows from regularity.

We first concentrate on this axiom for the free updates. This requires that for any partitioning Π of Ω,

$$V(f) \in \text{range}\{V_E(f_E) \mid E \in \Pi\}.$$

This indeed poses a substantial restriction on initial preferences, although much weaker than Savage's Sure Thing Principle.

Axiom 3a *(Equal Level Principle (ELP))* If $f_E c \sim c$ for all events E in a partitioning of Ω, then $f \sim c$.

[6] This is quite different from the notion in Sarin and Wakker (1998), where it is defined as the meta principle of model closedness. We explain below how we meet this principle.

Lemma 4.1 *Let be given a regular preference ordering $\succsim$ in $\mathcal{P}$ that satisfies Axiom 2a, and a partion Π of Ω. The collection of fixed point updates $\succsim_E$ with $E \in \Pi$, satisfies Axiom 3 if and only if $\succsim$ satisfies Axiom 3a, otherwise $\succsim$ has no free regular updates on Π that satisfy Axiom 3.*

Proof Consider an act f with $f \succsim_E c$ on Π. Then (3.1) implies that also $f_E c \succsim c$ on Π, and hence, by Axiom 3a, also $f \succsim c$. By an obvious symmetry argument, Axiom 3 follows.

Necessity of Axiom 3a: Consider an act $f \in \mathcal{A}$ with $f_E c \sim c$ on Π. We have to prove that $f \sim c$ from Axiom 2a and 3. Fix $E' \in \Pi$. Since $\succsim_E$ is regular for all $E \in \Pi$ by assumption, there exists c' such that $f_E c' \sim_E c'$ for all $E \in \Pi$. Then $f_{E'} c' \sim c'$ by Axiom 3, while also $f_{E'} c \sim c$ by assumption, so that Axiom 2a implies $c' = c$. Since $E' \in \Pi$ was arbitrary, $f_E \sim_E c$ for all $E \in \Pi$, and, again by Axiom 3, indeed $f \sim c$. The last claim is now obvious. □

This further underpins the fixed point update rule, as the only possibility to avoid that value predictably increases or decreases. It turns out that embedded updates inherit sequential consistency from free updates, since both updates always lie at the same side of the overall value of an act.

Lemma 4.2 *Assume Axiom 2 and 3a. For act f and event E, let $c, d, r \in X$ be defined by $f \sim c$, $f_E d \sim d$, and $r_E f \sim f$. Then $r - c$ and $d - c$ are both positive, both zero, or both negative.*

Proof Axiom 3a for $\Pi = \{E, \bar{E}\}$ requires $c \in \text{range}\{d, d'\}$ with d' such that $d'_E f \sim f$. So if $d > c$, $d' \leq c$, and by Axiom 2b in fact $d' < c$. Applying Axiom 3a again, now on act $r_E f$, yields $r > c$. Similarly, $d < c$ implies $r > c$. Finally, when $d = c$, then also $d' = c$, and hence $r = c$. □

Axiom 3 is trivial for Γ. Thus we arrive at the following class of preference indices.

Definition 4.3 *The* sequentially consistent class $\mathcal{S}$ *consists of regular preference indices* $(\succsim, \Gamma) \in \mathcal{G}$ *that satisfy Axiom 2 and 3a. Equivalently, it is the subclass of* $\mathcal{G}$ *with well-defined regular free updates satisfying Axiom 3 (hence also Axiom 1a), and unique embedded updates defined by Axiom 1b.*

Many controversies on rational choice reveal themselves in lack of a reasonable update rule, making consistent updating an important litmus test for normative models: (i) updating should be intuitive, (ii) conditional choice should be of the same type as initial choice, and (iii) preference reversals should be avoided. We believe the class $\mathcal{S}$ passes this test: (i) Axiom 1 has a direct intuition, not relying on a specific assumption on taste and belief, (ii) model closedness holds, since the relevance of bygone exposure is not excluded for the initial state, and, conversely, a neutral initial state without notion of bygone exposure is propagated by free updating, and (iii) Machina's rule directly guarantees the stick-to-your-choice principle.[7]

Sequential consistency adds to this the intuition of balance in conditional thinking: present value can always be seen as arising from relatively good and bad future states, both in terms of free updates and replacement values,

[7]Hanany and Klibanoff (2007, 2009) also adopt 'non-consequentialist' updating as a way to achieve dynamic consistency, in line with Machina (1989). They reject, however, Axiom 1b as a general update rule, because it lacks the closedness property. However, the distinction we make between free and embedded updates resolves the issue in Bayesian updating for the class of max-min expected utility (MEU), addressed in Hanany and Klibanoff (2007, Prop. 3). We conjecture that it also reconciles Axiom 1b with model closedness in ambiguity averse models without probabilistic sophistication, but a thorough analysis of this topic is beyond the scope of this paper.

and these two types in fact agree on this qualification of states.

We therefore propose class $\mathcal{S}$ as a general starting point for rational choice. Of course, descriptive models may further restrict this class by concepts for consistency of taste, belief, and/or compass for the value and degree of goodness of different deals. In particular, it should be emphasized that we do not model *how* a DM arrives at an sqrf γ in every decision, and what consistency rules this should obey among different decisions. Our point is that deals arise as natural units of choice, that humans have a strong compass at that level for good reasons, and that it can be reflected in γ to resolve some longstanding 'context-free' issues in updating. In class $\mathcal{S}$, there is no dilemma between forward or backward induction, no counterintuitive non-consequentialism, no issue of modelclosedness, no need to enforce consistent planning, since there is nothing to overrule.

5 The class with γ-equivalents

We have stressed that conditional choice is influenced by bygone exposure, through the conditional degree of goodness inherited from the initial state. Now we zoom in on the class of preference indices in which this is the only counterfactual memory of conditional choice. Consequently, replacement values of a sub-act g_E in $g_E f$ only depend on g_E and the initial degree of goodness, $\Gamma(g_E f)$.

Axiom 4 *(Γ-decomposability)* For all events $E \subset \Omega$ and acts $g, f, f' \in \mathcal{A}$ with $\Gamma(g_E f) = \Gamma(g_E f')$: $g_E f \sim r_E f$ if and only if $g_E f' \sim r_E f'$.

This is stronger than Axiom 3a. The following notion captures the essential feature of Axiom 4.

Definition 5.1 (γ-equivalent) *For a preference index (V, Γ) satisfying Axiom 4, the γ-equivalent of a sub-act is its replacement value in all acts of degree of goodness γ. Formally, for a given sub-act $g : E \to X$, and $\gamma \in [-1, 1]$,*

$$r_E^\gamma(g) := \begin{cases} r \text{ such that } r_E x_* \sim g_E x_* & \gamma < \Gamma(g_E x_*) \\ r \text{ such that } r_E x^* \sim g_E x^* & \gamma > \Gamma(g_E x^*) \\ r \text{ such that } r_E f \sim g_E f \text{ for (any) } f \text{ s.t. } \gamma = \Gamma(g_E f) & \text{otherwise.} \end{cases}$$

Only in the last case we call the replacement value r proper*, and we call the corresponding interval for γ the proper domain of the* Γ-profile *of g in E. The vector of γ-equivalents of an act $f \in \mathcal{A}$ with respect to a partition Π of Ω, denoted as $r_\Pi^\gamma(f)$, is called proper if all its entries are proper replacement values.*

Obviously, $r_\Pi^\gamma(f)$ is always a proper replacement vector for $\gamma = \Gamma(f)$, and hence, for this value of γ,

$$V(f) = V(r_\Pi^\gamma(f)). \tag{5.1}$$

In case Γ is constant, the axiom amounts to backward recursion, in line with the STP. On other hand, in case $\Gamma(c)$ is strictly increasing in c, the axiom is equivalent to the following.

Axiom 5 *(V-decomposability)* For all events $E \subset \Omega$ and acts $g, f, f' \in \mathcal{A}$ with $V(g_E f) = V(g_E f')$: $g_E f \sim r_E f$ if and only if $g_E f' \sim r_E f'$.

The recursion (5.1) then strengthens to a test for the value of f being at, or above, any given value $c \in X$,

$$V(f) \overset{(>)}{=} c \text{ if and only if } V(r_\Pi^{\gamma(c)}(f)) \overset{(>)}{=} c. \tag{5.2}$$

In other words, when the degree of goodness of an act is strictly monotone in value, Γ-decomposability implies a qualitative form of the STP and induced backward recursion, per level c under consideration.

5.1 Relation with weak-decomposability

Axiom 5 is equivalent to the condition of complementary replacement,

$$\text{if } f \sim r_E f \sim f_E \bar{r} \text{ then } f \sim r_E \bar{r}. \tag{5.3}$$

It means that serial and parallel replacement of sub-acts coincide. This, in turn, can be rewritten as the condition of *weak decomposability*, defined in Grant et al. (2000),

$$\text{if } f \sim g_E f \sim f_E g \text{ then } f \sim g. \tag{5.4}$$

They explain how this captures the original motivation of Savage for the STP, describe its equivalence with the notion of *dynamic programming solvability* in Gul and Lantto (1990), and provide necessary and sufficient conditions for weak decomposability in terms of what we call a *fixed point representation*, taking the form

$$V(f) \overset{(>)}{=} c \Leftrightarrow u_{c,1}(x_1) + \cdots + u_{c,n}(x_n) \overset{(>)}{=} 0, \tag{5.5}$$

where $u_{c,i}(x)$ is the (strictly increasing and continuous) utility of outcome x in the i-th state of Ω, in an act of value c, normalized to zero for the balance point $x = c$. Sufficient for Axioms 2 and 5 is the additional condition that the utility functions are strictly decreasing and continuous in c, but this is probably not necessary, see Grant et al. (2000, footnote 17). We describe a characterization in Section A.2 of the appendix.

In addition, they show how weak decomposability entails the well-known betweenness axiom (Dekel, 1986; Chew, 1983), when objects of choice can be identified with probability measures Q on X (see Section A.1 for our axioms in that setting). Under betweennes, value functions ϕ of probability measures take the form

$$\phi(Q) \overset{(>)}{=} c \Leftrightarrow E^Q u_c(x) \overset{(>)}{=} 0. \tag{5.6}$$

These representation results are straightforwardly adjusted for strengthening Axiom 5 to Axiom 4. As explained in Theorem A.1, we arrive at the Γ-*representation*

$$V(f) \overset{(>)}{=} c \Leftrightarrow \hat{u}_{\gamma,1}(x_1) + \cdots + \hat{u}_{\gamma,n}(x_n) \overset{(>)}{=} \hat{u}_{\gamma,1}(c) + \cdots + \hat{u}_{\gamma,n}(c), \qquad (5.7)$$

with parameter γ equal to $\Gamma(c)$. The result for betweenness is analogous.

6 Comparison with Prospect Theory

How far is the proposed framework from PT? We first concentrate on a technical issue, namely that capacities are not closed under fixed point updating. Secondly, we address probability weighting.

6.1 A refinement of comonotonicity

In PT, preference orderings are modeled in terms of capacities, in line with the Rank Dependent Utility (RDU) model, also called Choquet Expected Utility (CEU), proposed in Quiggin (1982) and Schmeidler (1989). For finite Ω, a (normalized) capacity ν is a mapping from subsets of Ω to the interval $[0, 1]$, characterized by the properties $\nu(\Omega) = 1$, $\nu(\emptyset) = 0$, and $\nu(A) \leq \nu(B)$ when $A \subseteq B$. For an act $f = (x_1, \ldots, x_N)$, now with indices rearranged so that $x_1 \geq x_2 \geq \cdots \geq x_N$, define

$$\nu \cdot f \ := \pi_1 x_1 + \pi_2 x_2 + \cdots + \pi_N x_N, \text{ with } \pi_j \ := \nu(\cup_j^N) - \nu(\cup_{j+1}^N), \quad (6.1)$$

where $\cup_j^k$ is the event corresponding to $(x_j, \ldots, x_k)$. The conjugate $\bar{\nu}$ is defined by $\bar{\nu}(A) = 1 - \nu(\bar{A})$, with $\bar{A}$ the complement of A in Ω. RDU consists of preference orderings representable by value functions V of the form $V(f) = \nu \cdot (u \cdot f)$, with u a strictly monotone continuous utility function.

In view of the fact that all our axioms are invariant under a strictly increasing utility transformation u of X,[8] we can restrict the attention to $u(x) = x$, and treat X as $\mathbb{R}$ in the analysis of the axioms.

Axiom 2a for a capacity ν amounts to

$$\nu(A \cap E) + \bar{\nu}(\bar{A} \cap E) > 0 \quad (A, E \subseteq \Omega).$$

For binary acts f of the form 1_A, the fpu V_E must therefore coincide with the conditional capacity ν_E defined by

$$\nu_E(A \cap E) = \frac{\nu(A \cap E)}{\nu(A \cap E) + \bar{\nu}(\bar{A} \cap E)} \quad (A, E \subseteq \Omega). \tag{6.2}$$

So if the update of ν in E is a capacity, then it is ν_E. As shown in Horie (2013), however, this is generally not the case, even when ν is convex (and hence V belongs to the class of Multiple Priors, cf. footnote 10). From our perspective, taking Axiom 1a as starting point, the class of capacities has to be adjusted, to meet the requirement of closedness under free updating.

To further analyse the issue, consider the outcome c for $V_E(f_E)$ prescribed by the fixed point update rule, $\nu \cdot f_E c = c$. The reason that V_E is generally not a capacity, is that a pair of comonotone sub-acts f_E, f'_E, need not have comonotone neutral embeddings $f_E c$, $f'_E c'$, since the rank of c in $f_E c$ need not be the same as the rank of c' in $f'_E c'$. This suggest to weaken the notion of comonotonicity, by imposing the property for pairs $(f, V(f)), (f', V(f'))$.[9]

[8] i.e., $\tilde{V}$ defined by $\tilde{V}(f) := u^{-1}V(u \cdot f) \in \mathcal{V}$ when $V \in \mathcal{V}$, and updates commute with such a transformation, i.e., $\tilde{V}_E$ is the transformation of V_E, and $\tilde{V}_E^{u \cdot f}$ that of V_E^f.

[9] Closedness can also be achieved by replacing c by a fixed embedding x_* or x^* for all acts, cf. (Gilboa and Schmeidler, 1993, Thm. 3.2), the latter one being the Dempster-Shafer rule for updating ambiguous beliefs, which amounts to maximum likelihood updating on the intersection with the class of Multiple Priors. These updates generally do not satisfy Axiom 3, but they turn out to play a natural role in the definition of γ-equivalents, Definition 5.1.

We then say that f, f' are *c-comonotone* (with respect to V).

The corresponding generalization of the comonotonic STP and its characterization in Chew and Wakker (1996) is straightforward, see Lemma A.2 in the appendix. As shown there, the corresponding extension of RDU is to apply two (not normalized) capacities inward, from both sides, until they 'meet' the balance point c. We call this a *centered capacity*, to be evaluated by a centered Choquet integral. This indeed restores modelclosedness under fixed point updating, and eliminates the need to impose gain-loss separability of PT. Axiom 3, however, turns out to leave hardly any room for nonlinear capacities, similar to the standard case, addressed in Sarin and Wakker (1998).

In brief, the refinement to c-comonotonicity makes the syntax of PT closed under fixed point updating, but still clashes with the principle that values should be in the range of their updates, Axiom 3. This leaves the accommodation of suitable forms of rank dependency in our framework a topic of future research.[10]

6.2 On probability weighting in PT

Together with the emphasis on a reference point, probability weighting determines the main characteristics of PT. We refer to Bernheim and Sprenger

[10] We note that Axiom 3a is well understood for the partially overlapping class of Multiple Priors (MP), also known as Maxmin Expected Utility (MEU), introduced in Gilboa and Schmeidler (1989) in the Anscombe-Aumann setting. We refer to Roorda and Schumacher (2007) for an extensive analysis. In brief, it is shown there that the rectangularity condition in Epstein and Schneider (2003) weakens to a *triangular* 'junctedness' condition for the priors: vectors of priors conditioned on a partition Π need not be combinable with *all* priors on Π, but only with *at least one*. Extensions to so-called variational preferences can be found in Roorda and Schumacher (2016).

(2020) for a critique on its descriptive validity. Our focus, however, is on its distance to the normative axioms we propose.

Our main observation at this point is that betweenness allows for precisely the S-shaped probability weighting in the standard PT model, for binary lotteries, as an artefact. For a binary lottery $L = (x, p; y, 1-p)$, with $x > y$, the test that L has at least value c, with $c \in [y, x]$, can be expressed as

$$\frac{p}{1-p} \geq \frac{\bar{u}_c(y)}{u_c(x)},$$

with u_c the utility function in the representation (5.6), and $\bar{u}_c$ denoting $-u_c$. This can be rewritten as

$$b \left(\frac{p}{1-p}\right)^a \geq b \, \frac{\bar{u}_c(y)^a}{u_c(x)^a},$$

which is in fact the only transformation that leaves the ratio at the right-hand side separable in y and x. In terms of probability weighting, the c-test for L takes the form

$$w(p)\ u_c(x)^a + (1 - w(p))\ b u_c(y)^a \overset{(\geq)}{=} 0 \text{ with } w(p) = \frac{bp^a}{(1-p)^a + bp^a}. \quad (6.3)$$

This corresponds to the 'linear in log odds' probability transformation (Gonzalez and Wu, 1999). In this sense, betweenness is compatible with probability weighting, for binary lotteries. The classic weighting functions in PT are of different functional form, but numerically nearly the same (Prelec, 1998). By choosing appropriate parameters a and b for each level c, the PT model in Tversky and Kahneman (1992), restricted to binary lotteries, can be approximated quite closely in the betweenness class, as follows.

Probability weighting in the standard PT model takes the form

$$w^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}} \text{ for gains; } w^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}} \text{ for losses,}$$

with $\gamma = 0.61, \delta = 0.69$. This is nearly the same as $w(p)$ in (6.3) with (i) $a = 0.543, b = 0.734$ for gains, and (ii) $a = 0.641, b = 0.834$ for losses. For mixes $L = (x, p; y, 1-p)$ with $x > 0 > y$, the ratio $w^+(p)/(w^+(p)+w^-(1-p))$ corresponds to (iii) $a = 0.651, b = 0,947$. This can be combined in one betweenness model by e.g. taking (iii) for $c = 0$, and a continuous transition towards (i) and (ii) for c sufficiently positive / negative.

So the PT model is practically indistinguishable from the betweenness class for the set comprised of all binary lotteries with gains only, losses only, or mixes with value (close to) zero. It is remarkable that the original empirical basis of PT, consisting of such binary lotteries only, also allows for linearity in probabilities, when utility is allowed to be dependending on the value of lotteries, and hence on their degree of goodness.

7 Conclusion

We conclude that rationality comes closer to behavioral evidence than generally believed. Although none of the elements in our framework are new, the contrast with the mainstream view on rationality is sharp. Quite strikingly, this is perfectly made clear by the way Tversky and Kahneman (1992) phrases the main conclusion, opposite to ours (italics are ours):

> The idealized assumption of rationality in economic theory is commonly justified on two grounds: the conviction that only rational behavior can sur*vive* in a competitive environment, and the fear that any treatment that abandons rationality will be chaotic and intractable. Both arguments are questionable. First, the evidence indicates that people can *spend a lifetime in a competitive environment* without acquiring a general ability to avoid framing

effects or to apply *linear* decision weights. Second, and perhaps more important, the evidence indicates that human choices are orderly, although not always rational in the *traditional* sense of this word.

Our diagnosis is that the tradition somehow lost attention for the notion of *how* good, believing that only *equally* good and *better* can be revealed by choice. However, *how* good does reveal its influence indirectly, but prominently, in updating. The proposed remedy is the rehabilitation of the role of degree of goodness, as central notion in the quest to advance the level at which we understand and apply aspects of rational choice.

We had to exclude several topics from the scope of this paper. We did not elaborate on the shape of the status quo reference function γ in Definition 2.1, in fact it is only relevant at which regions of X it is constant in the Γ-representation (5.7). How to model the quantitative relationship between degree of goodness Γ and the ordinal aspect V of a preference index, is a topic of future research. In particular, we only indicated that the slope of the corresponding status quo reference function γ at the left of where it crosses zero, can be taken as measure of loss alertness, independent of loss aversion, but this requires further modeling. Fairness could be modeled by allowing Γ to decrease for excessively high value, overruling monotonicity.

A more comprehensive topic of future research is the application to game theory. From our perspective, the degree of goodness should be given a more prominent role in concepts of common knowledge of rationality. For instance, in the beautiful paradox of the centipede game (Rosenthal, 1981), it turns out to be problematic to understand (true) rationality, when it is supposed to include backward induction, as consequence of subgame perfectness (formalized 'rationality'). The question in Binmore (1997) is spot on: how rational is it to be 'rational'? It is enough to escape 'rationality' by deeming all strategies to stop before step 92 *bad* right away, and understanding that the opponent must be that wise too. The extra value it brings proves how truly rational it is to play games *on the qui vive*.

A Appendix

A.1 Axioms for probability distributions (Section 5.1)

Our axioms translate to preference orderings on probability measures as follows. Let $\mathcal{D}$ denote the set of all probability distributions Q on X, and let $\phi : \mathcal{D} \to \mathbb{R}$ be a continuous valuation (in the topology of weak convergence). Define, for $K \in \mathbb{N}$, the value function $V_K^\phi(f) = \phi(Q^f)$, with Q^f the law of an act $f : \Omega_K \to X$ when a uniform distribution is assumed on Ω_K, consisting of K elements. This collection of value functions determines ϕ completely, and our axioms induce the following class.

Definition A.1 *$\mathcal{B}$ is the class of continuous valuations $\phi : \mathcal{D} \to X$ for which V_K^ϕ is regular and satisfies Axioms 2 and 3a, for all $K \in \mathbb{N}$.*

When extended with a status quo reference function in the obvious way, this exactly matches the definition of class $\mathcal{S}$, and hence all Axioms 1-3 apply.

Regularity (Definition 2.1) amounts to normalization ($\phi(\delta_x) = x$), continuity, and first order stochastic dominance, so that ϕ maps to X. Axiom 3a then amounts to betweenness,

$$\phi(Q) = c = \phi(Q') \Rightarrow \phi(\alpha Q + (1-\alpha)Q') = c \quad (\alpha \in [0,1]), \tag{A.1}$$

and Axiom 2a is then equivalent to the sensitivity condition

$$\phi(\alpha Q + (1-\alpha)\delta_c) = c \text{ for some } \alpha \in (0,1) \Rightarrow \phi(Q) = c.$$

Axiom 2b is the condition of *strict* first order stochastic dominance.

The fixed point rule (5.5) for free updates takes the form

$$\phi_A(Q) = c \quad :\Leftrightarrow \quad \phi(1_A Q + 1_{\bar{A}} c) = c,$$

which is equivalent to Bayesian updating. The replacement rule (3.2) for embedded updates becomes, in obvious notation,

$$\phi_A^R(Q) = r \quad :\Leftrightarrow \quad \phi(1_A Q + 1_{\bar{A}} r) = \phi(1_A Q + 1_{\bar{A}} R).$$

Axiom 5 (and hence weak decomposability) is now redundant, since it is equivalent to betweenness under law invariance. From the results in Dekel (1986); Chew (1989), the betweenness representation for class $\mathcal{B}$ follows:

$$\phi(Q) \overset{(>)}{=} c \Leftrightarrow E^Q u_c(x) \overset{(>)}{=} 0,$$

for a c-parametrized family of utility functions u_c that are strictly monotone, continuous, and normalized at $u_c(c) = 0$, and with $c \mapsto u_c(x)$ continuous and strictly decreasing.

A.2 Γ-representations (Section 5.1)

The characterization of Axiom 2 and 5, which is equivalent to (5.4), is largely the same as in Grant et al. (2000). We first rephrase their results for our simpler setting, addressing some details to close the gap between sufficient and necessary conditions, and then formulate a representation theorem for the subclass defined by Axiom 4.

We refer to the right-hand side in (5.5) as a *c-test* for f, induced by the fixed point representation (FPR). In more compact notation, with $\Omega = \{1, \ldots, n\}$, it takes the form

$$B_c(f) \overset{(>)}{=} 0, \text{ with } B_c(f) := u_{c,1}(x_1) + \ldots u_{c,n}(x_n). \tag{A.2}$$

The following two properties are necessary and sufficient to define a regular $V \in \mathcal{V}$: the *basic sign property*

$$B_c(f) = 0 \;\Rightarrow\; B_d(f) < 0 \text{ for } d > c \text{ and } B_d(f) > 0 \text{ for } d < c, \tag{A.3}$$

and the *continuity property*:

$$\{c \in X \mid B_c(f) \geq 0\} \text{ and } \{c \in X \mid B_c(f) \leq 0\} \text{ are closed sets.} \tag{A.4}$$

An FPR (5.5) with these two properties is called regular. The utility functions in state i are unique on their essential domain,

$$D_{c,i} = \{x \in X \mid u_{c,i}(x) + \Sigma_{j \in \Omega \setminus i}\, u_{c,j}(y) = 0 \text{ for some } y \in X\}, \tag{A.5}$$

modulo a free c-dependent scalar for each collection $\{u_{c,i}\}_{i \in \Omega}$.

Axiom 2a amounts to the extension of the sign property (A.3) to partial sums over events E, denoted as $B_{c,E}(f) := \Sigma_{i \in E} u_{c,i}(x_i)$:

$$B_{c,E}(f) = 0 \;\Rightarrow\; B_{d,E}(f) < 0 \text{ for } d > c \text{ and } B_{d,E}(f) > 0 \text{ for } d < c. \tag{A.6}$$

Axiom 2 and 5 are now straightforwardly verified, for preference orderings represented by a regular FPR with this *partial sign property.* Necessity of both axioms follows along the lines of Chateauneuf and Wakker (1993) and Segal (1992), and in particular the application of these results in (Grant et al., 2000, Proposition 7). Their proof relies on $\sharp\Omega \geq 4$, see also below.

Axiom 4 now requires, in addition, that all utility functions $u_{c,i}$ in the FPR with $c \in \Gamma^{-1}(\gamma)$ must be essentially the same, i.e., all pairs $u_{c,i}$ and $u_{c',i}$ coincide on $D_{c,i} \cap D_{c',i}$, modulo a constant (in fact $c' - c$) and a free scalar (independent of i). The FPR then can be transformed into a Γ-representation (5.7), by taking utility $\hat{u}_{\gamma,i}$ equal to $u_{c,i}$ on $D_{c,i}$ for some $c \in \Gamma^{-1}(\gamma)$, and then extending it such that it is an affine transformation of $u_{d,i}$ for *all* $d \in \Gamma^{-1}(\gamma)$. The theorem below now follows.

Theorem A.1 (Γ-representation) *A regular preference index (V, Γ) satisfies Axiom 2 and Axiom 4 if, and only if in case $\sharp\Omega \geq 4$, it is representable by (5.7), with utility $\hat{u}_{\gamma,i}$ such that the collection $u_{c,i}$ defined by $u_{c,i}(x) := \hat{u}_{\Gamma(c),i}(x) - \hat{u}_{\Gamma(c),i}(c)$ forms a regular FPR satisfying (A.6).*

The analysis in Chateauneuf and Wakker (1993) makes clear why an exception has to be made for the case $n = 3$. For $n > 3$, Axiom (5.4) entails the so-called Thomsen condition, which can be expressed as a condition on c-equivalents,

$$\text{If } (x, y) \sim^c (x', y') \text{ and } (x', y'') \sim^c (x'', y) \text{ then } (x, y'') \sim^c (x'', y'), \tag{A.7}$$

where $\sim^c$ denotes equality of c-equivalents on the corresponding pair of states. For $n = 3$, however, (5.4) is void, but the Thomsen condition is still required for the existence of an FPR. So, the exception for $n = 3$ can be cancelled if the Thomsen condition for $n = 3$ is added to (5.4), Axiom 4 and 5.

A.3 Lemma on the c-Comonotonic STP (Section 6.1)

We extend some of the results in Chew and Wakker (1996) to obtain a characterization of the c-comonotonic STP in our relatively simple setting with finite Ω. The essential idea is to refine comonotonic cones Π by pairs (Π, m), in which m marks the rank of c, i.e.,

$$(\Pi, m) = \{(x_1, \ldots, x_N) \,|\, x_1 \geq x_m \geq c \geq x_{m+1} \geq x_n\}.$$

We call $W : \mathbb{R} \times 2^S \to \mathbb{R}$ an outcome-dependent capacity if (i) $W(x, \emptyset) = 0$, (ii) $x \mapsto W(x, A)$ is continuous and (iii) $W(x, B) - W(x, A) - W(y, B) + W(y, A) > 0$ for all $x > y$ and $A \subsetneq B$.[11] Define $W^c(x, A) := W(x, A) -$

[11]Taking $A = \emptyset$ implies strict monotonicity of $x \mapsto W(x, B)$. The capacities induced by W are $\nu_{x,y}$, defined by $\nu_{x,y}(A) := (W(x, A) - W(y, A)/(W(x, \Omega) - W(y, \Omega))$.

$W(c, A)$, and, following their notation, $V(x, B, A) := W(x, B) - W(x, A)$.

A pair $(\hat{W}, \check{W})$ of outcome-dependent capacities represents $\succsim$ if $f = (x_1, \ldots, x_N) \sim c$ for the unique c such that there exists m with

$$\Sigma_{j=1}^{m} \hat{W}^c(x_j, \cup_1^j) - \hat{W}^c(x_j, \cup_1^{j-1}) \; + \; \Sigma_{j=m+1}^{N} \check{W}^c(x_j, \cup_j^N) - \check{W}^c(x_j, \cup_{j+1}^N) = 0. \tag{A.8}$$

More compactly,

$$f \sim c \text{ when } \Sigma_{j=1}^{n} V^c(x_j, \cup_1^j, \cup_1^{j-1}) = 0, \tag{A.9}$$

with V^c defined by, in obvious notation,

$$V^c(x, B, A) := \begin{cases} \hat{V}(x, B, A) - \hat{V}(c, B, A) & (x \geq c) \\ \check{V}(x, \bar{A}, \bar{B}) - \check{V}(c, \bar{A}, \bar{B}) & (x \leq c). \end{cases}$$

Lemma A.2 *A regular preference ordering satisfies c-comonotonicity if it is representable as above. For $n \geq 4$, this is also a necessary condition.*

Proof As in Chew and Wakker (1996, Thm. 1) for simple acts, with the following adjustments. Replace V^Π by $V^{\Pi,m}$, the additive representation on (Π, m) of the form $\Sigma_{j=1}^N V_j^{\Pi,m}$. Sufficiency follows as in their Lemma 1. Apply the uniqueness argument also to the intersection of comoncones $(\Pi, m) \cap (\Pi, m+1)$, i.e., with $x_m = c$, to deduce that $V_j^{\Pi,m} = V_j^{\Pi,m'}$ in case $j \geq m, m'$ or $j \leq m, m'$ (the argument requires that the intersection is at least of dimension 3, hence the condition $n \geq 4$). So the union of comoncones (Π, m) over m have a common additive representation as in (A.9), for each Π, say $V^{c,\Pi}$ corresponding to pairs $\hat{V}^\Pi, \check{V}^\Pi$. They satisfy their refinement condition (A4, p17). Since we need not impose that they agree on constants (their condition (A5)), but rather normalize $V^{c,\Pi}$ to zero in c for any $c \in X$, we can simply glue together $\hat{V}^\Pi, \check{V}^\Pi$ for all fully refined Π's , to arrive at (A.9) as the analog of (2) in their paper. □

References

R. J. Aumann. A synthesis of behavioural and mainstream economics. *Nature Human Behaviour*, 3(7):666–670, 2019.

B. D. Bernheim and C. Sprenger. On the empirical validity of cumulative prospect theory: Experimental evidence of rank-independent probability weighting. *Econometrica*, 88(4):1363–1409, 2020.

K. Binmore. Rationality and backward induction. *Journal of Economic Methodology*, 4(1):23–41, 1997.

A. Chateauneuf and P. Wakker. From local to global additive representation. *Journal of Mathematical Economics*, 22(6):523–545, 1993.

S. H. Chew. A generalization of the quasilinear mean with applications to the measurement of income inequality and decision theory resolving the allais paradox. *Econometrica*, 51:1065–1092, 1983.

S. H. Chew. Axiomatic utility theories with the betweenness property. *Annals of Operations Research*, 19(1):273–298, 1989.

S. H. Chew and P. Wakker. The comonotonic sure-thing principle. *Journal of Risk and Uncertainty*, 12(1):5–27, 1996.

E. Dekel. An axiomatic characterization of preferences under uncertainty: Weakening the independence axiom. *Journal of Economic Theory*, 40(2):304–318, 1986.

J. Eichberger, S. Grant, and D. Kelsey. Updating Choquet beliefs. *Journal of Mathematical Economics*, 43(7):888–899, 2007.

L. G. Epstein and M. Schneider. Recursive multiple-priors. *Journal of Economic Theory*, 113(1):1–31, 2003.

G. Gigerenzer. Bounded and rational. In *Contemporary debates in cognitive science*, pages 115–133. Blackwell, 2006.

I. Gilboa. Rationality and the Bayesian paradigm. *Journal of Economic Methodology*, 22(3):312–334, 2015.

I. Gilboa and D. Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153, 1989.

I. Gilboa and D. Schmeidler. Updating ambiguous beliefs. *Journal of Economic Theory*, 59(1):33–49, 1993.

R. Gonzalez and G. Wu. On the shape of the probability weighting function. *Cognitive Psychology*, 38(1):129–166, 1999.

S. Grant, A. Kajii, and B. Polak. Decomposable choice under uncertainty. *Journal of Economic Theory*, 92(2):169–197, 2000.

F. Gul and O. Lantto. Betweenness satisfying preferences and dynamic choice. *Journal of Economic Theory*, 52(1):162–177, 1990.

Y. Halevy. The possibility of speculative trade between dynamically consistent agents. *Games and Economic Behavior*, 46(1):189–198, 2004.

E. Hanany and P. Klibanoff. Updating preferences with multiple priors. *Theoretical Economics*, 2(3):261–298, 2007.

E. Hanany and P. Klibanoff. Updating ambiguity averse preferences. *The BE Journal of Theoretical Economics*, 9(1), 2009.

D. M. Hausman. *Preference, value, choice, and welfare.* Cambridge University Press, 2012.

M. Horie. Reexamination on updating Choquet beliefs. *Journal of Mathematical Economics*, 49(6):467–470, 2013.

D. Kahneman and G. Klein. Conditions for intuitive expertise: a failure to disagree. *American Psychologist*, 64(6):515, 2009.

D. Kahneman and A. Tversky. Prospect theory: An analysis of decision under risk. *Econometrica: Journal of the Econometric Society*, pages 263–291, 1979.

D. Kahneman, J. L. Knetsch, and R. H. Thaler. Anomalies: The endowment effect, loss aversion, and status quo bias. *Journal of Economic Perspectives*, 5 (1):193–206, 1991.

M. J. Machina. Dynamic consistency and non-expected utility models of choice under uncertainty. *Journal of Economic Literature*, 27(4):1622–1668, 1989.

H. Markowitz. The utility of wealth. *Journal of Political Economy*, 60(2):151–158, 1952.

T. O'Donoghue and C. Sprenger. Reference-dependent preferences. In *Handbook of Behavioral Economics: Applications and Foundations 1*, volume 1, pages 1–77. Elsevier, 2018.

O. Peters and M. Gell-Mann. Evaluating gambles using dynamics. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 26(2):023103, 2016.

C. P. Pires. A rule for updating ambiguous beliefs. *Theory and Decision*, 53(2): 137–152, 2002.

D. Prelec. The probability weighting function. *Econometrica*, 66:497–527, 1998.

J. Quiggin. A theory of anticipated utility. *Journal of Economic Behavior & Organization*, 3(4):323 – 343, 1982.

B. Roorda and R. Joosten. A universal update rule for consistent choice. Working paper, 2019.

B. Roorda and J. M. Schumacher. Time consistency conditions for acceptability measures, with an application to Tail Value at Risk. *Insurance: Mathematics and Economics*, 40:209–230, 2007.

B. Roorda and J. M. Schumacher. Membership conditions for consistent families of monetary valuations. *Statistics & Risk Modeling*, 30:255–280, 2013.

B. Roorda and J. M. Schumacher. Weakly time consistent concave valuations and their dual representations. *Finance and Stochastics*, 20(1):123–151, 2016.

R. W. Rosenthal. Games of perfect information, predatory pricing and the chain-store paradox. *Journal of Economic Theory*, 25(1):92–100, 1981.

R. Sarin and P. P. Wakker. Dynamic choice and nonexpected utility. *Journal of Risk and Uncertainty*, 17(2):87–120, 1998.

D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, 57:571–587, 1989.

U. Segal. Additively separable representations on non-convex sets. *Journal of Economic Theory*, 56(1):89–99, 1992.

A. Tversky and D. Kahneman. Loss aversion in riskless choice: A reference-dependent model. *The Quarterly Journal of Economics*, 106(4):1039–1061, 1991.

A. Tversky and D. Kahneman. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5(4):297–323, 1992.

B. M. van Praag. Ordinal and cardinal utility: an integration of the two dimensions of the welfare concept. *Journal of Econometrics*, 50(1-2):69–89, 1991.

P. P. Wakker. Justifying bayesianism by dynamic decision principles. In *plenary paper presented at FUR IX, Marrakesh*, 1999.