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An average case analysis of the minimum spanning tree heuristic for the power assignment problem

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Abstract

We present an average case analysis of the minimum spanning tree heuristic for the power assignment problem. The worst-case approximation ratio of this heuristic is 2. We show that in Euclidean d-dimensional space, when the vertex set consists of a set of i.i.d. uniform random independent, identically distributed random variables in $[0, 1]^d$, and the distance power gradient equals the dimension d, the minimum spanning tree-based power assignment converges completely to a constant depending only on d.

KEYWORDS

ad-hoc networks; analysis of algorithms; approximation algorithms; average case analysis; point processes; power assignment; range assignment

1 | INTRODUCTION

Ad hoc wireless networks have received significant attention in recent years due to their potential applications in battlefield, emergency disaster relief, and other scenarios (see for example [14, 19, 21]). In an ad hoc wireless network, a communications session is achieved either through single-hop transmission or by relaying through intermediate nodes. The topology of a multihop wireless network is given by the set of communication links between node pairs. It may depend on uncontrollable factors such as node mobility, interference, as well as on controllable parameters such as transmit power. In this paper, we assume an idealized propagation model, where omnidirectional antennas are used. We consider the case that for the purpose of energy conservation, each node can adjust its transmit power. For assigning the transmit powers, two conflicting effects have to be taken into account: if the transmit powers are too low, the resulting topology may be too sparse and the network may get disconnected. On the other extreme, if the transmit powers are too high, the nodes may run out of energy quickly. The goal of the power assignment problem is to assign transmit powers such that the resulting network is connected and the sum of transmit powers is minimized (see eg, [14]). This problem is, in general, NP-hard, but for some special cases there are polynomial solutions. An intuitive approximation approach is the minimum spanning tree (MST)-heuristic. This is known to have a worst-case approximation ratio of 2 (see e.g., [11]). The main result of this paper is that for the Euclidean d-dimensional space, when the distance power gradient is equal to the dimension d (corresponding to the free-space model for radio transmissions) and for a vertex set V of n uniform i.i.d. random variables in $[0,1]^d$ the total power of the minimum spanning tree-based power assignment, $P(V)$, converges completely (c.c.) to a constant $\mu_P(d)$, depending only on d.

1.1 | The power assignment problem

Let V be a finite vertex set, and let K_V denote the complete graph on V. Endow each (undirected) edge $e = \{u, v\}$ of K_V with a weight $c(e) \in [0, \infty)$. A *power assignment* is a function $\mathbf{p} : V \to [0, \infty)$. The weight $c(e)$ for $\{u, v\}$ represents the *transmit power threshold,* with the following meaning: a signal transmitted by the transceiver u can be received by v only when the transmit power $\mathbf{p}(u)$ is at least $c(e)$, and similarly u can receive from v only when $\mathbf{p}(v) \geq c(e)$. By including only edges where transmission is possible in both directions, a power assignment $\mathbf{p}$ defines an undirected graph $G_\mathbf{p} = (V, E_\mathbf{p})$, where $e = \{u, v\} \in E_\mathbf{p}$ if and only if $\min\{\mathbf{p}(u), \mathbf{p}(v)\} \geq c(e)$. The *power assignment problem* asks, for a given V and c, for a power assignment $\mathbf{p}$ such that $G_\mathbf{p}$ is connected and the total power $\sum_{v\in V} \mathbf{p}(v)$ is minimized. The so-called *MST-heuristic* gives an approximate solution to this problem, constructed as follows:

1. Compute a MST T for V using $c(e)$ as the weight for each pair $e = \{u, v\}$.
2. For each node $v \in V$ define $\mathbf{p}(v) = \max\{c(\{u, v\}) \mid u \text{ is adjacent to } v \text{ in } T\}$.

Let $P(V) = \sum_{v\in V} \mathbf{p}(v)$ denote the total power assignment from the MST-heuristic. The aim of this paper is to study the performance of the MST-heuristic on random points in Euclidean space. Specifically, we take vertex set $V_n = \{U_1, U_2, \ldots, U_n\}$, a set of n independent uniform random points on $[0,1]^d$. For the thresholds we take $c(e) = \|u-v\|^p$, where $\|u-v\|$ is the Euclidean distance between u and v, and $p \in [0, \infty)$. This reflects a power attenuation model where the signal power decreases with the distance r as r^{-p}. The *distance-power gradient* $p \in \mathbb{R}^+$ depends on the wireless environment and realistic values of p vary from 1 to more than 6 [15]; here we take $p = d$. Our main result is the following.

Theorem 1.1 *Let $V_n = \{U_1, \ldots, U_n\}$ denote a set of n uniform i.i.d. random variables. Then there exists a constant $\mu_P(d)$, depending only on d, such that for the power assignment $P(V_n)$ resulting from the MST-heuristic, with edge weights $\|e\|^d$:*

$$P(V_n) \xrightarrow{c.c.} \mu_P(d). \tag{1}$$

We say that a sequence of r.v.'s $\{X_n\}_n$, *converges completely* (c.c.) to a constant c (notation: $X_n \xrightarrow{c.c.} c$) if and only if for all $\varepsilon > 0$: $\sum_{n=1}^{\infty} \mathbb{P}(|X_n - c| > \varepsilon) < \infty$. Complete convergence implies almost sure convergence. Though our main result focusses on the case where $p = d$, in the process we formulate initial results on superadditivity for general p and d.

Given $V_n \subset [0,1]^d$, let $W_{\text{opt}}(V_n)$ denote the optimal power assignment on V_n. The approximation ratio $\tau_n(V_n)$ of the power assignment resulting from the MST-heuristic is defined as

$$\tau_n(V_n) = \frac{P(V_n)}{W_{\text{opt}}(V_n)}.$$

The following corollary is implied by the fact that complete convergence of $P(V_n)$ follows from Theorem 1.1, and complete convergence of $W_{\text{opt}}(V_n)$ and $\tau < 2$ follows from Corollary 4.9. and Theorem 5.1 in [9], respectively.

Corollary 1.1 *The approximation ratio $\tau_n(V_n)$ of the MST heuristic for power assignments converges completely to a constant τ, which depends only on d and is strictly smaller than 2:*

$$\tau_n(V_n) \xrightarrow{c.c.} \tau, \textit{where } \tau < 2. \tag{2}$$

1.2 | Previous work and contribution

The power assignment problem is NP-hard in all dimensions $d \geq 2$ for all values of the distance-power gradient p. The first NP-hardness result for power assignment in $\mathbb{R}^3$ was presented in [11]. NP hardness in 2 dimensions was shown in [6]. In [4, 7] the complexity of various other variants of the problem is analyzed. Motivated by these complexity results, polynomial time approximation algorithms have been studied. The first approximation algorithm to the range assignment problem is the MST-heuristic (see [5, 7]). Given V and c, it is well established (see eg, [3, 5]) that

$$T(V) \leq W_{\text{opt}}(V) \leq P(V) \leq 2T(V), \tag{3}$$

where $W_{\text{opt}}(V)$ denotes the weight of the optimal total power assignment. In [3] it is shown that the factor 2 is tight. While the worst-case performance ratio of 2 might discourage use of the MST-heuristic, numerical results indicate that the MST-heuristic is often rather close (ie, within 6%) of the optimal solution [3].

A probabilistic analysis of the power assignment problem is performed in [22] focusing on upper-bounds and lower-bounds for connectedness in the special case that all nodes have the same transmit power. In [8], the average case behavior of the MST heuristic is analyzed for the case $d = 1$ and for the nongeometric case where the edge weights are uniform [0, 1] distributed random variables. In [9] for the weight of the *optimal* power assignment, concentration of measure and complete convergence for all combinations of d and $p \geq 1$ is obtained. However, in [9], it is not shown that $P(V)$ converges completely. This latter is the objective of this paper (for $p = d$). The difficulty in showing complete convergence for $P(V)$ is that, unlike for the optimal power assignment, it is unknown whether $P(V)$ fulfills the requirements to apply Yukich's general framework for Euclidean functionals [26]. The underlying reason for this is that an optimal power assignment may be associated to a nonminimum spanning tree. Or stated differently, decreasing the weight of a spanning tree may increase the weight of the associated power assignment. Bounding the potential increase of this weight is a basic obstacle for proving sub-additivity and smoothness. In this paper we provide a much more detailed relation between the boundary MST (introduced in Section 3) and the MST, which enables us to bound the potential weight increase. We believe this general relation may be useful in the analysis of other heuristics as well. The paper is organized as follows. In Section 2 we provide preliminary results on MST's and on the power assignment by the MST-heuristic. Section 3 shows that the power assignment functional is superadditive. Section 4 presents convergence results for the d-dimensional Euclidean case with distance power gradient $p = d$. Finally, Section 5 presents conclusions and directions for further research.

2 | PRELIMINARIES

For later reference, we provide some general (ie, nongeometric) results on MST's.

2.1 | Preliminaries on MST's

For a vertex v, let $G\backslash\{v\}$ denote the graph resulting from G by deleting v and all edges incident to v. For an edge e, $G\backslash\{e\}$ denotes the graph resulting from G by deleting edge e. Well-known properties for (minimum) spanning trees are the *extension property*, and the *creek crossing criterion*. The *extension property* (see eg, [10]) states that if T_1 and T_2 are two spanning trees on V, then for each $e \in T_1$ there is an $f \in T_2$ so that $T_1\backslash\{e\} \cup \{f\}$ is again a spanning tree. The *creek crossing criterion* (see eg, [1]) states that if T is an MST for $G = (V, E, c)$ then $e = \{u, v\} \in T$ with $u \neq v$ if and only if there is no path in G connecting u and v of which the edges weights are all strictly smaller than $c(e)$.

Next, we state a relation between the MST of a graph and its extension by a single vertex and additional edges. We formulate it in a slightly more general way than the similar Lemma 2.1 in [23]. It is a reverse of the "add and delete algorithm" (see [10, 13]), so actually it is a "delete" and "add" algorithm.

Lemma 2.1 *$G = (V, E, c)$ be a connected weighted graph and let $H = (V \cup \{z\}, E \cup E', c')$ be an extension of G, where $E' = \{\{v, z\}$ with $v \in V\}$ and $c' : E \cup E' \to \mathbb{R}^+$ has the property that $c'(e) = c(e)$ for all $e \in E$. Furthermore, let $T[H]$ be an MST on the graph H and let $F = T[H]\backslash\{z\}$. Then there exists an MST T' of G, with $F \subset T'$.*

Proof Let T' be an MST on G with the property that the number of edges in $T' \cap F$ is maximal. Assume there exists an edge $e = \{u, w\} \in F\backslash T' = T[H]\backslash(\{z\} \cup T')$. As $e \in T[H]$ it follows that $T[H]\backslash\{e\}$ consists of two components, say, K_1 and K_2. Let P denote the path in T' connecting u and w. P must contain an edge $f = \{x, y\}$ with $x \in K_1$ and $y \in K_2$. Now $T[H]\backslash\{e\} \cup \{f\}$ is a spanning tree of H, and consequently $c'(e) \leq c'(f)$. As $f \in P$, also $T'\backslash\{f\} \cup \{e\}$ is a spanning tree of G, and we get $c'(e) \geq c'(f)$. It follows that $c'(e) = c'(f)$, and thus $T'\backslash\{f\} \cup \{e\}$ is also an MST of G, but with a larger intersection with F. This contradicts the choice of T' and therefore no edge $e \in F\backslash T'$ exists, implying that $F \subset T'$ and completing the proof. ■

We will apply this lemma in the following way: if we have an MST in an extended weighted graph (corresponding to H in the lemma), and we remove a vertex v from this graph, then the original MST will break down into degree(v) components. The lemma shows that if the remaining graph (corresponding to $G = H\backslash\{v\}$ in the lemma) is still connected, then an MST of G can be constructed by maintaining the components of the MST, and adding a minimum weight set of edges connecting these components.

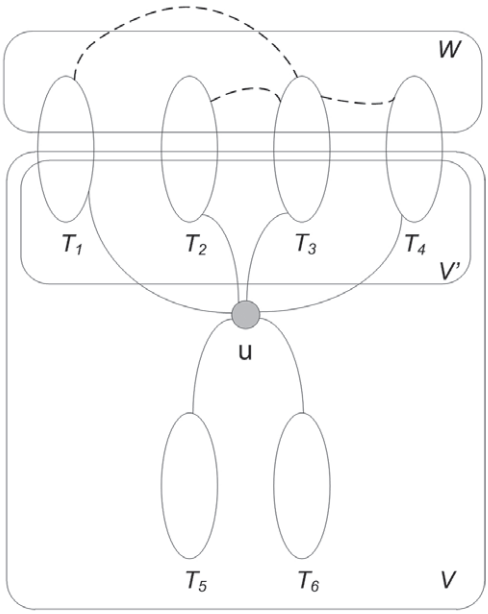

FIGURE 1 Situation sketch for the proof of Lemma 2.2, showing $T_1, \dots, T_6$, V, V' and W. The normal edges are part of $T[V \cup W]$, the dashed edges are part of $T[(W]$ but not of $T[V \cup W]$, V' is obtained from V by removing u, T_5, and T_6.

2.2 | Lipschitz continuity of $P(V)$

In the rest of this paper we consider a set of points $V \subset \mathbb{R}^d$, and a complete graph $G[V]$ where the weight of edge $\{u, v\}$ is given by: $c(u, v) = \|u - v\|^p$. To simplify notation, we assume the MST on $G[V]$ is uniquely determined for a given set of points V, note that in view of our application of this result for V i.i.d uniformly selected in $[0, 1]^d$, this is true with probability 1. We denote the unique MST with $T[V] = T[G[V]]$, and the weight of $T[V]$ with $T(V)$. For the results in this section we need an extension of Lemma 2.6 given in [27]. From [2] it is known that the maximum degree $d(v)$ of any vertex v in the (unique) MST is bounded by a constant $c(d)$ depending only on the dimension d. For example, for $d = 2$ it is known that $c(d) = 6$. The following lemmas show that if two point sets V and W are "close" then $|P(V) - P(W)|$ is bounded, showing Lipschitz continuity of $P(\cdot)$.

Lemma 2.2 *Let V and W be sets of points in $\mathbb{R}^d$. Moreover, suppose that the edges in $T[W]$ and $T[V \cup W]$ have weight at most C. Then*

$$P(W) \leq P(V \cup W) + 2(c(d) - 1)C|V \setminus W|.$$

Proof We prove this lemma by induction on $|V \setminus W|$. The case where $|V \setminus W| = 0$ is obvious. Suppose the lemma holds for all sets V, W with $|V \setminus W| < p_0, p_0 > 0$. To prove the inequalities for $|V \setminus W| = p_0$, consider any vertex $u \in V \setminus W$, and let d be the degree of u in $T(V \cup W)$. By deletion of u, the MST $T[V \cup W]$ is subdivided into a forest F consisting of d connected components, which we denote by $T_1, \dots, T_d$ (see Figure 1). By Lemma 2.1, the MST T on $G((V \cup W) \setminus \{u\})$ contains $T_1, \dots, T_d$. As the maximum weight of edges of $T[V \cup W]$ is at most C, this bound holds for all edges contained in one of the T_i, $i = 1, \dots, d$. We now work towards a bound on edge weights of edges between different T_i and T_j, $i, j \in \{1, \dots, \ell\}$.

Let ℓ denote the number of components that contain vertices of W, and number the components in such a way that $T_1, \ldots T_\ell$ contain vertices of W, and $T_{\ell+1}, \ldots, T_d$ contain only vertices of V. So all vertices in W belong to $T_1 \cup \ldots \cup T_\ell$. Define V' as the set of vertices of V which are part of one of the trees $T_1, \ldots, T_\ell$. Because $u \notin V'$ we get $|V' \backslash W| \leq |V \backslash W| - 1$. First, we show that all edges in the MST T' on $G(V' \cup W)$ have weight at most C. By Lemma 2.1 and the assumption that the MST is unique, T' consists of edges contained in one of the T_i, $i = 1, \ldots, \ell$, and of edges between different components T_i and T_j with $1 \leq i < j \leq \ell$. Note that by definition $T_{\ell+1}, \ldots, T_d$ are not part of V', and hence not of T'. The edges between different components T_i are the only edges in T' that are not part of $T[V \cup W]$. In order to bound their weight, note that the MST $T[W]$ on W has maximum edge weight at most C. From this tree we select $\ell - 1$ edges, such that the addition of these edges to $T_1, \ldots, T_\ell$ is a spanning tree on $G(V' \cup W)$ with maximum edge weight at most C. By the well-known fact that the MST also minimizes the maximum edge weight of a spanning tree, the spanning tree T' also has maximum edge weight at most C. Next, we bound $P(V' \cup W)$ in terms of $P(V \cup W)$. As each edge between different T_i and T_j contributes at most $2C$ to the power assignment $P(V' \cup W)$, we get $P(V' \cup W) \leq P(V \cup W) + 2(\ell - 1)C$. Since for V' and W we have $|V' \backslash W| < p_0$, we get by the induction hypothesis that:

$$\begin{aligned} P(W) &\leq P(V' \cup W) + 2(c(d) - 1)C|V' \backslash W| \\ &\leq P(V \cup W) + 2(\ell - 1)C + 2(c(d) - 1)C|V' \backslash W| \\ &\leq P(V \cup W) + 2(c(d) - 1)C(1 + |V' \backslash W|) \\ &\leq P(V \cup W) + 2(c(d) - 1)C|V \backslash W|, \end{aligned}$$

where the second inequality follows from the fact that the number of components is bounded by the maximum degree of a vertex in a spanning tree, that is, $\ell \leq d \leq c(d)$. ■

Note that in Lemma 2.2 we do not require a bound on the maximum edge weight of $T[V]$, but only on that of $T[W]$ and $T[V \cup W]$. Moreover, in the proof we do not use a bound of C on the weight of the edges from T_i to T_j, for $1 \leq i \leq \ell$ and $\ell + 1 \leq j \leq d$. The following argument shows that all edges from T_i to T_j could have length exceeding C. Assume that there is only one edge between V and W with weight at most C. If u would be chosen incident to this edge, then all edges from T_i to T_j, for $1 \leq i \leq \ell$ and $\ell + 1 \leq j \leq k$ have length exceeding C.

The next lemma bounds $P(V \cup W)$ in terms of $P(V)$ and $|W \backslash V|$.

Lemma 2.3 *Let V and W be sets of points in $\mathbb{R}^d$. Moreover, suppose that the edges in $T[V]$, $T[W]$ and $T[V \cup W]$ have weight at most C. Then,*

1. $P(V \cup W) \leq P(V) + (c(d) + 1)C|W \backslash V|$,
2. *If* $|V| = |W|$ *then* $|P(V) - P(W)| \leq 3c(d)C|W \backslash V|$.

Proof First note that for $V = W$ the lemma is obviously true. Thus, we assume $V \neq W$. Now, the proof consists of two steps. Since all edges in $T[V \cup W]$ have weight at most C, there is at least one edge $e = \{x, y\}$ with $x \in V$, $y \in W \backslash V$ and $c(e) \leq C$. Let $H = V \cup \{y\}$ and consider a minimum spanning tree $T[H]$ on H. Let $d(y)$ denote the degree of y in $T[H]$. By Lemma 2.1, $T[H] \backslash \{y\} \subset T[V]$, or stated otherwise: $T[H]$ consists of $d(y)$ subtrees of $T[V]$ that are connected to y. By the power assignment according to the MST Algorithm, it follows that $P(H) \leq P(V) + (d(y) + 1)C \leq P(V) + (c(d) + 1)C$. (By the heuristic, the power $P(H)$ might increase at most $d(y)$ times with at most C due to the fact that the connection between a subtree of $T[V]$ and y has larger weight in $V \cup \{y\}$ than in the original graph, based on V, where it was connected to another subtree. By assumption, this potential weight increase is bounded by C. The "+1" is explained by the weight of y in the power assignment of H, which is at most C.) Now the first statement follows iteratively. To see the second statement, assume in addition that $|V| = |W|$, then by Lemma 2.2 we have that :

$$P(W) \leq P(V \cup W) + 2(c(d) - 1)C|V \backslash W|. \tag{4}$$

Combining the bounds, it follows that

$$\begin{aligned} P(W) &\leq P(V \cup W) + 2(c(d) - 1)C|V \backslash W| \\ &\leq P(V) + (c(d) + 1)C|W \backslash V| + 2(c(d) - 1)C|V \backslash W| \\ &= P(V) + 3c(d)C|W \backslash V|, \end{aligned}$$

using that $|V \backslash W| = |W \backslash V|$. By reversing the roles of V and W, a bound follows for $P(V) - P(W)$. ■

3 | SUPER ADDITIVITY OF THE BOUNDARY POWER ASSIGNMENT FUNCTIONAL

The MST-based power assignment functional $P(V)$ is a Euclidean functional. We use notation and results from Yukich [26]. Let $R \subset \mathbb{R}^d$ be a hyperrectangle, let $V \subset \mathbb{R}^d$ be a point set, and let $T[V \cap R]$ denote the MST of the complete graph on $V \cap R$, and define the boundary minimum spanning tree $T_B[V \cap R]$ on $V \cap R$ by identifying the boundary B of R with a single point v_0 and calculating the MST of $G[(V \cap R) \cup \{v_0\}]$ with edge weights $c(\{v_i, v_j\}) = \|v_i - v_j\|^p$ for $i, j = 1, \dots, n$ and weights $c(\{v_i, v_0\}) = \min_{x \in B} \|v_i - x\|^p$ (see [27], page 14). We refer to the weight of $T_B[V \cap R]$ as $T_B(V, R)$. Similarly $P[V \cap R] : V \to \mathbb{R}$ denotes the power assignment according to the MST heuristic on $V \cap R$ and $P(V, R)$ denotes the total weight of $P[V \cap R]$. We define the weight $P_B(V, R)$ of the power assignment associated to the boundary MST on $V \cap R$ as the result of the following *Boundary MST power assignment* algorithm:

Boundary MST Power Assignment Algorithm (V, R)

1. Compute a boundary MST T_B by using $c(e) = \|e\|^p$ as edge for each $e \in T_B$.
2. For each node $v \in V$ assign

$$\mathbf{p}_B(v) = \max\{c(\{u, v\}) | \{u, v\} \text{ in } T_B \text{ and } u, v \in V\}. \tag{5}$$

In the boundary power assignment, no power is assigned to the boundary. Distances to the boundary are not taken into account as well. Note that when $T \neq T_B$, the spanning forest resulting from the boundary T_B may be thought of as a collection of small trees connected via the boundary of R into a single spanning tree, where the connections over the boundary of R incur no cost. We show that the boundary power assignment is a lower bound for $P(V)$ and that the boundary power assignment functional is *superadditive* as defined in [26] (3.3); that is, $P_B(V, R) \geq P_B(V, R_1) + P_B(V, R_2)$ for every partitioning of R into hyperrectangles R_1 and R_2.

Lemma 3.1 *Let $V \subset \mathbb{R}^d$, $R = [0, 1]^d$, and let $\mathbf{p}(v)$ for all $v \in V \cap R$ denote the powers assigned by $P[V \cap R] : V \to \mathbb{R}$. Likewise, let $\mathbf{p}_B(v)$ for all $v \in V \cap R$ denote the powers assigned by $P_B(V, R)$. Furthermore, let R_1 and R_2 be a partition of $R = [0, 1]^d$, into two hyperrectangles R_1 and R_2, and let*

$P_B(V, R_i)$ be the corresponding boundary power assignments for $i = 1, 2$, leading to powers $\boldsymbol{p}_{B,i}(v)$ for all $v \in V \cap R_i$. Then

(a) $\boldsymbol{p}_B(v) \leq \boldsymbol{p}(v)$ *and*
(b) $\boldsymbol{p}_{B,i}(v) \leq \boldsymbol{p}_B(v)$, *for all $v \in V \cap R_i$, $i = 1, 2$.*

Proof To prove statement (a) let T, B, and T_B be defined as above, and suppose $e = \{u, v\}$ is in T_B with $u, v \in V$ (so u, v are not on the boundary). We show $e \in T$, which implies the result. Let G denote the complete graph on $(V \cap R)$ with edge weights $c(\{v_i, v_j\}) = \|v_i - v_j\|^p$ for $i, j = 1, \dots, n$ and G_B the complete graph on $(V \cap R) \cup \{v_0\}$ with edge weights $c(\{v_i, v_j\}) = \|v_i - v_j\|^p$ for $i, j = 1, \dots, n$ and weights $c(\{v_i, v_0\}) = \min_{x \in B} \|v_i - x\|^p$ (see [27], page 14). So T_B is the MST of G_B. By the creek crossing criterion, in G_B there is no path connecting u and v with all edge weights strictly smaller than $c(e)$. As a consequence, there is also no such path in G, hence $e \in T$. The proof of statement (b) follows similarly: clearly, in the graph G_B, as defined above, partitioning R into R_1 and R_2 has the effect of enlarging B and hence decreasing weights $c(\{v_i, v_0\})$ for some of the vertices $v_i \in V \cap R_i$, $i = 1, 2$ (where v_0 corresponds to the extended boundary). Let $G_{B,i}$ denote the complete graph on $(V \cap R_i) \cup \{v_0\}$ with edge weights $c(\{v_i, v_j\}) = \|v_i - v_j\|^p$ for $i, j = 1, \dots, n$ and weights $c(\{v_i, v_0\}) = \min_{x \in B} \|v_i - x\|^p$. Let $T_{B,i}$ denote the associated boundary spanning trees on $V \cap R_i$ ($i = 1, 2$). Suppose $e = \{u, v\}$ is in $T_{B,1}$ with $u, v \in V\cap$ (so u, v are in the interior of R_1). By the creek crossing criterion, in $G_{B,1}$ there is no path connecting u and v with all edge weights strictly smaller than $c(e)$. Also in $G_{B,1} \cup G_{B,2}$ there can be no such path, because such a path should leave R_1 and enter R_1 again, and hence cross v_0 twice. As a consequence, there is also no such path in G_B, hence $e \in T_B$. ■

The following corollary is straightforward, and we omit the proof.

Corollary 3.1 *The boundary power assignment P_B, obtained by the boundary MST Power Assignment algorithm is superadditive.*

4 | PROOF OF THE MAIN THEOREM

Before proving the main Theorem 1.1, we provide some intermediate results. Throughout, $V_n = \{U_1, \dots, U_n\}$ denotes a set of uniform i.i.d. random variables, $U_1, \dots, U_n$ on $[0, 1]^d$. Following [26] (4.11), when $p = d$, we say that $P(V_n)$ is *close in mean* to $P_B(V_n)$ if

$$\mathbb{E}[|P(V_n) - P_B(V_n)|] = o(1).$$

We call $P(\cdot)$ *smooth in mean* (as defined in [26] (4.13)) if there exists a constant $\gamma < 1/2$ such that for all $n \geq 1$ and $0 \leq k \leq n/2$ we have $\mathbb{E}[|P(U_1, \dots, U_n) - P(U_1, \dots, U_{n \pm k})|] \leq Ckn^{-1+\gamma}$. We start by showing convergence in mean.

Theorem 4.1 *For all $d \geq 1$, there exists a constant $\mu_P(d)$ such that for the MST-based power assignment $P(V_n)$, with weights $c(e) = \|e\|^d$ we have:*

$$\lim_{n \to \infty} \mathbb{E}[P(V_n)] = \mu_P(d). \tag{6}$$

Proof We first note that by the space filling curve heuristic (see eg, [26], equation (3.7)), we have that $T(V_n) \leq C'$, for some constant C', and consequently by (3) $P(V_n) \leq C$, for some constant $C > 0$.

A fortiori, $\mathbb{E}[P(V_n)] \leq C$. If in addition, $P_B(V_n)$ is close in mean to $P(V_n)$ and $P(\cdot)$ is smooth in mean, then the theorem follows from [26] Theorem 4.5.

First, we show $P_B(V_n)$ is close in mean to $P(V_n)$, following the analogous proof for MSTs in [26] pp. 44 and 45. Let P_B denote the power assignment associated to $T_B[V]$. We enumerate the components of T_B by $T_1, \ldots, T_Q$, where Q is a random variable and where each T_i represents a tree which is rooted to the boundary of the unit cube (the connection to the boundary is not considered to be part of T_i). From the proof in [26], it follows that for any $\beta > 0$, with a probability exceeding $1 - n^{-\beta}$, $T_1, \ldots, T_Q$ can be connected into a spanning tree T_S (not necessarily minimum) by adding a set of at most $C(\beta)n^{(d-1)/d}$ edges, where each edge has weight at most $C(\beta)n^{-1/d}(\log n)^d$. We claim that $T_1, \ldots, T_Q$ can be connected into a *minimum* spanning tree by a set of at most $C(\beta)n^{(d-1)/d}$ edges, where each edge has weight at most $C(\beta)n^{-1/d}(\log n)^d$. First, by Lemma 2.1 $T_1, \ldots, T_Q$ can be connected into a MST T by adding only edges connecting T_i and T_j (with $1 \leq i < j \leq Q$). The number of edges required to do this will not exceed the number of edges required to connect $T_1, \ldots, T_Q$ into a tree. Moreover, suppose in T there is an edge e connecting T_i and T_j with $c(e) > C(\beta)n^{-1/d}(\log n)^d$. Consider the two components H_1 and H_2 of $T \backslash \{e\}$. Let $f \in T_S$ connect these two components. Then we could exchange e for f reducing the weight of T which is a contradiction. Hence $c(e) \leq C(\beta)n^{-1/d}(\log n)^d$ as required.

By Lemma 3.1 (a) and the fact that each additional edge increases the weight of $P_B(V_n)$ with at most two times the edge weight, it follows that with a probability of at least $1 - n^{-\beta}$, we have:

$$P_B(V_n) \leq P(V_n) \leq P_B(V_n) + 2C(\beta)n^{-\frac{1}{d}}(\log n)^d. \tag{7}$$

Combined with the fact that $P(V_n)$ is bounded above by a constant, this shows that $P(V_n)$ is close in mean to $P_B(V_n)$, for all d. Second, we have to show that $P(\cdot)$ is smooth in mean, so we must show that there is a constant C' and $\gamma < 1/2$ such that for all $n \geq 1$ and $0 \leq k \leq n/2$ we have:

$$\mathbb{E}[|P(\{U_1, \ldots, U_n\}) - P(\{U_1, \ldots, U_{n\pm k}\})|] \leq C'(\beta)kn^{-1+\gamma}. \tag{8}$$

We will show the following stronger statement, which implies (8). For any $\beta > 0$, there is a constant $C'(\beta)$, depending only on β, so that with a probability of at least $1 - n^{-\beta}$

$$|P(\{U_1, \ldots, U_n\}) - P(\{U_1, \ldots, U_{n\pm k}\})| \leq C'(\beta)k(\log n/n), \tag{9}$$

To see (9), note that for any $\beta > 0$, there is a third constant $C''(\beta)$ such that with a probability of at least $1 - n^{-\beta}$ the edges in the MST on V have length at most $C''(\beta)(\log n/n)^{1/d}$. This follows from the fact that the maximum length of the longest MST-edge is the same as the connectivity threshold, see [12, 16] that is, the minimal number r such that G_r is connected, where G_r denotes the graph obtained by joining two vertices if and only if their distance $\leq r$. In the setting of a ball in $\mathbb{R}^d$ this is with a probability of at least $1 - n^{-\beta}$ at most $C''(\beta)(\log n/n)^{1/d}$. In this case, the maximum weight of an edge is $C''(\beta)^d(\log n/n)$. To compensate for the fact that we may also remove (at most $n/2$) points from $\{U_1, \ldots, U_n\}$, we set $C'(\beta) = 2^d C''(\beta)^d$. It follows from the first statement of Lemma 2.3 (taking $V = \{U_1, \ldots, U_n\}$ and $W = \{U_1, \ldots, U_{n+k}\}$) that

$$P(\{U_1, \ldots, U_{n+k}\}) \leq P(\{U_1, \ldots, U_n\}) + (c(d) + 1)kC'(\beta)(\log n/n).$$

To see that

$$P(\{U_1, \ldots, U_n\}) \leq P(\{U_1, \ldots, U_{n+k}\}) + (2c(d) - 1)kC'(\beta)(\log n/n),$$

apply Lemma 2.2 with $V = \{U_{n+1}, \dots, U_{n+k}\}$ and $W = \{U_1, \dots, U_n\}$. This shows the "plus" case of (9). By similar reasoning we deal with the remaining case $|P(\{U_1, \dots, U_n\}) - P(\{U_1, \dots, U_{n-k}\})|$. ■

In order to prove Theorem 1.1, we provide some further definitions. Throughout, μ denotes the volume measure in $\mathbb{R}^d$ and μ^n denotes the n-fold product measure on $([0,1]^d)^n$. (Effectively, μ^n is the volume in $[0,1]^{dn}$.)

Let $x = (x_1, \dots, x_n)$ be an n-tuple in $([0,1]^d)^n$. Recall that the Hamming distance H on $([0,1]^d)^n$ measures the distance between x and y by the number of coordinates in which x and y disagree: $H(x,y) = \text{card}\{i : x_i \neq y_i\}$. Note that with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, we can associate the unordered sets $x' = \{x_1, \dots, x_n\}$ and $y' = \{y_1, \dots, y_n\}$. Now with $l(x', y') = |x' \backslash y'| = |\{x \in x' | x \notin y'\}| = |x'| - |x' \cap y'|$, we immediately have: $H(x,y) \geq l(x', y')$.

For all $t > 0$, the t-enlargement of a set $A \subset ([0,1]^d)^n$ is the set of tuples with Hamming distance at most t to A, defined by

$$A_t := \{x \in ([0,1]^d)^n : \exists y \in A \text{ such that } H(x,y) \leq t\}.$$

The following theorem is due to Talagrand [24]. It implies in particular that if $t = O(n/\log n)$ then $\mu^n(A_t^c) \to 0$. The importance of this theorem in obtaining general concentration results for subadditive Euclidean functional was first recognized by Rhee in [20] (Proposition 3 and its application to Theorem 1).

Theorem 4.2 ([24]) *Let $A \subset ([0,1]^d)^n$, and let A_t^c denote the complement of A_t. Then for all $t > 0$,*

$$\mu^n(A_t^c) \leq \frac{1}{\mu^n(A)} e^{\frac{-t^2}{n}}.$$

First we show that, in a way, the following set of grid points closely approximates an arbitrary set of n points. We call $\{g_i\}_{i=1}^n$ a collection of grid points in $[0,1]^d$ if the g_i are the intersections of $n^{1/d}$ hyperplanes in $[0,1]^d$ parallel to each axis with $n^{-1/d}$ spacing) between the hyperplanes. Below π_d denotes the volume of a hyperball of unit radius in $\mathbb{R}^d$.

Lemma 4.1 *Let $\{g_i\}_{i=1}^n$ denote a collection of grid points in $[0,1]^d$ and, for a fixed $C > 0$, let D denote the subset of n-tuples in $([0,1]^d)^n$, with the property that for each grid point there is some point in D that is "close":*

$$D := \{x = (x_1, \dots, x_n) \in ([0,1]^d)^n : \max_{1 \leq j \leq n} dist(g_j, \{x_i\}_{i=1}^n) \leq C(\log n/n)^{1/d}\}.$$

Then

$$\mu^n(D^c) \leq n^{1-\pi_d 2^{-d} C^d}.$$

Proof Consider D^c and note that $D^c \subset \cup_{j=1}^n D_j^c$, where D_i^c is defined as: $D_i^c = \{x \in ([0,1]^d)^n : \text{dist}(g_i, \{x_j\}_{j=1}^n) > C(\log n/n)^{1/d}\}$. Clearly, $\mu^n(D^c) \leq \sum_{i=1}^n \mu^n(D_i^c)$. We also have, for any $i = 1, \dots, n$, with

$$\mu^n(D_i^c) \leq \left(1 - 2^{-d} \pi_d C^d (\log n/n)\right)^n.$$

To see this, note that with $D_{i,j}^c = \{x \in ([0,1]^d)^n : \text{dist}(g_i, x_j) > C(\log n/n)^{1/d}\}$, we have $D_i^c = \cap_{j=1}^n D_{i,j}^c$, and $\mu(D_{i,j}^c) = 1 - \pi_d C^d (\log n/n)$, if the hyperball centered at g_i with radius $(C \log n/n)^{1/d}$ is fully contained in $[0,1]^d$. As at least a fraction of 2^{-d} of this hyperball is contained in $[0,1]^d$, it follows that, $\mu(D^c_{i,j}) \le 1-2^{-d}\pi_d C^d(\log n/n)$. As $1-x \le e^{-x}$ for $x \ge 0$, we have, $\mu^n(D^c_i) \le (1-2^{-d}\pi_d C^d \log n/n)^n \le e^{-2^{-d}\pi_d C^d \log n} = n^{-2^{-d}\pi_d C^d}$. Hence $\mu^n(D^c) \le \sum_{i=1}^n \mu^n(D^c_i) \le n^{1-\pi_d 2^{-d} C^d}$. ∎

First, we provide an outline of the proof of Theorem 1.1. This approach follows the approach developed by Rhee in [20]. For fixed n, with $V = \{U_1, \dots, U_n\}$ we let A denote all sets of n vertices on $[0,1]^d$, for which $P(V)$ exceeds the median weight $M(n)$, and define the t-enlargement A_t of A, where $t = t(n)$. The definition of A_t and some technical constructions ensures that if $y \in A_t$ then $P(y)$ is "close" to $M(n)$. It follows that with a probability of at least $1-n^{-\beta}$ any set of vertices has $P(V)$ "close" to $M(n)$.

In order to bring us in a position to apply Lemma 2.3 we need to make sure that if $y \in A_t$ then there is some $x \in A$ so that x and y are "close" and so that the maximum weight of the MST edges on x resp. y is "small". That is the role of the set D of grid points defined in Lemma 4.1. They provide a point of reference: if x and y are both close to D then x must be close to y. A set B of configurations for which the MSTs with "short" edge lengths will be defined to make sure that we are only dealing with MSTs that do not violate the assumptions for Lemma 2.3. We show:

Theorem 4.3 *For the MST-based power assignment $P(V_n)$, with weights $c(e) = \|e\|^d$:*

$$\lim_{n\to\infty} \mathbb{E}[|P(V_n) - M(n)|] = 0, \text{ and} \tag{10}$$

$$|P(V_n) - M(n)| \xrightarrow{c.c.} 0, \tag{11}$$

where $M(n)$ denotes a median of $P(V_n)$.

In order to prove statements (10) and (11), we first show:

Lemma 4.2 *For any $\epsilon, \beta > 0$:*

$$\mathbb{P}(|P(V_n) - M(n)| > \epsilon) \le O(n^{-\beta}) + 6exp\left(-\frac{n}{(\log n)^2}\frac{\epsilon^2}{D_0^2}\right), \tag{12}$$

where $D_0 = (3c(d)-1)C(\beta)^d$, $C(\beta)$ is chosen so that with a probability of $1-n^{-\beta}$ the edges in the MST on $\{U_i\}_{i=1}^n$ have length at most $C(\beta)(\log n/n)^{1/d}$, and the implicit constant in $O(\cdot)$ does not depend on ϵ.

Proof Fix $\epsilon > 0$ and $\beta > 0$ and let $A \subseteq ([0,1]^d)^n$ consist of those n-tuples $V_n := (u_1, \dots, u_n) \in ([0,1]^d)^n$ for which

$$P(V_n) \ge M(n).$$

By definition of $M(n)$, $\mu^n(A) \ge 1/2$. Choose $C(\beta)$, so that $2^{-d}\pi_d C(\beta)^d > \beta$, and so that with a probability of at least $1-n^{-\beta}$, the edges in the MST on $\{x_i\}_{i=1}^n$ have length at most $C(\beta)(\log n/n)^{1/d}$. Let $B \subset ([0,1]^d)^n$ consist of all n-tuples $x = (x_1, \dots, x_n)$ such that the edges in the MST on $\{x_i\}_{i=1}^n$ have length at most $C(\beta)(\log n/n)^{1/d}$. By further increasing $C(\beta)$ we can ensure that with D as in Lemma 4.1, it follows that $\mu^n(B) \ge 1-n^{-\beta}$, and $\mu^n(D) \ge 1-n^{-\beta}$. Thus, we easily have $\mu^n(A \cap B \cap D) \ge 1/3$.

For n large enough, it follows by Talagrand's theorem Theorem 4.2 (see [24], Proposition 5.1), that if we define $t = t(n) = D_0^{-1}\epsilon\ (n/\log n)$ with $D_0 = 3c(d)C(\beta)^d$, the volume of the enlarged set $\mu^n(A \cap B \cap D)_t^c \le O(n^{-\beta})$:

$$\mu^n((A \cap B \cap D)_t^c) \le 3\exp\left(-\frac{n}{(\log n)^2}\frac{\epsilon^2}{D_0^2}\right). \tag{13}$$

Now define $E := (B \cap D) \cap (A \cap B \cap D)_t$, so E is the set of points "close" to a grid point with "short" edges in the MST and not deviating "too much" from $A \cap B \cap D$. Note that, $\mu^n(E^c) \le \mu^n(B^c) + \mu^n(D^c) + \mu^n((A \cap B \cap D)_t^c)$,

$$\mu^n(E^c) \le O(n^{-\beta}) + 3\exp(-f(n)),$$

where $f(n)$ is shorthand notation for the expression in (13). We now show that if $x \in E$ then $|P(x) - M(n)|$ is bounded. Suppose $x := (x_1, \dots, x_n) \in E$, then $x \in (A \cap B \cap D)_t$ and so there is a point $y := y(x) = (y_1, \dots, y_n) \in A \cap B \cap D$ such that $H(x, y) \le t$. Since x and y are both in B, the edges in the graph of the minimal spanning tree on x and y have length bounded by $C(\beta)(\log n/n)^{1/d}$. Since x and y are both in D, y is close to x in the sense that $\max_{1\le i\le n} \text{dist}(x_i, \{y_j\}_{j=1}^n) \le C(\beta)(\log n/n)^{1/d}$ and $\max_{1\le i\le n} \text{dist}(y_i, \{x_j\}_{j=1}^n) \le C(\beta)(\log n/n)^{1/d}$. By Lemma 2.3, the fact that $H(x, y) \ge l(x', y')$, and weights are the dth powers of the distances, our definition of $t(n)$ implies:

$$|P(y) - P(x)| \le 3c(d)t(n)C(\beta)^d\left(\frac{\log n}{n}\right) = \epsilon.$$

Therefore, for all $x \in E$ and $y = y(x) \in A \cap B \cap D$ as above, we have

$$P(x) \ge P(y) - |P(x) - P(y)| \ge M(n) - \epsilon.$$

Thus it follows for an n-tuple of uniform i.i.d. random variables $V_n = (U_1, \dots, U_n) \in ([0, 1]^d)^n$ that

$$\mathbb{P}\left(P(V_n) < M(n) - \epsilon\right) \le \mu^n(E^c) \le O(n^{-\beta}) + 3\exp(-f(n)).$$

By a similar argument, defining $A \subseteq ([0, 1]^d)^n$ as the set of those n-tuples $V_n := (u_1, \dots, u_n) \in ([0, 1]^d)^n$ for which $P(V_n) \le M(n)$, we find for an n-tuple of uniform i.i.d. random variables $V_n = (U_1, \dots, U_n) \in ([0, 1]^d)^n$ that

$$\mathbb{P}(P(V_n) > M(n) + \epsilon) \le O(n^{-\beta}) + 3\exp(-f(n)),$$

which shows the Lemma. ■

We proceed by showing the next step in the proof of Theorem 1.1.

Proof of Theorem 4.3 We first show (10). For $V_n = \{x_1, \dots, x_n\}$, consider $|P(V_n) - M(n)|$. First note that $|P(V_n) - M(n)| \le C$ for an appropriate constant $C > 0$ and all V_n. To see this, as in the proof of Theorem 4.1, observe that $0 \le P(x) \le C$, for some constant C, depending on d. For $M(n)$ clearly the same is true. Combining the identity $\mathbb{E}[|P(V_n) - M(n)|] = \int_0^C \mathbb{P}(|P(V_n) - M(n)| > t)\mathrm{d}t$ with the estimate (12) implies

$$\mathbb{E}[|P(V_n) - M(n)|] \le O(n^{-\beta}) + \epsilon, \tag{14}$$

As ϵ is arbitrarily small and $O(n^{-\beta})$ tends to 0 when $n \to \infty$, this shows (10). To see (11), consider (12) with $\epsilon > 0$, and $\beta > 1$. So, for all $\epsilon > 0$ we have,

$$\sum_{n=2}^{\infty} \mathbb{P}\left(|P(V_n) - M(n)| > \epsilon\right) \leq \sum_{n=2}^{\infty} O(n^{-\beta}) + 6 \sum_{n=2}^{\infty} \exp\left(-\frac{n}{(\log n)^2} \frac{\epsilon^2}{D_0^2}\right), \tag{15}$$

which is bounded by a constant, as $\beta > 1$. This implies complete convergence. ∎

Finally, we are in the position to give the proof of the main theorem.

Proof of Theorem 1.1 By Theorem 4.1 $\lim_{n\to\infty} \mathbb{E}[P(V_n)] = \mu_P^d$ and by (10) $\lim_{n\to\infty} \mathbb{E}[|P(V_n) - M(n)|] = 0$. This implies that also $\lim_{n\to\infty} | \mathbb{E}[P(V_n)] - M(n) | = 0$. In order to show (1), note that by the triangle inequality,

$$\mathbb{P}(|P(V_n) - \mu_P(d)| > \varepsilon) \leq \mathbb{P}(|P(V_n) - M(n)| + |M(n) - \mathbb{E}[P(V_n)]| + |\mathbb{E}[P(V_n)] - \mu_P(d)| > \varepsilon)$$

by the previous convergence results we have, for large n, that both $|M(n) - \mathbb{E}[P(V_n)]| \leq \varepsilon/3$ and $|\mathbb{E}[P(V_n)] - \mu_P(d)| \leq \varepsilon/3$ so,

$$\mathbb{P}(|P(V_n) - M(n)| + |M(n) - \mathbb{E}[P(V_n)]| + |\mathbb{E}[P(V_n)] - \mu_P(d)| > \varepsilon) \leq \mathbb{P}(|P(V_n) - M(n)| \geq \varepsilon/3).$$

Now (1) follows from (11). ∎

5 | CONCLUSIONS AND FURTHER RESEARCH

This paper presents an average case analysis of the minimum spanning tree heuristic for the power assignment problem on a graph with power weighted edges. The worst-case approximation ratio of this heuristic is 2. We show that in Euclidean d-dimensional space, when the distance power gradient equals the dimension and the vertex set V consists of a set of n uniform i.i.d. random variables in $[0, 1]^d$ the minimum spanning tree-based power assignment $P(V)$ converges completely c.c. to a constant. In order to show this, we used extensions of results of Yukich [27], that require general results on minimum spanning trees in graphs that are of interest by itself. It would be interesting to investigate whether the methods of Penrose and Yukich [18] which show L^2 convergence rather than complete convergence for any $p > 0$, with a more general class of probability density functions could be applied to extend our results. To this end, results from Penrose [17] may be invoked to obtain an almost sure convergence result.

Also interesting would be to further investigate the approximation ratio for power assignments. This paper shows that the ratio of the optimal power assignment to the power assignment based on the MST-heuristic c.c. to a constant $1 \leq \tau < 2$, but it is an open question to obtain stronger bounds.

Concerning the MST heuristic, further research into heuristics as presented in [3], and a further extension of this type of results to power assignments resulting in general k-connected graphs are interesting next steps.

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