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A successive censoring algorithm for a system of connected LDQBD-processes

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Abstract

We consider a Markov Chain in which the state space is partitioned into sets where both transitions within sets and between sets have a special structure. Transitions *within* each set constitute a finite level dependent quasi-birth-and-death-process (LDQBD), and transitions *between* sets are restricted to six types of transitions. These latter types are needed to preserve the sets structure in the reduction step of our algorithm. Specifically, we present a successive censoring algorithm, based on matrix analytic methods, to obtain the stationary distribution of this system of connected LDQBD-processes.

Keywords Successive censoring algorithm · Matrix analytic methods · Connected level dependent QBD-processes · Steady state analysis · Exact aggregation/disaggregation

1 Introduction

We consider a class of continuous time Markov chains with hysteresis. The hysteresis phenomenon is a common problem in physics that is due to magnetic fields, see, e.g., Hassani et al. (2014) for a detailed survey. Hysteresis is also encountered in other applications. For example, Baer et al. (2019) modelled road traffic as a queueing model with hysteresis. Their model captured the empirical shape of the relation between speed, flow and density, the so-called fundamental diagram, for a real-world uninterrupted traffic system on a Danish highway. They showed the power of their model with hysteresis by mimicking a wide range of fundamental diagram shapes. Tournaire et al. (2019) analyzed an auto-scaling cloud system using a queueing model with hysteresis. This model is used to optimize the system's power

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consumption while satisfying the required quality of service. They applied their model on a real case of a cloud system and showed its relevance in optimizing the cost and determining the pricing strategy of the instances of cloud service providers. Additional application areas of models with hysteresis include image processing, see Medina-Carnicer et al. (2010), systems control, see Chen et al. (2008), and communication networks, see Le Ny and Tuffin (2002). For additional work on queueing models with hysteresis we refer to, e.g., Dikong and Dshalalow (1999) and Lu and Serfozo (1984).

In this paper, we focus on a class of Markov chains in which the state space can be partitioned into sets. Transitions within each set constitute a finite Level Dependent Quasi-Birth-and-Death process (LDQBD), and the transitions between sets follow special types. To obtain the stationary distribution of such a Markov chain, we present a successive censoring algorithm building on the censoring algorithm by Kemeny and Snell (1960) for discrete time Markov chains. The basic censoring algorithm consists of three main steps: reduction step, intermediate step, and expansion step. In our case, we repeat the reduction step multiple times, where in each step an LDQBD set is eliminated from the state space. The considered special types of the transitions between the sets preserve the tri-diagonal structure of the LDQBD sets after the reduction steps. We prove that these types of transitions are the only ones to satisfy the preservation property. These findings help in deriving the fundamental matrix of the transient LDQBD chain needed in the reduction and the expansion steps. Furthermore, we propose a new algorithm to automatically check whether a Markov chain has the required type of transitions. Finally, we show that the successive censoring algorithm reduces the complexity of the computation of the stationary distribution compared to the standard algorithm in the literature.

Our work extends the work of Gaver et al. (1984) to determine the stationary distribution of a LDQBD with a finite number of levels to more general transitions, see Remark 1. The censoring algorithm Kemeny and Snell (1960) also forms the basis for the folding algorithm in Ye and Li (1994) and Li and Sheng (1996), where the stationary distribution of a finite QBD was obtained by sequentially splitting (and renumbering) the state space in odd and even numbered sets, followed by application of the censoring algorithm to the two resulting subsets. In the literature, the censoring algorithm is also called exact aggregation/disaggregation algorithm in which the state space is aggregated to obtain a smaller (and easier to solve) Markov chain. The stationary distribution for this aggregated Markov chain is then disaggregated to obtain the stationary distribution of the full Markov chain. The recent work in this area is of Katehakis and Smit (2012) and Katehakis et al. (2015). In Katehakis and Smit (2012), a Markov chain is studied in which the state space is partitioned in sets, without any restrictions on the transitions within a set. In their successive lumping procedure it is crucial that a set contains a single entrance state, i.e., a single state through which the set can be reached from other sets. Our work extends this aggregation method by allowing multiple entrance states, under the restriction that the transitions within a set form an LDQBD. The work in Katehakis and Smit (2012) is applied to Quasi-Skip Free Processes to the left in Katehakis et al. (2015), where it is assumed that lower levels are entered via a single entrance state only. The single entrance states in Katehakis and Smit (2012), Katehakis et al. (2015) are called *mandatory* states in Kim and Smith (1989) and *input* states in Feinberg and Chiu (1987) in which a *parallel* lumping procedure was introduced. Most recently, Ertiningsih et al. (2019) studied a class of Quasi–Skip Free processes where the transition rate submatrices in the Skip Free direction, either to the right or the left, have a column times row structure. For this class of Markov chains they derived the stationary distributions and its properties. For a thorough overview and comparison of several aggregation/disaggregation algorithms see Cao and Stewart (1985), Haviv (1987), Kafeety et al. (1992) and Rogers and Plante (1993).

Our contributions in this paper are as follow:

- We extend several results of the literature: (1) In Gaver et al. (1984), a successive censoring algorithm is presented to find the stationary distribution of a LDQBD with number of sets equal to 1. In our case, the number of sets is a finite positive integer. (2) In Katehakis and Smit (2012) a successive lumping procedure is presented for a Markov chain with a state space that can be partitioned into sets where each set has only one single state through which the set is entered. In our case, each set can have multiple entrance states. (3) In Ertiningsih et al. (2019) a two dimensional ski-free process is considered with up transition matrices having a rank of one, e.g., equal to the product of a column and a row vector. In our case, we consider a three-dimensional process with the rank of matrices larger than one.
- We give a simplified algorithm to check for the applicability of our main censoring algorithm on Markov chains with complex structure.
- We show that our algorithm has a reduced complexity compared to the standard algorithm in the literature, see Sect. 5.

Our paper is structured as follows. Section 2 introduces the system of connected LDQBDs as a three-dimensional Markov chain and specifies the exact restrictions on the transitions between the LDQBDs. In Sect. 3, we present the successive censoring algorithm to determine the stationary distribution of the system of connected LDQBDs. In Sect. 4, we give an algorithm which determines if the successive censoring algorithm can be applied for a given Markov chain. We perform a complexity analysis in Sect. 5 and the algorithm is demonstrated with an example in Sect. 6. Section 7 gives concluding remarks.

2 Model description

We consider a three-dimensional Markov chain $\mathcal{X}$, describing a system of connected Level Dependent Quasi-Birth-and-Death processes (LDQBD), with states (s, l, p) where s denotes the *set*, l denotes the *level* and p denotes the *phase* of a single state. Let $\boldsymbol{Q}$ be its infinitesimal generator. Each set ω_s, $s = 1, \dots, S$, has L_s levels labelled L_l^s, $l = 1, \dots, L_s$, and each level L_l^s in ω_s has $P_{s,l}$ phases labelled $P_p^{s,l}$, $p = 1, \dots, P_{s,l}$. The transitions *within* each set ω_s, $s = 1, \dots, S$, constitute an LDQBD and are described by the generator $\boldsymbol{Q}_{s,s}$, a submatrix of $\boldsymbol{Q}$, in which the states are ordered lexicographically, i.e., $(s, 1, 1)$,$(s, 1, 2)$,...,$(s, 1, P_{s,1})$, $(s, 2, 1)$,...,$(s, 2, P_{s,2})$,...,$(s, L_s, 1)$,...,(s, L_s, P_{s,L_s}). The (transient) generator $\boldsymbol{Q}_{s,s}$ has a tri-diagonal block structure and is given by

$$\boldsymbol{Q}_{s,s} = \begin{bmatrix} \boldsymbol{L}_s^{(1)} & \boldsymbol{F}_s^{(1)} & 0 & \cdots & \cdots & 0 \\ \boldsymbol{B}_s^{(2)} & \boldsymbol{L}_s^{(2)} & \boldsymbol{F}_s^{(2)} & \ddots & & \vdots \\ 0 & \boldsymbol{B}_s^{(3)} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \boldsymbol{L}_s^{(L_s-1)} & \boldsymbol{F}_s^{(L_s-1)} \\ 0 & \cdots & \cdots & 0 & \boldsymbol{B}_s^{(L_s)} & \boldsymbol{L}_s^{(L_s)} \end{bmatrix}, \qquad (1)$$

where, $\boldsymbol{F}_s^{(l)}$ describes the transitions from L_l^s to L_{l+1}^s, $\boldsymbol{B}_s^{(l)}$ describes the transitions from L_l^s to L_{l-1}^s, and $\boldsymbol{L}_s^{(l)}$ describes the transitions within L_l^s.

The transitions *between* two sets ω_i and ω_j, $i \neq j$, are governed by two sets of conditions, referred to as, *direct* and *indirect* conditions. These conditions ensure that the tri-diagonal structure within each set is maintained throughout the successive censoring algorithm that is introduced in Sect. 3. A specific part of the algorithm will require the determination of the inverse of the generator $\boldsymbol{Q}_{s,s}$. If the tri-diagonal block structure is maintained throughout the algorithm, this inverse can be obtained using the results by Shin (2009) on the fundamental matrix of a transient LDQBD, or the results by Choi et al. (2003) on the fundamental matrix of a transient QBD. The latter holds in the case of a system of connected QBD processes. These results are also presented in Chapter 5 of Baër (2015).

We denote by $\boldsymbol{Q}_{i,j}$, $i, j = 1, \dots, S$, the submatrix of $\boldsymbol{Q}$ with transitions from ω_i to ω_j, and by $\left[\boldsymbol{Q}_{i,j}\right]_{a,b}$, $a = 1 \dots, L_i$ and $b = 1 \dots, L_j$, the submatrix of $\boldsymbol{Q}_{i,j}$ with transitions from L_a^i to L_b^j.

Definition 1 The direct conditions describe the one step transitions between ω_i and ω_j. We define six sets of transitions that can occur between ω_i and ω_j for ($i < j$ and fixed positive integer z):

$T^{(1)}(z)$: Transitions from any level $L_l^i, l = 1, \dots, L_i$, to only L_z^j and L_{z+1}^j, and back, i.e.,

$$\begin{aligned} \left[\boldsymbol{Q}_{i,j}\right]_{a,b} &= 0, && \text{if } b \neq z \text{ and } b \neq z+1, \\ \left[\boldsymbol{Q}_{j,i}\right]_{b,a} &= 0, && \text{if } b \neq z \text{ and } b \neq z+1, \end{aligned}$$

for $z = 1, \dots, L_j - 1$.

$T^{(2)}(z)$: Transitions from any level L_l^i, $l = 1, \dots, L_i$, to only L_{z-1}^j, L_z^j, and L_{z+1}^j, and transitions from level L_z^j to any level $L_l^i, l = 1, \dots, L_i$, i.e.,

$$\begin{aligned} \left[\boldsymbol{Q}_{i,j}\right]_{a,b} &= 0, && \text{if } b \neq z-1, b \neq z \text{ and } b \neq z+1, \\ \left[\boldsymbol{Q}_{j,i}\right]_{b,a} &= 0, && \text{if } b \neq z, \end{aligned}$$

for $z = 2, \dots, L_j - 1$.

$T^{(3)}(z)$: Transitions from any level $L_l^i, l = 1, \dots, L_i$, to only L_z^j, and transitions from levels L_{z-1}^j, L_z^j, and L_{z+1}^j to any level $L_l^i, l = 1, \dots, L_i$, i.e.,

$$\begin{aligned} \left[\boldsymbol{Q}_{i,j}\right]_{a,b} &= 0, && \text{if } b \neq z, \\ \left[\boldsymbol{Q}_{j,i}\right]_{b,a} &= 0, && \text{if } b \neq z-1, b \neq z \text{ and } b \neq z+1, \end{aligned}$$

for $z = 2, \dots, L_j - 1$.

$T^{(4)}$: Only transition from ω_i to ω_j, i.e., $\boldsymbol{Q}_{j,i} = 0$.

$T^{(5)}$: Only transition from ω_j to ω_i (reversed $T^{(4)}$ transition), i.e., $\boldsymbol{Q}_{i,j} = 0$.

$T^{(6)}$: No transitions between ω_i and ω_j, i.e., $\boldsymbol{Q}_{i,j} = 0$ and $\boldsymbol{Q}_{j,i} = 0$. □

Note that $T^{(4)}$ and $T^{(5)}$ are mutually exclusive, except for the trivial case of all zero, however, other condition sets may have a non-empty intersection, for example $T^{(2)}(x)$ and $T^{(3)}(y)$ for $x = y$, and $T^{(1)}(x)$ and $T^{(4)}$.

These six sets of transitions are shown in an example in Fig. 1. In this small example we consider a network of connected LDQBDs and focus on ω_i and ω_j, each with 5 levels, and their one-step transitions. For each of the six sets of transitions from Definition 1 we present a schematic view of the generator. In this schematic view, we depict a (possibly) non-zero submatrix by a light gray square. The dark gray squares depict the one-step transitions between ω_i and ω_j. The white squares depict zero-submatrices. In Fig. 1 it is shown that

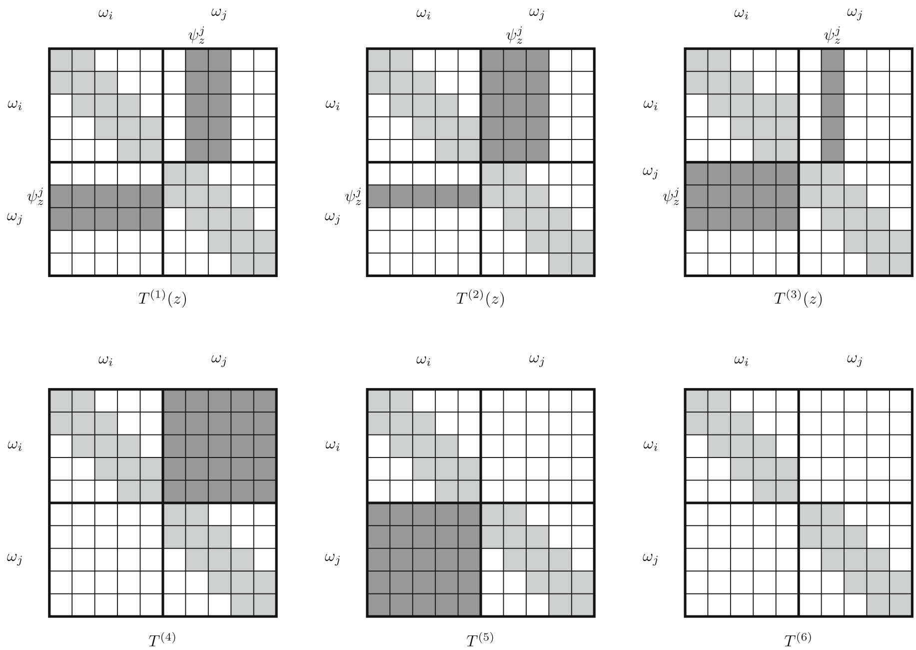


Fig. 1 Schematic representation of the generators corresponding to each of the six types of transitions between ω_i and ω_j

in a $T^{(1)}(z)$ transition there are transitions from any level in ω_i to L_z^j and L_{z+1}^j, and back. Figure 1 also shows that a $T^{(5)}$ transitions only consist of transitions from ω_j to ω_i. Figure 1 makes it easy to visualise how the intersection of $T^{(1)}(z)$ and $T^{(5)}$, with transitions from L_z^j and L_{z+1}^j to any level in ω_i but none back, looks like. Finally, observe that a $T^{(6)}$ transition is the trivial all-zero intersection of the other five sets of transitions.

Definition 2 Indirect conditions describe the multiple step paths between ω_i and ω_j. We define a **lower path** from ω_i to ω_j as a path from ω_i to ω_j only passing through sets with index less than $\min\{i, j\}$. Based on the one step transitions between ω_i and ω_j, $(i < j)$ in Definition 1, we define the following indirect conditions

i. If there is a $T^{(1)}(z)$ transition between ω_i and ω_j, $z = 1, \ldots, L_j - 1$, then:

a. Each lower path from ω_i to ω_j is enabled only if it ends with a $T^{(1)}(z)$ transition, and,
b. Each lower path from ω_j to ω_i is enabled only if it starts with a $T^{(1)}(z)$ transition.

ii. If there is a $T^{(2)}(z)$ transition between ω_i and ω_j, $z = 2, \ldots, L_j - 1$, then:

a. Each lower path from ω_i to ω_j is enabled only if it ends with a $T^{(2)}(z)$ transition, and,
b. Each lower path from ω_j to ω_i is enabled only if it starts with a $T^{(2)}(z)$ transition.

iii. If there is a $T^{(3)}(z)$ transition between ω_i and ω_j, $z = 2, \ldots, L_j - 1$, then:

a. Each lower path from ω_i to ω_j is enabled only if it ends with a $T^{(3)}(z)$ transition, and,

b. Each lower path from ω_j to ω_i is enabled only if it start with a $T^{(3)}(z)$ transition.

iv. If there is a $T^{(4)}$ transition between ω_i and ω_j, then there cannot be a lower path from ω_j to ω_i.
v. If there is a $T^{(5)}$ transition between ω_i and ω_j, then there cannot be a lower path from ω_i to ω_j.
vi. If there is a $T^{(6)}$ transition between ω_i and ω_j, then either:

a. For some $z \in \{1, \ldots, L_j - 1\}$, all lower paths from ω_j to ω_i start with a $T^{(1)}(z)$ transition and all lower paths from ω_i to ω_j end with a $T^{(1)}(z)$ transition, or,
b. For some $z \in \{2, 3, \ldots, L_j - 1\}$, all lower paths from ω_j to ω_i start with a $T^{(2)}(z)$ transition and all lower paths from ω_i to ω_j end with a $T^{(2)}(z)$ transition, or,
c. For some $z \in \{2, 3, \ldots, L_j - 1\}$, all lower paths from ω_j to ω_i start with a $T^{(3)}(z)$ transition and all lower paths from ω_i to ω_j end with a $T^{(3)}(z)$ transition, or,
d. There can be one or more lower paths from ω_i to ω_j, but none from ω_j to ω_i, or,
e. There can be one or more lower paths from ω_j to ω_i, but none from ω_i to ω_j, or,
f. There are no lower paths between ω_i and ω_j.

□

In Sect. 6, we give an example satisfying the above conditions. In general, there are many lower paths from ω_i to ω_j which makes it difficult to check whether or not Definition 2 holds for a given Markov chain. Algorithm 2 is a simplified version of the successive censoring algorithm, given by Algorithm 1, which quickly checks if Definition 2 holds for a given Markov chain.

Remark 1 (Special cases)
We will briefly discuss the relation between our model and the models discussed in Gaver et al. (1984), Katehakis and Smit (2012), and Ertiningsih et al. (2019).

In Gaver et al. (1984), a successive censoring algorithm is presented to find the stationary distribution of a LDQBD. Assuming $S = 1$ we restrict our model to a single set and we obtain a LDQBD. In this special case, our successive censoring algorithm is the same as the successive censoring algorithm of Gaver et al. (1984).

In Katehakis and Smit (2012) a successive lumping procedure is presented for a special class of Markov chains. Important is that the state space can be partitioned into sets and that in each set there is only one single entrance state, a state through which the set is entered. Note that there are no restrictions for the transitions within a set. By assuming that all levels consists of a single phase, and by restricting to transition from and to a single level in ω_j, i.e., the intersection of a $T^{(2)}(x)$ and $T^{(3)}(x)$ transition, $x = 1, \ldots, L_j$, for each set ω_j, $j = 1, \ldots, S$, we obtain a special case of both our model and the model by Katehakis and Smit (2012).

Finally, it is worth noting that in this paper there is no restriction that the up transition matrices should have a rank of one, e.g., equal to the product of a column and a row vector, as assumed in Ertiningsih et al. (2019).

3 Successive censoring algorithm

Let $\boldsymbol{\pi} = [\,\boldsymbol{\pi}_1 \ \boldsymbol{\pi}_2 \ \cdots \ \boldsymbol{\pi}_S\,]$ denote the stationary distribution of the three-dimensional Markov chain $\mathcal{X}$ such that $\boldsymbol{\pi}\boldsymbol{Q} = \boldsymbol{0}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$ and let $\boldsymbol{\pi}_i$ denote the probability vector of states in ω_i, $i = 1, \ldots, S$. We obtain $\boldsymbol{\pi}$ by using a successive censoring algorithm based on the censoring algorithm in Kemeny and Snell (1960) and its extension to continuous time Markov Chains, described in Ye and Li (1994). In the censoring algorithm the state space of an arbitrary Markov Chain $\mathcal{Y}$ is first split into subsets A and B such that its generator $\boldsymbol{T}$ and stationary distribution $\boldsymbol{\nu}$ can be partitioned as follows:

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{T}_A & \boldsymbol{T}_{AB} \\ \boldsymbol{T}_{BA} & \boldsymbol{T}_B \end{bmatrix}, \qquad \boldsymbol{\nu} = [\, \boldsymbol{\nu}_A \;\; \boldsymbol{\nu}_B \,].$$

Then a reduction step occurs in which transitions from B to B via A are projected onto transitions within B creating the generator $\widetilde{\boldsymbol{T}}_B$:

$$\widetilde{\boldsymbol{T}}_B = \boldsymbol{T}_B + \boldsymbol{T}_{BA} \left[-\boldsymbol{T}_A\right]^{-1} \boldsymbol{T}_{AB}. \tag{2}$$

During an intermediate step the stationary distribution $\widetilde{\boldsymbol{\nu}}_B$ is determined by solving:

$$\widetilde{\boldsymbol{\nu}}_B \widetilde{\boldsymbol{T}}_B = \boldsymbol{0},$$

and then used in the expansion step the determine $\widetilde{\boldsymbol{\nu}}_A$

$$\widetilde{\boldsymbol{\nu}}_A = \widetilde{\boldsymbol{\nu}}_B \boldsymbol{T}_{BA} \left[-\boldsymbol{T}_A\right]^{-1}. \tag{3}$$

Normalising $[\widetilde{\boldsymbol{\nu}}_A \; \widetilde{\boldsymbol{\nu}}_B]$ gives the stationary distribution $\boldsymbol{\nu}$.

The successive censoring algorithm consists of $S-1$ reduction steps (2), one intermediate step, and $S-1$ expansion steps (3). In reduction step k, $k = 1, \ldots, S-1$, the generator $\boldsymbol{Q}^k$ is reduced to $\boldsymbol{Q}^{k+1}$ by removing ω_k from the state space (censoring). Observe that following this definition, $\boldsymbol{Q}^1 = \boldsymbol{Q}$. In the intermediate step, the stationary distribution of $\boldsymbol{Q}^S$ is determined. Next, in expansion step k, $k = 1, \ldots, S-1$, the stationary distribution is expanded by adding ω_{S-k}, the set with highest index still censored, back to the state space. Finally, by normalising the resulting vector, we obtain the stationary distribution $\boldsymbol{\pi}$. The successive censoring algorithm is formally described in Algorithm 1.

Algorithm 1 (The Successive Censoring Algorithm)

1. Reduce the state space in $S-1$ reduction steps. In reduction step k, ω_k is removed from the state space and the generator $\boldsymbol{Q}^k$ is reduced to $\boldsymbol{Q}^{k+1}$ using the following equation

$$\boldsymbol{Q}^{k+1}_{i,j} = \boldsymbol{Q}^k_{i,j} + \boldsymbol{Q}^k_{i,k} \left[-\boldsymbol{Q}^k_{k,k}\right]^{-1} \boldsymbol{Q}^k_{k,j}. \tag{4}$$

2. Determine the stationary distribution Π^S of $\boldsymbol{Q}^S$, such that $\Pi^S \boldsymbol{Q}^S = 0$ and $\Pi^S \boldsymbol{e} = 1$.
3. Expand Π^S in $S-1$ expansion steps. In expansion step k, ω_{S-k} is added to the state space and the vector $\Pi^{S-k+1} = \left[\, \boldsymbol{p}^{S-k+1} \;\; \boldsymbol{p}^{S-k+2} \;\; \cdots \;\; \boldsymbol{p}^S \,\right]$ is expanded to Π^{S-k} where the vector $\boldsymbol{p}^k$ is obtained using the following equation

$$\boldsymbol{p}^{S-k} = \sum_{i=1}^{k} \boldsymbol{p}^{S-k+i} \boldsymbol{Q}^{S-k}_{S-k+i,S-k} \left[-\boldsymbol{Q}^{S-k}_{S-k,S-k}\right]^{-1}. \tag{5}$$

4. Normalise Π^1 to obtain the stationary distribution $\boldsymbol{\pi}$ of the Markov Chain $\mathcal{X}$, i.e.,

$$\boldsymbol{\pi} = \frac{\Pi^1}{|\Pi^1|}.$$

□

Each $\boldsymbol{Q}_{j,j}$, $j = 1, \ldots, S$ describes a transient LDQBD and due to the irreducibility assumption the negative inverse $\left[-\boldsymbol{Q}_{j,j}\right]^{-1}$, or fundamental matrix of $\boldsymbol{Q}_{j,j}$, exists and describes the sojourn time in ω_j before transition to some other ω_i, $i \neq j$. Let $\left[-\boldsymbol{Q}_{j,j}\right]^{-1}_{a,b}$, $a, b = 1 \ldots, L_j$, denote the submatrix of $\left[-\boldsymbol{Q}_{j,j}\right]^{-1}$ describing the average time spent in L^j_b before the Markov process leaves ω_j, given that it entered ω_j through L^j_a. The fundamental matrix of $\boldsymbol{Q}_{j,j}$ is given by Shin in Shin (2009) and is discussed in detail in Section 5.5 of Baër (2015). It will be proven in Theorem 1 and Theorem 2 that $\boldsymbol{Q}^k_{j,j}$, $k = 1, \ldots, S-1$ and $j = k, \ldots, S$, is also a transient LDQBD such that the results of Shin in Shin (2009) on the fundamental matrix of a transient LDQBD can be used throughout the algorithm.

In the special case where $\boldsymbol{Q}^k_{j,j}$ would describe a Quasi-Birth-and-Death process (QBD), the results on the fundamental matrix of Choi et al. in Choi et al. (2003) and in Section 5.4 of Baër (2015), can be applied.

The remainder of this section discusses the first three steps of Algorithm 1 in detail.

3.1 Reduction step *k*

In reduction step k, the generator $\boldsymbol{Q}^k$ is reduced to $\boldsymbol{Q}^{k+1}$ by removing ω_k from the state space. Observe that ω_k is the set with the smallest index in $\boldsymbol{Q}^k$. Following the reduction step (2) we obtain for $i, j > k$

$$\boldsymbol{Q}^{k+1}_{i,j} = \boldsymbol{Q}^k_{i,j} + \boldsymbol{Q}^k_{i,k}\left[-\boldsymbol{Q}^k_{k,k}\right]^{-1}\boldsymbol{Q}^k_{k,j}.$$

Decomposing these submatrices by their levels, for $i = j > k$, gives:

$$\left[\boldsymbol{Q}^{k+1}_{i,i}\right]_{x,y} = \left[\boldsymbol{Q}^k_{i,i}\right]_{x,y} + \sum_{a=1}^{L_k}\sum_{b=1}^{L_k}\left[\boldsymbol{Q}^k_{i,k}\right]_{x,a}\left[-\boldsymbol{Q}^k_{k,k}\right]^{-1}_{a,b}\left[\boldsymbol{Q}^k_{k,i}\right]_{b,y}. \tag{6}$$

In this reduction step transitions from ω_i to ω_i via ω_k are projected onto transitions within ω_i. For example, a $T^{(1)}(z)$ transition from ω_k to ω_i is projected onto transitions within and between L^i_z and L^i_{z+1} and in this case (6) can be reduced since

$$\left[\boldsymbol{Q}^{k+1}_{i,i}\right]_{x,y} = \left[\boldsymbol{Q}^k_{i,i}\right]_{x,y}, \qquad \text{if } x, y \notin \{z, z+1\}.$$

We rewrite (6) as

$$\left[\boldsymbol{Q}^{k+1}_{i,i}\right]_{x,y} = \begin{cases}\left[\boldsymbol{Q}^k_{i,i}\right]_{x,y} + \sum_{a=1}^{L_k}\sum_{b=1}^{L_k}\left[\boldsymbol{Q}^k_{i,k}\right]_{x,a}\left[-\boldsymbol{Q}^k_{k,k}\right]^{-1}_{a,b}\left[\boldsymbol{Q}^k_{k,i}\right]_{b,y}, & \text{if } x \in R_1 \text{ and } y \in R_2,\\ \left[\boldsymbol{Q}^k_{i,i}\right]_{x,y}, & \text{otherwise.}\end{cases} \tag{7}$$

Here, $R_1, R_2 \subseteq \{1, \ldots, L_k\}$ depend on the type of transition between ω_k and ω_i and are given in Table 1. When $R_1 = R_2 = \{z, z+1\}$ all transitions between ω_k and ω_i are projected onto transitions within and between L^i_z and L^i_{z+1}. Observe that $T^{(4)}$, $T^{(5)}$ and $T^{(6)}$ transitions are not projected onto transitions within ω_i since there are no transitions from ω_i to ω_i via ω_k and

$$\left[\boldsymbol{Q}^{k+1}_{i,i}\right]_{x,y} = \left[\boldsymbol{Q}^k_{i,i}\right]_{x,y}.$$

Table 1 The subsets R_1 and R_2 for each type of transition from ω_k to ω_i $(k < i)$

$T^{(1)}(z)$	$T^{(2)}(z)$	$T^{(3)}(z)$
$R_1 = \{z, z+1\}$	$R_1 = \{z\}$	$R_1 = \{z-1, z, z+1\}$
$R_2 = \{z, z+1\}$	$R_2 = \{z-1, z, z+1\}$	$R_2 = \{z\}$
$T^{(4)}$	$T^{(5)}$	$T^{(6)}$
$R_1 = \emptyset$	$R_1 = \emptyset$	$R_1 = \emptyset$
$R_2 = \emptyset$	$R_2 = \emptyset$	$R_2 = \emptyset$

A similar decomposition as (6) applies for transitions between two sets ω_i and ω_j, with $k < i < j$,

$$\left[\boldsymbol{Q}_{i,j}^{k+1}\right]_{x,y} = \begin{cases} \left[\boldsymbol{Q}_{i,j}^{k}\right]_{x,y} + \sum_{a=1}^{L_k}\sum_{b=1}^{L_k}\left[\boldsymbol{Q}_{i,k}^{k}\right]_{x,a}\left[-\boldsymbol{Q}_{k,k}^{k}\right]_{a,b}^{-1}\left[\boldsymbol{Q}_{k,j}^{k}\right]_{b,y}, & \text{if } x \in S_1 \text{ and } y \in S_2, \\ \left[\boldsymbol{Q}_{i,j}^{k}\right]_{x,y}, & \text{otherwise,} \end{cases} \tag{8}$$

and

$$\left[\boldsymbol{Q}_{j,i}^{k+1}\right]_{x,y} = \begin{cases} \left[\boldsymbol{Q}_{j,i}^{k}\right]_{x,y} + \sum_{a=1}^{L_k}\sum_{b=1}^{L_k}\left[\boldsymbol{Q}_{j,k}^{k}\right]_{x,a}\left[-\boldsymbol{Q}_{k,k}^{k}\right]_{a,b}^{-1}\left[\boldsymbol{Q}_{k,i}^{k}\right]_{b,y}, & \text{if } x \in T_1 \text{ and } y \in T_2, \\ \left[\boldsymbol{Q}_{j,i}^{k}\right]_{x,y}, & \text{otherwise.} \end{cases} \tag{9}$$

The subsets $S_1, T_2 \subseteq \{1, \ldots, L_i\}$ and $S_2, T_1 \subseteq \{1, \ldots, L_j\}$ depend on the transitions between ω_k and ω_j and between ω_k and ω_i. For $i < j$ these ranges are given in Table 2. For example, suppose there are $T^{(1)}(x)$ transitions from ω_k to ω_i and $T^{(4)}$ transitions from ω_k to ω_j $(i < j)$. In reduction step k, these transitions will be projected onto transitions from L_x^i and L_{x+1}^i to any level L_l^j, $l = 1, \ldots, L_j$ ($S_1 = \{x, x+1\}$ and $S_2 = \{1, \ldots, L_j\}$), and no transitions from ω_j to ω_i ($T_1 = \emptyset$ and $T_2 = \emptyset$).

Note that during reduction step k the transitions from (or via) ω_k are projected onto existing transitions (including $T^{(6)}$ transitions) between sets ω_i and ω_j, $i, j > k$. Using this we can now formulate the following theorem relating the indirect conditions in Definition 2 to the direct conditions in Definition 1.

Theorem 1 *The indirect conditions in Definition 2 ensure that the direct conditions in Definition 1 are preserved in each reduction step.*

Proof Observe that a lower path from ω_i to ω_j, $i < j$, is projected onto a direct transition from ω_i to ω_j in reduction steps $1, \ldots, i-1$. Therefore, following the order in Definition 2, we can easily state that after reduction step $i-1$:

i. The lower paths in Def. **2.i.a.** and Def. **2.i.b.** will be projected onto transitions from any level L_l^i, $l = 1, \ldots, L_i$, to L_z^j and L_{z+1}^j, and back, i.e.,

$$\left[\boldsymbol{Q}_{i,j}^{i-1}\right]_{a,b} = \mathbf{0}, \qquad \text{if } b \neq z \text{ and } b \neq z+1,$$
$$\left[\boldsymbol{Q}_{j,i}^{i-1}\right]_{b,a} = \mathbf{0}, \qquad \text{if } b \neq z \text{ and } b \neq z+1,$$

for $z = 1, \ldots, L_j - 1$, preserving the $T^{(1)}(z)$ transitions.

Table 2 Ranges r_1, r_2, s_1 and s_2 for different types of transition between ω_k and ω_j and between ω_k and ω_i for $k < i < j$

	Transition between ω_k and ω_j $(k < j)$				
	$T^{(1)}(y)$	$T^{(2)}(y)$	$T^{(3)}(y)$	$T^{(4)}$	$T^{(5)}$
Transition between ω_k and ω_i $(k < i)$					
$T^{(1)}(x)$	$S_1 = \{x, x+1\}$	$S_1 = \{x, x+1\}$	$S_1 = \{x, x+1\}$	$S_1 = \{x, x+1\}$	$S_1 = \emptyset$
	$S_2 = \{y, y+1\}$	$S_2 = \{y-1, y, y+1\}$	$S_2 = \{y\}$	$S_2 = \{1, \ldots, L_j$	$S_2 = \emptyset$
	$T_1 = \{y, y+1\}$	$T_1 = \{y\}$	$T_1 = \{y-1, y, y+1\}$	$T_1 = \emptyset$	$T_1 = 1, \ldots, L_j$
	$T_2 = \{x, x+1\}$	$T_2 = \{x, x+1\}$	$T_2 = \{x, x+1\}$	$T_2 = \emptyset$	$T_2 = \{x, x+1\}$
$T^{(2)}(x)$	$S_1 = \{x\}$	$S_1 = \{x\}$	$S_1 = \{x\}$	$S_1 = \{x\}$	$S_1 = \emptyset$
	$S_2 = \{y, y+1\}$	$S_2 = \{y-1, y, y+1\}$	$S_2 = \{y\}$	$S_2 = \{1, \ldots, L_j\}$	$S_2 = \emptyset$
	$T_1 = \{y, y+1\}$	$T_1 = \{y\}$	$T_1 = \{y-1, y, y+1\}$	$T_1 = \emptyset$	$T_1 = \{1, \ldots, L_j\}$
	$T_2 = \{x-1, x, x+1\}$	$T_2 = \{x-1, x, x+1\}$	$T_2 = \{x-1, x, x+1\}$	$T_2 = \emptyset$	$T_2 = \{x-1, x, x+1\}$
$T^{(3)}(x)$	$S_1 = \{x-1, x, x+1\}$	$S_1 = \{x-1, x, x+1\}$	$S_1 = \{x-1, x, x+1\}$	$S_1 = \{x-1, x, x+1\}$	$S_1 = \emptyset$
	$S_2 = \{y, y+1\}$	$S_2 = \{y-1, y, y+1\}$	$S_2 = \{y\}$	$S_2 = \{1, \ldots, L_j\}$	$S_2 = \emptyset$
	$T_1 = \{y, y+1\}$	$T_1 = \{y\}$	$T_1 = \{y-1, y, y+1\}$	$T_1 = \emptyset$	$T_1 = \{1, \ldots, L_j\}$
	$T_2 = \{x\}$	$T_2 = \{x\}$	$T_2 = \{x\}$	$T_2 = \emptyset$	$T_2 = \{x\}$
$T^{(4)}$	$S_1 = \emptyset$	$S_1 = \emptyset$	$S_1 = \emptyset$	$S_1 = \emptyset$	$S_1 = \emptyset$
	$S_2 = \emptyset$	$S_2 = \emptyset$	$S_2 = \emptyset$	$S_2 = \emptyset$	$S_2 = \emptyset$
	$T_1 = \{y, y+1\}$	$T_1 = \{y\}$	$T_1 = \{y-1, y, y+1\}$	$T_1 = \emptyset$	$T_1 = \{1, \ldots, L_j\}$
	$T_2 = \{1, \ldots, L_i\}$	$T_2 = \{1, \ldots, L_i\}$	$T_2 = \{1, \ldots, L_i\}$	$T_2 = \emptyset$	$T_2 = \{1, \ldots, L_i\}$
$T^{(5)}$	$S_1 = \{1, \ldots, L_i\}$	$S_1 = \{1, \ldots, L_i\}$	$S_1 = \{1, \ldots, L_i\}$	$S_1 = \{1, \ldots, L_i\}$	$S_1 = \emptyset$
	$S_2 = \{y, y+1\}$	$S_2 = \{y-1, y, y+1\}$	$S_2 = \{y\}$	$S_2 = \{1, \ldots, L_j\}$	$S_2 = \emptyset$
	$T_1 = \emptyset$	$T_1 = \emptyset$	$T_1 = \emptyset$	$T_1 = \emptyset$	$T_1 = \emptyset$
	$T_2 = \emptyset$	$T_2 = \emptyset$	$T_2 = \emptyset$	$T_2 = \emptyset$	$T_2 = \emptyset$

ii. The lower paths in Def. 2.**ii.a.** and Def. 2.**ii.b.** will be projected onto transitions from any level L_l^i, $l = 1, \dots, L_i$, to L_{z-1}^j, L_z^j and L_{z+1}^j, and from L_z^j to any level L_l^i, $l = 1, \dots, L_i$, i.e.,

$$\left[\boldsymbol{Q}_{i,j}^{i-1}\right]_{a,b} = \mathbf{0}, \qquad \text{if } b \neq z,$$
$$\left[\boldsymbol{Q}_{j,i}^{i-1}\right]_{b,a} = \mathbf{0}, \qquad \text{if } b \neq z-1, b \neq z \text{ and } b \neq z+1,$$

for $z = 2, \dots, L_j - 1$, preserving the $T^{(2)}(z)$ transitions.
iii. The lower paths in Def. 2.**iii.a.** and Def. 2.**iii.b.** will be projected onto transitions from any level L_l^i, $l = 1, \dots, L_i$, to L_z^j, and from L_{z-1}^j, L_z^j and L_{z+1}^j to any level L_l^i, $l = 1, \dots, L_i$, i.e.,

$$\left[\boldsymbol{Q}_{i,j}^{i-1}\right]_{a,b} = \mathbf{0}, \qquad \text{if } b \neq z-1, b \neq z \text{ and } b \neq z+1,$$
$$\left[\boldsymbol{Q}_{j,i}^{i-1}\right]_{b,a} = \mathbf{0}, \qquad \text{if } b \neq z,$$

for $z = 2, \dots, L_j - 1$, preserving the $T^{(3)}(z)$ transitions.
iv. There are no lower paths from ω_k to ω_j so $\boldsymbol{Q}_{k,j}^{j-1} = \mathbf{0}$ and $T^{(4)}$ transitions are preserved.
v. There are no lower paths from ω_j to ω_k so $\boldsymbol{Q}_{j,k}^{j-1} = \mathbf{0}$ and $T^{(5)}$ transitions are preserved.
vi. Following the same reasoning as above we immediately state that the lower paths are projected onto:

a. a $T^{(1)}(z)$ transition.
b. a $T^{(2)}(z)$ transition.
c. a $T^{(3)}(z)$ transition.
d. a $T^{(4)}$ transition.
e. a $T^{(5)}$ transition.
f. a $T^{(6)}$ transition.

Since $T^{(6)}$ transitions can be considered as special cases of the other five transitions we can conclude that the direct conditions are maintained in each reduction step by the indirect conditions. □

Theorem 1 ensures that the six types of transitions in Definition 1 are maintained through all the reduction steps. We can therefore state the following relation between the direct conditions and the tri-diagonal block structure of each set.

Theorem 2 *The direct conditions between ω_i and ω_j, $i < j$, in Definition 1 ensure that the original tri-diagonal block structure of ω_j is preserved in reduction step i. Moreover, these six sets of transitions are the only transitions that preserve the tri-diagonal block structure.*

Proof From (6) and Table 1 it can be seen that $T^{(1)}(z)$, $T^{(2)}(z)$ and $T^{(3)}(z)$ transitions are projected onto transitions within L_z^j or onto transition to and from one of the adjacent levels, i.e., L_{z-1}^j and L_{z+1}^j (assuming these levels exist). It also follows from Table 1 that the remaining three types of transitions are not projected onto transitions in ω_j and we conclude that the tri-diagonal block structure of ω_j is preserved by the direct conditions.

Suppose there exist a $T^{(7)}$ transition which is not included in any of the six sets in Definition 1. A $T^{(7)}$ must have transitions in both directions, otherwise it is merely a special case of a $T^{(4)}$ or a $T^{(5)}$ transition. Note that since ω_i is removed from the state space in reduction step i, we can assume that transitions occur from, and to, any level in ω_i, i.e., $L_l^i, l = 1, \ldots, L_i$. To preserve the tri-diagonal block structure, a $T^{(7)}$ transition must be projected onto transitions within a certain level and between directly adjacent levels, meaning that the transitions from ω_j to ω_i cannot originate from more than three levels. Similarly, transitions from any level in ω_i cannot go to more than three (adjacent) levels in ω_j.

Suppose that a $T^{(7)}$ transition contains transitions from L_{z-1}^j, L_z^j, and L_{z+1}^j, for some $z \in \{2, \ldots, L_i - 1\}$, to any level in ω_i. Then, the only possibility to preserve the tri-diagonal block structure is to only allow transitions from any level in ω_i to L_z^j, making it a $T^{(2)}(z)$ transition.

Next, suppose that a $T^{(7)}$ transitions contains transition from L_a^j and L_b^j, $b \neq a$, to any level in ω_i. If $|b - a| > 1$ the two levels are not adjacent and to preserve the tri-diagonal block structure $|b - a| = 2$ must hold. In this case a $T^{(7)}$ transition is again a special case of a $T^{(2)}(z)$ transition. If $|b - a| = 1$ the levels are adjacent and, to preserve the tri-diagonal structure, transitions may occur from any level in ω_i to L_a^j and L_b^j, $|b - a| = 1$, making it a $T^{(1)}(z)$ transition.

Finally suppose that a $T^{(7)}$ transitions contains transition from L_z^j, for some $z \in \{1, \ldots, L_i\}$, to any level in ω_i then, to preserve the tri-diagonal block structure, transition can occur from any level in ω_i to only three levels (two levels in case $z = 1$ or $z = L_i$) in ω_j, namely, L_{z-1}^j, L_z^j, and L_{z+1}^j. Even if some of these transitions are zero, we find that $T^{(7)}$ must be a $T^{(1)}(z)$, or a $T^{(3)}(z)$ transition.

We can thus conclude that the six sets of transitions in Definition 1 are the only types that preserve the tri-diagonal block structure of ω_j. □

Theorem 2 guarantees that $\boldsymbol{Q}_{i,i}^k$, for $i = k, \ldots, S$, has a tri-diagonal block structure after reduction step $k - 1$.

Remark 2 (Ordering of the sets) Note that in Definition 1 and Theorem 2 each type of transition specifies that there are transition from *any* level in ω_i to *some* levels in ω_j, and/or transitions from *some* levels in ω_j to *any* level in ω_i, with $i < j$. This ordering of sets is important since it determines whether or not the tri-diagonal block structure is preserved. If, for instance, a $T^{(1)}(z)$ transition would be reversed, i.e., there are transitions from any level in ω_j to L_z^i and L_{z+1}^i, and back, with $i < j$, then in reduction step i, this $T^{(1)}(z)$ transition will be projected onto transitions within and between all levels in ω_j, making $\boldsymbol{Q}_{j,j}^{i+1}$ a full matrix instead of a tri-diagonal block structured matrix. It is therefore important that the sets are ordered correctly such that Definition 1 holds.

Remark 3 Without loss of generality, we assume that all the LDQBD sets are internally connected sets, that is all the F_i and B_i are non zero matrices. If in an LDQBD set, one of the F_i or B_i is zero, we can split this LDQBD set into two subsets with either a direct $T^{(2)}$ or direct $T^{(3)}$ transition. By the latter assumption, we cannot have the following situation given Fig. 2a, where we have a nonstandard transition which still preserves the LDQBD structure during the reduction step. Besides, these previous transitions do not preserve the original LDQBD, see Fig. 2b. However, the LDQBD structure could be preserved if we split the bottom-right set into subsets according the white lines in Fig. 1b. (that is combine the levels 1 and 2 and combine the levels 3 and 4) to get the structure in Fig. 2c. In this case, the transitions between the subsets is of type $T^{(1)}(1)$.

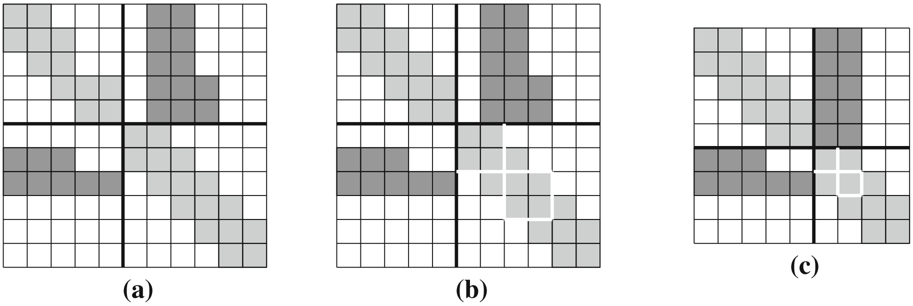


Fig. 2 Non standard transitions

3.2 Intermediate step

Theorem 2 guarantees that $\boldsymbol{Q}^S$ describes a finite LDQBD of L_S levels:

$$\boldsymbol{Q}^S = \begin{bmatrix} \boldsymbol{L}_S^{(1)} & \boldsymbol{F}_S^{(1)} & 0 & \cdots & \cdots & 0 \\ \boldsymbol{B}_S^{(2)} & \boldsymbol{L}_S^{(2)} & \boldsymbol{F}_S^{(2)} & \ddots & & \vdots \\ 0 & \boldsymbol{B}_S^{(3)} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \boldsymbol{L}_S^{(L_S-1)} & \boldsymbol{F}_S^{(L_S-1)} \\ 0 & \cdots & \cdots & 0 & \boldsymbol{B}_S^{(L_S)} & \boldsymbol{L}_S^{(L_S)} \end{bmatrix}.$$

The stationary distribution $\boldsymbol{p}^S = \begin{bmatrix} \boldsymbol{p}_1^S & \boldsymbol{p}_2^S & \cdots & \boldsymbol{p}_{L_S}^S \end{bmatrix}$ of $\boldsymbol{Q}^S$, i.e., $\boldsymbol{p}^S$ such that $\boldsymbol{p}^S\boldsymbol{Q}^S = 0$ and $\boldsymbol{p}^S\boldsymbol{e} = 1$, is given by Gaver et al. (1984), see also Section 5.5 of Baër (2015),

$$C_1 = \boldsymbol{L}_S^{(1)},$$
$$C_i = \boldsymbol{L}_S^{(i)} + \boldsymbol{B}_S^{(i)}\left[-C_{i-1}\right]^{-1}\boldsymbol{F}_S^{(i-1)}, \qquad 2 \le i \le L_S.$$

and

$$\boldsymbol{p}_{L_S}^S C_{L_S} = 0,$$
$$\boldsymbol{p}_i^S = \boldsymbol{p}_{i+1}^S \boldsymbol{B}_S^{(i+1)}\left[-C_i\right]^{-1}, \qquad 1 \le i \le L_S - 1,$$

such that

$$\sum_{i=1}^{L_S} \boldsymbol{p}_i^S \boldsymbol{e} = 1.$$

3.3 Expansion step *k*

Let $\boldsymbol{p}_{S-k} = \begin{bmatrix} \boldsymbol{p}^{S-k} & \boldsymbol{p}^{S-k+1} & \cdots & \boldsymbol{p}^S \end{bmatrix}$ be the vector obtained after expansion step k. By normalising this vector we obtain the stationary distribution of $\boldsymbol{Q}^{S-k}$. Let $\boldsymbol{p}_j^i$ denote the

Table 3 Subset U for each type of transition from ω_{S-k} to ω_{S-k+i}

$T^{(1)}(z)$	$T^{(2)}(z)$	$T^{(3)}(z)$
$U = \{z, z+1\}$	$U = \{z\}$	$U = \{z-1, z, z+1\}$
$T^{(4)}$	$T^{(5)}$	$T^{(6)}$
$U = \emptyset$	$U = \{1, \ldots, L_{S-k+i}\}$	$U = \emptyset$

subvector of $\boldsymbol{p}^i$ corresponding to L^i_j. Following the expansion step (3) we obtain

$$\boldsymbol{p}^{S-k} = \left[\, \boldsymbol{p}^{S-k+1} \cdots \boldsymbol{p}^{S} \,\right] \begin{bmatrix} \boldsymbol{Q}^{S-k}_{S-k+1,S-n} \\ \vdots \\ \boldsymbol{Q}^{S-n}_{S,S-k} \end{bmatrix} \left[-\boldsymbol{Q}^{S-k}_{S-k,S-k} \right]^{-1}$$
$$= \sum_{i=1}^{k} \boldsymbol{p}^{S-k+i} \boldsymbol{Q}^{S-k}_{S-k+i,S-k} \left[-\boldsymbol{Q}^{S-k}_{S-k,S-k} \right]^{-1}.$$

By decomposing the submatrices by their levels gives

$$\boldsymbol{p}^{S-k}_j = \sum_{i=1}^{k} \sum_{b=1}^{L_{S-k}} \sum_{a=1}^{L_{S-k+i}} \boldsymbol{p}^{S-k+i}_a \left[\boldsymbol{Q}^{S-k}_{S-k+i,S-k} \right]_{a,b} \left[-\boldsymbol{Q}^{S-k}_{S-k,S-k} \right]^{-1}_{b,j}. \tag{10}$$

Utilising the type of transition between ω_{S-k+i} and ω_{S-k} we can write the inner sum as

$$\sum_{a \in U} \left[\boldsymbol{p}_{S-k+i} \right]_a \left[\boldsymbol{Q}^{S-k}_{S-k+i,S-k} \right]_{a,b},$$

where $U \subseteq \{1, \ldots, L_{S-k+i}\}$ follows from the type of transition and is given in Table 3.

The stationary distribution $\boldsymbol{\pi}$ of $\boldsymbol{Q}$ is obtained by normalising the vector obtained after expansion step $S-1$.

3.4 Inverse of $-\boldsymbol{Q}^{k}_{k,k}$

In reduction step k and expansion step $S-k$ the negative inverse of the generator $\boldsymbol{Q}^k_{k,k}$ needs to be determined. It follows from Theorems 1 and 2 that $\boldsymbol{Q}^k_{k,k}$ has a tri-diagonal structure for $k = 1, \ldots, S$, and, moreover, that it is described by a LDQBD. In Shin (2009), direct formulas are given to determine the fundamental matrix of a transient LDQBD, see also Section 5.5 of Baër (2015).

4 Simplified successive censoring algorithm

In Sect. 2 we introduced the three-dimensional Markov chain $\mathcal{X}$ with states (s, l, p). Next, we have shown that under the conditions stated in Definition 1 and Definition 2, the stationary distribution can be obtained using the successive censoring algorithm in Algorithm 1. The direct conditions in Definition 1 are easy to check for the Markov chain $\mathcal{X}$ but checking the indirect conditions in Definition 2 is a difficult task since all lower paths must be checked. To solve this problem, we introduce a simplified version of the successive censoring algorithm which tells if the indirect conditions hold and consequently, if the successive censoring algorithm can be applied.

Let $\mathcal{C}^k(i,j), i<j$, denote the collection of transition types from ω_i to ω_j after reduction step $k-1$ (in which ω_{k-1} is removed). For example, suppose that after reduction step $k-1$, L_3^j, L_4^j, and L_5^j can be reached from ω_i, $i<j$, but ω_i cannot be reached from ω_j, then the transitions between ω_i and ω_j are a special case of both a $T^{(3)}(4)$ and $T^{(4)}$ transition and

$$\mathcal{C}^k(i,j)=\{T^{(3)}(4),T^{(4)}\}.$$

Next, we define the iteration

$$\mathcal{C}^{k+1}(i,j)=\mathcal{C}^k(i,j)\cap\mathcal{W}\left[\mathcal{C}^k(k,i)\times\mathcal{C}^k(k,j)\right],\qquad i<j<k, \tag{11}$$

where the Cartesian product $\mathcal{C}^k(k,i)\times\mathcal{C}^k(k,j)$ consists of all ordered pairs describing the type of transitions from ω_k to ω_i and from ω_k to ω_j. Upon removing ω_k these transitions are projected to transitions from ω_i to ω_j according to Table 4. The function $\mathcal{W}$ is then the union of the projections of each pair in $\mathcal{C}^k(k,i)\times\mathcal{C}^k(k,j)$, given in Table 4.

Recall that a $T^{(6)}$ transition is the trivial all-zero special case of all other transitions. Therefore, we state that if there is a $T^{(6)}$ transitions from ω_i to ω_j, then $\mathcal{C}^k(i,j)$ is the union of all possible transition types, i.e.,

$$\mathcal{C}^k(i,j)=\left\{\left[\bigcup_{z=1}^{L_j-1}T^{(1)}(z)\right]\cup\left[\bigcup_{z=2}^{L_j-1}T^{(2)}(z)\right]\cup\left[\bigcup_{z=2}^{L_j-1}T^{(3)}(z)\right]\cup T^{(4)}\cup T^{(5)}\right\}.$$

In case of a $T^{(6)}$ transition, the intersection in (11) comes down to

$$\mathcal{C}^k(i,j)\cap\{T^{(6)}\}=\mathcal{C}^k(i,j).$$

To demonstrate the iteration in (11) we use the following example.

Example 1 Suppose that $\mathcal{C}^k(i,j)=\{T^{(3)}(4),T^{(4)}\}$, $\mathcal{C}^k(k,i)=\{T^{(2)}(3),T^{(5)}\}$, and $\mathcal{C}^k(k,j)=\{T^{(3)}(4)\}$ then

$$\begin{aligned}\mathcal{W}\left[\mathcal{C}^k(k,i)\times\mathcal{C}^k(k,j)\right]&=\mathcal{W}\left[\{T^{(5)},T^{(3)}(4)\},\{T^{(2)}(3),T^{(3)}(4)\}\right]\\&=\{T^{(1)}(3),T^{(1)}(4),T^{(2)}(3),T^{(2)}(4),T^{(2)}(5),T^{(3)}(4),T^{(4)}\}\cup\{T^{(3)}(4)\}\\&=\{T^{(1)}(3),T^{(1)}(4),T^{(2)}(3),T^{(2)}(4),T^{(2)}(5),T^{(3)}(4),T^{(4)}\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}^{k+1}(i,j)&=\{T^{(3)}(4),T^{(4)}\}\cap\{T^{(1)}(3),T^{(1)}(4),T^{(2)}(3),T^{(2)}(4),T^{(2)}(5),T^{(3)}(4),T^{(4)}\}\\&=\{T^{(3)}(4),T^{(4)}\}.\end{aligned}$$

□

Note, if $\mathcal{C}^{k+1}(i,j)=\emptyset$ for any two sets ω_i and ω_j, $i<j$, after reduction step k ($k=1,\ldots,\omega-1$), then the direct conditions in Definition 1 are violated and the successive censoring algorithm can no longer be applied. Observe that the first term in (11) describes the possible types of transitions from ω_i to ω_j before removing ω_k, whereas the second term

Table 4 Projections onto transitions from ω_i to ω_j, $k < i < j$

$\mathcal{C}^k(k, j), k < j$	$T^{(1)}(y)$	$T^{(2)}(y)$	$T^{(3)}(y)$	$T^{(4)}$	$T^{(5)}$	$T^{(6)}$
$\mathcal{C}^k(k, i), k < i$						
$T^{(1)}(x)$	$T^{(1)}(y)$	$T^{(2)}(y)$	$T^{(3)}(y)$	$T^{(4)}$	$T^{(5)}$	$T^{(6)}$
$T^{(2)}(x)$	$T^{(1)}(y)$	$T^{(2)}(y)$	$T^{(3)}(y)$	$T^{(4)}$	$T^{(5)}$	$T^{(6)}$
$T^{(3)}(x)$	$T^{(1)}(y)$	$T^{(2)}(y)$	$T^{(3)}(y)$	$T^{(4)}$	$T^{(5)}$	$T^{(6)}$
$T^{(4)}$	$\left\{T^{(1)}(y), T^{(3)}(y), T^{(3)}(y+1), T^{(5)}\right\}$	$\left\{T^{(1)}(y-1), T^{(1)}(y), T^{(2)}(y), T^{(3)}(y-1), T^{(3)}(y), T^{(3)}(y+1), T^{(5)}\right\}$	$\{T^{(3)}(y), T^{(5)}\}$	$T^{(6)}$	$T^{(5)}$	$T^{(6)}$
$T^{(5)}$	$\left\{T^{(1)}(y), T^{(2)}(y), T^{(2)}(y+1), T^{(4)}\right\}$	$\{T^{(2)}(y), T^{(4)}\}$	$\left\{T^{(1)}(y-1), T^{(1)}(y), T^{(2)}(y-1), T^{(2)}(y), T^{(2)}(y+1), T^{(3)}(y), T^{(4)}\right\}$	$T^{(4)}$	$T^{(6)}$	$T^{(6)}$
$T^{(6)}$	$T^{(6)}$	$T^{(6)}$	$T^{(6)}$	$T^{(6)}$	$T^{(6)}$	$T^{(6)}$

describes the projection as a result of removing ω_k. The direct conditions only hold if these projections correspond to the existing types of transitions. If $\mathcal{C}^{k+1}(i,j)=\emptyset$ the projections are different than the existing transitions and the direct conditions are violated after reduction step k. We close this section by presenting the simplified successive censoring algorithm

Algorithm 2 (Simplified successive censoring algorithm)

1. Determine $\mathcal{C}^1(i,j)$, $i<j$, by charaterising the types of transitions in the Markov chain $\mathcal{X}$.
2. Perform the iteration in (11) until $\mathcal{C}^{S-1}(S-1,S)$ is obtained. Determine in each iteration step

   a. the Cartesian product $\mathcal{C}^k(k,i)\times\mathcal{C}^k(k,j)$ specifying all possible combinations of transitions between ω_k and ω_i, and between ω_k and ω_j, $i<j<k$,
   b. the resulting projection of each element in $\mathcal{C}^k(k,i)\times\mathcal{C}^k(k,j)$ using Table 4,
   c. the union of these projection, i.e., determining $\mathcal{W}\left[\mathcal{C}^k(k,i)\times\mathcal{C}^k(k,j)\right]$, $i<j<k$,
   d. and finally the intersection of the original transitions, $\mathcal{C}^k(i,j)$, and the union of the projections.

3. If $\mathcal{C}^{S-1}(S-1,S)\neq\emptyset$ then the successive censoring algorithm can be applied to obtain the stationary distribution. If, on the other hand, in some iteration $\mathcal{C}^k(i,j)=\emptyset$, for $i<j<k$, the successive censoring algorithm cannot be applied.

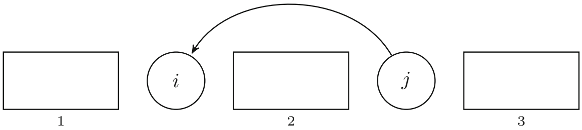
5 Complexity analysis

Let us consider a system of connected LDQBDs with S sets, each with L levels ($L=\max_k L_k$) of P phases ($P=\max_{j,k} P_{j,k}$) phases each. The successive censoring algorithm consists of $S-1$ reduction steps, one intermediate step, and $S-1$ expansion steps. Since the intermediate step is performed only once while both the reduction and expansion steps are performed $S-1$ times, we can ignore the effect of the intermediate step on the complexity. Furthermore, some of the operations needed in the reduction steps are also needed in the expansions steps, the product, for example, $\boldsymbol{Q}^k_{i,k}[-\boldsymbol{Q}^k_{k,k}]^{-1}$ is used in both the reduction step as well as the expansion step. However, in reduction step k, we need to multiply this product with a matrix (on the right) $(S-k)^2$ times, while in expansion step $S-k$ we must multiply this product with a vector (on the left) $S-k$ times. Therefore, the reduction step requires more computations than an expansion step and we can focus on the reduction steps alone to determine the complexity of the algorithm.

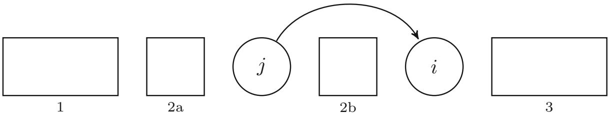
The complexity of each reduction step depends on the type of transitions between the sets. Obviously, a $T^{(6)}$ transition does not have any projections and will not contribute to the complexity, therefore, we will assume that in this worst-case scenario there are no $T^{(6)}$ transitions. Furthermore, a $T^{(1)}(z)$ requires 4 projections, whereas censoring a $T^{(2)}(z)$ or $T^{(3)}(z)$ would only require 3 projections, making $T^{(1)}(z)$ the worst-case among these three transitions. Next, we show that by reordering the sets, any $T^{(5)}$ transition can be changed into a $T^{(4)}$ transition without disobeying the indirect conditions.

Corollary 1 *For any system of connected LDQBDs which satisfies both the direct and indirect conditions with $T^{(4)}$ and $T^{(5)}$ transitions, the sets can be reordered such that there are only $T^{(4)}$ transitions.*

Proof Consider the (schematically represented) Markov chain in Fig. 3a with a $T^{(5)}$ transition from ω_i to ω_j, $i<j$. Recall that this $T^{(5)}$ transition only contains transitions from ω_j to



(a)



(b)

Fig. 3 Schematic representation of system of connected LDQBDs, before (**a**) and after (**b**) reordering of the sets

ω_i, therefore, we depict is with a black arrow from ω_j to ω_i. The blocks $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ are collections of sets with appropriate index, e.g., subset $\mathcal{D}_1$ contains all sets ω_y with $y < i$.

Suppose $|j - i| = 1$, then $\mathcal{D}_2 = \emptyset$. By definition, there is no lower path from ω_j to ω_i and the sets can be switched without violation Definition 2.

Next, suppose $\mathcal{D}_2 \neq \emptyset$, then $|j - i| > 1$. Since both the direct and indirect conditions hold there cannot be a lower path from ω_i to ω_j. If there would be a lower path, then after reduction step $j-1$ there would be transitions from any level in ω_j (due to the $T^{(5)}$ transition) to an some level in ω_j (due to the lower path from ω_i to ω_j). This suggests that $\mathcal{D}_2$ can be partitioned into the subsets

- $\mathcal{D}_{2a}$, with sets ω_x, $i < x < j$, with at least one lower path from ω_x to ω_j and none from ω_i to ω_x,
- $\mathcal{D}_{2b}$, with sets ω_y, $i < y < j$, with at least one lower path from ω_i to ω_y and none from ω_y to ω_j,
- $\mathcal{D}_{2c}$, with sets ω_z, $i < z < j$, with no lower paths from ω_z to ω_j and no lower paths from ω_i to ω_z.

Note that there cannot be any lower paths between $\mathcal{D}_{2c}$ and $\mathcal{D}_{2a}$ or $\mathcal{D}_{2b}$ since this would imply that there would be a lower path from $\mathcal{D}_{2c}$ to ω_j or from ω_i to $\mathcal{D}_{2c}$, which is not possible. Furthermore, note that there can be lower paths from ω_j to any set in $\mathcal{D}_{2a}$, $\mathcal{D}_{2b}$, or $\mathcal{D}_{2c}$, also, there can be lower paths from any set in $\mathcal{D}_{2a}$, $\mathcal{D}_{2b}$, or $\mathcal{D}_{2c}$ to ω_i.

We can switch ω_i and ω_j and relocate the subsets $\mathcal{D}_{2a}$, $\mathcal{D}_{2b}$, and $\mathcal{D}_{2c}$ such that the Markov chain in Fig. 3b is formed. Let $\sigma(j)$ be the index of ω_j in this new Markov chain, then $\sigma(j) < \sigma(i)$ and the $T^{(5)}$ transition from ω_i to ω_j is transformed in a $T^{(4)}$ transition from $\omega_{\sigma(j)}$ to $\omega_{\sigma(i)}$. Since there are no lower paths from ω_i to ω_j, there are no lower paths from $\omega_{\sigma(i)}$ to $\omega_{\sigma(j)}$ and Definition 2 still holds. Furthermore, by making sure that $\mathcal{D}_{2a}$ is placed before $\omega_{\sigma(j)}$ and $\mathcal{D}_{2b}$ is placed after $\omega_{\sigma(i)}$, no new lower paths were created that might violate Definition 2. □

Due to Corollary 1 we will consider a system of connected LDQBDs with only $T^{(1)}(z)$ and $T^{(4)}$ transitions. By only considering non-zero transitions, we can conclude that a projection of 2 $T^{(1)}(z)$ transitions is a vector-matrix-vector multiplication with $\mathcal{O}(L^2P^3)$, and that a projection of a $T^{(1)}(z)$ and a $T^{(4)}$ transition is a vector-matrix-matrix multiplication, also with $\mathcal{O}(L^2P^3)$. So to determine the complexity of the algorithm we must maximise the number of projections made in each step (instead of the size of the projections). Since the projection of 2 $T^{(1)}(z)$ transitions results in 4 projections while the projection of a $T^{(1)}(z)$ and a $T^{(4)}$ transition results in 2 projections, we will consider a system of connected LDQBD-processes with only $T^{(1)}(z)$ transitions, for some $z \in \{1, \ldots, L\}$.

In reduction step k, we must perform $(S-k)^2$ projections of 2 $T^{(1)}(z)$ transitions. A single projection of 2 $T^{(1)}$ transitions is a vector-matrix-vector multiplication with $\mathcal{O}(L^2P^3)$. Therefore, the total complexity, including the inverse following Shin (2009), of reduction step k is $\mathcal{O}(S^2L^2P^3)$. Finally, there are $S-1$ reduction steps resulting in a complexity of the successive censoring algorithm of $\mathcal{O}(S^3L^2P^3)$. Comparing this to directly solving $\boldsymbol{\pi}\boldsymbol{Q} = \mathbf{0}$ which has complexity $\mathcal{O}(S^3L^3P^3)$ we conclude that we decrease complexity by a factor L. Typically, the number of levels is much larger than the number of sets and phases, therefore, the complexity reduction by L is representative.

6 Demonstration of the successive censoring algorithm

To demonstrate the successive censoring algorithm we use Algorithm 1 to obtain the stationary distribution for a Markov chain with generator $\boldsymbol{Q}$ given in (12). Before that, we check using Algorithm 2 whether our successive censoring algorithm can be applied.

Example 2 Let us consider a Markov chain with generator $\boldsymbol{Q}$ in (12). Here, $S = 3$, $L_1 = L_2 = L_3 = 4$, and all levels consists of a single phase.

$$\boldsymbol{Q} = \left[\begin{array}{cccc|cccc|cccc}
-1 & 1 & & & & & & & & & & \\
9 & -16 & 2 & & 5 & & & & & & & \\
 & 8 & -19 & 3 & & 8 & & & & & & \\
 & & 7 & -11 & & & & & & & 4 & \\
\hline
2 & & & & -6 & 4 & & & & & & \\
 & & 3 & & 6 & -14 & 5 & & & & & \\
 & & & & & 5 & -11 & 6 & & & & \\
 & & & & & & 4 & -8 & & & 4 & \\
\hline
 & & & & & & & & -7 & 7 & & \\
 & 3 & & & & & 3 & & 3 & -17 & 8 & \\
 & & 3 & & & & & & & 2 & -14 & 9 \\
 & & & 3 & & & & 3 & & & 1 & -7
\end{array}\right]. \tag{12}$$

In (12), the solid lines denote three distinctive sets, each representing a LDQBD and $\boldsymbol{Q}$ is partitioned as follows

$$\boldsymbol{Q} = \boldsymbol{Q}^1 = \begin{bmatrix} \boldsymbol{Q}^1_{1,1} & \boldsymbol{Q}^1_{1,2} & \boldsymbol{Q}^1_{1,3} \\ \boldsymbol{Q}^1_{2,1} & \boldsymbol{Q}^1_{2,2} & \boldsymbol{Q}^1_{2,3} \\ \boldsymbol{Q}^1_{3,1} & \boldsymbol{Q}^1_{3,2} & \boldsymbol{Q}^1_{3,3} \end{bmatrix}.$$

We apply Algorithm 2 to the generator $\boldsymbol{Q}$. Based on Step 1 of Algorithm 2 we see in (12) that

$$\mathcal{C}^1(1,2) = \{T^{(1)}(1)\}, \quad \mathcal{C}^1(1,3) = \{T^{(3)}(2)\}, \quad \mathcal{C}^1(2,3) = \{T^{(3)}(2)\}.$$

In Step 2 of Algorithm 2, the iteration (11) and Table 4 now give us

$$\mathcal{C}^2(2,3) = \{T^{(3)}(2)\} \cap \mathcal{W}\left[\{T^{(1)}(1), T^{(3)}(2)\}\right] = \{T^{(3)}(2)\}.$$

From this step we conclude that $\mathcal{C}^2(1,2) \neq \emptyset$, therefore, Algorithm 1 can be applied.

Next, we apply Step 1 of Algorithm 1 by performing 2 reduction steps (2) to remove ω_1 and ω_2 from the state space. The generators obtained in these 2 reduction step are $\boldsymbol{Q}^2$

$$\boldsymbol{Q}^2 = \left[\begin{array}{cccc|cccc} -\frac{248}{57} & \frac{1228}{285} & 0 & 0 & 0 & 0 & \frac{4}{95} & 0 \\ \frac{136}{19} & -\frac{1176}{95} & 5 & 0 & 0 & 0 & \frac{21}{95} & 0 \\ 0 & 5 & -11 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -8 & 0 & 0 & 4 & 0 \\ \hline 0 & 0 & 0 & 0 & -7 & 7 & 0 & 0 \\ \frac{47}{19} & \frac{44}{95} & 3 & 0 & 3 & -17 & \frac{766}{95} & 0 \\ \frac{22}{19} & \frac{154}{95} & 0 & 0 & 0 & 2 & -\frac{1309}{95} & 9 \\ \frac{14}{19} & \frac{98}{95} & 0 & 3 & 0 & 0 & \frac{212}{95} & -7 \end{array}\right],$$

and $\boldsymbol{Q}^3$

$$\boldsymbol{Q}^3 = \begin{bmatrix} -7 & 7 & 0 & 0 \\ 3 & -17 & 13 & 0 \\ 0 & 2 & -11 & 9 \\ 0 & 0 & 7 & -7 \end{bmatrix}.$$

The intermediate step, Step 2 of Algorithm 1, gives us the stationary distribution $\boldsymbol{p}^3$ of $\boldsymbol{Q}^3$

$$\boldsymbol{p}^3 = \left[\, \tfrac{3}{122} \;\; \tfrac{7}{122} \;\; \tfrac{49}{122} \;\; \tfrac{63}{122} \,\right].$$

Finally, we execute Step 3 of Algorithm 1 and perform 2 expansion steps (3) to obtain $\boldsymbol{p}^2$

$$\boldsymbol{p}^2 = \left[\, \tfrac{2043279}{873520} \;\; \tfrac{280371}{218380} \;\; \tfrac{4021}{43676} \;\; \tfrac{38619}{43676} \,\right].$$

and $\boldsymbol{p}^1$

$$\boldsymbol{p}^1 = \left[\, \tfrac{1755243}{109190} \;\; \tfrac{553077}{436760} \;\; \tfrac{43827}{87352} \;\; \tfrac{24255}{87352} \,\right].$$

Step 4 of Algorithm 1 gives the normalisation of the vector $\left[\, \boldsymbol{p}^1 \;\; \boldsymbol{p}^2 \;\; \boldsymbol{p}^3 \,\right]$ that is equal to $\boldsymbol{\pi} = \left[\, \boldsymbol{\pi}^1 \;\; \boldsymbol{\pi}^2 \;\; \boldsymbol{\pi}^3 \,\right]$ where

$$\begin{aligned} \boldsymbol{\pi}^1 &= \frac{1}{21443881}\left[14041944 \; 1106154 \; 438270 \; 242550\right], \\ \boldsymbol{\pi}^2 &= \frac{1}{21443881}\left[2043279 \; 1121484 \; 804300 \; 772380\right], \\ \boldsymbol{\pi}^3 &= \frac{1}{21443881}\left[21480 \; 50120 \; 350840 \; 451080\right]. \end{aligned}$$

Finally, it is now easy to check that $\boldsymbol{\pi}\boldsymbol{Q} = 0$.

7 Summary and conclusion

We introduced a successive censoring algorithm to find the stationary distribution of a general class of Markov chains consisting of multiple Level Dependent Quasi-Birth-and-Death processes (LDQBD) connected by special types of transitions. The successive censoring algorithm consists of reduction steps, in which the state space is reduced by removing a LDQBD in each step, an intermediate step, in which the stationary distribution of the reduced Markov chain is determined, and expansion steps, in which the stationary distribution is expanded by adding a (previously removed) LDQBD back to the state space. By applying the well-known results on the inverse of the LDQBD-generator we determine the stationary probability distribution. Moreover, we show that the complexity of the successive censoring algorithm is $\mathcal{O}(S^3 L^2 P^3)$ with a decrease by a factor L. This is a considerable reduction because L is typically much larger than S and P. Finally, in an example we apply the successive censoring algorithm to a Markov chain with three sets.

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