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Combining $\mathcal{DRA}$ and $\mathcal{CYC}$ into a Network Friendly Calculus

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Abstract. Qualitative spatial reasoning is usually performed using spatial calculi specially designed to represent certain aspects of spatial knowledge. However most calculi must be adapted for use in applications where additional constraints are at play. This paper combines the $\mathcal{DRA}$ and $\mathcal{CYC}$ algebras into a new calculus for reasoning within network structures. In the process we prove some interesting results about some properties of the standard operations, converse and composition, for versions of $\mathcal{CYC}$ and $\mathcal{DRA}$.

1 INTRODUCTION

Qualitative spatial representation and reasoning (QSR) research investigates methods for representing and reasoning about configurations of spatial entities using a limited number of distinctions. The distinctions induce, quite naturally, a finite set of relations over the domain of the representation. Many previous theories in QSR (e.g. [4, 6, 8]) have addressed representation of spatial relations for unconstrained entities embedded in a Euclidean space. However most of these *qualitative calculi* must be adapted for use in applications where additional constraints are at play [7]. As in the case of developing a new qualitative calculus, a major challenge in performing such adaptations is finding the appropriate refinements, coarsening, or extensions of the spatial relations expressed by the calculus being adapted [2].

In this paper we present a calculus for representing weak relative position information about line segments. We make detailed analysis of some aspects of the $\mathcal{CYC}$ algebras for cyclic ordering of planar orientations (Isli and Cohn [4]) and the $\mathcal{DRA}$ (dipole relation algebra) of Moratz et al. [6] and show how the new ternary calculus is generated from their product. The *target* domain of the new calculus presented here is the set of line segments in a network (or edges in a plane embedded graph). We are especially interested in representing and reasoning on the connectivity of segments in the network and the relative orientations between segments within the network. In the next section we overview background material and present some results that will be used in the paper. A substantial part of this paper is spent on the coarse version of $\mathcal{DRA}$ which we call $\mathcal{DRA}_7$. It was first introduced by Wallgrün et al. in [10] but they did not provide an analysis of its structure and standard operations (converse and composition).[2]

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[2] Tables for the operations of the calculi discussed in this paper can be downloaded at `http://ifgi.uni-muenster.de/~m_chip02/ecai2012/calculi.zip`.

2 PRELIMINARIES

In our analysis of qualitative calculi we follow the characterization given by Ligozat and Renz in [5]. In that work a qualitative calculus is "constructed" from a partition scheme defined on the domain of interest (e.g. the set of oriented line segments in $\mathbb{R}^2$) usually called the universe of the calculus. The calculus itself is defined as a relationship between the partition scheme and an algebraic structure known as a non-associative algebra [5]. Due to space limitations we will not concern ourselves with those algebraic structures. Instead we will focus on a more rudimentary aspect, namely, the partition scheme and the operations defined on it, which we will still refer to as a qualitative calculus.

Definition 1 *A partition scheme is a pair $(U, (r_i)_{i\in I})$, where U is a non-empty universe, I is a finite set and $(r_i)_{i\in I}$ is a partition of $U \times U = U^2$ satisfying*

1. *There is an identity element $r_0 \in (r_i)_{i\in I}$ given by $r_0 = \{(u, v) \in U^2 \mid u = v\}$*
2. *$(\forall i \in I)$ $(\exists j \in I)$ such that $r_i^{\smile} = r_j$ where $r_i^{\smile} = \{(u, v) \in U^2 \mid (v, u) \in r_i\}$*

In this case the classes $(r_i)_{i\in I}$ are atomic binary relations, called the base relations of the calculus, because they form a so called jointly exhaustive pairwise disjoint (JEPD) set. For any calculus $\mathfrak{C}$ we will denote by $\mathcal{B}_{\mathfrak{C}}$ the set of base relations of $\mathfrak{C}$ and by $U_{\mathfrak{C}}$ its universe or domain. The full calculus generated by $\mathcal{B}_{\mathfrak{C}}$ is given by the set $2^{\mathcal{B}_{\mathfrak{C}}}$ to which all operations are extended in a set theoretic way. Unless explicitly stated all relations referred to in this paper are atomic. Given two relations r_i and r_j their composition and weak composition are given (respectively) by:

1. $r_i \circ_{\mathfrak{C}} r_j = \{(x, y) \in U_{\mathfrak{C}}^2 \mid \exists z \in U_{\mathfrak{C}} : (x, z) \in r_i, (z, y) \in r_j\}$
2. $r_i \diamond_{\mathfrak{C}} r_j = \bigcup r_l$ *for* r_l *satisfying* $(r_i \circ_{\mathfrak{C}} r_j) \cap r_l \neq \emptyset$

If $(\forall i, j \in I), r_i \circ_{\mathfrak{C}} r_j = r_i \diamond_{\mathfrak{C}} r_j$ then $\circ_{\mathfrak{C}}$ is said to be strong. Otherwise it is weak. For ternary calculi the partition scheme is defined on $U_{\mathfrak{C}}^3$ so that the partition classes are now atomic ternary relations, the operations are ternary, and the identity is defined more precisely as an identity element for composition.

Definition 2 *Let $(U_{\mathfrak{C}}, (r_i)_{i\in I})$ be a partition scheme denoted by $\mathfrak{C}$ where $(r_i)_{i\in I}$ is a partition of $U_{\mathfrak{C}}^3$. The following define the composition, weak composition, identity for composition, converse, and rotation of relations on $(U_{\mathfrak{C}}, (r_i)_{i\in I})$ respectively:*

1. $r_i \circ_{\mathfrak{C}} r_j = \{(x, y, w) \in U_{\mathfrak{C}}^3 \mid \exists z \in U_{\mathfrak{C}} : (x, y, z) \in r_i, (x, z, w) \in r_j\}$
2. $r_i \diamond_{\mathfrak{C}} r_j = \bigcup r_l$ *for* r_l *satisfying* $(r_i \circ_{\mathfrak{C}} r_j) \cap r_l \neq \emptyset$

3. $\exists r_0 \in (r_i)_{i\in I}$, *such that* $\forall i \in I$, $r_0 \circ_{\mathfrak{C}} r_i = r_i = r_i \circ_{\mathfrak{C}} r_0$
4. $(\forall i \in I)$ $(\exists j \in I)$ *such that* $r_i^{\smile} = r_j$ *where* $r_i^{\smile} = \{(u, v, w) \in U_{\mathfrak{C}}^3 \mid (u, w, v) \in r_i\}$
5. $(\forall i \in I)$ $(\exists j \in I)$ *such that* $r_i^{\frown} = r_j$ *where* $r_i^{\frown} = \{(u, v, w) \in U_{\mathfrak{C}}^3 \mid (w, u, v) \in r_i\}$

A ternary calculus, $\mathfrak{C}$, is said to be an *induced ternary calculus* if every relation of $\mathfrak{C}$ has a decomposition into binary relations of some binary calculus, say $\mathfrak{B}$, such that if $r_i \in \mathcal{B}_{\mathfrak{C}}$ then there are three relations $r_{i_1}, r_{i_2}, r_{i_3} \in \mathcal{B}_{\mathfrak{B}}$ such that $r_i = \{(a, b, c) \in U_{\mathfrak{B}}^3 \mid (a, b) \in r_{i_1}, (b, c) \in r_{i_2}, (a, c) \in r_{i_3}\}$. We will write $r_i = (r_{i_1}, r_{i_2}, r_{i_3})$ whenever $(r_{i_1}, r_{i_2}, r_{i_3})$ is the decomposition of r_i.

Proposition 1 *Let $\mathfrak{C}$ be ternary calculus induced by a binary calculus $\mathfrak{B}$. Let $r_i, r_i' \in \mathcal{B}_{\mathfrak{C}}$ and let $r_{i_1}, r_{i_2}, r_{i_3}, r_{i_1}', r_{i_2}', r_{i_3}' \in \mathcal{B}_{\mathfrak{B}}$ such that $r_i = (r_{i_1}, r_{i_2}, r_{i_3})$ and $r_i' = (r_{i_1}', r_{i_2}', r_{i_3}')$. Then*

1. $r_i^{\smile} = (r_{i_3}, r_{i_2}^{\smile}, r_{i_1})$,
2. $r_i^{\frown} = (r_{i_2}, r_{i_3}^{\smile}, r_{i_1}^{\smile})$,
3. *if composition is strong in $\mathfrak{B}$ then* $r_i \circ_{\mathfrak{C}} r_i' = \bigcup u_i$ *where* $u_i = (u_{i_1}, u_{i_2}, u_{i_3})$ *with* $u_{i_1} \subseteq r_{i_1}, u_{i_2} \subseteq (r_{i_2} \circ_{\mathfrak{B}} r_{i_2}') \cap (r_{i_1}^{\smile} \circ_{\mathfrak{B}} r_{i_3}')$, *and* $u_{i_3} \subseteq r_{i_3}'$. *If* $r_{i_3} \neq r_{i_1}'$ *then* $r_i \circ_{\mathfrak{C}} r_i' = r_i \diamond_{\mathfrak{C}} r_i' = \emptyset$,
4. *if composition is strong in $\mathfrak{B}$ then composition is strong in $\mathfrak{C}$.*

Proof *1.*, *2.*, and *3.* all follow directly from the definitions of the operations and the definition of an induced ternary calculus. For *4.* assume to the contrary that composition is weak in $\mathfrak{C}$. Then $\exists\, r_k \in \mathcal{B}_{\mathfrak{C}}$ with $r_k \subseteq r_i \diamond_{\mathfrak{C}} r_i'$ and $\exists\,(x, y, w) \in r_k$ such that $\forall z \in U_{\mathfrak{C}}$, $(x, y, z) \notin r_i$ or $(x, z, w) \notin r_i'$. By *3.* $r_{k_2} \cap [(r_{i_2} \circ_{\mathfrak{B}} r_{i_2}') \cap (r_{i_1}^{\smile} \circ_{\mathfrak{B}} r_{i_3}')] \neq \emptyset$. Since $\circ_{\mathfrak{B}}$ is strong, $\forall (a, d) \in r_{k_2}, \exists\, b, c \in U_{\mathfrak{B}}$ such that $(a, b) \in r_{i_2}$, $(b, d) \in r_{i_2}'$, $(a, c) \in r_{i_1}^{\smile}$, and $(c, d) \in r_{i_3}'$. Setting $a = y, d = w, b = z$, and $c = x$ contradicts the initial assumption that no such z exists. □

3 ORIENTATION RELATIONS OF DIRECTED LINE SEGMENTS

3.1 Cyclic ordering of 2D orientations: $\mathcal{CYC}_t$

$\mathcal{CYC}_t$ is a ternary calculus induced by a binary calculus $\mathcal{CYC}_b$. A $\mathcal{CYC}_b$ relation between an orientation X and another orientation Y is one of the following: (1) $e \equiv$ "$X = Y$", (2) $l \equiv$ "the angle (X, Y) is in $(0, \pi)$", (3) $o \equiv$ "$X = (Y + \pi)$", (4) $r \equiv$ "the angle (X, Y) is in $(\pi, 2\pi)$". A ternary relation is then induced for each triple of orientations as described above resulting in 24 relations: $eee, ell, eoo, err, lel, lll, llo, llr, lor, lre, lrl, lrr, oeo, olr, ooe, orl, rer, rle, rll, rlr, rol, rrl, rro, rrr$ (see [3] for details).

3.2 A coarse cyclic order for orientations: $\mathcal{CYC}_{tc}$

We now introduce a calculus of relative orientations that happens to be a coarsening of the $\mathcal{CYC}_t$ algebra. Following [4] we assume that there is a reference system (O, x, y) associated with $\mathbb{R}^2$. We will refer to the unit circle in $\mathbb{R}^2$ centred at O by $\mathcal{C}_{O,1}$, and to the set of 2D orientations as $2\mathcal{DO}$. For every point on $\mathcal{C}_{O,1}$, say w, the radius Ow of $\mathcal{C}_{O,1}$ is a segment of the directed line incident with O having the same orientation as Ow. The natural isomorphism h_2 as given in [2] maps each orientation in $2\mathcal{DO}$ to a unique point on the unit circle $\mathcal{C}_{O,1}$.

$h_2(W) = w \in \mathcal{C}_{O,1}$ whenever the orientation of the radius Ow is $W \in 2\mathcal{DO}$

Henceforth we will say the point $w \in \mathcal{C}_{O,1}$ is associated with the orientation W whenever $h_2(W) = w$. Let w_1, w_2, and w_3 be points on $\mathcal{C}_{O,1}$ associated with orientations W_1, W_2, and W_3 respectively. Further, for x, y on $\mathcal{C}_{O,1}$, let $\triangle(Ox, Oy)$ denote the angle at O subtended by Ox and Oy measured counter-clockwise from x to y. Then exactly one of the following is true:

For $w_1 \neq w_2$,
$l(W_1, W_2, W_3)$: $\triangle(Ow_1, Ow_3) < \triangle(Ow_1, Ow_2)$;
$r(W_1, W_2, W_3)$: $\triangle(Ow_1, Ow_3) > \triangle(Ow_1, Ow_2)$;
$e_{23}(W_1, W_2, W_3)$: $\triangle(Ow_1, Ow_3) = \triangle(Ow_1, Ow_2)$;
$e_{13}(W_1, W_2, W_3)$: $\triangle(Ow_1, Ow_2) = \triangle(Ow_3, Ow_2)$;
For $w_1 = w_2$,
$e_{123}(W_1, W_2, W_3)$: $\triangle(Ow_1, Ow_3) = \triangle(Ow_1, Ow_2)$;
$e_{12}(W_1, W_2, W_3)$: $\triangle(Ow_1, Ow_3) > \triangle(Ow_1, Ow_2)$.

The set $\{e_{123}, e_{12}, e_{23}, e_{13}, l, r\}$ of relations defined by the configurations above is the base for the coarsened calculus $\mathcal{CYC}_{tc}$. Those familiar with the $\mathcal{LR}$ calculus of [9] can see the parallels between our relations and the $\mathcal{LR}$ relations eq, e_{12}, e_{13}, e_{23}, l, and r. To see that $\mathcal{CYC}_{tc}$ is a coarsening of $\mathcal{CYC}_t$ notice that each $\mathcal{CYC}_{tc}$ relation is the set union of one or more $\mathcal{CYC}_t$ relations (see table 1). In particular, note that r is the unique CYCORD relation [8, 4]. The identity relation for $\mathcal{CYC}_{tc}$, $\{e_{23}, e_{123}\}$, is also the identity for $\mathcal{CYC}_t$, $\{eee, oeo, lel, rer\}$.

Table 1. Every $\mathcal{CYC}_{tc}$ relation has a unique partition into $\mathcal{CYC}_t$ relations.

$\mathcal{CYC}_{tc}$ relation	$\mathcal{CYC}_t$ relations
e_{123}	eee
e_{23}	oeo, lel, rer
e_{12}	eoo, ell, err
e_{13}	ooe, rle, lre
l	$rrr, rro, rrl, rol, rll, orl, lrl$
r	$lll, llo, llr, lor, lrr, olr, rlr$

Moreover, for each relation r in $\mathcal{CYC}_{tc}$, the relations $T = \{t \in \mathcal{B}_{\mathcal{CYC}_t} \mid t \subseteq r\}$ partition r. The operations of conversion, rotation, and composition in $\mathcal{CYC}_{tc}$ are defined as follows:

Definition 3 *Let $s, s_1, s_2 \in \mathcal{B}_{\mathcal{CYC}_{tc}}$ and let $T = \{t \in \mathcal{B}_{\mathcal{CYC}_t} \mid t \subseteq s\}$. Then*

1. $s^{\smile} = \{t^{\smile} \mid t \in T\}$
2. $s^{\frown} = \{t^{\frown} \mid t \in T\}$
3. $s_1 \circ_{\mathcal{CYC}_{tc}} s_2 = \bigcup_{t_1 \subseteq s_1, t_2 \subseteq s_2, t_1, t_2 \in \mathcal{B}_{\mathcal{CYC}_t}} (t_1 \circ_{\mathcal{CYC}_t} t_2)$

The map from $\mathcal{B}_{\mathcal{CYC}_t}$ onto $\mathcal{B}_{\mathcal{CYC}_{tc}}$ is closed under conversion and rotation. The following are easily verifiable from the preceding definition, the $\mathcal{CYC}_t$ tables in [3] for conversion and rotation and [4] for composition, and tables 1, 2 and 3 in this paper.

Table 2. The converse and rotation table of $\mathcal{CYC}_{tc}$.

r	e_{123}	e_{23}	e_{12}	e_{13}	l	r
$r^{\smile}$	e_{123}	e_{23}	e_{13}	e_{12}	r	l
$r^{\frown}$	e_{123}	e_{12}	e_{13}	e_{23}	l	r

Proposition 2 *If $r \in \mathcal{B}_{\mathcal{CYC}_{tc}}$ then so are $r^{\smile}$ and $r^{\frown}$.*

Table 3. Composition table of $\mathcal{CYC}_{tc}$. '-' represents the empty relation.

	e_{123}	e_{23}	e_{12}	e_{13}	l	r
e_{123}	e_{123}	-	e_{12}	-	-	-
e_{23}	-	e_{23}	-	e_{13}	l	r
e_{12}	-	e_{12}	-	e_{123}	e_{12}	e_{12}
e_{13}	e_{13}	-	r, l, e_{23}	-	-	-
l	-	l	-	e_{13}	l	r, l, e_{23}
r	-	r	-	e_{13}	r, l, e_{23}	r

In addition, composition of $\mathcal{CYC}_{tc}$ is complete over $\mathcal{B}_{\mathcal{CYC}_t}$ in the following sense:

Proposition 3 *Let $r_1, r_2, r_3 \in \mathcal{B}_{\mathcal{CYC}_{tc}}$ and let $t_1 \subseteq r_1$ and $t_2 \subseteq r_2$. If $t_1 \circ_{\mathcal{CYC}_t} t_2 \neq \emptyset$ then $\forall s_1 \subseteq r_1, \exists s_2 \subseteq r_2$ such that $s_1 \circ_{\mathcal{CYC}_t} s_2 \neq \emptyset$. Moreover if $r_3 \subseteq r_1 \circ_{\mathcal{CYC}_{tc}} r_2$ then $\forall s_3 \subseteq r_3, \exists s_1 \subseteq r_1$ and $\exists s_2 \subseteq r_2$ such that $s_3 \subseteq s_1 \circ_{\mathcal{CYC}_t} s_2$. Here $t_1, t_2, s_1, s_2, s_3 \in \mathcal{B}_{\mathcal{CYC}_t}$.*

4 DIPOLE RELATION ALGEBRAS

In $\mathcal{DRA}$ the relative positions of directed line segments, also called dipoles in [6], are based on the relations of their vertices. Formally, a dipole is an ordered pair of points in $\mathbb{R}^2$ which can be written as $\mathbf{A} = (A_s, A_e)$, where A_s and A_e are the start- and end-point of $\mathbf{A}$ respectively. We refer to A_s and A_e as nodes of $\mathbf{A}$. The dipole $\mathbf{A}$ induces a partition of the plane $\mathcal{P}(\mathbf{A})$ with the following classes: the open left (l) and right (r) half-planes bounded by the directed line l incident with A_s and A_e and having the same orientation as $\mathbf{A}$, the points of l lying after A_e (f) and those lying before A_s (b) in the traversal of l in the positive direction, A_s (s), A_e (e), and the interior points of $\mathbf{A}$ (i). We call l the embedding line of $\mathbf{A}$.

A basic $\mathcal{DRA}$ relation between two dipoles $\mathbf{A}$ and $\mathbf{B}$ is given by the 4-tuple of dipole-point relations (dpt relation) $\mathbf{s}_B\mathbf{e}_B\mathbf{s}_A\mathbf{e}_A$ where $\mathbf{s}_B$ is the label of the region of $\mathcal{P}(\mathbf{A})$ in which B_s lies. The other three elements of the relation $\mathbf{e}_B$, $\mathbf{s}_A$, and $\mathbf{e}_A$ are interpreted analogously with $\mathcal{P}(\mathbf{A})$ replaced by $\mathcal{P}(\mathbf{B})$ for $\mathbf{s}_A$ and $\mathbf{e}_A$. The resulting system of 72 relations is called $\mathcal{DRA}_f$ [6]. Apparently composition in $\mathcal{DRA}_f$ is weak. However, the refinement of $\mathcal{DRA}_f$ called $\mathcal{DRA}_{fp}$ has strong composition. $\mathcal{DRA}_{fp}$ refines $\mathcal{DRA}_f$ by introducing orientation descriptors '+', '-', 'A', and 'P' that describe the orientation of $\mathbf{B}$ with respect to $\mathbf{A}$ where the orientation is '+' if $\mathbf{B}$ is oriented towards the left of $\mathbf{A}$, '-' if it is oriented towards the right, 'P' if $\mathbf{A}$ and $\mathbf{B}$ are parallel and 'A' if they are anti-parallel. The descriptors are defined only for the four relations $rrrr, rrll, llll, llrr$ [6].

The "coarsened" $\mathcal{DRA}_c$ [6] demands that points be in general position (i.e. no three points can be collinear). This means that only the values l, r, s, e are allowed. The resulting calculus has 24 relations (see [6] for details). The version of $\mathcal{DRA}$ that is the focus of this paper is $\mathcal{DRA}_7$, so called because only 7 relations can be realised between any pair of dipoles.

4.1 $\mathcal{DRA}_7$

The distinctive feature of $\mathcal{DRA}_7$ is that only the distinctions possible are which (if any) nodes of a pair of dipoles coincide. This is because the semantics of the relations are derived from a coarser planar partition than that introduced above. $\mathcal{P}(\mathbf{A})$ is a partition of the plane into three parts, namely, A_s (s), A_e (e), and the rest of plane (x). The resulting seven relations are $sese$ (coincidence), $sxsx$ (share same start point), $xsex$ (start point of first dipole coincident with end point of second dipole), $xxxx$ (disjoint), $exxs$ (start point of second dipole is coincident with end point of first dipole), $xexe$ (share same end point), and $eses$ (reverse of each dipole is coincident with the other).

Note that there are at least three different ways to interpret the dpt relations of $\mathcal{DRA}_7$ depending on whether it is considered as a coarsening of $\mathcal{DRA}_{fp}$, $\mathcal{DRA}_f$, or $\mathcal{DRA}_c$ which will be called $\mathcal{DRA}_{7fp}$, $\mathcal{DRA}_{7f}$, and $\mathcal{DRA}_{7c}$ respectively. $\mathcal{DRA}_{7c}$ like $\mathcal{DRA}_c$ forbids collinear triples of dipole nodes while the first two are essentially the same.

Interestingly, the composition tables of $\mathcal{DRA}_{7c}$, $\mathcal{DRA}_{7f}$ and $\mathcal{DRA}_{7fp}$ have the same entries. This is because $\mathcal{B}_{\mathcal{DRA}_c}$ is a proper subset of both $\mathcal{B}_{\mathcal{DRA}_f}$ and $\mathcal{B}_{\mathcal{DRA}_{fp}}$ and the fact that the finer $\mathcal{DRA}$ versions only *add* dpt relations that are subsumed by the x relation in $\mathcal{DRA}_{7fp}$ - i.e. b, i, f. What this means is that it makes no difference which version is used to derive the composition table of $\mathcal{DRA}_7$ in general because, the entries and relations involved only differ in their interpretations. In order to understand this relationship we introduce the following characterisation of dipole configurations.

Given any configuration of dipoles in the plane, a **free node** of a dipole A is a node that is not coincident with any other node in the configuration. The number of free nodes over all dipoles in the configuration is denoted by **FN**. We will call a configuration of n dipoles a **n-cfg**. The number of free nodes of a $\mathcal{DRA}$ relation is the FN of the 2-cfg it describes. Two $\mathcal{DRA}$ relations t_1 and t_2 will be said to **bind** the same nodes if they are subsumed by the same $\mathcal{DRA}_7$ relation. i.e. for every dpt relation that has the value s or e in t_1, the corresponding dpt relation in t_2 has the same value and vice-versa. The number of bound nodes will be denoted by **BN**: BN $= 4-$ FN. For any 3-cfg, say D = $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$, in the configuration, the pair formed by associating the FN of D and the set of FNs of the doubletons $\{\mathbf{A}, \mathbf{B}\}, \{\mathbf{B}, \mathbf{C}\}$, and $\{\mathbf{A}, \mathbf{C}\}$ of D will be called the **canonical shape** (or just **shape**) of D with respect to connectivity. If D is the shape of some 3-cfg then that 3-cfg is a realization of D and the triple of relations that describe it will be said to have the same shape as every triple of relations that describes a realization of D. In particular D is also the shape of the triple of relations.

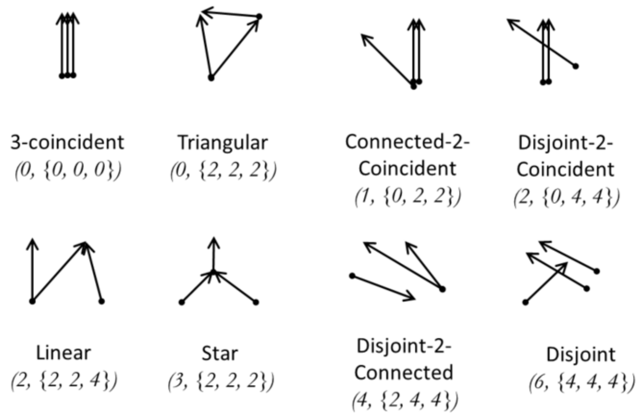

Figure 1. Examples of configurations of three dipoles (3cfgs) each corresponding to one of the eight possible shapes.

Any 2-cfg can have a FN of either 0, 2, or 4. Any 3-cfg can have a FN between 0 and 6 except the value 5 for which no configuration on three dipoles can be realized. In particular, there are only eight possible shapes for any triple of dipoles (figure 1). We now claim the following:

Proposition 4 *The respective converse and composition tables of $\mathcal{DRA}_{7c}$, $\mathcal{DRA}_{7f}$ and $\mathcal{DRA}_{7fp}$ have the same entries.*

Proof All dpt relations in the set $\{l, b, i, f, r\}$ are subsumed by the x dpt relation in $\mathcal{DRA}_7$. Computing the converse of a relation in $\mathcal{DRA}$ simply involves swapping the first two characters and the last two characters of the relation's name to get the name of the converse relation [6]. The result of applying the converse operation in $\mathcal{DRA}_7$ is therefore the same regardless of which dpt relations from the set above are involved in the underlying relation. Similarly, all 3-cfgs with the same shape and involving relations that bind the same nodes (in a pairwise correspondence) are subsumed by the same triple of $\mathcal{DRA}_7$ relations. So these relations map to the same $\mathcal{DRA}_7$ composition table entry regardless of which dpt relations from the set above are involved in the underlying relation triple. Moreover, the $\mathcal{DRA}_7$ relations distinguished by the orientation descriptors '+', '-', 'A', and 'P' are subsumed by the relation $xxxx$. □

Definition 4 *Given a 3-cfg of dipoles, the replacement of one or more dipoles in the 3-cfg by another dipole(s) such that the shape of the resulting 3-cfg is the same as the shape of the initial 3-cfg is called a* ***shape preserving (SP) transformation*** *between the initial and final configurations.*

SP transformations can be thought of as discreet steps that rewrite a 3-cfg in a way that preserves its shape. One way to think of it is to consider three elastic bands tied to pegs at their ends. Binding the same node is the same as tying two bands to the same pegs. The pegs can be moved around freely but it is not allowed to remove any end of a band from one peg to another peg. Any moving of a peg that changes its relative position to any other peg(s) is a SP transformation. For the forthcoming part it will be useful consider the replacement of a dipole by its reverse. This is the dipole $rev(\mathbf{A})$ defined by $rev(\mathbf{A}) = \mathbf{B}$ if and only if $(\mathbf{A}, \mathbf{B}) \in eses$. The replacement of $\mathbf{A}$ with $rev(\mathbf{A})$ is a SP transformation.

Definition 5 *Given a $\mathcal{DRA}_k$ relation t ($k \in \{7, c, f, fp\}$), the first, second, and complete reversals of t are given by :*

1. $rev_1(t) = q \in \mathcal{B}_{\mathcal{DRA}_k} \;:\; \forall(\boldsymbol{A}, \boldsymbol{B}) \in t, (rev(\boldsymbol{A}), \boldsymbol{B}) \in q$
2. $rev_2(t) = q \in \mathcal{B}_{\mathcal{DRA}_k} \;:\; \forall(\boldsymbol{A}, \boldsymbol{B}) \in t, (\boldsymbol{A}, rev(\boldsymbol{B})) \in q$
3. $rev_c(t) = q \in \mathcal{B}_{\mathcal{DRA}_k} \;:\; \forall(\boldsymbol{A}, \boldsymbol{B}) \in t, (rev(\boldsymbol{A}), rev(\boldsymbol{B})) \in q$

Proposition 5 *Let $q, t \in \mathcal{B}_{\mathcal{DRA}_{fp}}$ such that q and t have the same FN. If there exists a 3-cfg involving the relation q and having a shape, say S, then the realization of S constructed by the relations of that 3-cfg can be transformed into a realization of S involving the relation t using a finite number of shape preserving transformations.*

Proof By construction. Given a 3-cfg involving two dipoles $(\mathbf{A}, \mathbf{B}) \in q$, the following procedure is a SP transformation of that 3-cfg to a 3-cfg involving t as a relation:

1. if q and t differ in the dpt relations of their bound nodes, apply the appropriate reversal operation on the dipoles involved in the relation q and verify if the resulting relation p equals t. In any case p binds the same nodes as t;
2. if $p \neq t$ then for one of the dipoles involved in p, say $\mathbf{A}$, pick in each of the regions in $\mathcal{P}(\mathbf{A})$ that correspond to the dpt relations $\mathbf{s}_B$ and $\mathbf{e}_B$ of relation t, a representative point that is not coincident to any node in the configuration except possibly a node of $\mathbf{A}$. Call these points D_s and D_e respectively and let $\mathbf{D}$ be a dipole with start-node D_s and end-node D_e;
3. let r be the relation between $\mathbf{A}$ and $\mathbf{D}$. If $r = t$ the we are done;
4. if $r \neq t$, then r and t differ only in one or both of the dpt relations $\mathbf{s}_A$ and $\mathbf{e}_A$. Moreover, at least one of these relations corresponds to a free node because otherwise $p = t$ since they bind the same nodes and involve no free nodes. There are three exhaustive cases
   (a) if exactly one of $\mathbf{s}_D$ or $\mathbf{e}_D$ is in $\{b, i, f\}$ then it must be the case that $r = t$. To see this suppose, w.l.o.g., $\mathbf{s}_D \in \{b, i, f\}$. Since any choice of points A_s and A_e must lie on the embedding line of $\mathbf{A}$ and must each be such that the order in which D_s, A_s, and A_e are encountered along the line (in any chosen direction) is preserved, no choice of A_s and A_e can change the dpt relations $\mathbf{s}_A$ and $\mathbf{e}_A$. So it is not the case that exactly one of $\mathbf{s}_D$ or $\mathbf{e}_D$ is in $\{b, i, f\}$;
   (b) if both $\mathbf{s}_D$ and $\mathbf{e}_D$ are in $\{b, i, f\}$ then both nodes of $\mathbf{A}$ are also free nodes with $\mathbf{s}_A$ and $\mathbf{e}_A$ in $\{b, i, f\}$. This so because D_s and D_e both lie on the embedding line of $\mathbf{A}$ and so $\mathbf{A}$ and $\mathbf{D}$ are collinear. In all cases for which $\mathbf{s}_D = \mathbf{e}_D$, $rev_2(r) = t$. Notice as above that in this case the order in which A_s, A_e and either of D_s and D_e are encountered is preserved. For $\mathbf{s}_D \neq \mathbf{e}_D$ it must be that $r = t$. Otherwise any choice of A_s and A_e that changes the values $\mathbf{s}_A$ and $\mathbf{e}_A$ also changes the value of $\mathbf{s}_D$ or $\mathbf{e}_D$ or both. So it not the case that $\mathbf{s}_D \neq \mathbf{e}_D$.
   (c) if both $\mathbf{s}_D$ and $\mathbf{e}_D$ are in $\{r, l\}$ then $\mathbf{A}$ and $\mathbf{D}$ are not collinear. Pick in each of the regions in $\mathcal{P}(\mathbf{D})$ that correspond to the dpt relations $\mathbf{s}_A$ and $\mathbf{e}_A$ of t, a representative point that lies on the embedding line of $\mathbf{A}$ and is not coincident with any other node in the configuration. Note that this pair of points, call them E_s and E_e for $\mathbf{s}_A$ and $\mathbf{e}_A$ respectively, induce dipole E with start node E_s and end node E_e that does not alter any of the dpt relations $\mathbf{s}_D$ or $\mathbf{e}_D$. So $(E, D) \in t$.
5. if the third dipole $\mathbf{C}$ of the original configuration has any bound nodes then each is coincident with a node of $\mathbf{A}$ or $\mathbf{B}$. Moreover, we can keep track of these nodes during the transformations above. So once s is transformed to t, $\mathbf{C}$ can be replaced by a dipole extending from any of its nodes to the other.

The new configuration has the same shape as the initial configuration because it preserves all the bound and free nodes of the initial configuration. □

Proposition 6 *Let $r_1, r_2, r_3 \in \mathcal{B}_{\mathcal{DRA}_{7fp}}$ such that $r_3 \subseteq r_1 \diamond_{\mathcal{DRA}_{7fp}} r_2$. Then $\forall t_3 \in \mathcal{B}_{\mathcal{DRA}_{fp}}$ such that $t_3 \subseteq r_3$ $\exists t_1, t_2 \in \mathcal{B}_{\mathcal{DRA}_{fp}}$ such that $t_1 \subseteq r_1, t_2 \subseteq r_2$ and $t_3 \subseteq t_1 \diamond_{\mathcal{DRA}_{fp}} t_2$.*

Proof If $r_3 \subseteq r_1 \diamond_{\mathcal{DRA}_{7fp}} r_2$ then by definition $\exists s_1, s_2, s_3 \in \mathcal{B}_{\mathcal{DRA}_{fp}}$ such that $s_1 \subseteq r_1, s_2 \subseteq r_2, s_3 \subseteq r_3$ and $s_3 \subseteq s_1 \diamond_{\mathcal{DRA}_{fp}} s_2$. Since $s_3 \subseteq r_3$ and $t_3 \subseteq r_3$, s_3 and t_3 have the same FN and bind the same nodes. So the shape of s_1, s_2, s_3 has a realization involving t_3. Moreover, the transformation into this realization does not involve any reversal operations since s_3 and t_3 bind the same nodes. So the resulting relations on the dipole pairs involved in the relations s_1 and s_2 bind the same nodes as s_1 and s_2. Therefore these relations are also subsumed by r_1 and r_2 in $\mathcal{B}_{\mathcal{DRA}_{7fp}}$ respectively. □

Theorem 1 *Composition of $\mathcal{B}_{\mathcal{DRA}_{7fp}}$ is strong.*

Proof Suppose not. Then there must be three relations $r_1, r_2, r_3 \in \mathcal{B}_{\mathcal{DRA}_{7fp}}$ with $r_3 \subseteq r_1 \diamond_{\mathcal{DRA}_{7fp}} r_2$ such that r_3 contains a pair of dipoles $(\mathbf{A}, \mathbf{C})$ for which there is no dipole $\mathbf{B}$ in $\mathcal{B}_{\mathcal{DRA}_{7fp}}$ satisfying

$(\mathbf{A},\mathbf{B}) \in r_1$ and $(\mathbf{B},\mathbf{C}) \in r_2$. Since r_3 has a partition into one or more relations in $\mathcal{B}_{\mathcal{DRA}_{fp}}$, $(\mathbf{A},\mathbf{C}) \in t_3$ for some $t_3 \in \mathcal{B}_{\mathcal{DRA}_{fp}}$. By proposition 6 $\exists t_1, t_2 \in \mathcal{B}_{\mathcal{DRA}_{fp}}$ such that $t_1 \subseteq r_1, t_2 \subseteq r_2$ and $t_3 \subseteq t_1 \diamond_{\mathcal{DRA}_{fp}} t_2$. Now, since composition of $\mathcal{DRA}_{fp}$ is strong (Theorem 54 in [6]) it follows that there is at least one dipole $\mathbf{B}$ satisfying $(\mathbf{A},\mathbf{B}) \in t_1$ and $(\mathbf{B},\mathbf{C}) \in t_2$. So $(\mathbf{A},\mathbf{B}) \in r_1$ and $(\mathbf{B},\mathbf{C}) \in r_2$ as well. A contradiction. □

4.2 The calculus of ternary dipole relations $\mathcal{DRA}_{7t}$

We computed the ternary calculus induced by $\mathcal{DRA}_7$ based on the definitions in section 2. The calculus has 87 relations corresponding to all entries of the composition table of $\mathcal{DRA}_7$. The base relations are closed under converse and rotation operations and by proposition 1 we can conclude that composition is strong in $\mathcal{DRA}_{7t}$ if we assume that it is induced by $\mathcal{DRA}_{7fp}$. The list of relations is given in table 4 organised by shape. The name of a relation is the concatenation of the first two characters of the names of the binary relations in its decomposition taken in the order they appear. For example, $(\mathbf{A},\mathbf{B},\mathbf{C}) \in exxsxe$ means $(\mathbf{A},\mathbf{B}) \in \mathbf{ex}xs$, $(\mathbf{B},\mathbf{C}) \in \mathbf{xs}ex$, and $(\mathbf{A},\mathbf{C}) \in \mathbf{xe}xe$.

Table 4. Base relations of the induced ternary calculus $\mathcal{DRA}_{7t}$ categorized according to their shapes.

Shape	$\mathcal{DRA}_{7t}$
3-coincident	$sesese, seeses, esesse, essees$
connected-2-coincident	$sesxsx, sexexe, sexsxs, seexex, sxsxse, xexese, xsexse, exxsse, sxsesx, xesexe, xssexs, exseex, sxxses, exsxes, xeexes, xsxees, sxesxs, xsessx, exesxe, xeesex, essxex, esxexs, esexsx, esxsxe$
disjoint-2-coincident	$sexxxx, xxxxse, xxsexx, xxxxes, xxesxx, esxxxx$
triangular	$sxxeex, xesxxs, xsxsex, exexxs, sxexxe, xexssx, xssxxe, exxesx$
star	$xsxexs, xsexsx, sxsxsx, sxxsxs, xeexex, exsxex, exxsxe, xexexe$
linear	$exxxxs, xexxsx, xxxesx, xxsxxs, xxsxxe, sxxexx, xxexxs, xxxeex, xsxxex, sxxxxe, xsxsxx, xxxsex, xexsxx, exxxsx, sxxxex, xxexxe, xssxxx, exxexx, xexxxs, exexxx, xsxxxe, sxexxx, xxxssx, xesxxx$
disjoint-2-connected	$xxxxex, xxexxx, exxxxx, xxxxxe, xxxexx, xexxxx, xxxxsx, xxsxxx, sxxxxx, xxxxxs, xxxsxx, xsxxxx$
disjoint	$xxxxxx$

5 TERNARY CALCULI OF DIRECTED LINE SEGMENTS

The ternary calculi of directed line segments are generated by associating relations of $\mathcal{DRA}_{7t}$ and with relations of $\mathcal{CYC}_t$ and $\mathcal{CYC}_{tc}$ respectively. In order to establish this association, $\mathcal{CYC}_b$ relations are used to evaluate the possible orientation relations between pairs of dipoles. Here, the orientation of a dipole is the orientation of its embedding line and the $\mathcal{CYC}_b$ relation of two dipoles is the $\mathcal{CYC}_b$ relation of their embedding lines. This is extended to the ternary case by evaluating the possible combinations of triples of $\mathcal{CYC}_b$ and $\mathcal{DRA}_7$ relations. A relation resulting from a combination of two relations from two different calculi will be written as a concatenation of the two relations separated by a dash. The order in which the relations are given is '$\mathcal{DRA}$ relation'-'$\mathcal{CYC}$ relation'. The range of possible combinations of $\mathcal{CYC}_b$ and $\mathcal{DRA}_7$ relations is determined by the value of FN and the dpt relations of the $\mathcal{DRA}_f$ relations underlying $\mathcal{DRA}_7$ relations involved (table 5).

Generating the corresponding ternary combinations simply involves evaluating the possible combinations of triples from relations described above. For example, the triangular relation $sxxeex$ if none of the binary orientation relations given is e or o then we know that none of the underlying $\mathcal{DRA}$ relations involves the dpt relations i, f, and b. So we can determine the validity of the combination by noting that the position of the free node of $\mathbf{B}$ is always on the same side of $\mathbf{A}$ as $\mathbf{B}$s orientation relative to $\mathbf{A}$ and similarly with the free node of $\mathbf{A}$. The same can be performed for all other relations and scenarios.

Table 5. Compatibility of $\mathcal{CYC}_b$ and $\mathcal{DRA}_7$ relations.

FN	dpt relations	Combined relations
0	only e and s	$sese$-e, $eses$-o
2	2 from $\{s,e\}$, 2 from $\{i,f,b\}$	$sxsx$-e, $sxsx$-o, $xsex$-e, $xsex$-o, $exxs$-e, $exxs$-o, $xexe$-e, $xexe$-o
	2 from $\{s,e\}$, 2 from $\{l,r\}$	$sxsx$-l, $sxsx$-r, $xsex$-l, $xsex$-r, $exxs$-l, $exxs$-r, $xexe$-l, $xexe$-r
4	i,f,b	$xxxx$-e, $xxxx$-o
	l,r	$xxxx$-e, $xxxx$-o, $xxxx$-l, $xxxx$-r
	2 from $\{l,r\}$, 1 from $\{i,f,b\}$	$xxxx$-l, $xxxx$-r

Table 6. Legal combinations $\mathcal{DRA}_{7t}$ relations with those of $\mathcal{CYC}_t$ if it is assumed that the relations of $\mathcal{DRA}_{7t}$ are induced by $\mathcal{DRA}_{7c}$.

$\mathcal{CYC}_t$ relations	$\mathcal{DRA}_{7t}$ relations
eee	$sesese$
eoo	$seeses$
ell	$sesxsx, sexexe, sexsxs, seexex, sexxxx$
err	$sesxsx, sexexe, sexsxs, seexex, sexxxx$
ooe	$esesse$
rle	$sxsxse, xexese, xsexse, exxsse, xxxxse$
lre	$sxsxse, xexese, xsexse, exxsse, xxxxse$
oeo	$essees$
lel	$sxsesx, xesexe, xssexs, exseex, xxsexx$
rer	$sxsesx, xesexe, xssexs, exseex, xxsexx$
rrr	$xxxxxx$, $sxxeex$, $xesxxs$, linear, star, D-2-c*
rrl	$xxxxxx$, $xsxsex$, $exexxs$, linear, star, D-2-c
rll	$xxxxxx$, $sxexxe$, $xexssx$, linear, star, D-2-c
lrl	$xxxxxx$, $xssxxe$, $exxesx$, linear, star, D-2-c
rro	$exsxes, xexses, xxxxes, sxexes, xsxees$
rol	$exesxe, xeesex, xxesxx, sxesxs, xsessx$
orl	$esexsx, esxexs, esxxxx, essxex, esxsxe$
lll	$xxxxxx$, $sxxeex$, $xesxxs$, linear, star, D-2-c
llr	$xxxxxx$, $xsxsex$, $exexxs$, linear, star, D-2-c
lrr	$xxxxxx$, $sxexxe$, $xexssx$, linear, star, D-2-c
rlr	$xxxxxx$, $xssxxe$, $exxesx$, linear, star, D-2-c
llo	$exsxes, xexses, xxxxes, sxexes, xsxees$
lor	$exesxe, xeesex, xxesxx, sxesxs, xsessx$
olr	$esexsx, esxexs, esxxxx, essxex, esxsxe$

* D-2-c stands for Disjoint-2-connected

As expected two different calculi can be generated. The first $\mathcal{DRA}_{7to}$ is based on the combination of $\mathcal{DRA}_{7t}$ relations with $\mathcal{CYC}_t$, as discussed above. The second $\mathcal{DRA}_{7toc}$ is based on the combination of $\mathcal{DRA}_{7t}$ relations with $\mathcal{CYC}_{tc}$. The two relations $r \in \mathcal{CYC}_{tc}$ and $t \in \mathcal{DRA}_{7t}$ is a relation, t-r, of $\mathcal{DRA}_{7toc}$ if and only if there is a relation $s \in \mathcal{CYC}_t$ such that $s \subseteq r$ and t-s is a relation of $\mathcal{DRA}_{7to}$. In both cases, the version of $\mathcal{DRA}_{7t}$ assumed may be induced by $\mathcal{DRA}_{7c}$ or $\mathcal{DRA}_{7fp}$. For the latter, every $\mathcal{DRA}_{7t}$ relation can be combined with every $\mathcal{CYC}_{tc}$ relation. This not true for $\mathcal{CYC}_t$ since, e.g., each triangular relation can only be combined with five different $\mathcal{CYC}_t$ relations. Table 6 gives for each $\mathcal{CYC}_t$ relation the compatible $\mathcal{DRA}_{7t}$ relations induced by $\mathcal{DRA}_{7c}$.

5.1 Matching line segment in spatial networks

The calculi presented in section 5 are quite suitable for representing and reasoning about links or edges in networks where the spatial layout is of primary importance. These requirements often arise in applications in robot navigation as well as other spatial sciences. One useful application (that mostly motivated this work) is the problem of matching sketch maps with metric maps based on their representations using qualitative spatial calculi.

Consider the network of three edges in frame I in figure 2 which is part of a larger network structure (e.g. a street network in a metric map). An interpretation of this subnetwork into $\mathcal{DRA}$ would yield six dipoles as shown in frame II. Now suppose the two configurations in frames III and IV are subnetworks from two sketch maps that are being matched our metric map. Empirical studies have shown that an algorithm for matching sketch maps must care about (i) street network topology and (ii) order of outgoing streets at a junction, among other things [1]. So we assume that the algorithm will pick from each map three non-coincident but jointly adjacent edges such that the $\mathcal{CYC}_{tc}$ component of their relation is same for both maps. If their $\mathcal{DRA}_{7t}$ relation is also the same, then we have a local match that with respect to orientation and ordering. In frames II-IV of figure 2 the triples $\mathbf{DA}, \mathbf{DB}, \mathbf{DC}$ have the relation $sxsxsx\text{-}r$ as do also all the triples $\mathbf{AD}, \mathbf{BD}, \mathbf{CD}$.

If inconsistencies arise between any matched triple of non-adjacent dipoles and a matched triple of jointly adjacent dipoles then either remove those matched dipoles adjacent two or less dipoles or expand the relation constraining the non-adjacent dipole. The power of the representation comes from the ability to distinguish different types of local configurations and using reversals to compare different configurations.

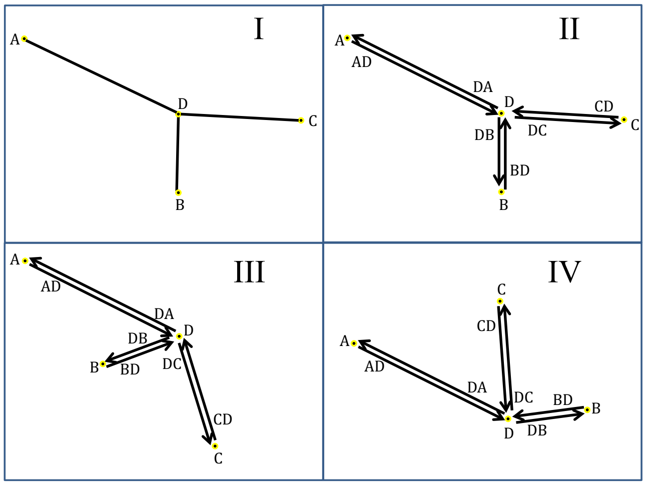

Figure 2. A configuration of three jointly adjacent edges (I) and three (II, III, IV) non-identical sets of information about six dipoles, possibly, describing the same configuration.

6 CONCLUSION

We have presented an analysis of different families of spatial calculi and used the results to generate a new family of calculi that integrates both of them. The main distinctive features of the knowledge representable by these calculi are the integration of connectivity and orientation information and ability to distinguish left and right within a network as opposed to between pairs of edges. This achieved using coarser ternary orientation relations.

In the preliminaries section we saw that the strength of composition is hereditary from inducing atomic binary relations to induced atomic ternary relations. We also showed that composition in $\mathcal{DRA}_{7fp}$ is strong. This was made possible by considering the notion of shape. We did not at all tackle issues concerning the algebraic (in a strict sense) structures associated with the calculi and, therefore, neither did we approach the question of complexity. Analysing the complexity problem for the calculi discussed in this paper is part of future work. However, it is noteworthy that as [7] showed, under certain conditions, some of which both of the coarse calculi presented herein meet, the portions of tractable subsets of relations of a finer calculus that lie in the coarse calculus become tractable subsets of the coarse calculus as well.

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