Identification of imperfections in thin plates based on the modified potential energy principle

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Abstract

A procedure to identify the imperfection in thin plates is proposed in this paper. The modified potential energy principle, which serves as the theoretical basis of the identification procedure, is improved to allow for the experimental measurements in static tests. Several typical examples are studied to illustrate the effectiveness of the procedure.

Keywords: Thin plate; modified potential energy principle; imperfection; parameter identification; static tests

1. Introduction

Imperfection identification is often of foremost concern for many structural systems in service. Non-destructive load tests are usually conducted to determine the unknown parameters for identification of the imperfections.

As a major class of techniques that has the merit of uninterrupted operation of systems, dynamic damage identification methods have been developed for many years (for example, Adams et al., 1987; Gudmundson, 1982; Hearn and Testa, 1991; Capecchi and Vestroni, 1999; Vestroni and Capecchi, 2000; Ren and Roeck, 2002; Gladwell, 2004). Among them, some are exclusively proposed to detect the imperfections in thin plates (see Cornwell et al., 1999; Lee and Shin, 2002; Lee et al., 2003). It has been recognized that the major deficiency of dynamic identification methods is the presence of uncertainties in masses and damping. In contrast, static identification procedures can bypass this deficiency and enjoy easy implementation for simple structural systems. Existing methods based on the static measurements are mostly represented as constrained non-convex optimization problems (Sanayei and Onipede, 1991; Banan and Hjelmstad, 1994; Hjelmstad and Shin, 1997; Di Paola and Bilello, 2004; Buda and Caddemi, 2007). Nevertheless, these static identification procedures have the disadvantage of lack of test repeatability or generality for various structural systems. In addition, neither existing dynamic nor static identification procedures give analytical expressions of identification parameters.

In the recent years, Caddemi and his coworkers' have conducted research on damage identification of beams by static tests (Caddemi and Greco, 2006; Caddemi and Morassi, 2007; Caddemi and Di Paola, 2008; Caddemi and Morassi, 2013), standing out due to the explicit expressions of the parameters to be identified. In particular, in the work of Caddemi and Di Paola (2008), a modified version of the Hu-Washizu variational principle was introduced to identify the imperfections in beams, shedding light on the development of a general procedure to obtain closed-form solutions of different identification problems according to the principle.

In this paper, attention is concentrated on the identification of imperfections in thin plates. The static response of a thin plate is governed by a fourth-order partial differential equation, being much more difficult than the beam problem. Alternatively, displacement-based approximate methods, such as the Ritz method, are usually used to acquire approximate analytical solutions, featuring concision and efficiency. However, in the above-mentioned identification procedure based on Hu-Washizu principle, the expressions of internal forces (or stresses) in the elastic body usually need to be assumed, which is not an easy matter for thin plates. Therefore, the modified potential energy principle, with the independent variables of displacements and tractions on the constrained boundary, is used in this paper to obtain displacement-based approximate analytical solutions, avoiding the variables of internal forces.

An improved version of modified potential energy functional is established for the identification purpose, allowing for the response measurements as the additional fictitious constraints. Expressions of the fictitious reactions, as functions of imperfection parameters to be identified, are derived from the stationary conditions of the functional, and nullification of these reactions leads to the identification of the unknown structural parameters.

The framework proposed in this paper is applicable to different cases of imperfections in thin plates, providing approximate analytical solutions to the identification problems. The procedure is exemplified by three typical applications, showing the

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generality in a class of inverse problems.

2. The modified potential energy principle

In this section, the modified potential energy principle (Tian and Pian, 2001) is introduced as the basic theory for the identification procedure, especially under for thin plates. An improved version of the principle is established for the purpose of identification, accounting for the response measurements via experimental tests as fictitious prescribed displacements.

2.1. The modified potential energy principle for linear elasticity

An elastic body in the orthogonal Cartesian coordinate system $x_i\,(i = 1, 2, 3)$, occupies a domain Ω bounded by the surface S. Denoted by S_u, the constrained part of S has prescribed displacement components $u_i = \bar{u}_i\,(i = 1, 2, 3)$; while the complementary part of S_u, where the tractions are given as $\bar{T}_i\,(i = 1, 2, 3)$, is denoted by S_σ. The well-known principle of minimum potential energy is given as

$$\begin{aligned}\Pi_{\mathrm{p}}(u_i) = \int_\Omega \frac{1}{2} D_{ijkl} u_{i,j} u_{k,l} \mathrm{d}\Omega - \int_\Omega \bar{f}_i u_i \mathrm{d}\Omega \\ - \int_{S_\sigma} \bar{T}_i u_i \mathrm{d}S = \min, \\ \text{subject to} \quad u_i = \bar{u}_i \quad \text{on } S_u,\end{aligned} \tag{1}$$

in which $\bar{f}_i\,(i = 1, 2, 3)$ are the assigned body force components, and D_{ijkl}, the Hooke stiffness tensor for isotropic elastic material, is positive definite.

With the Lagrange multiplier method, one can relax the constraints $u_i = \bar{u}_i$ on S_u by introducing Lagrange multipliers and construct an augmented functional, termed the modified potential energy functional

$$\Pi_{\mathrm{mp}}(u_i, \lambda_i) = \Pi_{\mathrm{p}}(u_i) - \int_{S_u} \lambda_i (u_i - \bar{u}_i)\,\mathrm{d}S. \tag{2}$$

The stationary condition of Π_{mp} gives

$$\begin{aligned}&\delta\Pi_{\mathrm{mp}} \\ &= -\int_\Omega \left[\left(D_{ijkl} u_{k,l}\right)_{,j} + \bar{f}_i\right] \delta u_i \mathrm{d}\Omega + \int_{S_\sigma} \left(D_{ijkl} u_{k,l} n_j - \bar{T}_i\right) \delta u_i \mathrm{d}S \\ &\quad + \int_{S_u} \left(D_{ijkl} u_{k,l} n_j - \lambda_i\right) \delta u_i \mathrm{d}S - \int_{S_u} (u_i - \bar{u}_i)\,\delta\lambda_i \mathrm{d}S = 0,\end{aligned} \tag{3}$$

where integration by parts and Gauss theorem are applied. In the above derivation, one can identify the Euler equations, i.e. equilibrium equations as,

$$\left(D_{ijkl} u_{k,l}\right)_{,j} + \bar{f}_i = 0 \quad \text{in } \Omega, \tag{4}$$

and the boundary conditions as

$$D_{ijkl} u_{k,j} n_j = \bar{T}_i \quad \text{on } S_\sigma, \tag{5}$$

$$u_i = \bar{u}_i, \ \lambda_i = D_{ijkl} u_{k,l} n_j \quad \text{on } S_u. \tag{6}$$

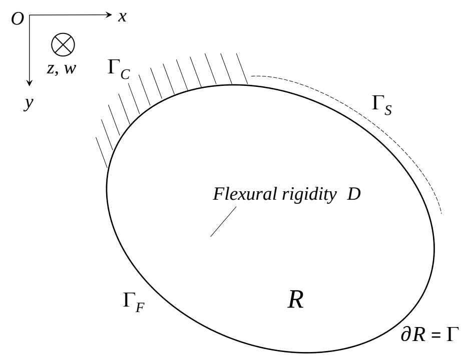

Figure 1: The thin plate model

The three Lagrange multipliers are identified as the boundary tractions T_i on S_u, thus the modified potential energy functional can be rewritten as

$$\Pi_{\mathrm{mp}}\left(u_i,\, T_i|_{S_u}\right) = \Pi_{\mathrm{p}}(u_i) - \int_{S_u} T_i (u_i - \bar{u}_i)\,\mathrm{d}S, \tag{7}$$

and the modified potential energy principle can be stated as: *The modified potential energy functional* Π_{mp} *takes stationary value for true* $\left(u_i,\, T_i|_{S_u}\right)$.

2.2. The modified potential energy principle for thin plates

Since the present work concentrates on the damage parameter identification of elastic thin plates, the modified potential functional in Eq. (7) should be rewritten in its formulation for thin plates.

Consider an isotropic elastic plate in Cartesian coordinate system $x_i\,(i = 1, 2, 3, \{x_1, x_2, x_3\} = \{x, y, z\})$, x_3 is the coordinate perpendicular to the mid-surface of the plate, which occupies the region R bounded with the curve Γ, as shown in Figure 1. The deflection of the plate along the x_3 axis is denoted by w. On the clamped part of Γ, denoted by Γ_C, the slope angle along the normal direction $w_{,n}$ is assigned as $w_{,n} = \bar{w}_{,n}$; while on Γ_C and the simply supported part Γ_S, the deflection is prescribed as $w = \bar{w}$. Besides, normal moment M_n and Kirchhoff shear force V_n are assigned as $M_n = \bar{M}_n$, $V_n = \bar{V}_n$ on the free part Γ_F complementary to $\Gamma_C \cup \Gamma_S$ (i.e. $\Gamma_C \cup \Gamma_S \cup \Gamma_F = \Gamma$), while $M_n = \bar{M}_n$ on Γ_S. Hence, the modified potential energy functional is given as

$$\begin{aligned}&\Pi_{\mathrm{mp}}\left(w,\, M_n|_{\Gamma_C},\, V_n|_{\Gamma_C \cup \Gamma_S}\right) \\ &= \int_R \frac{D}{2} \left[(1-\mu) w_{,\alpha\beta} w_{,\alpha\beta} + \mu (\nabla^2 w)^2\right] \mathrm{d}A - \int_R \bar{q} w \mathrm{d}A \\ &\quad - \int_{\Gamma_F} \bar{V}_n w \mathrm{d}\Gamma - \int_{\Gamma_F \cup \Gamma_S} \bar{M}_n w_{,n} \mathrm{d}\Gamma \\ &\quad - \int_{\Gamma_C \cup \Gamma_S} V_n (w - \bar{w})\,\mathrm{d}\Gamma - \int_{\Gamma_C} M_n (w_{,n} - \bar{w}_{,n})\,\mathrm{d}\Gamma,\end{aligned} \tag{8}$$

where $\alpha, \beta = 1, 2$ and q is the distributed load per unit area normal to the plate, and the flexural rigidity is defined as

$D = Eh^3/12(1-\mu^2)$ with E, μ being the Young's modulus, Poisson's ratio of the material and h being the thickness of the plate. Moreover, M_n and V_n can be expressed in terms of the deflection w as $M_n = -D\left(\frac{\partial^2 w}{\partial n^2} + \mu\frac{\partial^2 w}{\partial s^2}\right)$ and $V_n = -D\left(\frac{\partial \nabla^2 w}{\partial n} + (1-\mu)\frac{\partial^3 w}{\partial n \partial s^2}\right)$, where n and s indicate the normal and tangent directions at the boundary of the plate.

2.3. The improved version for the purpose of parameter identification

In Eq. (8), the flexural stiffness D can be considered to depend on some structural parameters expressed by a vector $\boldsymbol{\beta}$, i.e. $D = D(\boldsymbol{\beta})$. In the solution to a parameter identification problem, one needs to search for $\boldsymbol{\beta}$ leading to the solution of the boundary-value problem by experimental tests. An improved version of modified potential energy principle is to be introduced, allowing for the response measurements by static tests and devoted to the identification of imperfection parameters $\boldsymbol{\beta} = \{\beta_1, \beta_2, \cdots, \beta_n\}^T$.

Assuming that the deflections at m points within the plate, denoted by $\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_m$, are acquired in the experimental tests, one can suppose that fictitious constraints are set at these m points with prescribed displacements $\tilde{w}_k \, (k = 1, 2, \cdots, m)$ and concentrated reactions $\tilde{R}_k$. The modified potential energy functional thus becomes

$$\begin{aligned}
&\Pi_{\mathrm{mp}}\left(w,\, M_n|_{\Gamma_C},\, V_n|_{\Gamma_C \cup \Gamma_S}, \tilde{R}_k\right) \\
&= \int_R \frac{D}{2}\left[(1-\mu)w_{,\alpha\beta}w_{,\alpha\beta} + \mu(\nabla^2 w)^2\right]\mathrm{d}A - \int_R \bar{q} w \mathrm{d}A \\
&\quad - \int_{\Gamma_F} \bar{V}_n w \mathrm{d}\Gamma - \int_{\Gamma_F \cup \Gamma_S} \bar{M}_n w_{,n} \mathrm{d}\Gamma \\
&\quad - \int_{\Gamma_C \cup \Gamma_S} V_n\,(w - \bar{w})\,\mathrm{d}\Gamma - \int_{\Gamma_C} M_n\,(w_{,n} - \bar{w}_{,n})\,\mathrm{d}\Gamma \\
&\quad - \sum_{k=1}^{m} \tilde{R}_k (w_k - \tilde{w}_k)\,,
\end{aligned} \tag{9}$$

in which the structural response measurements by real load tests and the corresponding fictitious reactions are included.

However, there exist no external constraints actually at the points where the experimental measurements are taken, so one should have $\tilde{R}_k = 0 \, (k = 1, 2, \cdots, m)$ in the solution to the problem, which are given by the stationary condition of the functional in Eq. (9) according to the improved version of modified potential energy principle, on the condition that the experimental measurements $\tilde{w}_k$ are exact.

Therefore, to identify the structural parameter vector $\boldsymbol{\beta}$, the expressions of fictitious concentrated reactions about these parameters, $\tilde{R}_k(\boldsymbol{\beta}) \, (k = 1, 2, \cdots, m)$, can be obtained via the stationary condition of the functional, i.e.

$$\partial_{w,\, M_n|_{\Gamma_C},\, V_n|_{\Gamma_C \cup \Gamma_S}, \tilde{R}_k} \Pi_{\mathrm{mp}}\left(w,\, M_n|_{\Gamma_C},\, V_n|_{\Gamma_C \cup \Gamma_S}, \tilde{R}_k\right) = 0\,. \tag{10}$$

According to zero fictitious reactions, the conditions for identification of $\boldsymbol{\beta}$ are given as

$$\tilde{R}_k(\boldsymbol{\beta}) = 0 \quad (k = 1, 2, \cdots, m)\,, \tag{11}$$

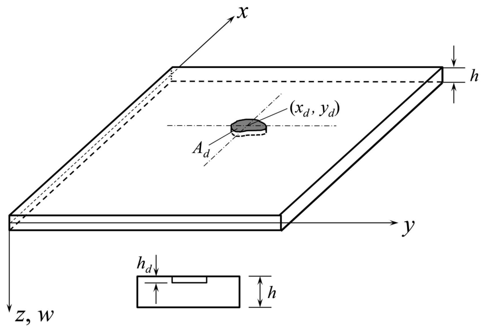

Figure 2: A plate with a notch-type defect

providing the expression of parameter vector $\boldsymbol{\beta}$ as functions of the measurements $\tilde{w}_k$.

In the practical calculation, several independent variables are adopted to discretize the problem described by the improved version of modified potential energy functional. Usually, the numbers of experimental measurements and structural parameters to be identified are the same, i.e. $m = n$, so that the identification problem is solvable and the condition for uniqueness can be satisfied.

This proposed identification procedure is applicable to various kinds of inverse problems about parameter identification of imperfections in thin plates. Three example problems of thin plates are presented in the following sections, showing the details of the procedure.

3. Application 1: Identification of the notch parameter of a damaged plate

In this section, notch-type defects in thin plates are under consideration. Described as localized reduction in the plate thickness, notch-type damage is analyzed according to the configuration shown in Figure 2.

Based on the Kirchhoff's theory for thin plates, the displacement components of points in a plate are given in the following form:

$$u_1 := u = -z\frac{\partial w}{\partial x},\ u_2 := v = -z\frac{\partial w}{\partial y},\ u_3 = w. \tag{12}$$

Then the elastic strain energy of a damaged plate with a notch is represented as

$$\begin{aligned}
\Pi_e &= \int_R \int_{-h/2}^{h(1-\mathbf{1}_{R_d}(x,y)\times 2h_d/h)/2} \frac{1}{2} D_{ijkl} u_{i,j}\, u_{k,l}\ \mathrm{d}z\mathrm{d}A \\
&= \int_R \frac{1}{2} D_d \left[(1-\mu)w_{,\alpha\beta}w_{,\alpha\beta} + \mu(\nabla^2 w)^2\right]\mathrm{d}A - \int_R \bar{q} w \mathrm{d}A,
\end{aligned} \tag{13}$$

in which $\mathbf{1}_{R_d}(x, y)$ denotes the membership function to R_d, i.e.

$$\mathbf{1}_{R_d}(x, y) = \begin{cases} 1, & (x, y) \in R_d; \\ 0, & (x, y) \notin R_d, \end{cases} \tag{14}$$

and D_d denotes the distribution of flexible rigidity, expressed as

$$D_d(x,y) = \frac{D_0}{2}\left[1+\left(1-\mathbf{1}_{R_d}(x,y)\frac{2h_d}{h}\right)^3\right], \tag{15}$$

with $D_0 = Eh^3/12(1-\mu^2)$ being the uniformly distributed rigidity in the undamaged region, h_d being the depth of the notch, and R_d being the damaged region.

Suppose the notch-type defect is small, i.e. $h_d/h \ll 1$, the distributed stiffness D_d in (15) can be approximated as

$$D_d(x,y) \simeq D_0\left(1-\mathbf{1}_{R_d}(x,y)\frac{3h_d}{h}\right) = D_0(1-\varepsilon\mathbf{1}_{R_d}(x,y)), \tag{16}$$

where $\varepsilon = 3h_d/h$ is called the notch parameter in this section.

Remark 1: The area of the damaged region, denoted by A_d, is assumed to be much smaller than the whole area A of the plate, i.e. $A_d/A \ll 1$. Let $F(w,_{\alpha\beta}(x,y))$ denote the expression in the bracket in (13), which is closely related to the strain energy density, let the diameter of region R_d be defined as $\mathrm{diam}(R_d) := \max_{(x,y),(x_1,y_1)\in R_d}[(x-x_1)^2+(y-y_1)^2]^{\frac{1}{2}}$, and assume that the second-order derivatives of F with respect to x and y exist in R_d, and $\|\nabla F\||_{(x_d,y_d)} \cdot \mathrm{diam}(R_d)/|F|_{(x_d,y_d)} \ll 1$. Then,

$$\int_R \mathbf{1}_{R_d}(x,y)F(w,_{\alpha\beta}(x,y))\,\mathrm{d}A \approx A_d \cdot F|_{(x_d,y_d)}, \tag{17}$$

based on the mean value theorems.

Due to the assumption that ε is a small parameter corresponding to a small notch depth h_d, the load response of a damaged plate can be considered as perturbation from that of the corresponding undamaged plate. Thus the analysis of the plate with a notch-type defect can be performed via the perturbation method (see Gopalakrishnan, 2011).

Based on the first-order perturbed formulations of the displacement w and the Kirchhoff shear V_n on $\Gamma_C \cup \Gamma_S$ and normal moment M_n on Γ_C (assuming that the linearized Taylor's series expansions of them exist), i.e.

$$w = w_0 + \varepsilon w_1,\; V_n = V_{n0} + \varepsilon V_{n1},\; M_n = M_{n0} + \varepsilon M_{n1}, \tag{18}$$

the stationary condition of the improved version of modified potential functional (9), i.e. $\delta\Pi_{\mathrm{mp}} = 0$, can be split into the following set of equations by collecting coefficients of the same power of ε and considering the independence between variations of different variables:

$$\begin{aligned}&\int_R D_0[(1-\mu)w_0,_{\alpha\beta}\,\delta w,_{\alpha\beta} + \mu(\nabla^2 w_0)(\nabla^2\delta w)]\,\mathrm{d}A \\ &- \int_R \bar{q}\delta w\,\mathrm{d}A - \int_{\Gamma_F}\bar{V}_n\delta w\,\mathrm{d}\Gamma - \int_{\Gamma_F\cup\Gamma_S}\bar{M}_n\delta w,_n\,\mathrm{d}\Gamma \\ &- \int_{\Gamma_C\cup\Gamma_S} V_{n0}\delta w\,\mathrm{d}\Gamma - \int_{\Gamma_C} M_{n0}\delta w,_n\,\mathrm{d}\Gamma - \sum_{k=0}^{m}\tilde{R}_k\delta w_k = 0\,,\end{aligned} \tag{19}$$

$$\begin{aligned}&\int_R D_0[(1-\mu)w_1,_{\alpha\beta}\,\delta w,_{\alpha\beta} + \mu(\nabla^2 w_1)(\nabla^2\delta w)]\,\mathrm{d}A \\ &+ \int_R D_1[(1-\mu)w_0,_{\alpha\beta}\,\delta w,_{\alpha\beta} + \mu(\nabla^2 w_0)(\nabla^2\delta w)]\,\mathrm{d}A \\ &- \int_{\Gamma_C\cup\Gamma_S} V_{n1}\delta w\,\mathrm{d}\Gamma - \int_{\Gamma_C} M_{n1}\delta w,_n\,\mathrm{d}\Gamma = 0\,,\end{aligned} \tag{20}$$

$$\begin{aligned}&\int_{\Gamma_C\cup\Gamma_S}\delta V_n(w-\bar{w})\,\mathrm{d}\Gamma + \int_{\Gamma_C}\delta M_n(w,_n - \bar{w},_n)\,\mathrm{d}\Gamma \\ &\qquad + \sum_{k=0}^{m}\delta\tilde{R}_k(w_k-\tilde{w}_k) = 0\,.\end{aligned} \tag{21}$$

As an example, let's consider a simply supported square plate defined over $[0,a]\times[0,a]$, with a single notch-type defect at $(x_d,y_d) = (a/2,a/2)$ which covers an area of A_d within the plate. A uniformly distributed load applied on the surface of the plate is $\bar{q}(x,y) = p$. In order to identify the parameter ε for the notch, a static measurement is conducted at $(\tilde{x},\tilde{y})$, obtaining the experimental value $\tilde{w}$ of the deflection here.

Thus, from Eq. (19) one has the analytical expression of w_0 (see Timoshenko and Woinowsky-Krieger, 1970) in the following form:

$$w_0(x,y) = p\varphi_p(x,y) + \tilde{R}\varphi_R(x,y), \tag{22}$$

where φ_p and φ_R are expressed as:

$$\begin{aligned}\varphi_p(x,y) &= \sum_{m,n=1,3,5,\cdots}^{\infty}\frac{16a^4}{\pi^6 D_0}\frac{1}{(m^2+n^2)^2 mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,, \\ \varphi_R(x,y) &= \sum_{m,n=1,2,3,\cdots}^{\infty}\frac{4a^2}{\pi^4 D_0}\frac{1}{(m^2+n^2)^2}\sin\frac{m\pi\tilde{x}}{a}\sin\frac{n\pi\tilde{y}}{a} \\ &\qquad\times\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,.\end{aligned} \tag{23}$$

Remark 2:

According to Weierstrass' M-test for uniform convergence of series (see Zorich, 2004), the double trigonometric series in the expressions of $\varphi_p(x,y)$, $\varphi_R(x,y)$ and up to the second order derivatives of $\varphi_p(x,y)$ are all absolutely and uniformly convergent due to the fact that $|\sin\alpha| \le 1$ for $\forall\alpha\in\mathbb{R}$.

$\varphi_R(x,y)$ is the influence function under the concentrated load $\tilde{R}$, and is the limitation of the influence function under a uniform patch load of resultant force $\tilde{R}$ applied on an area of $u\times v$ that is centered at $(\tilde{x},\tilde{y})$ when $u\to 0$ and $v\to 0$ (see Ventsel and Krauthammer, 2001). Up to the second order derivatives of the influence function under the patch load can be expressed by absolutely and uniformly convergent double trigonometric series, which ensures the availability to represent the derivatives of $\varphi_R(x,y)$ as the sum of corresponding derivatives of the terms in the double trigonometric series of $\varphi_R(x,y)$.

Moreover, w_1 can be obtained by substituting Eq. (22) into Eq. (20):

$$w_1(x,y) = p\psi_p(x,y) + \tilde{R}\psi_R(x,y), \tag{24}$$

in which ψ_p is given by:

$$\begin{aligned}\psi_p(x,y) =&\frac{4A_d}{\pi^2}\left(\varphi_{p,xx}\left(\frac{a}{2},\frac{a}{2}\right)+\mu\varphi_{p,yy}\left(\frac{a}{2},\frac{a}{2}\right)\right)\times\\ &\sum_{m,n=1,3,5,\cdots}^{\infty}\frac{m^2(-1)^{\frac{m+n}{2}-1}}{(m^2+n^2)^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\\ &+\frac{4A_d}{\pi^2}\left(\varphi_{p,yy}\left(\frac{a}{2},\frac{a}{2}\right)+\mu\varphi_{p,xx}\left(\frac{a}{2},\frac{a}{2}\right)\right)\times\\ &\sum_{m,n=1,3,5,\cdots}^{\infty}\frac{n^2(-1)^{\frac{m+n}{2}-1}}{(m^2+n^2)^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\\ &+\frac{8A_d}{\pi^2}(1-\mu)\varphi_{p,xy}\left(\frac{a}{2},\frac{a}{2}\right)\times\\ &\sum_{m,n=2,4,6,\cdots}^{\infty}\frac{mn(-1)^{\frac{m+n}{2}-1}}{(m^2+n^2)^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a},\end{aligned} \tag{25}$$

and ψ_R can be similarly given by replacing all the φ_p in Eq. (25) by φ_R. Note that $\varphi_{p,xy}\left(\frac{a}{2},\frac{a}{2}\right)=0$ and

$$\begin{aligned}&\varphi_{p,xx}\left(\frac{a}{2},\frac{a}{2}\right)=\varphi_{p,yy}\left(\frac{a}{2},\frac{a}{2}\right)=\\ &\frac{8a^2}{\pi^4D_0}\sum_{m,n=1,3,5,\cdots}^{\infty}\frac{(-1)^{\frac{m+n}{2}}}{(m^2+n^2)mn}=-0.448516\frac{8a^2}{\pi^4D_0},\end{aligned} \tag{26}$$

ψ_p can be further simplified into

$$\begin{aligned}\psi_p(x,y)=&0.448516(1+\mu)\frac{32A_da^2}{\pi^6D_0}\times\\ &\sum_{m,n=1,3,5,\cdots}^{\infty}\frac{(-1)^{\frac{m+n}{2}}}{m^2+n^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}.\end{aligned} \tag{27}$$

Then, Eq. (21) gives the following identity:

$$p\left(\varphi_p(\tilde{x},\tilde{y})+\varepsilon\psi_p(\tilde{x},\tilde{y})\right)+\tilde{R}\left(\varphi_R(\tilde{x},\tilde{y})+\varepsilon\psi_R(\tilde{x},\tilde{y})\right)=\tilde{w}, \tag{28}$$

which leads to the explicit expression of $\tilde{R}$

$$\tilde{R}=\left[\tilde{w}-p\left(\varphi_p(\tilde{x},\tilde{y})+\varepsilon\psi_p(\tilde{x},\tilde{y})\right)\right]/\left(\varphi_R(\tilde{x},\tilde{y})+\varepsilon\psi_R(\tilde{x},\tilde{y})\right). \tag{29}$$

According to the energy theorem, the external work by a single concentrated transverse load upon the corresponding displacement caused by the load is positive, implying that $\varphi_R(\tilde{x},\tilde{y})>0$. Thus the denominator of the right-hand side in Eq. (29) does not vanish for sufficiently small ε.

Furthermore, it can be concluded from Eq. (27) that $\psi_p(x,y)$ is the solution of the following boundary-value problem:

$$\begin{aligned}\nabla^2\psi_p(x,y)=0.448516(1+\mu)\frac{8A_da^2}{\pi^4D_0}\delta\left(x-\frac{a}{2},y-\frac{a}{2}\right)\\ \text{in } R=(0,a)\times(0,a),\quad \psi_p=0\quad\text{on }\partial R.\end{aligned} \tag{30}$$

Thus, one can always choose a point $(\tilde{x},\tilde{y})\in R$ which satisfies $0\neq\psi_p(\tilde{x},\tilde{y})<\infty$. Then, the parameter ε can be identified by setting the fictitious reaction $\tilde{R}$ to be zero:

$$\varepsilon=\left(\tilde{w}/p-\varphi_p(\tilde{x},\tilde{y})\right)/\psi_p(\tilde{x},\tilde{y}). \tag{31}$$

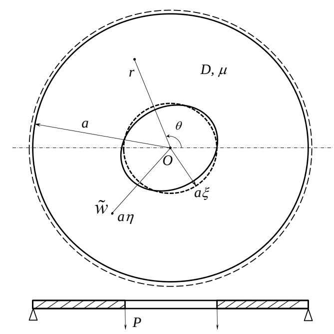

Figure 3: A simply supported circular plate with an imperfect circular hole, uniformly loaded along the inner edge

Remark 3: The location of the static measurement should not coincide exactly with that of the damage, i.e. $(\tilde{x},\tilde{y})\neq(x_d,y_d)=(a/2,a/2)$. There are two reasons: one is that the distributions of $\varphi_{R,xx}$ and $\varphi_{R,yy}$ are singular at $(\tilde{x},\tilde{y})$, which would lead to infinite terms in the expression of ψ_R if $(\tilde{x},\tilde{y})=(a/2,a/2)$; the other reason is that the solution to Eq. (30) is in direct proportion to the Green's function of operator ∇^2 with homogenous Dirichlet boundary conditions, indicating that ψ_p is singular at $(a/2,a/2)$.

4. Application 2: Identification of the equivalent radius of an imperfect circular hole in a circular plate

As shown in Figure 3, a simply supported circular plate, with the outer radius a and a slightly ovalized concentric hole, is under consideration in this section. The plate is loaded by shear force uniformly distributed along the inner edge, whose resultant force has the magnitude of P. Moreover, the uniform flexural rigidity of this plate is denoted by D and the Poisson's ratio is μ.

According to Saint Venant's principle (see Timoshenko and Goodier, 1951), if the hole is considered as a perfect circular one with the equivalent radius $a\xi$, and the shear force P is uniformly distributed along the perfect circular inner edge, there will be little influence on the load response at sufficiently large distances from the hole. Since the distinction between the slightly oval hole and the equivalent standard circle is small, a difference is only made in a very limited part of the place, i.e. the region near the hole.

The coefficient of equivalent radius of the hole, denoted by ξ, can be identified via static measurements. Due to the slight ovalization of the circular hole, it is reasonable to assume that $0<\xi<1$, and the load effect is almost axisymmetric. Thus, the deflection can be approximately represented in the following axisymmetric form according to Timoshenko and Woinowsky-Krieger (1970):

$$w(r)=-c_1+c_1\frac{r^2}{a^2}+c_2\ln\frac{r}{a}+c_3\frac{r^2}{a^2}\ln\frac{r}{a}, \tag{32}$$

in which c_1,c_2,c_3 are constants to be determined. The constraint $w=0$ at $r=a$ is considered in Eq. (32).

Remark 4: Consider a concentric circle with radius $a\zeta$ satisfying that $\xi \le \zeta < 1$. The imperfect hole is completely surrounded by this circle. After minimization of factor ζ, a smallest circle with radius $a\zeta_0$ will be obtained. Since the imperfection of the hole is slight, the values of ξ and ζ_0 are close to each other. Neglecting the eccentricity of resultant force of the distributed shear along inner edge of the imperfect hole, the tractions exposed at the radius $a\zeta_0$ in both cases, the one with imperfect hole and that with the circular one of radius $a\xi$, will be statically equivalent. This point shows the usability of the Saint Venant's principle.

For the purpose of identification, deflection measurement $\tilde{w}$ is obtained at a point with the polar radius $a\eta$. Now, the modified potential energy functional is expressed as

$$
\begin{aligned}
&\Pi_{\mathrm{mp}}(c_1, c_2, c_3, \tilde{R}) \\
&= \int_0^{2\pi} \int_{a\xi}^{a} \frac{D}{2} \left[\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right)^2 - \frac{2(1-\mu)}{r} \frac{\partial w}{\partial r} \frac{\partial^2 w}{\partial r^2} \right] r \mathrm{d}r \mathrm{d}\varphi \\
&\quad - \oint_{r=a\xi} \frac{P}{2\pi a\xi} w \mathrm{d}l - \tilde{R} \left(w(a\eta) - \tilde{w} \right) ,
\end{aligned}
\tag{33}
$$

where the fictitious concentrated reaction $\tilde{R}$ acts as a Lagrange multiplier to introduce the constraint $w(a\eta) = \tilde{w}$.

Remark 5: In fact the existence of concentrated reaction $\tilde{R}$ violates the axisymmetry of the problem to be solved, so the displacement under both the linearly distributed load P and the concentrated load $\tilde{R}$ is actually in a different form from that in Eq. (32). However, the fictitious reaction $\tilde{R}$ will eventually be set to zero. Thus, it is proper to perform calculation and identification under the displacement mode given in Eq. (32).

Subsequently, the stationary condition $\partial_{c_1, c_2, c_3, \tilde{R}} \Pi_{\mathrm{mp}} = 0$ is applied again, leading to a system of algebraic equations. Then, the following explicit expression of the fictitious concentrated reaction $\tilde{R}$ is obtained

$$
\tilde{R} = 2P(1-\mu^2) q(\xi) \frac{g(\xi,\eta) - 8\pi D\tilde{w}/Pa^2}{b(\xi,\eta)} , \tag{34}
$$

in which

$$
q(\xi) = (1-\xi^2) \left[(1-\xi^2)^2 - 4\xi^2 \ln^2 \xi \right] , \tag{35}
$$

$$
g(\xi,\eta) = (1-\eta^2) \left[\frac{3+\mu}{2(1+\mu)} - \frac{\xi^2}{1-\xi^2} \ln \xi \right] + \eta^2 \ln \eta + \frac{2\xi^2}{1-\xi^2} \frac{1+\mu}{1-\mu} \ln \xi \ln \eta , \tag{36}
$$

$$
\begin{aligned}
b(\xi,\eta) = &\left(\eta^2 - 1\right)^2 [\left(\mu^2 + 2\mu - 3\right)\left(\xi^2 - 1\right)^2 + 4\xi^2 \left(\mu^2 \xi^2 - 2\mu - \xi^2 + 2\right) \ln^2 \xi \\
&+ 4\left(\mu^2 - 1\right)\left(\xi^2 - 1\right) \xi^2 \ln \xi] - 4\left(\eta^2 - 1\right)\left(\mu^2 - 1\right) \ln \eta \left(\xi^2 + 2\xi^2 \ln \xi - 1\right) \\
&\left[\eta^2 \left(\xi^2 - 1\right) + 2\xi^2 \ln \xi\right] + 4 \ln^2 \eta \{ -\left(\xi^2 - 1\right)^2 \left[2(\mu+1)\xi^2 - \eta^4 \left(\mu^2 - 1\right)\right] \\
&+ 4\eta^2 \left(\mu^2 - 1\right)\left(\xi^2 - 1\right) \xi^2 \ln \xi + 4(\mu+1)^2 \xi^4 \ln^2 \xi \} .
\end{aligned}
\tag{37}
$$

Noticing $-1 < \mu < 0.5$ and $0 < \xi, \eta < 1$, one has

$$
q(\xi) > 0 , \tag{38}
$$

as shown in Figure 4, and

$$
b(\xi,\eta) < 0 , \tag{39}
$$

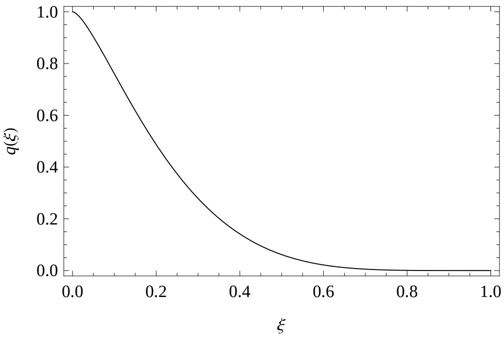

Figure 4: $q(\xi)$ with $0 < \xi < 1$

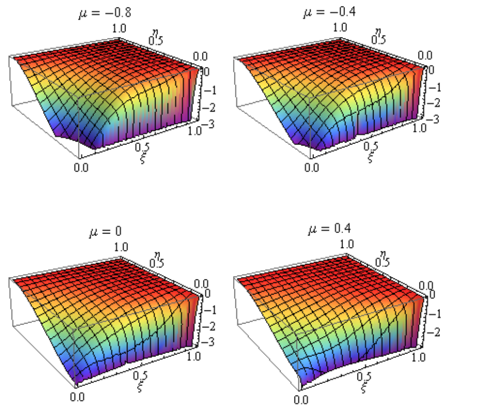

Figure 5: $b(\xi,\eta)$ with $0 < \xi, \eta < 1$ and $\mu = \{-0.8, -0.4, 0, 0.4\}$

as shown in Figure 5 with μ set to be -0.8, -0.4, 0 and 0.4.

Thus, by setting $\tilde{R}(\xi) = 0$, one can identify the coefficient ξ of the equivalent hole radius via solving the following transcendental equation:

$$
g(\xi,\eta) = \frac{8\pi D\tilde{w}}{Pa^2} , \tag{40}
$$

where η and $\tilde{w}$ are acquired from the static measurement. Figure 6 shows the values of g versus ξ with η taken as 0.2, 0.4, 0.6, 0.8 and μ set to be -0.8, -0.4, 0 and 0.4, where the range of ξ is $0 < \xi < \eta$ according to the fact that the measurement is conducted within the plate.

5. Application 3: Identification of the parameters of variable stiffness

Let us consider a cantilever square plate defined over $[0, a] \times [0, a]$ (see Figure 7), clamped along the edge $y = 0$ and subject to a linearly varying line load px/a along the edge $y = a$.

As an example of the cases with variable stiffness, the flexural rigidity of the cantilever plate is a linear function of x and y expressed in the form

$$
D = D_0 \left(1 + \alpha \frac{x}{a} + \beta \frac{y}{a} \right) , \tag{41}
$$

in which α and β are the two dimensionless constants to be identified, which describe the varying stiffness. The Poission ratio of the plate is taken as $\mu = 0.2$.

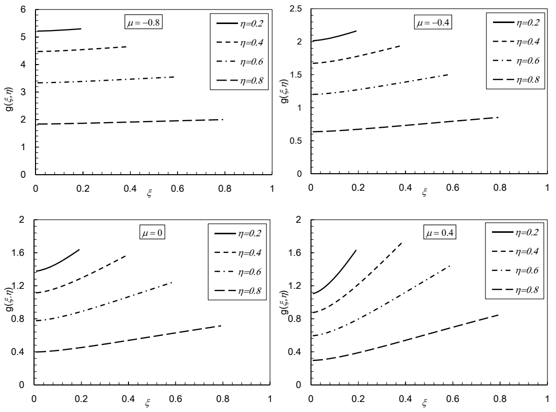

Figure 6: Values of g versus ξ with $\eta = \{0.2, 0.4, 0.6, 0.8\}$ and $\mu = \{-0.8, -0.4, 0, 0.4\}$

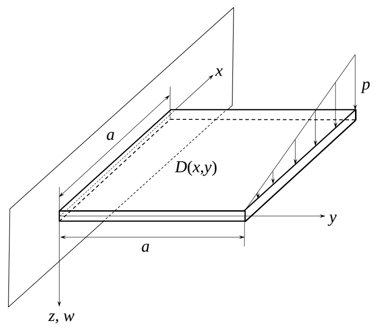

Figure 7: A cantilever square plate under linearly varying line load

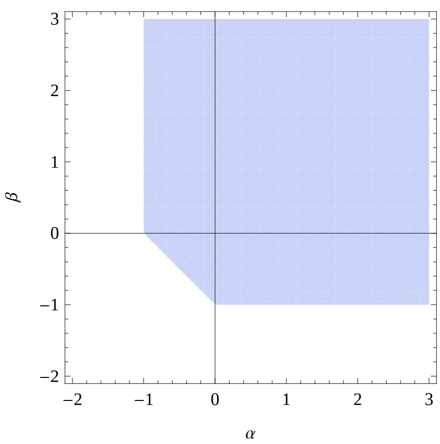

Figure 8: Range of (α, β) to ensure $D(x, y) > 0$ with $(x, y) \in [0, a] \times [0, a]$

For the identification purpose, two experimental tests are taken at specific points $\tilde{x}_1 = \tilde{y}_1 = a/4$ and $\tilde{x}_2 = \tilde{y}_2 = 3a/4$, acquiring the deflection measurements $\tilde{w}_1$ and $\tilde{w}_2$ at the two points. For the field of deflection and resultant tractions on the clamped edge, the following expressions are assumed:

$$w = c_0 + c_1\frac{x}{a} + c_2\frac{y}{a} + c_3\frac{x^2}{a^2} + c_4\frac{xy}{a^2} + c_5\frac{y^2}{a^2} + c_6\frac{x^2y}{a^3} + c_7\frac{xy^2}{a^3}\,,$$
$$V_n|_{y=0} = f_0 + f_1\frac{x}{a}\,, \qquad M_n|_{y=0} = b_0 + b_1\frac{x}{a}\,, \tag{42}$$

in which c_i, f_i, b_i are unknown constants.

Hence the improved version of modified potential energy principle is written as

$$\begin{aligned}&\Pi_{\text{mp}}(c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, b_0, b_1, f_0, f_1, \tilde{R}_1, \tilde{R}_2) = \\ &\int_0^a\int_0^a \frac{1}{2}D_0\left(1+\alpha\frac{x}{a}+\beta\frac{y}{a}\right)\left\{(\nabla^2 w)^2 - 2(1-\mu)\left(w_{,xx}\,w_{,yy} - w_{,xy}^2\right)\right\}\mathrm{d}x\mathrm{d}y \\ &-\int_0^a p\frac{x}{a}\,w|_{y=a}\mathrm{d}x - \int_0^a\left(f_0+f_1\frac{x}{a}\right)w|_{y=0}\mathrm{d}x - \int_0^a\left(b_0+b_1\frac{x}{a}\right)w_{,y}\Big|_{y=0}\mathrm{d}x \\ &-\tilde{R}_1\left(w(\tilde{x}_1,\tilde{y}_1)-\tilde{w}_1\right) - \tilde{R}_2\left(w(\tilde{x}_2,\tilde{y}_2)-\tilde{w}_2\right).\end{aligned} \tag{43}$$

Applying the stationary condition $\partial_{c_0,c_1,c_2,c_3,c_4,c_5,c_6,c_7,b_0,b_1,f_0,f_1,\tilde{R}_1,\tilde{R}_2}\Pi_{\text{mp}} = 0$ yields the expressions of $\tilde{R}_1(\alpha,\beta)$ and $\tilde{R}_1(\alpha,\beta)$ as follows

$$\tilde{R}_1(\alpha,\beta) = \frac{G_1(\alpha,\beta)}{B(\alpha,\beta)}\,, \quad \tilde{R}_2(\alpha,\beta) = \frac{G_2(\alpha,\beta)}{B(\alpha,\beta)}\,, \tag{44}$$

where

$$\begin{aligned}B(\alpha,\beta) &= 62071\alpha^2 + 132\alpha(1022\beta + 1829) \\ &\quad + 3(21219\beta^2 + 83056\beta + 76318)\,,\end{aligned} \tag{45}$$

$$\begin{aligned}G_1(\alpha,\beta) &= 19.2D_0\{\tilde{w}_1[2229745\alpha^3 + \alpha^2(8940551\beta + 15517730) \\ &\quad + \alpha(10428175\beta^2 + 38436204\beta + 33833736) + 3560537\beta^3 \\ &\quad + 21443762\beta^2 + 39770760\beta + 23434512] - \tilde{w}_2[241147\alpha^3 \\ &\quad + \alpha^2(950645\beta + 1621454) + \alpha(1091653\beta^2 + 3971460\beta + 3430440) \\ &\quad + 370859\beta^3 + 2202350\beta^2 + 4003752\beta + 2304240]\}/a^2 \\ &\quad - 0.5pa[119997\alpha^2 + 4\alpha(52348\beta + 107235) \\ &\quad + 76923\beta^2 + 376260\beta + 387432)]\,,\end{aligned} \tag{46}$$

$$\begin{aligned}G_2(\alpha,\beta) &= 19.2D_0\{\tilde{w}_1[241147\alpha^3 + \alpha^2(950645\beta + 1621454) \\ &\quad + \alpha(1091653\beta^2 + 3971460\beta + 3430440) + 370859\beta^3 \\ &\quad + 2202350\beta^2 + 4003752\beta + 2304240] - \tilde{w}_2[45097\alpha^3 \\ &\quad + \alpha^2(160703\beta + 279458) + \alpha(173335\beta^2 + 635340\beta + 556776) \\ &\quad + 57185\beta^3 + 335186\beta^2 + 611688\beta + 356496]\}/a^2 \\ &\quad - 0.5pa[94785\alpha^2 + 4\alpha(52874\beta + 92961) \\ &\quad + 101871\beta^2 + 390252\beta + 353736]\,.\end{aligned} \tag{47}$$

To ensure that $D(x, y) > 0$ with $\forall(x, y) \in [0, a] \times [0, a]$, the range of (α,β) in the parameter space is constrained by

$$(\alpha,\beta) \in \mathfrak{B} = \{(\alpha,\beta) : \alpha > -1,\ \beta > -1,\ \alpha+\beta > -1\}\,, \tag{48}$$

which is schematically illustrated in Figure 8. Then the corresponding range of $B(\alpha,\beta)$ is given by

$$B(\alpha,\beta) > 43444.7\,, \quad (\alpha,\beta) \in \mathfrak{B}. \tag{49}$$

Above all, by nullification of the fictitious reactions $\tilde{R}_1$ and $\tilde{R}_2$, the parameters α and β can be identified as the solution to the following equations:

$$G_1(\alpha,\beta) = 0\,, \quad G_2(\alpha,\beta) = 0\,. \tag{50}$$

Remark 6: The deformation values $\tilde{w}_1$ and $\tilde{w}_2$ are experimental data, thus there is possibility that the parameter pair (α,β) obtained from Eq. (50) does not belong to the set $\mathfrak{B}$ in Eq. (48). This circumstance is mainly due to the poor accuracy of the assumed forms of w, V_n and M_n in Eq. (42). More accurate forms with higher-order terms can be counted on to obtain satisfactory solution of the problem.

6. Conclusions

Based on the improved version of the modified potential energy principle allowing for static experimental measurements, parameters describing the imperfections in thin plates can be identified via the stationary condition of the functional and nullification of the fictitious reactions. In the identification problems for thin plates, approximate analytical solutions can be obtained by the proposed procedure, which offers a general framework for various inverse problems. Three typical identification problems are solved by the proposed procedure, showing the effectiveness of the procedure. Due to the variational basis, analog of the proposed framework can be introduced into the hybrid finite elements, and a computational approach to the imperfection identification problem can then be established. However, the 'gap' between the numerical treatment for a certain case and the analytical solution to the problem is the main difficulty to be overcome.

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