Received: 16 February 2017 | Revised: 26 November 2019 | Accepted: 4 December 2019

DOI: 10.1002/jgt.22536

ARTICLE

WILEY

Vertex-disjoint properly edge-colored cycles in edge-colored complete graphs

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Funding information
China Scholarship Council, Grant/Award Number: 201506290097; National Natural Science Foundation of China, Grant/Award Number: 11271300; CSC, Grant/Award Number: 201506290097; NSFC, Grant/Award Numbers: 11671320, 11901459; The Fundamental Research Funds for the Central Universities of China, Grant/Award Numbers: 31020180QD124, 3102019GHJD003

Abstract

It is conjectured that every edge-colored complete graph G on n vertices satisfying $\Delta^{mon}(G) \leq n - 3k + 1$ contains k vertex-disjoint properly edge-colored cycles. We confirm this conjecture for $k = 2$, prove several additional weaker results for general k, and we establish structural properties of possible minimum counterexamples to the conjecture. We also reveal a close relationship between properly edge-colored cycles in edge-colored complete graphs and directed cycles in multipartite tournaments. Using this relationship and our results on edge-colored complete graphs, we obtain several partial solutions to a conjecture on disjoint cycles in directed graphs due to Bermond and Thomassen.

KEYWORDS

complete graph, edge-colored graph, multipartite tournament, properly edge-colored cycle, vertex-disjoint cycles

JEL CLASSIFICATION

05C15; 05C20; 05C38

1 | INTRODUCTION

All graphs considered in this paper are finite and simple. For terminology and notation not defined here, we refer the reader to [6].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a nonempty subset S of $V(G)$, let $G[S]$ denote the subgraph of G induced by S, and let $G - S$ denote the subgraph of G induced by $V(G) \setminus S$. When $S = \{v\}$, we use $G - v$ instead of $G - \{v\}$. An *edge-coloring* of G is a mapping col: $E(G) \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. A graph G with an edge-coloring is called an *edge-colored* graph or simply a *colored* graph. We say that a colored graph G is a *properly colored* graph or simply a *PC* graph if each pair of adjacent edges (ie, edges incident with one common vertex) in G are assigned distinct colors. A PC graph G is called a *rainbow* graph if all the edges of G are assigned different colors.

Let G be a colored graph. For a vertex $v \in V(G)$, the *color degree* of v, denoted by $d_G^c(v)$ is the number of different colors appearing on the edges incident with v. For an edge $e \in E(G)$, let $col_G(e)$ denote the color of e. For a subgraph H of G, let $col_G(H)$ denote the set of colors appearing on $E(H)$. For two vertex-disjoint subgraphs F and H of G, let $col_G(F, H)$ denote the set of colors appearing on the edges between F and H. If $V(F) = \{v\}$, then we often write $col_G(v, H)$ instead of $col_G(F, H)$. For two disjoint nonempty subsets S and T of $V(G)$, we use $col_G(S, T)$ as shorthand for $col_G(G[S], G[T])$. When there is no ambiguity, we often write $d^c(v)$ for $d_G^c(v)$, $col(e)$ for $col_G(e)$, $col(H)$ for $col_G(H)$, $col(F, H)$ for $col_G(F, H)$, $col(v, H)$ for $col_G(v, H)$, and $col(S, T)$ for $col_G(S, T)$. For each color $i \in col(G)$, we use G^i to denote the spanning subgraph of G induced by the edges of color i in G. Let $\Delta^{\text{mon}}(G)$ denote the *maximum monochromatic degree* of G, that is, $\Delta^{\text{mon}}(G) = \max\{\Delta(G^i): i \in col(G)\}$. Throughout this paper, we use C_3 and C_4 to denote cycles of length 3 and 4, respectively. We also frequently use the term triangle for a C_3.

Research problems related to PC cycles and rainbow cycles in colored graphs have attracted a lot of attention during the past decades, not only because of the many challenging open problems and conjectures and interesting results, but also because of the relation to problems on cycles in digraphs. Here we adopt the terminology of [4] and use digraph for directed graph and cycle for a directed cycle in a digraph. We refer the reader to [10] and Chapter 16 in [4] for surveys on rainbow cycles and PC cycles, respectively. We also recommend proof techniques in [11,12], constructions in [7], and Chapter 16 in [4] for a glance of the deep relation between edge-colored graphs and digraphs. Here, we are mainly interested in the existence of vertex-disjoint PC cycles (called disjoint PC cycles for simplicity in the sequel) in colored complete graphs. Our first easy observation implies that having k disjoint PC cycles is equivalent to having k disjoint PC cycles of length at most 4 in colored complete graphs.

Observation 1. Let G be a colored K_n and let C be a PC cycle in G. Then, for each vertex $v \in V(C)$, there exists a PC cycle C' of length at most 4 in G containing v and with $V(C') \subseteq V(C)$.

Proof. Let $G' = G[V(C)]$. It is equivalent to prove that each vertex in G' is contained in a PC cycle of length at most 4.

By contradiction. Suppose that $C' = v_1v_2...v_rv_1$ is a PC cycle of minimal length in G' containing v_1, and assume that $r \geq 5$. Assume without loss of generality that $col(v_1v_2) = 1$ and $col(v_1v_r) = 2$. If $col(v_3v_4) = 1$, then, using the chord v_1v_4, either $v_1v_2...v_4v_1$ or $v_1v_rv_{r-1}...v_4v_1$ is a PC cycle in G' containing v_1 and shorter than C', a contradiction. So, $col(v_3v_4) \neq 1$. Similarly, using the chord v_1v_3, we can show that $col(v_3v_4) \neq 2$. Without loss of generality, assume that $col(v_3v_4) = 3$. If $col(v_1v_3) = 3$, then $v_1v_2v_3v_1$ is a PC cycle in G' containing v_1 and shorter than C', a contradiction. So, $col(v_1v_3) \neq 3$. Similarly, we can show that $col(v_1v_4) \neq 3$ by considering the cycle $v_1v_rv_{r-1}...v_4v_1$. Noting that C' is a shortest PC cycle containing v_1 in G', the cycles $v_1v_2v_3v_4v_1$ and $v_1v_rv_{r-1}v_4v_3v_1$ are not PC cycles. So we have $col(v_1v_4) = 1$ and $col(v_1v_3) = 2$. This implies that $v_1v_3v_4v_1$ is a PC C_3, our final "21"contradiction. □

Before turning to disjoint PC cycles, we first recall the following fundamental result on the existence of PC cycles in colored graphs.

Theorem 2 (Grossman and Häggkvist [9] and Yeo [13]). *Let G be a colored graph containing no PC cycles. Then G contains a vertex v such that no component of $G - v$ is joined to v with edges of more than one color.*

Combining Theorem 2 and Observation 1 for colored complete graphs, we immediately obtain a maximum monochromatic degree condition for the existence of a PC C_3 or C_4.

Observation 3. Let G be a colored K_n. If $\Delta^{\text{mon}}(G) \leq n - 2$, then G contains a PC cycle of length at most 4.

The observation follows from the simple fact that in a complete graph G on at least two vertices, for every vertex v of G, $G - v$ consists of only one component.

Using Observations 1 and 3, and repeatedly deleting the vertices of PC cycles of length at most 4, it is easy to obtain the following sufficient condition for the existence of k disjoint PC cycles.

Observation 4. Let G be a colored K_n. If $\Delta^{\text{mon}}(G) \leq n - 4k + 2$, then G contains k disjoint PC cycles of length at most 4.

Motivated by the above observations, our aim is to find a (best possible) positive function $g(k)$ (only depending on k) such that every colored complete graph G with $\Delta^{\text{mon}}(G) \leq n - g(k)$ contains k disjoint PC cycles. We conjecture that the following holds.

Conjecture 5. *Let G be a colored K_n. If $\Delta^{\text{mon}}(G) \leq n - 3k + 1$, then G contains k disjoint PC cycles of length at most 4.*

We confirm this conjecture for the case that $k = 2$.

Theorem 6. *Let G be a colored K_n. If $\Delta^{\text{mon}}(G) \leq n - 5$, then G contains two disjoint PC cycles of length at most 4.*

We postpone the proof of Theorem 6 to Section 4. In Section 2, we give several additional results related to Conjecture 5. The proofs of these results can be found in Sections 5 and 6.

We continue here with some examples to discuss the tightness of the bounds in Conjecture 5. First of all, note that for a PC complete graph G on $n = 3k - 1$ vertices, $\Delta^{\text{mon}}(G) = 1 \leq n - 3k + 2$, whereas it cannot have k disjoint PC cycles. This implies that the upper bound on $\Delta^{\text{mon}}(G)$ in Conjecture 5 would be best possible, in a weak sense: for each k, this provides only one example. When $k = 2$, except for a PC K_5, Example 1 also implies the tightness of the bound $n - 5$.

Example 1. Let G be a colored complete graph with $V(G) = \{v_1, v_2, ..., v_6\}$. Decompose $G - v_1$ into two Hamilton cycles and color them by α and β, respectively. Color the edge v_1v_i with c_i for $i \in [2, 6]$. Then $\Delta^{\text{mon}}(G) = 6 - 4 = 2$, but G cannot contain two disjoint PC cycles.

For $k = 2$, we have no other examples to support the tightness of the bound in Conjecture 5. For $k \geq 3$, we cannot find other examples to support the bound in Conjecture 5 except for a PC K_{3k-1}. It is not unlikely that the bound in Conjecture 5 can be improved for large n. The next example shows that for arbitrarily large n, we can construct a colored complete graph G on n vertices with $\Delta^{\text{mon}}(G) = n - \frac{3}{2}k$, but containing at most $k - 1$ disjoint PC cycles.

Example 2. Given integers $k \geq 2$ (k is even) and $n \geq \frac{9}{2}k - 3$, let $G_1 \cong K_{3k-3}$ with $V(G_1) = \{v_i: 1 \leq i \leq 3k - 3\}$. Decompose G_1 into $\frac{3}{2}k - 2$ Hamilton cycles. Arbitrarily choose a direction for each Hamilton cycle. For all $i, j \in [1, 3k - 3]$ and $i \neq j$, color the edge $v_i v_j$ with a color c_j if and only if v_j is the successor of v_i in one of the Hamilton cycles. Let $G_2 \cong K_{n-3k+3}$ with $V(G_2) = \{u_i: 1 \leq i \leq n - 3k + 3\}$ and $col(G_2) = \{\alpha\}$. Let G be an edge-colored K_n constructed by joining G_1 and G_2 such that $col(v_i u_{n-3k+3}) = \beta$ for all $i \in [1, 3k - 3]$ and $col(v_i u_j) = c_i$ for all i, j with $1 \leq i \leq 3k - 3$ and $1 \leq j \leq n - 3k + 2$. Then $\Delta^{\text{mon}}(G) = n - \frac{3}{2}k$, but G contains at most $k - 1$ disjoint PC cycles.

Since cycles in edge-colored graphs are closely related to cycles in digraphs, here we naturally think of disjoint cycles in tournaments. In fact, Bang-Jensen et al [3] proved that for every $\epsilon > 0$, when k is large enough, every tournament with minimum out-degree at least $\left(\frac{3}{2} + \epsilon\right)k$ contains k disjoint cycles. And the linear factor $\frac{3}{2}$ is better than the factor 2 that was conjectured by Bermond and Thomassen [5] in digraphs. In light of the close relationship between PC cycles in colored complete graphs and cycles in multipartite tournaments that we are going to discuss later, this could serve as supporting evidence that maybe the bound in Conjecture 5 can be improved when n is sufficiently large.

2 | ADDITIONAL RESULTS RELATED TO CONJECTURE 5

For the case that $k \geq 3$, our first additional result implies the existence of k disjoint PC cycles if there exists a vertex in G that is not contained in any PC cycle.

Theorem 7. *Let G be a colored K_n. If $\Delta^{\text{mon}}(G) \leq n - 3k + 1$, then either G contains k disjoint PC cycles of length at most 4, or each vertex of G is contained in a PC C_3 or C_4.*

Under some specific conditions, the bound for $\Delta^{\text{mon}}(G)$ in Theorem 7 can be improved to $n - 2k$.

Theorem 8. *Let G be a colored K_n satisfying $\Delta^{\text{mon}}(G) \leq n - 2k$. If G has a Gallai partition,[1] then either G contains k disjoint PC cycles of length at most 4, or each vertex of G is contained in a PC C_3 or C_4.*

With the same upper bound on $\Delta^{\text{mon}}(G)$, we can prove the following closely related result.

[1] See Definition 2 and Lemma 16 in Section 3 for more information on Gallai partitions.

Theorem 9. *Let G be a colored K_n satisfying $\Delta^{mon}(G) \le n - 2k$. Then either G contains k disjoint PC cycles of length at most 4, or each vertex of G with color degree at most 3 is contained in a PC C_3 or C_4.*

Using some transformation techniques that we are going to specify later, it turns out that the results of Theorems 7 to 9 are closely related to a problem on disjoint cycles in multipartite tournaments. In 1981, Bermond and Thomassen posed the following conjecture on the existence of k disjoint cycles in digraphs. Here, $\delta^+(D)$ denotes the minimum out-degree of the digraph D.

Conjecture 10 (Bermond and Thomassen [5]). *Let D be a digraph. If $\delta^+(D) \ge 2k - 1$, then D contains k disjoint cycles.*

This conjecture has been confirmed for tournaments [3] and for bipartite tournaments [1] (for other progress on this conjecture, we refer to the introductory sections in [1,3]). We can state an equivalent of Conjecture 10 in terms of disjoint PC cycles when D is a multipartite tournament, using Theorem 11. We first define two classes of graphs. For convenience, we say a vertex in a colored graph is *bad* if there is no PC cycle passing through this vertex.

Definition 1. Let $k \ge 1$, $\ell \ge 2$, $f(k) \ge 2k - 1$, and $I \subseteq \{a: a \ge 3, a \in \mathbb{N}\}$. Define

$$\mathcal{MT}(I, f(k), \ell) = \left\{ MT \,\middle|\, \begin{array}{l} MT \text{ is an } \ell\text{-partite tournament with } \delta^+(MT) \ge f(k) \\ \text{and containing no cycle of length } i \in I \end{array} \right\}$$

and

$$\mathcal{G}(I, f(k), \ell) = \left\{ G \,\middle|\, \begin{array}{l} G \text{ is a colored complete graph containing a bad vertex } v \\ \text{with } d^c_G(v) \le \ell, \text{ satisfying } \Delta^{mon}(G) \le |V(G)| - f(k) - 1 \\ \text{and containing no PC cycle of length } i \in I \end{array} \right\}.$$

Theorem 11. *$\mathcal{MT}(I, f(k), \ell) \ne \emptyset$ if and only if $\mathcal{G}(I, f(k), \ell) \ne \emptyset$. Furthermore, every digraph in $\mathcal{MT}(I, f(k), \ell)$ has k disjoint cycles if and only if every graph in $\mathcal{G}(I, f(k), \ell)$ has k disjoint PC cycles.*

Theorems 7 to 9, respectively, imply that every graph in $\bigcup_{\ell \ge 2} \mathcal{G}(\emptyset, 3k - 2, \ell)$, $\bigcup_{\ell \ge 2} \mathcal{G}(\{3\}, 2k - 1, \ell)$, and $\bigcup_{\ell = 2,3} \mathcal{G}(\emptyset, 2k - 1, \ell)$ has k disjoint PC cycles. By directly using Theorem 11, we immediately obtain the following three corollaries.[2]

Corollary 12. *Let D be a multipartite tournament. If $\delta^+(D) \ge 3k - 2$, then D contains k disjoint cycles.*

[2]During the process of writing this paper, we became aware of the fact that Bai and Li [2] have obtained Corollaries 12 to 14 in 2015 using techniques in digraphs. This study is still in progress.

Corollary 13. *Let D be a multipartite tournament containing no triangles. If $\delta^+(D) \geq 2k-1$, then D contains k disjoint cycles.*

Corollary 14. *Let D be a 2-partite or 3-partite tournament. If $\delta^+(D) \geq 2k-1$, then D contains k disjoint cycles.*

Finally, we present some structural properties of a possible minimum counterexample (G, k) to Conjecture 5. Here, a minimum counterexample (G, k) satisfies that k is as small as possible, and subject to this, $|V(G)|$ is as small as possible, and subject to this, $|col(G)|$ is as small as possible.

Theorem 15. *Let (G, k) be a minimum counterexample to Conjecture 5. Then the following statements hold.*
(a) $k \geq 3$;
(b) $|col(G)| = 2$ or 3;
(c) G contains no rainbow triangle;
(d) G contains no monochromatic edge-cut;
(e) for each set $S \subseteq V(G)$ with $|S| \leq k-1$ and each vertex $v \in V(G)\backslash S$, there exists a PC C_4 in $G-S$ containing v.

All the omitted proofs of the above results (except for the corollaries) can be found in Sections 4 to 6, but we start with some additional terminology and auxiliary lemmas in Section 3.

3 | TERMINOLOGY AND LEMMAS

Let G be a colored complete graph. A *partition* of G is a family of subsets $V_1, V_2, \ldots, V_q$ of $V(G)$ satisfying $\bigcup_{1\leq i\leq q} V_i = V(G)$ and $V_i \cap V_j = \emptyset$ for $1 \leq i < j \leq q$ (In the proofs, we sometimes allow that V_i is an empty set.) For each partition $V_1, V_2, \ldots, V_q$ of G and a vertex $x \in V(G)$, we use V_x to denote the unique set V_i $(1 \leq i \leq q)$ containing x. The following type of partition plays a key role in some of the proofs that follow. In this definition, the sets U_i are supposed to be nonempty.

Definition 2 (Gallai partition). Let G be a colored K_n. A partition $U_1, U_2, \ldots, U_q$ of G is called a *Gallai partition* if $q \geq 2$, $|\bigcup_{1\leq i<j\leq q} col(U_i, U_j)| \leq 2$ and $|col(U_i, U_j)| = 1$ for $1 \leq i < j \leq q$.

The following result shows that Gallai partitions exist in colored complete graphs without a PC C_3.

Lemma 16 (Gallai [8]). *Let G be a colored K_n with $n \geq 2$. If G contains no rainbow triangles, then G has a Gallai partition.*

Let G be a colored K_n. Clearly, each monochromatic edge-cut in G corresponds to a special Gallai partition of G. Actually, in the presence of a monochromatic edge-cut, the degree condition $\Delta^{mon}(G) \leq n-2k$ easily implies the existence of k disjoint PC cycles of length 4. In fact, we prove a slightly stronger result.

Lemma 17. *Let G be a colored K_n satisfying $\Delta^{\text{mon}}(G) \le n - 2k$. If G contains a monochromatic edge-cut, then for each vertex $v \in V(G)$, there is a set of k disjoint PC cycles of length 4 in G containing v.*

Proof. Supposing that G contains a monochromatic edge-cut, let V_1, V_2 be a partition of G with only one color (say *red*) appearing on the edges between V_1 and V_2, and let $v \in V_1$. The condition $\Delta^{\text{mon}}(G) \le n - 2k$ implies that $k \ge 1$ and that each vertex of V_1 is joined to at least $2k - 1$ vertices of V_1 with edges of colors distinct from *red*. Using induction on k, it is straightforward to see that this implies that there are k disjoint edges $x_1x'_1, x_2x'_2, \ldots, x_kx'_k$ in $G[V_1]$ with colors distinct from *red* and with $v = x_1$. By symmetry, there are also k disjoint edges $y_1y'_1, y_2y'_2, \ldots, y_ky'_k$ in $G[V_2]$ with colors distinct from *red*. Thus, $\{x_ix'_iy'_iy_ix_i\colon 1 \le i \le k\}$ is a set of k disjoint PC cycles of length 4 containing v. □

Let $k = 1$ and let G be a colored complete graph. Lemma 17 implies that if G contains a bad vertex and satisfies $\delta^c(G) \ge 2$, then G has no monochromatic edge-cut. In fact, a new partition result can be obtained in the presence of a bad vertex. Before delivering the result, we first give a description of this partition.

Definition 3 (v-partition). Let G be a colored K_n and let v be a vertex of G. We say $V_0, V_1, \ldots, V_p$ is a *v-partition* of G if $V_0, V_1, \ldots, V_p$ is a partition of G such that the following statements hold for some distinct colors $c_1, c_2, \ldots, c_p \in col(G)$.

(a) $2 \le p \le d^c(v)$, $v \in V_0$, and $|V_i| \ge 1$ for $0 \le i \le p$;
(b) $col(V_0, V_i) = \{c_i\}$ for $1 \le i \le p$;
(c) $col(V_i, V_j) \subseteq \{c_i, c_j\}$ for $1 \le i < j \le p$;
(d) $col(G[V_i]) \subseteq \{c_i\}$ (ie, $col(G[V_i]) = \{c_i\}$ when $|V_i| \ge 2$) for $1 \le i \le p$.

Lemma 18. *Let G be a colored K_n ($n \ge 2$) with $\delta^c(G) \ge 2$. If G contains a bad vertex v_0, then G admits a v_0-partition $V_0, V_1, \ldots, V_p$.*

Proof. Since v_0 is a bad vertex and $\delta^c(G) \ge 2$, by Lemma 17, G contains no monochromatic edge-cut. Let $N^c(v_0) = \{c_1, c_2, \ldots, c_{d^c(v_0)}\}$ and let $S_i = \{v \in V(G)\colon col(vv_0) = c_i\}$ for $1 \le i \le d^c(v_0)$. Since v_0 is not contained in any PC triangle, we have $col(S_i, S_j) \subseteq \{c_i, c_j\}$. Thus, the sets $\{v_0\}, S_1, S_2, \ldots, S_{d^c(v_0)}$ form a partition of G satisfying (a), (b), and (c) of Definition 3.

Let $V_0, V_1, \ldots, V_p$ be a partition of G satisfying (a), (b), and (c), and with $|V_0|$ as large as possible. We will prove that this partition also satisfies (d). Suppose it does not. Then, without loss of generality, assume that there exist vertices $x, y \in V_1$ such that $col(xy) \ne c_1$. For each vertex $v_j \in V_j$ ($2 \le j \le p$), on the one hand, by (c), we have $col(xv_j) \in \{c_1, c_j\}$; on the other hand, since $v_0yxv_jv_0$ is not a PC cycle, we have $col(xv_j) \in \{col(xy), c_j\}$. This forces that $col(xv_j) = c_j$. Similarly, we can prove that $col(yv_j) = c_j$. This implies that $col(x, V_j) = col(y, V_j) = \{c_j\}$ for all j with $2 \le j \le p$. Now define

$$T_1 = \{x \in V_1\colon \exists\, y \in V_1 \text{ s.t. } col(xy) \ne c_1\}.$$

Then, $col(T_1, V_j) = \{c_j\}$ for $2 \le j \le p$. Let $V'_1 = V_1 \setminus T_1$ and $V'_0 = V_0 \cup T_1$. Then, $V'_0, V'_1, V_2, \ldots, V_p$ is a new partition of G. If $V'_1 \ne \emptyset$, then by the definition of T_1, we have $col(V'_1, T_1) = \{c_1\}$ and $col(G[V_1']) \subseteq \{c_1\}$. Thus, $V'_0, V'_1, V_2, \ldots, V_p$ is a partition of G

satisfying (a), (b), and (c) with $|V'_0| > |V_0|$. This contradicts the choice of $V_0, V_1, ..., V_p$. If $V'_1 = \emptyset$, then $p \geq 3$ (otherwise, $p = 2$, $T_1 = V_1$ and the edges between V'_0 and V_2 form a monochromatic edge-cut of G, a contradiction). Thus, $V'_0, V_2, ..., V_p$ is a partition of G satisfying (a), (b), and (c) with $|V'_0| > |V_0|$, a contradiction. So (d) also "13"holds. □

In the cases that a colored complete graph G does and does not admit a Gallai partition, respectively, the next two lemmas give extra structural results beyond the v-partition.

Lemma 19. *Let G be a colored K_n ($n \geq 2$) with $\delta^c(G) \geq 2$. If G has a Gallai partition and contains a bad vertex v_0, then G has a v_0-partition V_0, V_1, V_2 with $v_0 \in V_0$ and a PC cycle $xyzwx$ with $x, z \in V_1$ and $y, w \in V_2$.*

Proof. By Lemma 17, G contains no monochromatic edge-cut. Choose $U_0, U_1, ..., U_q$ as a Gallai partition of G with $v_0 \in U_0$. By Definition 2, there are at most two colors (say *red* and *blue*) between the sets $\{U_i\colon\ 0 \leq i \leq q\}$ and exactly one color (say *red* or *blue*) between each pair of distinct sets U_i and U_j ($0 \leq i < j \leq q$). Since there is no monochromatic edge-cut in G, for each $i \in [0, q]$, there exist $s, t \in [0, q]$ with $i \neq s$, $i \neq t$, and $s \neq t$ such that $col(U_i, U_s) = \{red\}$ and $col(U_i, U_t) = \{blue\}$. Let $G' = G - U_0 \setminus \{v_0\}$ (it is possible that $U_0 = \{v_0\}$ and $G' = G$). Thus, $d^c_{G'}(v_0) = 2$ and v_0 is also bad in G' with $\delta^c(G') \geq 2$. Hence, by Lemma 18, G' has a v_0-partition $V'_0, V'_1, ..., V'_p$ with $2 \leq p \leq d^c_{G'}(v_0)$ and $v_0 \in V'_0$. Recall that $d^c_{G'}(v_0) = 2$. We have $p = 2$. Let $V_0 = V'_0 \cup U_0, V_1 = V'_1$, and $V_2 = V'_2$. By the property of Gallai partitions, for each vertex $u \in \cup_{i=1}^{q} U_i$, we have $col(U_0, u) = \{col(v_0 u)\}$. Note that $V_1 \cup V_2 \subseteq \cup_{i=1}^{q} U_i$. We can easily see that V_0, V_1, V_2 is a v_0-partition of G. In this case, we are left to prove the existence of a specific PC C_4. If $G[V_1 \cup V_2]$ contains a PC cycle, then it must be a PC C_4 (because of Observation 1 and $|col(G[V_1 \cup V_2])| = 2$) with two vertices in V_1 (say x, z) and two vertices in V_2 (say y, w). Since $col(xz) \neq col(yw)$, this C_4 must be $xyzwx$. So, it is sufficient to prove that $G[V_1 \cup V_2]$ contains a PC cycle. Suppose the contrary. Then, by Theorem 2, we may assume that there exists a vertex $x \in V_1$ joined to all the other vertices in $V_1 \cup V_2$ with edges of the same color. If this unique color is c_1, then $d^c_G(x) = 1$, a contradiction; otherwise, this unique color is c_2. This forces that $V_1 = \{x\}$. Then, the edges between $V_0 \cup \{x\}$ and V_2 form a monochromatic edge-cut of G, again a "21"contradiction. □

Lemma 20. *Let G be a colored K_n ($n \geq 2$) and let v_0 be a bad vertex of G. If G has no Gallai partition and has a v_0-partition $V_0, V_1, ..., V_p$, then G contains a rainbow triangle $xyzx$ such that V_x, V_y, and V_z are three distinct sets with $V_0 \notin \{V_x, V_y, V_z\}$.*

Proof. Supposing that G contains no Gallai partition, by Lemma 16, G must contain a rainbow triangle. Let $V_0, V_1, ..., V_p$ be a v_0-partition of G. Let $G' = G - V_0 \setminus \{v_0\}$ (it is possible that $V_0 = \{v_0\}$ and $G' = G$). Then, $\{v_0\}, V_1, ..., V_p$ is a v_0-partition of G'. Now we are left to prove the existence of a specific rainbow triangle in G'. Assume that G' contains a rainbow triangle $xyzx$. Since v_0 is bad and $|col(G[V_i \cup V_j])| \leq 2$ for $1 \leq i < j \leq p$, the vertices x, y, z must come from different sets in $\{V_1, ..., V_p\}$. So it is sufficient to prove that G' contains a rainbow triangle. Suppose the contrary. Since $|V(G')| \geq p + 1 \geq 3$, by Lemma 16, G' has a Gallai partition $U_0, U_1, U_2, ..., U_q$ ($q \geq 1$). Without loss of generality, assume that $v_0 \in U_0$. On the one hand, by the property of v_0-partitions, for each pair of vertices $v \in V_0$ and $u \in \cup_{i=1}^{p} V_i$, we have $col(vu) = col(v_0 u)$; on the other hand, by the property of Gallai partitions, for each pair of vertices $v \in U_0$ and $u \in \cup_{i=1}^{q} U_i$, we have $col(vu) = col(v_0 u)$. Note that $\cup_{i=1}^{q} U_i \subseteq \cup_{i=1}^{p} V_i$. So for each pair of vertices $v \in U_0 \cup V_0$ and $u \in \cup_{i=1}^{p} U_i$, we have $col(vu) = col(v_0 u)$. Thus, $U_0 \cup V_0, U_1, U_2, ..., U_q$ is a Gallai partition of G, which contradicts that G has no Gallai partition. □

We now have all the necessary ingredients to prove our main theorem and the additional results. In Section 4, we present our proof of Theorem 6.

4 | PROOF OF THEOREM 6

Proof. By contradiction. Let G be a colored complete graph satisfying $\Delta^{\text{mon}}(G) \leq n - 5$ but containing no two disjoint PC cycles. Since $\Delta^{\text{mon}}(G) \geq 1$, we have $n \geq 6$. If G contains a rainbow triangle $uvwu$, then by deleting vertices u, v, and w from G, we obtain a graph G' with $|V(G')| = n - 3 \geq 3$ and $\Delta^{\text{mon}}(G') \leq n - 5 = (n - 3) - 2$. So, by Observation 3, G' contains a PC cycle C of length 3 or 4. Thus, the cycles $uvwu$ and C form two disjoint PC cycles of length at most 4, a contradiction. Hence G contains no rainbow triangles, and, by Lemma 16, G has a Gallai partition. Let $U_1, U_2, ..., U_q$ be a Gallai partition of G with q as small as possible. By Lemma 17, G contains no monochromatic edge-cut. So, we have $q \geq 4$. Assume that the two colors appearing between U_i and U_j $(1 \leq i < j \leq q)$ are *red* and *blue*.

We proceed by proving six claims.

Claim 1. There exists a PC C_4 in G with vertices from distinct sets of $U_1, U_2, ..., U_q$.

Proof. Construct an auxiliary colored complete graph H with $V(H) = \{x_1, x_2, ..., x_q\}$ and for $1 \leq i < j \leq q$, color the edge $x_i x_j$ with the color that appears on the edges between U_i and U_j. Since G contains no monochromatic edge-cut, $col(H) = \{red, blue\}$ and $col(x, H - x) = \{red, blue\}$ for each vertex $x \in V(H)$. Thus, by Observation 3, H contains a PC C_4, which corresponds to a PC C_4 in G with vertices from different sets of $U_1, U_2, ..., U_q$. □

Without loss of generality, assume that the PC C_4 in Claim 1 is $C^* = v_1 v_2 v_3 v_4 v_1$ with $v_i \in U_i$ for $1 \leq i \leq 4$, and satisfying that $col(v_1 v_2) = col(v_3 v_4) = red$ and $col(v_2 v_3) = col(v_1 v_4) = blue$ (see Figure 1). Let $G' = G - V(C^*)$. Since $|V(G')| \geq n - 4 \geq 2$, G' is nonempty. If $\Delta^{\text{mon}}(G') \leq |V(G')| - 2$, then, by Observation 3, G' contains a PC C_4. Combining this PC cycle with $v_1 v_2 v_3 v_4 v_1$, we get two disjoint PC cycles of length 4, a contradiction. So, there exists a vertex $v \in V(G')$ with $d_{G'}^{c}(v) = 1$ (see Figure 1). Define

$$S_1 = \left\{v \in V(G') : d_{G'}^{c}(v) = 1\right\}.$$

Clearly, there is only one color in $col(S_1, G' - S_1) \cup col(G[S_1])$. We assert that this color must be *red* or *blue*. Suppose not. Then, by the definition of Gallai partition, $V(G')$ is a subset of U_i for some i with $1 \leq i \leq q$. Let v_j be a vertex in U_j for some j with $1 \leq j \leq q$ and $j \neq i$. Then, the unique color in $col(U_i, U_j)$ appears at least $|V(G')| = n - 4$ times at v_j. This contradicts that

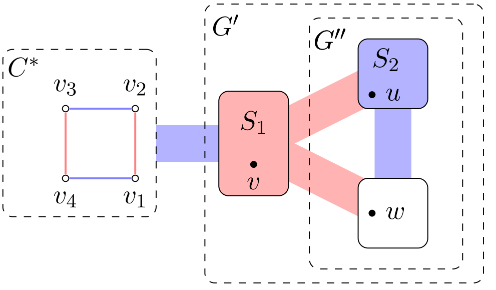

FIGURE 1 The coloring of G [Color figure can be viewed at wileyonlinelibrary.com]

$\Delta^{\text{mon}}(G) \le n-5$. Now, without loss of generality, assume that $col(S_1, G' - S_1) \cup col(G[S_1]) = \{red\}$.

Claim 2. For each vertex $v \in S_1$, $v \notin \cup_{1\le i\le 4} U_i$, and $col(v, C^*) = \{blue\}$.

Proof. Let v be an arbitrary vertex of S_1. By the assumption that $col(S_1, G' - S_1) \cup col(G[S_1]) = \{red\}$, we have $col(v, G' - v) = \{red\}$. Then, $col(vv_i) \ne red$ for $1 \le i \le 4$ (otherwise, the color red would appear more than $n-5$ times at v, a contradiction). We further assert that $v \notin \cup_{1\le i\le 4} U_i$. Suppose this is not the case. Then, $v \in U_i$ for some $1 \le i \le 4$, and $col(vv_{i+1}) = red$ or $col(vv_{i-1}) = red$ (where the indices are taken module 4), a contradiction. This implies that $v \notin \cup_{1\le i\le 4} U_i$ and $col(v, C^*) = \{blue\}$ (see Figure 1). □

Claim 3. $U_i = \{v_i\}$ for $1 \le i \le 4$.

Proof. Claim 2 shows that $S_1 \cap \{\cup_{1\le i\le 4} U_i\} = \emptyset$. We are left to prove that $u \notin \cup_{1\le i\le 4} U_i$ for each vertex $u \in V(G')\backslash S_1$. Note that for each vertex $u \in V(G')\backslash S_1$ and any vertex $v \in S_1$, we have $col(vu) = red \notin col(v, C^*)$. This implies that $u \notin \cup_{1\le i\le 4} U_i$. □

Now, for convenience, we call a cycle *special* if it is a PC cycle and its vertices come from different sets of $U_1, U_2, \ldots, U_q$. We say a vertex $z \in V(G)\backslash V(C)$ is a *companion vertex* of a special cycle C if z is joined to C with color *blue* (*red*) and joined to other vertices with color *red* (*blue*). By Claims 2 and 3, we know that

(a) each special cycle of length 4 in G has a companion vertex;
(b) if a vertex $v_i \in U_i$ is contained in a special cycle, then $U_i = \{v_i\}$.

Claim 4. $|S_1| \le 3$, and for each vertex v_i ($1 \le i \le 4$), there exist two distinct vertices $x_i, y_i \in V(G)\backslash(V(C^*) \cup S_1)$ such that $col(v_i x_i) = col(v_i y_i) = red$.

Proof. Suppose that $|S_1| \ge 4$. Let x, y, z, w be four distinct vertices in S_1. Then, xyv_1v_2x and zwv_3v_4z are two disjoint PC cycles, a contradiction. So, we have $|S_1| \le 3$. For each i with $1 \le i \le 4$, by Claim 3 and the definition of Gallai partition, we know that all the edges incident with v_i are colored in *red* or *blue*. Since each color appears at most $n-5$ times at v_i, we know that both *red* and *blue* appear at least 4 times at v_i. Note that there are at most 2 vertices in $V(C^*) \cup S_1$ joined to v_i by an edge with color *red*. Hence, there exist another two vertices $x_i, y_i \in V(G)\backslash(V(C^*) \cup S_1)$ joined to v_i by an edge with color *red*. □

Let $G'' = G' - S_1$. By Claim 4, we have $|V(G'')| \geq 2$. If $\Delta^{\text{mon}}(G'') \leq |V(G'')| - 2$, then, by Observation 3, G'' contains a PC C_4. Combining this cycle with C^*, we obtain two disjoint PC C_4's, a contradiction. Thus, there exists a vertex $u \in V(G'')$ such that $d^c_{G''}(u) = 1$ (see Figure 1). Define

$$S_2 = \left\{u \in V(G''): d^c_{G''}(u) = 1\right\}.$$

Claim 5. For each vertex $u \in S_2, col(u, G'' - u) = \{blue\}, U_u \cap S_1 = \varnothing$, and $col(u, C^*) = \{red\}$.

Proof. Let u be a vertex in S_2. Then $d^c_{G''}(u) = 1$.

Suppose that $col(u, G'' - u) = \{red\}$. Then, $col(u, G' - u) = col(u, G'' - u) \cup col(u, S_1) = \{red\}$. This implies that $u \in S_1$, a contradiction. Suppose that the unique color in $col(u, G'' - u)$ is neither *red* nor *blue*. Then, by Claim 3 and the definition of Gallai partition, we have $V(G'') \subseteq U_j$ for some j with $5 \leq j \leq q$. By Claim 4, there are vertices x_1, x_2, x_3, x_4 in $V(G'')$ such that $col(v_i x_i) = red$ for all i with $1 \leq i \leq 4$. This implies that $col(v_i, U_j) = \{red\}$ for all i with $1 \leq i \leq 4$. Let

$$T_1 = \{v_1, v_2, v_3, v_4\}, \quad T_2 = S_1, \quad \text{and} \quad T_3 = V(G'').$$

It is easy to check that $col(T_1, T_2) = \{blue\}, col(T_2, T_3) = \{red\}$, and $col(T_1, T_3) = \{red\}$. Thus, the edges between $T_1 \cup T_2$ and T_3 form a *red* edge-cut of G, a contradiction. So, we have $col(u, G'' - u) = \{blue\}$.

Suppose that there exists a vertex $v \in S_1$ such that $v \in U_u$. Then, $u \in U_v$ and $col(u, C^*) = col(v, C^*) = \{blue\}$. Thus, $col(u, G - S_1) = col(u, G'' - u) \cup col(u, C^*) = \{blue\}$. This implies that the color *blue* appears at least $n - 1 - |S_1| \geq n - 4$ times at the vertex u (by Claim 4), a contradiction. Thus, we have $U_u \cap S_1 = \varnothing$.

Now we need to prove that $col(uv_i) = \{red\}$ for all i with $1 \leq i \leq 4$. Suppose, to the contrary, that there exists a vertex (say v_1) on C^* such that $col(uv_1) = blue$. Then, choose a vertex $v \in S_1$. Thus, the cycle $C = vuv_1v_2v$ is a special C_4 (see Figure 2). Recall that each special cycle of length 4 has a companion vertex. Let z be a companion vertex of C. For vertices $x \in V(G'') - u$ and $y \in S_1 - v$, we have $col(xu) \neq col(xv)$ and $col(yv_1) \neq col(yv)$. This implies that $z \notin V(G'') \cup S_1$. Thus, z is either v_3 or v_4. If $z = v_3$, then $col(z, C) = \{col(v_3v_2)\} = \{blue\}$. By the definition of z, we know that $col(z, V(G') - u - v) = \{red\}$. Note that $col(z, S_1) = col(v_3, S_1) = \{blue\}$. This forces that $S_1 = \{v\}$. Now, for each vertex $x \in V(G)\backslash\{v_2, v_4, v, u\}$, we have $col(ux) = blue$. The color *blue* appears at least $n - 4$ times at u, a contradiction. So $z \neq v_3$. Similarly, we can prove that $z \neq v_4$. Thus, there is no choice for z, a contradiction. This implies that $col(uv_i) = red$ for all $1 \leq i \leq 4$. □

Claim 6. $V(G'')\backslash S_2 \neq \varnothing$, and there exists a vertex $w \in V(G'')\backslash S_2$ such that $col(w, C^*) = \{red, blue\}$ and $w \notin U_v \cup U_u$ for any vertices $v \in S_1$ and $u \in S_2$.

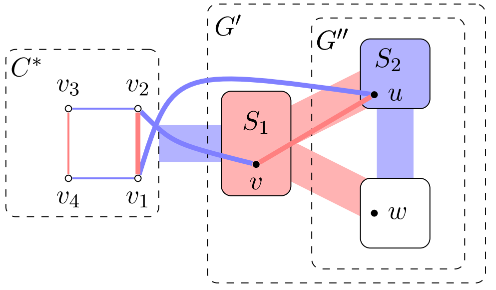

FIGURE 2 $col(uv_1) = blue$ [Color figure can be viewed at wileyonlinelibrary.com]

Proof. If $V(G'')\backslash S_2 = \varnothing$, then the edges between S_2 and $G - S_2$ form a *red* edge-cut of G, a contradiction. So, we have $V(G'')\backslash S_2 \neq \varnothing$. Suppose that for each vertex $w \in V(G'')\backslash S_2$, we have $col(wv_i) = col(wv_j)$ for all $1 \leq i < j \leq 4$. Recall that $col(S_1, C^*) = \{blue\}$ and $col(S_2, C^*) = \{red\}$ (by Claim 5). We have $col(xv_i) = col(xv_j)$ for each vertex $x \in V(G)\backslash V(C^*)$ and $1 \leq i < j \leq 4$. Thus,

$$\{v_1, v_2, v_3, v_4\}, U_5, U_6, \ldots, U_q$$

is also a Gallai partition of G. This contradicts that q is as small as possible. Thus, we can choose a vertex $w \in V(G'')\backslash S_2$ such that $col(w, C^*) = \{red, blue\}$. Since $col(S_1, C^*) = \{blue\}$ and $col(S_2, C^*) = \{red\}$, by the definition of Gallai partition, $w \notin U_v \cup U_u$ for any vertices $v \in S_1$ and $u \in S_2$.

Since $col(w, C^*) = \{red, blue\}$, without loss of generality, assume that $col(wv_1) = red$. Choose vertices $v \in S_1$ and $u \in S_2$. Then, the cycle $C = vuwv_1v$ is a special cycle of length 4 (see Figure 3). Let z be a companion vertex of C. Since $col(vx) \neq col(ux)$ for each vertex x in $G - (S_1 \cup \{u\})$, we have $z \in S_1 - v$. However, for each vertex $y \in S_1 - v$, we have $col(yv_1) = blue$ and $col(yu) = red$. Thus, $z \notin S_1 - v$. So there is no choice for z, a contradiction. This completes the proof of Theorem 6. □

5 | PROOFS OF THEOREMS 7, 8, 9, AND 15

By Observation 1, the existence of k disjoint PC cycles is equivalent to the existence of k disjoint PC C_3's or C_4s. In this section, for convenience, we also use the term *short PC cycle(s)* instead of PC cycle(s) of length at most 4.

Proof of Theorem 7. By contradiction. Let G be a colored K_n. We say (G, k) is a counterexample to Theorem 7 if $\Delta^{mon}(G) \leq n - 3k + 1$, but there are no k disjoint short PC cycles in G and not every vertex of G is contained in a short PC cycle. Let (G, k) be a counterexample to Theorem 7 with k as small as possible. By Observation 3 and Theorem 6, we know that $k \geq 3$. If G contains a rainbow triangle $xyzx$, then let $H = G - \{x, y, z\}$. Then, $\Delta^{mon}(H) \leq \Delta^{mon}(G) = n - 3 - 3(k - 1) + 1$. Hence, by the choice of (G, k), H either contains $k - 1$ disjoint short PC cycles, or each vertex of H is contained in a short PC cycle. This in turn implies that either G contains k disjoint short PC cycles, or each vertex of G is contained in a short PC cycle, a contradiction. Thus, G contains no rainbow triangles and, due to Lemma 16, has a Gallai partition. Note that $\Delta^{mon}(G) \leq n - 3k +$

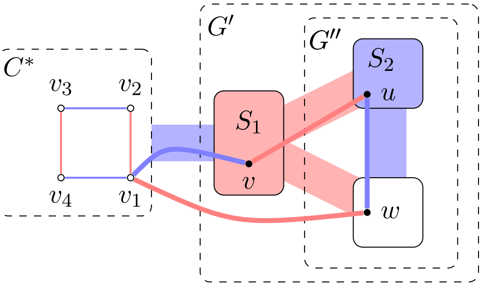

FIGURE 3 $col(wv_1) = red$ [Color figure can be viewed at wileyonlinelibrary.com]

$1 < n - 2k$. By Theorem 8, G either contains k disjoint short PC cycles, or each vertex is contained in a short PC cycle. This completes the proof. □

Proof of Theorem 8. By contradiction. Let (G, k) be a counterexample to Theorem 8 with k as small as possible. Then G contains a bad vertex v_0. By Observation 3, we have $k \geq 2$. By Lemma 17, G contains no monochromatic edge-cut. Since G contains a Gallai partition, by Lemma 19, G admits a v_0-partition V_0, V_1, V_2 (see Figure 4) such that

$$v_0 \in V_0, \quad col(V_0, V_1) = \{c_1\}, \quad col(V_0, V_2) = \{c_2\},$$
$$col(V_1, V_2) \subseteq \{c_1, c_2\}, \quad col(G[V_1]) = \{c_1\}, \quad col(G[V_2]) = \{c_2\}$$

and G contains a PC cycle $xyzwx$ with

$$x, z \in V_1, \quad y, w \in V_2.$$

This implies that

$$|V_1| \geq 2 \quad \text{and} \quad |V_2| \geq 2.$$

Without loss of generality, assume that $col(xy) = col(zw) = c_1$ and $col(xw) = col(zy) = c_2$. Since $d_{G^{c_1}}(x) \leq n - 2k$ and $d_{G^{c_2}}(y) \leq n - 2k$, we have

$$|V_0| + (|V_i| - 1) + 1 \leq n - 2k \quad \text{for} \quad i = 1, 2.$$

Thus,

$$2 \leq |V_i| \leq n - 2k - 1 \quad \text{for} \quad i = 1, 2.$$

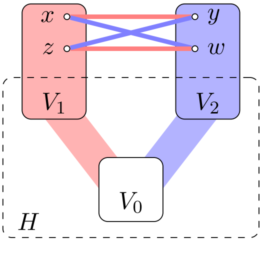

FIGURE 4 G in the proof of Theorem 8 [Color figure can be viewed at wileyonlinelibrary.com]

Let $H = G - \{x, y, z, w\}$. We will show that $\Delta^{mon}(H) \le n - 2k - 2$. For each vertex $v_1 \in V_1 \backslash \{x, z\}$, by the partition, we know that $col(v_1, G - v_1) \subseteq \{c_1, c_2\}$, $d_{H^{c_1}}(v_1) \le d_{G^{c_1}}(v_1) - 2 \le n - 2k - 2$, and $d_{H^{c_2}}(v_1) \le |V_2| - 2 < n - 2k - 2$. Similarly, for each vertex $v_2 \in V_2 \backslash \{y, w\}$, we have $col(v_2, G - v_2) \subseteq \{c_1, c_2\}$, $d_{H^{c_2}}(v_2) \le d_{G^{c_2}}(v_2) - 2 \le n - 2k - 2$, and $d_{H^{c_1}}(v_2) \le |V_1| - 2 < n - 2k - 2$. For each vertex $u \in V_0$, we have $d_{H^{c_1}}(u) \le d_{G^{c_1}}(u) - 2 \le n - 2k - 2$, $d_{H^{c_2}}(u) \le d_{G^{c_2}}(u) - 2 \le n - 2k - 2$, and $d_{H^{c}}(u) \le |V_0| - 1 \le n - 2k - |V_1| - 1 < n - 2k - 2$ for each color $c \in col(G) \backslash \{c_1, c_2\}$. This implies that $\Delta^{mon}(H) \le n - 2k - 2 = |V(H)| - 2(k - 1)$. Note that (G, k) is a counterexample with k as small as possible, and that the vertex v_0 is also bad in H. We know that H contains $k - 1$ disjoint short PC cycles. Together with the PC cycle $xyzwx$, there exist k disjoint short PC cycles in G, a contradiction. This completes the proof of Theorem 8. □

Proof of Theorem 9. By contradiction. Let (G, k) be a counterexample to Theorem 9 with k as small as possible. Then G contains a bad vertex v_0 with $d^c(v_0) \le 3$. By Observation 3, $k \ge 2$. By Theorem 8, G admits no Gallai partition. Furthermore, by Lemmas 18 and 20, G has a v_0-partition $V_0, V_1, \ldots, V_p$ and a rainbow triangle $xyzx$ such that V_x, V_y, V_z are distinct sets with $V_0 \notin \{V_x, V_y, V_z\}$. So we have $3 \le p \le d^c(v_0) \le 3$. This forces $p = 3$ (see Figure 5). Without loss of generality, assume that

$$x \in V_1, \quad y \in V_2, \quad z \in V_3,$$

and

$$col(xy) = c_1, \quad col(yz) = c_2, \quad col(zx) = c_3.$$

Since colors c_1, c_2, and c_3 appear at most $n - 2k$ times at x, y, and z, respectively, we have

$$|V_0| + (|V_i| - 1) + 1 \le n - 2k \quad \text{for } i = 1, 2, 3.$$

Thus

$$1 \le |V_i| \le n - 2k - 1 \quad \text{for } i = 1, 2, 3.$$

Let $H = G - \{x, y, z\}$. We will show that $\Delta^{mon}(H) \le n - 2k - 1$.

For each vertex $v_1 \in V_1 \backslash \{x\}$, by the partition, we know that $col(v_1, G - v_1) \subseteq \{c_1, c_2, c_3\}$, $d_{H^{c_1}}(v_1) \le d_{G^{c_1}}(v_1) - 1 \le n - 2k - 1$, and $d_{H^{c_i}}(v_1) \le |V_i| - 1 < n - 2k - 1$ for $i = 2, 3$.

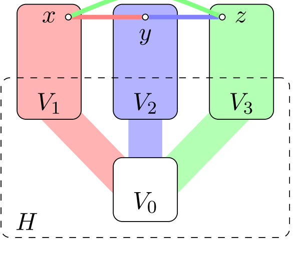

FIGURE 5 G in the proof of Theorem 9 [Color figure can be viewed at wileyonlinelibrary.com]

Similarly, for vertices $v_2 \in V_2 \backslash \{y\}$ and $v_3 \in V_3 \backslash \{z\}$, we have $col(v_2, G - v_2) \cup col(v_3, G - v_3) = \{c_1, c_2, c_3\}$ and $d_{H^{c_i}}(v_j) \leq n - 2k - 1$ for $i = 1, 2, 3$ and $j = 2, 3$. For each vertex $u \in V_0$, we have $d_{H^{c_i}}(u) \leq d_{G^{c_i}}(u) - 1 \leq n - 2k - 1$ for $i = 1, 2, 3$, and $d_{H^c}(u) \leq |V_0| - 1 < n - 2k - 1$ for each color $c \in col(G) \backslash \{c_1, c_2, c_3\}$. This implies that $\Delta^{\mathrm{mon}}(H) \leq n - 2k - 1 = |V(H)| - 2(k - 1)$. By the choice of (G, k), and since v_0 is bad, we conclude that H contains $k - 1$ disjoint short PC cycles. Together with the PC cycle $xyzx$, there exist k disjoint short PC cycles in G, a contradiction. This completes the proof of Theorem 9. □

Proof of Theorem 15. By Observation 3 and Theorem 6, we have $k \geq 3$. If G contains a rainbow triangle $xyzx$, then one easily checks that $G - \{x, y, z\}$ is a smaller counterexample to Conjecture 5, a contradiction. So, G contains no rainbow triangles, and thus has a Gallai partition $U_1, U_2, \ldots, U_q$ by Lemma 16. By Lemma 17, G contains no monochromatic edge-cut. So $q \geq 4$, and we can assume that the two colors appearing between the parts in this Gallai partition are *red* and *blue*. Since $\Delta^{\mathrm{mon}}(G) \leq n - 3k + 1$, by the definition of Gallai partition, we have $|U_i| \leq n - 3k + 1$ for all i with $1 \leq i \leq q$. This implies that $\sum_{c \in col(G), c \neq red, blue} d_{G^c}(v_i) \leq |U_i| - 1 < n - 3k + 1$ for each $v_i \in U_i$ $(1 \leq i \leq q)$. If $|col(G)| \neq 2$ or 3, then $|col(G)| \geq 4$. In this case, let H be the colored graph obtained from G by recoloring all the edges which are neither *red* nor *blue* in G with the color *green*. Clearly, H contains no k disjoint short PC cycles (otherwise, there exist k disjoint short PC cycles in G), and $\Delta^{\mathrm{mon}}(H) \leq |V(H)| - 3k + 1$. So, (H, k) is a counterexample to Conjecture 5 with $|V(H)| = |V(G)|$ and $|col(H)| < |col(G)|$, a contradiction. Hence, we conclude that $|col(G)| = 2$ or 3. This completes the proof of (a), (b), (c), and (d) of Theorem 15.

Finally, we prove (e) of Theorem 15, by contradiction. Suppose to the contrary, that there exists a set $S \subseteq V(G)$ with $|S| \leq k - 1$, and a vertex $v_0 \in V(G) \backslash S$ such that v_0 is a bad vertex in $G - S$. Let $H = G - S$. Then,

$$\begin{aligned}\Delta^{\mathrm{mon}}(H) &\leq \Delta^{\mathrm{mon}}(G) = n - 3k + 1 = (n - |S|) - 2k + (|S| - k + 1) \\ &\leq |V(H)| - 2k.\end{aligned}$$

By Theorem 8 and the fact that v_0 is bad in H, the colored complete graph H contains k disjoint short PC cycles, which are also contained in G, a contradiction. This completes the proof of Theorem 15. □

6 | PROOF OF THEOREM 11

Before delivering the proof of Theorem 11, we first give the following two constructions.

Construction 1. *Let* $MT_1 \in \mathcal{MT}(I, f(k), \ell)$ *with partite sets* $V_1, V_2, \ldots, V_\ell$. *Define a colored complete graph* G_1 *with* $V(G_1) = \bigcup_{1 \leq i \leq \ell} V_i \cup \{v_0\}$, *for* $1 \leq i \leq j \leq \ell, v_i \neq v_j$, $v_i \in V_i, v_j \in V_j, col(v_0 v_i) = c_i$ *and*

$$col(v_i v_j) = \begin{cases} c_j & \text{if } v_i v_j \in A(MT), \\ c_i & \text{otherwise}. \end{cases}$$

Construction 2. *Let $G_2 \in \mathcal{G}(I, f(k), \ell)$ and let v_0 be a bad vertex in G_2 with $d^c(v_0) \leq \ell$. Then G_2 contains no monochromatic edge-cut (otherwise, it is easy to see that every vertex in G_2 is contained in a PC cycle). By Lemma 18, G_2 admits a v_0-partition $V_0, V_1, V_2, ..., V_p$ such that $v_0 \in V_0$ and $p \leq d^c(v_0) \leq \ell$. Then we define a p-partite tournament MT with vertex set $V(MT) = V_1 \cup V_2 \cup \cdots \cup V_p$ and arc set*

$$A(MT) = \{xy\colon V_x \neq V_y,\ col(xy) = col(v_0 y)\}.$$

Define an ℓ-partite tournament MT_2. If $p = \ell$, we take $MT_2 = MT$; otherwise, let

$$V(MT_2) = V(MT) \cup \{u_1, u_2, ..., u_{\ell - p}\}$$

and

$$A(MT_2) = A(MT) \cup \{u_i x\colon x \in V(MT),\ 1 \leq i \leq \ell - p\} \cup \{u_j u_i\colon 1 \leq i < j \leq \ell - p\}.$$

Proof of Theorem 11. Let MT_1 and G_2 be arbitrarily chosen graphs from $\mathcal{MT}(I, f(k), \ell)$ and $\mathcal{G}(I, f(k), \ell)$, respectively. Define G_1 and MT_2 by Constructions 1 and 2. To prove Theorem 11, it is sufficient to prove the following statements: (i) $G_1 \in \mathcal{G}(I, f(k), \ell)$, and if G_1 contains k disjoint PC cycles, then MT_1 contains k disjoint cycles; (ii) $MT_2 \in \mathcal{MT}(I, f(k), \ell)$, and if MT_2 contains k disjoint cycles, then G_2 contains k disjoint PC cycles.

(i) By Construction 1, we have $col(G_1[V_i]) \subseteq \{c_i\}$, that is, $col(G_1[V_i]) = \{c_i\}$ when $|V_i| \geq 2$ for all i with $1 \leq i \leq \ell$. Let $n = |V(G_1)|$. Then $n = \sum_{1 \leq i \leq \ell} |V_i| + 1$. For a vertex $v_i \in V_i$ $(1 \leq i \leq \ell)$, denote by $N^+_{MT_1}(v_i)$ the set of out-neighbors of v_i in MT_1. Since $|N^+_{MT_1}(v_i)| = d^+_{MT_1}(v_i) \geq f(k)$ and $N^+_{MT_1}(v_i) \cap V_i = \emptyset$, we have $|V_i| \leq n - f(k) - 1$. Note that each vertex in $N^+_{MT_1}(v_i)$ is joined to v_i by an edge with color distinct from c_i in G_1. So, the color c_i appears at most $n - f(k) - 1$ times at v_i, and any color c_j $(j \neq i)$ may appear at most $|V_j| \leq n - f(k) - 1$ times at v_i. For the vertex v_0, each color appears at most $|V_j| \leq n - f(k) - 1$ times at v_0. Thus, we have $\Delta^{\text{mon}}(G_1) \leq n - f(k) - 1$.

Claim 1. v_0 is a bad vertex in G_1 and each edge xy is not contained in any PC cycle in G_1 for $x, y \in V_i$ $(1 \leq i \leq \ell)$.

Proof. Suppose that C is a PC cycle in G_1 containing v_0. Orient the edges of C in one of the two directions along C. Choose a vertex $u \in V(C) \setminus \{v_0\}$, and assume that $u \in V_i$ for some i with $1 \leq i \leq \ell$. Then, we obtain $col(u^- u) = c_i$ and $col(u^+ u) = c_i$, by following the paths $v_0 \overrightarrow{C} u^- u$ and $v_0 \overleftarrow{C} u^+ u$, respectively. (Here, u^+ and u^- denote the immediate successor and predecessor of u on C in the direction specified by the orientation of C, respectively, and $\overrightarrow{C}$ and $\overleftarrow{C}$ denote the traversal of C in the direction of the orientation, and in the opposite direction, respectively.) Thus $col(u^+ u) = col(u^- u)$, a contradiction. Similarly, we can prove that xy is not contained in any PC cycles for $x, y \in V_i$ $(1 \leq i \leq \ell)$.

Claim 1 implies that each PC cycle in G_1 corresponds to a cycle in MT_1. So G_1 contains no PC cycle of length $i \in I$ and v_0 is a bad vertex in G_1 with $d^c_{G_1}(v_0) \leq \ell$. In summary, $G_1 \in \mathcal{G}(I, f(k), \ell)$, and if G_1 contains k disjoint PC cycles, then MT_1 contains k disjoint cycle. Hence, (i) holds.

(i) Let $V_0, V_1, V_2, \ldots, V_p$ be the v_0-partition of G_2 in Construction 2. Note that for each vertex $v_i \in V_i$ $(1 \leq i \leq p)$, there are at least $f(k)$ vertices in $G_2 - V_0$ joined to v_i by edges with colors different from c_i. This implies that $\delta^+(MT_2) \geq f(k)$. Note that every cycle in MT_2 corresponds to a PC cycle in G_2. So MT_2 contains no cycle of length $i \in I$. In summary, $MT_2 \in \mathcal{MT}(I, f(k), \ell)$ and if MT_2 contains k disjoint cycles, then G_2 contains k disjoint PC cycles. Hence (ii) holds. This completes the proof of Theorem 11. □

ACKNOWLEDGMENTS

The authors are grateful to the referees for their valuable comments and suggestions. The research is supported by CSC (No. 201506290097), NSFC (Nos. 11671320 and 11901459), and the Fundamental Research Funds for the Central Universities of China (Nos. 31020180QD124 and 3102019GHJD003), and partially by the group of Formal Methods and Tools when the first author visited University of Twente.

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How to cite this article: Li R, Broersma H, Zhang S. Vertex-disjoint properly edge-colored cycles in edge-colored complete graphs. *J Graph Theory*. 2020;94:476–493. https://doi.org/10.1002/jgt.22536