Optimality criteria and optimisation procedure for 2.5D triangulations

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Abstract.
Triangulating a set of points is a key technique in solving problems in Surface Reconstruction. Let V be a finite point set in $3D$ and let $ST(V)$ be the set of closed triangulated polyhedral surfaces with vertex set V. Those surfaces can be defined as $2.5D$ (closed) triangulations of the given discrete data set V. We discuss possible approaches to construct an optimal 2.5D triangulation. We present and compare several optimality criteria, among them two curvature criteria, both proposed by Alboul and van Damme: the Tight criterion and the criterion of minimising total Mean curvature. Then we discuss the procedure of optimisation for 2.5D (closed) triangulations. As a transformation operation we use the so-called *extended diagonal flip*, or simply *EDF* that generalises the 'conventional' flip operation. The authors have recently introduced the EDF by omitting the usual restriction that a flip operation should not produce a self–intersecting triangulation. We use the EDF procedure to obtain an optimal triangulation for the data taken from the surface of an object of the simplest topological type, *i.e.*, with the boundary topologically equivalent to the $2D$ sphere.

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1 Introduction

Triangulating a set of points (a data set) is a key technique in many scientific fields, for example, in scattered data interpolation and approximation, computational geometry, computer–aided geometric design, and finite–element computation. It has many practical applications. In real life one often deals with reconstructing or modelling the surface of a three-dimensional object from an initial data set; for instance, in computer graphics, cartography, geology, stereology, architecture, visual perception, medicine and so on. One of the important problems in medicine is to build a 3D model of an organ from $2D$ cross–sectional images, obtained, for example, by means of CT (computerised tomography)[vHBKSTL95] (see Fig. 1).

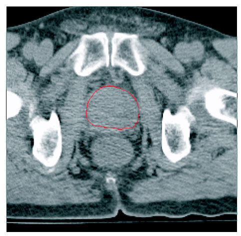

Figure 1: $2D$ cross–sectional image of the prostate motion

The first step in evaluating a surface is to obtain a triangulation of the data sites. A 'good' triangulation can help to solve many problems. It is the quickest way to obtain an initial look at the data before using higher–order interpolation/approximation methods. If the data are taken from an irregular (non–smooth) surface, a triangulation (a polyhedral surface) might be the only possible way to view the object, determined by the data. The problem of finding a suitable triangulation of the given data consists, in general, in two steps: first, in constructing some initial triangulation, and second, in its optimisation with respect to a chosen criterion.
We discuss possible approaches to construct an optimal triangulation. We present and compare several optimality criteria; among them the Tight criterion and the criterion based on minimising the total Mean curvature, both introduced by L. Alboul and R. van Damme (see, for example, [AKTvD00]). These two criteria seem the most promising criteria at the moment. Then we discuss the procedure of optimisation for triangulations of closed surfaces in $3D$. Such triangulations can be defined as $2.5D$ triangulations. Optimisation usually consists of transforming an initial triangulation via a sequence

of transformations to some final triangulation which is better with respect to the given criterion. The operation of transformation should preferably be simple as well as general enough in order to be able to reach the optimal triangulation from any initial one. For triangulations of points in the plane such a simple operation of transformation exists, known as Lawson's procedure [Law72]. The operation consists of swapping a diagonal of the convex quadrilateral formed by two adjacent triangles to the other diagonal, thus replacing one edge by a new one and obtaining another triangulation of the given data. Recently Alboul and van Damme have introduced an innovative generalisation of Lawsons's procedure for triangulations of surfaces in $3D$, by allowing self–intersections of a triangulation and defining a local swapping procedure for non–flat quadrilaterals ([AvD01]). We discuss this procedure in the case the data are taken from the surface of an object of the simplest topological type, *i.e.*, the boundary of the object is topologically equivalent to the $2D$ sphere. We show, for example, that we are able to recover the optimal triangulation of convex data *i.e.*, a *convex triangulation*, starting from almost any initial triangulation of the data.

2 Triangulation

The concept of triangulation is known in different scientific areas, such as topology, differential geometry, computational geometry, approximation and interpolation theory. We deal with triangulations in Surface Reconstruction, aiming to reconstruct a surface. Research on the triangulation problem in Surface Reconstruction lies in the intersection of the above–mentioned areas, and therefore, involves the use of notions from these fields, sometimes appropriately adapted.

A *triangulation* T is a partition of a geometric domain into simplices (triangles, tetrahedra and so on) that meet only at shared faces. In topological graph theory therefore under a *triangulation* on a surface one understands a simple graph G embedded on the surface so that each face is triangular and any two faces share at most one edge. In the field of computational geometry as well as in applications a geometric domain can be a point set, a polygon or a polyhedron. In the planar case a triangulation represents a collection of triangles. The difference with respect to topological graph theory is that the points (vertices) have fixed positions and the edges are straight line segments.

We consider a triangulated surface in three-dimensional space. In general, one presupposes that this surface is embedded in 3D, *i.e.*, has no self–

intersections. The problem of constructing a triangulation can be considered then as the 3D boundary construction problem which can be stated as follows (see, for example, [Vel94]):
Boundary Construction 3D. Given is a set of vertices in 3–dimensional space. Find a closed polyhedron of triangular faces through all vertices.
The above–mentioned problem can be addressed in one of two ways:

1. By directly constructing a triangulation of the surface defined by the data. This approach is called the *surface–based approach*.

2. By intermediately constructing some $3D$ structure, obtained by filling the interior of the object with tetrahedra, and then deriving a triangulated representation of the object from such an auxiliary structure. This approach can be called the *solid–based approach*.

The data can be either a set of irregularly distributed points, or a set of straight–line segments, or a set of polygonal cross sections.
As we are dealing with a surface–reconstruction problem, we follow the first approach, which seems more logical, and also because the theory of surfaces is well–developed in the field of global (differential) geometry. Using the language of computational geometry we can define the problem of reconstructing a surface embedded in three–dimensional space is as the *two–and–a–half–dimensional problem*. This problem has two forms: the first deals with interpolating surfaces for point set data with elevation, or *functional surfaces*, and the second deals with triangulated (polyhedral) surfaces for three-dimensional models, or *bounding surfaces of solid objects from* $3D$ *data*. Actually, these latter surfaces are two-dimensional closed surfaces, which are situated in three-space. Consequently, we will refer to triangulations of points in the plane, as to ***2D* triangulations**, to triangulated polyhedral surfaces in space as to ***2.5D* triangulations**, and triangulations of points in space (as in the solid–based approach) as to ***3D* triangulations**.
The solid–based approach has several drawbacks, for example, the non–existence of a Hamiltonian polyhedron for some triangulations. This means that one can not recover a $2D$ valid boundary (a surface) of the object from the constructed collection of tetrahedra. This situation can occur for the well–known Delaunay triangulations [Vel94]. There is also such a phenomenon as non–tetrahedralisable polyhedron (see Fig. 2).
Let us define the concept of a triangulation more precise. The following definition is standard:

Definition 2.1 *A triangulation T is a collection of triangles, that satisfies the following properties:*

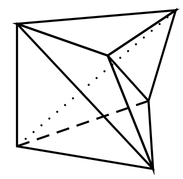

Figure 2: Schönhardt's prism.

1. *Two triangles are either disjoint, or have one vertex in common, or have two vertices and consequently the entire edge joining them in common.*

2. *T is connected.*

In the case of compact surfaces (2.5D triangulations) a triangulation T consists of a finite number of triangles, and we can conclude that the two following conditions are valid [Mass91]:

1. Each edge is an edge of exactly two triangles.

2. For every vertex V of a triangulation T, we may arrange the set of all triangles with V as a vertex in cyclic order, $T_0, T_1, ..., T_{n-1}, T_n = T_0$, such that T_i and T_{i+1} have an edge in common for $0 \leq i \leq n-1$.

The last condition means, that our triangulated surface is a manifold, *i.e.*, the neighbourhood of every point, as well as a vertex, is topologically the same as the open unit ball in $\mathbf{R}^\mathbf{n}$ (in our case $n = 2$).
For more details about different types of triangulations, see, for example, [BE95, AvD01].

3 Construction of a triangulation

The problem of finding some (an initial) triangulation clearly represents a difficult problem. Moreover, as this triangulation can be very 'bad', the next problem, which can be even more difficult, arises: to optimise this initial triangulation with respect to a chosen criterion.
There is not yet a general, formal and practical criterion to measure the quality of a triangulation, applicable to a wide class of data. The main idea governing the different approaches is that long thin triangles should be avoided and that the triangles should be as equiangular as possible. The arguments are:

1. Aesthetic justification. The Piecewise Linear Interpolating Surface (PLIS) constructed from thin triangles is not, in general, visually pleasing.

2. Numerical justification. In spline approximation theory thin triangles are undesirable because general expressions for the approximation error depend on the 'thinness' of the triangle in the sense that the error bound grows if the triangles become thinner.

3. Geometrical justification. This can be easily motivated in the framework of the theory of surfaces. Roughly speaking, thin triangles might deviate considerably from tangent planes to the surface (if these planes exist) than thick triangles.

The most popular triangulation for functional data, which satisfies the above criteria is the well–known Delaunay triangulation. This triangulation is to be preferred if one wants to approximate different functions, using the same triangulation: the triangulation does not depend on the function values, but only on the location of the data sites. However, if one wants to approximate a specific surface the Delaunay triangulation can give a very unsatisfactory representation of the object, especially if it is used for a closed surface (see the left picture in Fig 3).

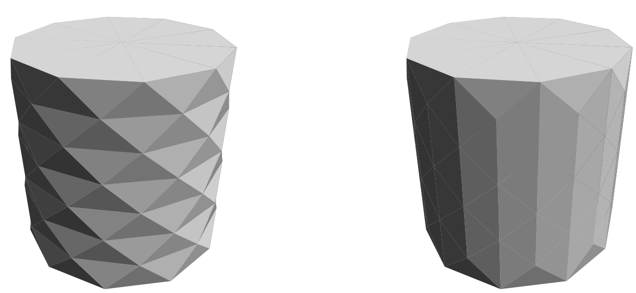

Figure 3: Left: Delaunay traingulation; Right: Tightest triangulation

4 Optimisation

Optimisation usually consists in transforming an initial triangulation via a sequence of transformations to some final triangulation which is better with respect to the given criterion. At each step of the optimisation process, the triangulation, obtained at the previous step, is transformed to a better triangulation by some, preferably simple, operation. The operation should also be general enough in order to be able to reach the optimal triangulation from any initial one.
Therefore the optimisation problem can be split into two sub–problems:

- Selection of an optimality criterion.
- Determination of an optimisation procedure.

4.1 Optimality criteria

The main goal of a triangulation is to give an initial representation of the surface which is to be reconstructed. In order to obtain a suitable representation different requirements to a triangulation are considered. For example, one can require that a triangulation must be as smooth as possible (*i.e.*, without unnecessary sharp edges), or reflect certain features of the shape of the object. These requirements determine the choice of one or another optimality criterion.
There are several known optimality criteria in the literature such as

- minimising the area of the resulting object [ORourke81], which can give a very strange final triangulation, even for very simple data (see, for instance, Fig. 4);

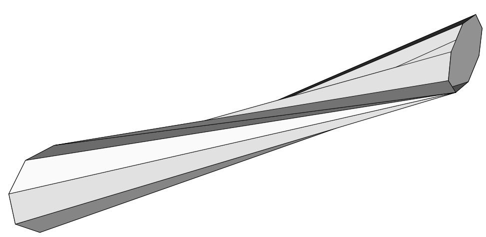

Figure 4: A skewed cylinder

- heuristic criteria, based on minimising a measure of roughness of the resulting object. One such measure is the jump in normal derivative (JND) and the other measure is the angle between normals (ABN) [DLR90, DLR90a];
- methods based on minimising a certain functional, like the energy of a bending plate, constrained by the interpolation conditions. Such a method can *e.g.*, be found in [QS89, QQ90, QS90a] (with the disadvantage of this latter method that it only works for *functional* data).

Some heuristic criteria do work well, but their main defect is the absence of a well–defined theoretical background. The above–mentioned criteria are discussed in detail in [AKTvD00].

Much work in Surface Reconstruction has been done either in the spline approximation theory, *i.e.*, in numerical analysis, or in applications thereof. However, surface reconstructing seems more apt as a geometric problem. Unfortunately, geometric methods are not widely used and have not yet been developed in their full strength. We can point out several reasons for this: first, geometry is not a very common discipline in fields dealing with applications, especially differential geometry is not known; second, the direct application of tools of classical differential geometry require high order differentiability of objects, and this can hardly cope with discrete schemes used in numerical analysis and applications, the third reason might be the popularity of Delaunay triangulations, which became even a field of intensive research in computational geometry.

4.1.1 Curvature criteria

Using pure geometric criteria, Alboul and van Damme have introduced new triangulations for irregularly-located 3D data [vDA95, AvD96, AKTvD00]. The first triangulation is based on minimising of a discrete analogue of the integral absolute Gaussian curvature. This triangulation was initially called the Tight triangulation. We have renamed the Tight triangulation to the Tightest triangulation. (The reasons for this are given in 4.1.2. Tightest triangulations are evidently better than Delaunay triangulations, as, for example, the Tightest triangulation automatically preserves convexity (see the right picture in Fig 3).

The second proposed optimality criterion is the minimisation of the total absolute Mean curvature.

Roughly speaking, the first criterion deals with the curvature determined around the vertices, and the second – along the edges. Below we give a

short review of main notions concerning the Curvature criteria.
Both the Gaussian and Mean curvatures are among central concepts in differential geometry and both related to the concept of angle. For triangulated polyhedral surfaces which are non–regular surfaces, appropriate analogues of curvatures are defined. As such a polyhedral surface is a $2.5D$ triangulation, and therefore represents a collection of triangles, we denote it by $\triangle$. The *star* $\mathrm{Str}(v)$ of a vertex v is by definition the union of all the faces and edges that contain the vertex, and the *link* of the star (the boundary of the star) is the union of all those edges of the faces of the star $\mathrm{Str}(v)$ that are not incident to v.
On the basis of the notion of the angle, the following curvatures for a triangulation $\triangle$ are determined:

1. The *(integral) curvature* ω (an analogue of the integral Gaussian curvature).
The *total angle* $\theta(v_i)$ around the vertex $v_i \in \triangle$ is the sum of those angles of the faces (*i.e.*, triangles)of $\mathrm{Str}(v_i)$, that are incident to v_i. For any point $x \in \triangle$: $\omega(x) = 2\pi - \theta(x)$. The quantity ω is also known as the *angle deficit.* Only for vertices we have: $\omega(x) \neq 0$. (see Fig. 5).

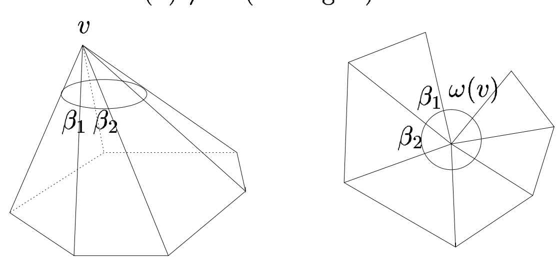

Figure 5: Curvature around a vertex: $\omega(v) = 2\pi - \sum \beta_i = 2\pi - \theta(v)$.

2. The *positive (extrinsic) curvature* $\omega^+(v_i)$.
Suppose, that through the vertex v_i there passes some (local) supporting plane of a triangulation $\triangle$. Then this vertex lies on the boundary of the convex hull of $\mathrm{Str}(v)$. We denote the star of v_i in the boundary of this convex hull by $\mathrm{Str}^+(v_i)$ and will call it the star of the convex cone of a vertex (or, simply, the convex cone of a vertex, if it does not lead to ambiguities). The curvature $\omega^+(v_i)$ of $\mathrm{Str}^+(v_i)$ is called the *positive (extrinsic) curvature* of v_i. If there is no supporting plane through v_i then $\omega^+(v_i)$ is equal to zero

by definition.
3. The *negative (extrinsic) curvature* $\omega^-(v_i)$.
$\omega^-(v_i) = \omega^+(v_i) - \omega(v_i)$.
4. The *absolute (extrinsic) curvature* $\hat{\omega}(v_i)$.
$\hat{\omega}(v_i) = \omega^+(v_i) + \omega^-(v_i)$

We can isolate the following types of vertices:
• *Convex vertices*: $\omega(v_i) = \omega^+(v_i) = \hat{\omega}(v_i)$, $(\omega^-(v_i) = 0)$.
Geometrically it means that $\text{Str}(v)$ coincides with $\text{Str}^+(v_i)$.
• *Saddle vertices*: $\hat{\omega}(v_i) = \omega^-(v_i) = -\omega(v_i)$ $(\omega^+(v_i) = 0)$.
The Gaussian curvature ω of a saddle vertex is less than zero and there exists no supporting plane.
• *Mixed vertices*:
1) $\omega(v_i) > 0$, $\omega^+(v_i) > \omega(v_i)$
or
2) $\omega(v_i) < 0$, $\omega^+(v_i) > 0$.
In Fig. 6 examples of all three types of vertices are presented.

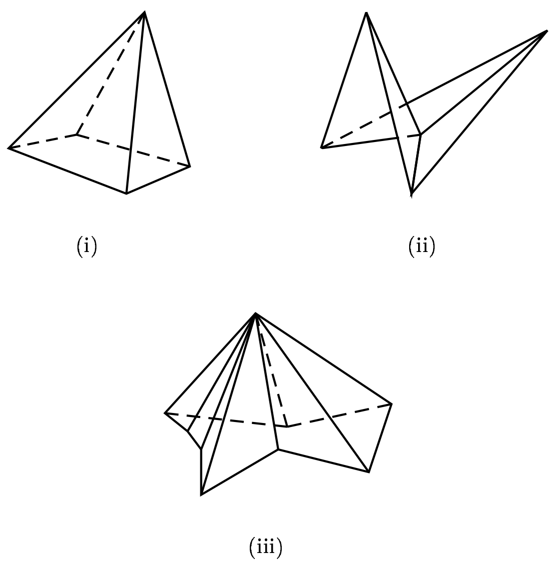

Figure 6: Types of vertices: (i) Convex (ii): Saddle (iii) Mixed.

You can see a mixed vertex and its correspondent convex star in Fig. 7.

(i) (ii)

Figure 7: A mixed vertex (i) and its convex star (ii).

The *total absolute (extrinsic) curvature* $\hat{\Omega}_{abs}(\triangle)$ of a triangulation $\triangle$ is given by the following expression:

$$\hat{\Omega}_{abs}(\triangle) = \sum_{v_\alpha \text{ convex}} \omega^+(v_\alpha) + \sum_{v_\beta \text{ saddle}} \omega^-(v_\beta) + \sum_{v_\gamma \text{ mixed}} (\omega^+(v_\gamma) + \omega^-(v_\gamma)). \quad (1)$$

For any triangulation convex, saddle and mixed vertices form three disjoint subsets of the vertices of the triangulation, and their union is the data set itself.

Another curvature that can be defined for a polyhedron and, therefore, for a triangulation, is the *Mean curvature.* This curvature measures how much a surface is 'bent', and for a polyhedron, it is logically defined along the edges.

> $H(e)$ equals half of the oriented exterior angle between the faces adjacent to the edge e and zero otherwise.

The sign of H depends on the orientation of the polyhedral surface (triangulation) $\triangle$.
The *integral mean curvature* (IMC) is determined for polyhedral surfaces as well. For a domain $U \subset \triangle$ it is defined as follows:

$$H(U)_{U \cap P} = \sum_e H(e) \cdot \text{length}(e \cap U),$$

where the sum ranges over all edges e of $\triangle$.

4.1.2 Triangulations based on minimising the Curvature criteria

Let us given a (closed) data set V: $\{v^i\}, i = 1, ..., N$. Let us note that if we have a given discrete point set in $3D$, we can, in general, construct triangulated surfaces of different genera with the given points as the vertices. Therefore, all triangulations $[\triangle]$ of the data set V fall into non-intersecting subclasses of triangulations in such a way that all triangulations of the same class have the same genus and orientability (*i.e.*, topologically equivalent to the $2D$ sphere, the torus, and so on). From the other side, the data are situated, in general, on an existing surface. Even if we cannot view the surface, nevertheless, we can presuppose its basic topological features as orientability and genus. Therefore, it makes sense to apply a criterion only to one subclass of the set $[\triangle(V)]$ of all triangulations.

Tightest triangulation.A triangulation $\bar{\triangle}$, that belong to some subclass of all possible triangulations of the data set is said to be *Tightest*, if $\hat{\Omega}_{abs}(\bar{\triangle})$ is minimal, *i.e.*,

$$\forall \;\; \triangle : \hat{\Omega}_{abs}(\triangle) \geq \hat{\Omega}_{abs}(\bar{\triangle}).$$

Some properties of the Tightest (ex-Tight) triangulation are given in [AvD96].

REMARK. *The notion of a tightness has its roots in the theory of embeddings (immersions) of differential manifolds. This concept arose when one attempted to generalise theorems about convex surfaces to surfaces that are topologically more complex (of other genera). A two–dimensional manifold M is tightly embedded in $\mathbf{R^3}$ if every hyperplane in $\mathbf{R^3}$ which contains a point of M and no nearby points is a global supporting hyperplane of M. In $\mathbf{R^3}$ a tight close surface has also the minimal total absolute curvature as well as it satisfies the* two–piece property, *i.e., any plane cuts it into two pieces at the most. The theory of tight embeddings is well–developed and has sense for polyhedral surfaces as well. A* tight triangulation *can be determined as a triangulated polyhedron which is* tight *in the ambient Euclidean space (for a detailed description of the notions concerning tight immersions (embeddings) see [Kui70, Kui80, Kuhn95]). Our triangulations with minimal absolute total extrinsic curvature $\hat{\Omega}_{abs}(\bar{\triangle})$ are not really tight in the classical sense, because they are dependent on the lie of the data. Actually, it seems that for any finite given data we can always find at least one tight triangulation (polyhedron) of a certain genus. If we stick to triangulations of the same genus, we might not have the Tight triangulation (in a classical sense), however we can determine a triangulation of the minimal absolute total extrinsic curvature, which might be not unique. (This topic was discussed in a scientific discussion of the first author with Wolfgang Kühnel). Therefore, to avoid further confusion, we refer now to triangulations with*

the minimal $\hat{\Omega}_{abs}(\bar{\triangle})$ *as to* **Tightest triangulations**.
Another optimal triangulation, introduced by the authors is the **triangulation of the minimal total (integral) absolute Mean curvature**. The absolute total mean curvature is given by the formula

$$H_{abs} = \sum_{e} |H(e)|,$$

where the sum ranges over all edges e of $\triangle$. Consequently a triangulation of the data set V (among all the triangulations of same topological type) is said to be the triangulation of the minimal absolute total mean curvature if it minimises H_{abs}.
An interesting fact that the above–mentioned heuristic criteria are all, in a certain sense, related to minimisation of the Mean curvature (for more details see [AvD97, AKTvD00]).

Up to now it has not yet been clear which of two 'curvature' criteria is the 'best' (if any), and how they are compared. This is a subject of our current research. For example, you can see an application of two criteria to an initial triangulation of the scalp (see Fig 8).
The first (initial) triangulation on the left is good, but it has been obtained mainly by hand. The second triangulation in the middle is the triangulation, obtained from the first one by applying the Tight criterion. The third triangulation is the triangulations obtained from the second one by applying the criterion of minimising the absolute total Mean curvature. This triangulation is almost identical to the first one and evidently better than the second one. However, the problem in the case of the Tight criterion might be of algorithmic character. The algorithm to obtain a triangulation of the minimal absolute extrinsic Gaussian curvature is not global if the data are not convex.

4.2 Transformation procedure

For triangulations of points in the plane a simple operation of optimisation exists, and this is a diagonal flip (called also a *swap*). The operation consists of swapping a diagonal of the convex quadrilateral formed by two adjacent triangles to the other diagonal, thus replacing one edge by a new one and obtaining another triangulation of the given data. For a non–convex quadrilateral in the plane this operation is not allowed, because it does not produce a triangulation (see Fig. 9).

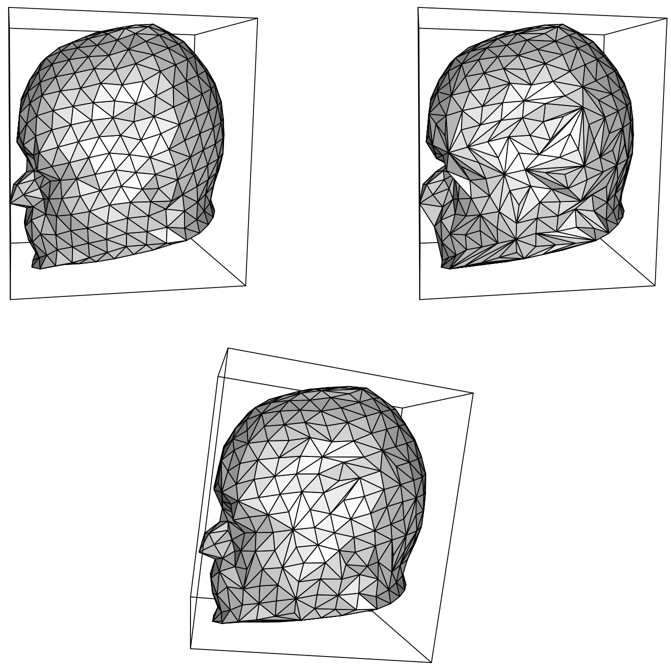

Figure 8: Transformation of an initial triangulation of the scalp

In Surface Reconstruction this operation is known as *Lawson's procedure*. Recently Alboul and van Damme have introduced a generalisation of Lawson's procedure for triangulations of surfaces in $3D$, by allowing self–intersections of a triangulation and defining a local swapping procedure for non–flat quadrilaterals ([AvD01]).

4.2.1 Review of the extended edge operation (EDF)

As we deal with the surface–based approach, it seems logical to consider a generalisation of Lawson's procedure, using the same idea as in $2D$: to swap just an edge. One of conventional assumptions in the Surface Reconstruction is that a reconstructed surface may not have self–intersections, or in other words must be embedded. Moreover, this property is also assumed for every intermediate surface which might occur during the transformation process.

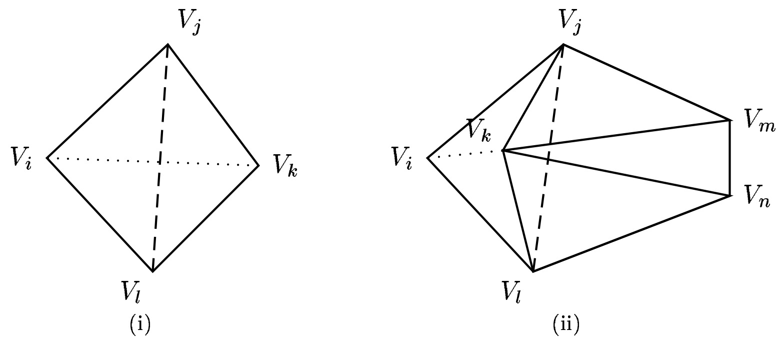

Figure 9: Swapping a diagonal in a quadrilateral: (i) Convex quadrilateral. (ii) Non–convex quadrilateral. In both cases the dashed edge swapped to the dotted one.

Indeed, the first condition of Defintion 2.1 excludes self–intersections. If we stick to Definition 2.1, then it can be shown that embedded triangulations (of the same topological type) of certain $3D$ data can be split into at least two non–empty sets, such that the intersection of these sets is empty; we cannot reach a triangulation of the first set from a triangulation of the second one and vice versa by swapping edges (for a detailed description see [AAH00]). However, the above–mentioned definition is the definition of an *abstract triangulation* of an abstract surface [Gibl77]. 'Real' surfaces can have self–intersections and/or have singularities, in which a surface is not a manifold. We should distinguish between the abstract topological representation of a surface S (as a 2-dimensional *manifold*) and its realisation in 3-space (as an image of this 'canonical' manifold).
The restriction on the flip operation that forbids self–intersections seems logical for plane triangulations, because self–intersections result in producing not a triangulation. However, in contrast to planar triangulations, a valid triangulation can still be determined even if a self–intersection has occurred after applying the flip operation.
We deal with triangulated compact surfaces. Compact surfaces are characterised by such properties as orientability and Euler-Poincaré characteristics χ. In our research we deal with orientable surfaces and mostly with surfaces without boundary (closed surfaces). (Reconstruction of non–orientable surface might be of interest as well).
We have a finite data set V and we want to reconstruct a 2.5D triangulation (closed surface) which spans these data. At every step of transformation

process the obtained triangulation represents, in general, a new polyhedral surface. Our goal is to reconstruct a final surface, which is optimal with respect to a given criterion. We presuppose that the reconstructed surface belongs to a certain topological type. Therefore, it seems logical to require that each intermediate triangulation remains in the subclass of the given topological class of all possible triangulations of data V. Therefore, a flip operation must preserve orientability and the genus of this subclass. Let us note that when we speak about a surface of a certain topological type we mean under the surface a 2-dimensional manifold and not as a 2–dimensional submanifold in three–space. For example, our surface can be not homeomorphic to a sphere in three–space (because of self–intersections), but as an abstract 2–manifold be topologically equivalent to the 2D sphere.
Orientation. Our triangulated surface is an orientable closed surface. For every triangle we can define an orientation such that two triangles with a common edge are always oriented coherently (see Fig. 10) [Gibl77]

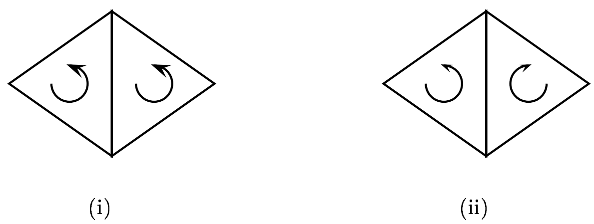

Figure 10: Two triangles with a common edge: (i) coherently oriented,(ii) - not coherently oriented.

Informally speaking, to assign an orientation to a triangle means that we determine the direction of the walk along its boundary. Going around the triangle can be clockwise or counter–clockwise. If one of two is fixed, then it can be taken as 'positive' and in the case of oriented surfaces, when all triangles have the same direction of going around, we can determine the positive side of the surface. The positive orientation is usually counter–clockwise. For oriented surfaces we can define two directions of normals. Usually, the direction of outwards pointing normals is said to be positive.
Quadrilaterals in the space. We specify some notations concerning quadrilaterals in space. In plane swapping means the exchange of the diagonals in a convex quadrilateral, determined by four vertices, or replacing two adjacent plane triangles forming the given plane quadrilateral by other two

adjacent plane triangles that form the same plane quadrilateral. In space, four vertices, form in general a tetrahedron. If we would like to stick to a surface ('plane') terminology, we prefer to consider two adjacent triangles, formed by edges connecting the vertices, instead of a tetrahedron. The figure, built of two adjacent triangles, will be called a *spatial quadrilateral*. The edges of two triangles except the common edge form a closed polygonal line. We call it the *boundary* of the quadrilateral. If we keep these edges fixed, then there are only two possible spatial quadrilaterals for every four vertices, if the data are situated in general position, *i.e.*, no three vertices lie in a line and no four in the plane. If a triangulation is given then the boundary of a quadrilateral is always fixed. The orientation of triangles forming the spatial quadrilateral should be coherent. The common side of two adjacent triangles in a spatial quadrilateral will be called a (spatial) diagonal. We can say that an edge (diagonal) is *concave*, if two lines, determined by the unit normals to the adjacent triangles, sharing this edge as the common side, intersect each other in the positive direction, otherwise, the edge is called *convex*. The edge is *flat* if the normals are parallel.
We assume that data are in general position. Then flipping in space means the exchange of a convex edge to a concave edge and vice versa.
At each step we work only with two triangles, which together form a spatial quadrilateral. As the data are situated in general position and if we connect two vertices of the quadrilateral that are not the end–points of the common edge, the new edge will lie outside the surface of the quadrilateral and we can determine two new triangles. We will call a spatial quadrilateral *convex*, if its development on the plane is a convex quadrilateral; otherwise, we call it *concave*. Therefore, we have four cases: convex quadrilateral with a convex diagonal, convex quadrilateral with a concave diagonal, concave quadrilateral with a convex diagonal, concave quadrilateral with a concave diagonal. If we instead of the word *concave* use the word *reflex* then we can denote the above–mentioned quadrilaterals by **CC, CR, RC, RR**. Let us determine the vertices of the quadrilateral as V_i, V_j, V_k, V_l. We choose as the diagonal edge V_iV_k. Our triangles are oriented counter–clockwise, as on the left diagram on the Fig. 10. Now if we swap edge V_iV_k, we get edge V_jV_l and triangles $V_lV_jV_i$ and $V_lV_kV_j$. The new obtained triangles should be oriented coherently with the orientation on the surface. This can be easily done. Indeed, if we start to walk from any vertex of the triangle along its boundary, the triangle itself must lie on our left side. So we can always determine the positive side of the new obtained part of the surface, determined by two new triangles. The above–given ordering of vertices in two new obtained triangles corresponds to their orientation

(counter–clockwise). See Fig. 11, where we present the swapping of the concave (reflex) diagonal in CR and RR quadrilaterals.

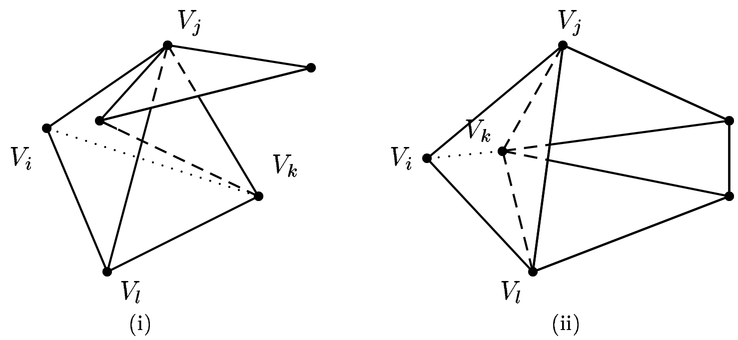

Figure 11: Swapping diagonal V_iV_k in a spatial quadrilateral: (i) CR quadrilateral (ii) RR quadrilateral. Vertex V_j on the left picture is a pinch point.

Therefore, at every swap our triangulated surface remains properly oriented. An intersection can occur, but, in contrast to the $2D$ situation, two new triangles are well–defined together with their adjacent triangles. At each step we work exactly with two triangles. We can always swap the common edge of two triangles and replace two initial triangles by two possible other ones which would again share a common (new) edge. From the definition of the triangulation it follows that our triangulated surface is strongly–connected *i.e.*, we can always find a path from one vertex to any other one in such a way that this path will lie inside a chain of m triangles, where $\triangle_i$ shares with $\triangle_{i+1}$ a common edge ($i = 0, ..., m-1$). If we now apply our flip operation, it will not violate this property. This also means that if we start with a triangulation with the Euler characteristic χ, the transformation process will not change it. Self–intersections do occur, but the graph of our triangulation remains planar. However, we need to consider the graph of a triangulation in an extended meaning and allow the existence of multiple graphs, as you can see below.

Degenerated case (DC)

We suppose that our data are in general position. As three points always lie in a plane, we can encounter a degenerated situation when a quadrilateral collapses into two glued together triangles. This situation can occur if we

swap an edge in the star of a vertex of valence 3, or in other words, the star of the vertex has only three faces.
Let us denote this vertex as V_1, the vertices in its star as V_2, V_3 V_4. Face $V_1V_2V_3$ shares a common edge V_2V_3 with face $V_2V_5V_3$. Let us consider the quadrilateral $V_1V_2V_4V_3$, formed by triangles $V_1V_4V_2$ and $V_1V_3V_4$. After having swapped edge V_1V_4 we obtain an edge, that connects V_2 and V_3. We have vertices V_1, V_2 and V_3 and two faces (taking into consideration the orientation) $V_1V_3V_2$ and $V_1V_2V_3$, but edge V_2V_3 is not their common edge. What does it mean? Actually the first triangle has the adjacent triangle $V_2V_3V_4$ and the second one – $V_2V_5V_3$: we have instead of one edge V_2V_3 **two different edges** with the same end–points. This degenerated case can be easily eliminated by swapping in turn the common edge of triangles $V_1V_2V_3$ and $V_2V_5V_3$. We would have now the star of vertex V_1 with vertices V_2, V_3 and V_5 in its link (see Fig. 12).

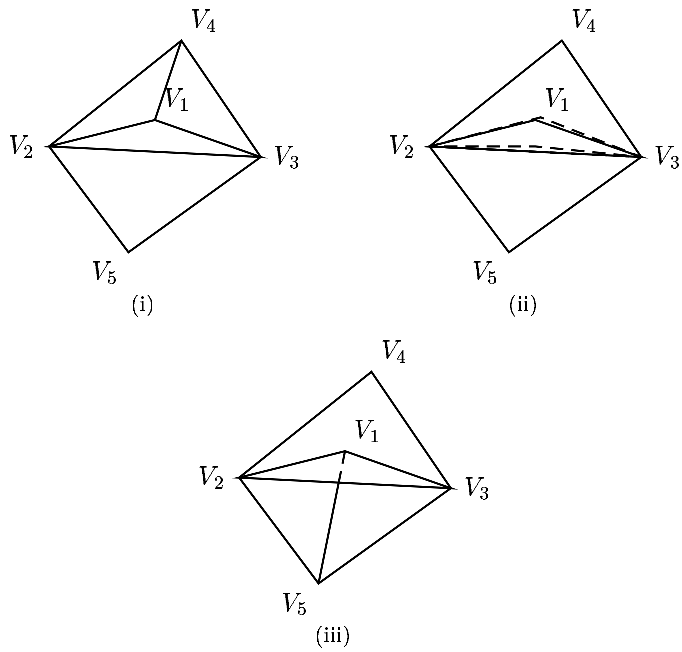

Figure 12: Degenerated case. (i) Before the first swap (ii) After the first swap (iii) After the next swap.

If we would draw the part of the graph of the triangulation that corresponds to the first swap (after having obtained two 'glued' triangles), we would have a multiple graph (see Fig. 13).

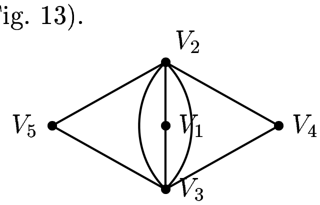

Figure 13: Part of the plane graph corresponding to the degenrated case.

The edges V_1V_2 and V_2V_1 (V_1V_3 and V_3V_1 respectively) are the common edges. These edges are the same. We still have a triangulation with the same Euler–Poincaré characteristics as the $2D$ sphere. The swapping of edges V_1V_2 or V_1V_3 is not allowed, because the quadrilateral is not defined in this case. Therefore, if the data in general position, the flip operation is completely defined. We can call this operation the *extended diagonal flip*, or *EDF*.
Other degenerated situations can occur if our data are not in general position. In this case we can imagine that partially–glued triangles might occur. This problem is now under investigation (see more in [AvD01].
Now we define the type of triangulated surfaces which would be allowed.

Definition 4.1 *(EXTENDED DEFINITION OF A TRIANGULATION).*

- *A triangulation T (triangulated surface, $2.5D$ triangulation) either corresponds to the properties of Definition 2.1 or, if not, then the following exceptions can occur:*

  1. *For each triangle we can still single out three adjacent triangles, with which it shares a common edge. Two of three adjacent triangles may coincide.*
  2. *Two triangles can have some points in common, besides vertices or edges. If these triangles are not adjacent then they can intersect each other along a line segment. Two adjacent triangles can have a triangular domain in common.*

In other words we consider now not only 'pure' simplicial complexes and we allow some singularities. We will call such triangulations generalised 2.5D triangulations, or shortly **2.5GD** ¿From all–above mentioned we can conclude:

Theorem 4.2 *The extended diagonal flip operation preserves two main topological characteristics (orientability and the genus) of the given class of triangulations of the same data set V.*

The following theorem is valid (for a proof see [AvD01]):

Theorem 4.3 *: All generalised 2.5D triangulations of the data which are in general position and that are topologically equivalent to the 2D sphere are connected under the EDF.*

5 Implementation

We test the EDF procedure on the convex data.

Definition 5.1 *A data set is called convex if all its data sites lie on the boundary of the convex hull of the data.*

We can also say that any vertex of convex data belongs to some global supporting plane of the data. As an optimisation criterion we use the Tight criterion. ¿From the theory of tight embeddings it follows that an optimal triangulation of convex data set should be convex. A convex triangulation is also unique up to *flat* edges. In a convex triangulation all edges are convex. If the data originate from a convex closed surface the convex triangulation coincides with the Tightest triangulation. The latter one is also *tight* in the classical sense. An algorithm based on minimising total absolute extrinsic Gaussian curvature (Tight criterion) is implemented on C^0–level and consists in the following:

1. Compute the Gaussian curvature $\omega(v_i)$ for each vertex.
2. Determine $\mathrm{Str}^+(v_i)$ for each vertex.
3. Calculate $\omega^+(v_i)$ for each vertex.
4. Minimise the value $\hat{\Omega}_{abs}(\bar{\Delta})$ using a local edge–swapping (flipping) procedure.

All the above-mentioned operations are easy; the second operation is slightly more difficult. First this algorithm was developed for triangulations that were conventionally presupposed to be without self–intersections. In that case it has also been proved that for convex data the algorithm is global

[AvD97]. The property of convex (embedded) triangulations were also defined. The algorithm has been now adapted to allow self–intersections (for example, to deal with the degenerate case). If we allow self–intersections then we can have two types of self–intersections: local (which occurs in the star of some vertex, a so–called *pinch point*) and global. If a self–intersection is global then some part of the 'negative' side of a triangulation must be seen from 'outside'. In the case of global self–intersections our algorithm is not global, because in this case a triangulation of convex data does not posses more properties of embedded convex triangulations, and therefore, does not differ from a triangulation of non–convex data. If we have only local self–intersections, then we can show that the algorithm does not yield global self–intersections at any step of the transformation process . It seems also that in this case the algorithm is global. We have been able to extract from different initial triangulations of various convex data (even not in general position) a final convex triangulation of the data (one of the examples is presented in Fig. 14).
The 'twisted double prism' in the left picture of Fig. 14, after a sequence of transformations, is transformed to the final convex polyhedron (on the right). One of the intermediate polyhedra is given in the middle picture.
The details of the algorithmic aspects of our approach will be given in [AvD01].

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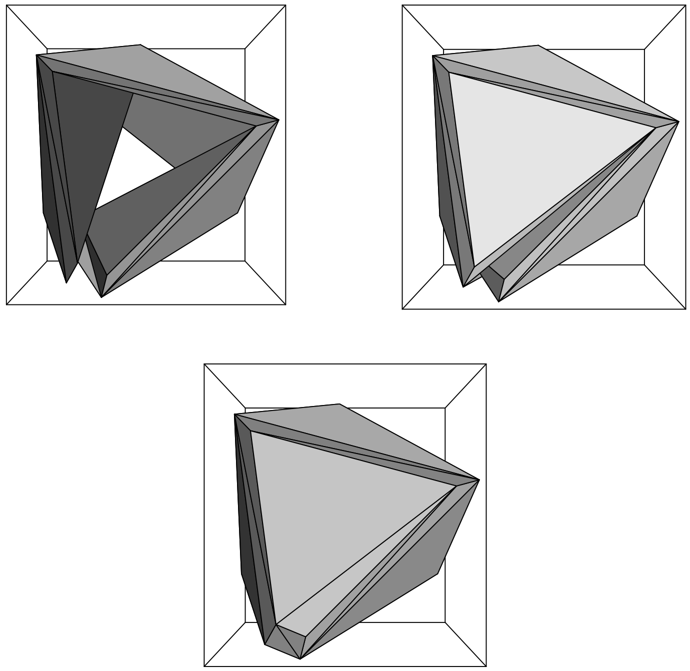

Figure 14: Transformation of the 'twisted prism'

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