On Flips in Polyhedral Surfaces: a new development

[Extended Abstract]

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ABSTRACT

Let V be a finite point set in $3D$ and let $ST(V)$ be the set of closed triangulated polyhedral surfaces with a vertex set V. Those surfaces can be defined as $2.5D$ (closed) triangulations of the given discrete data set V. We generalise the operation of diagonal flip for $2.5D$ triangulations by omitting the usual restriction that the flip operation should not produce a self–intersecting triangulation. We denote this flip operation by *EDF* (*extended diagonal flip*). Among all possible $2.5D$ triangulations with the vertex set V we first single out those that are topologically equivalent to the $2D$ sphere. We show that any two such $2.5D$ triangulations (if V is situated in general position), are equivalent under EDF, *i.e.,* they can be transformed into each other via a finite sequence of EDF.

Keywords

Triangulations, diagonal flips, polyhedral surfaces

1. INTRODUCTION

This paper contributes to the study on diagonal flips in triangulations. We study diagonal flips in non–planar triangulations of discrete data.

Much research on this subject is carried out in the field of computational geometry as well as in topological graph theory. Whereas in topological graph theory an extended study has been done on diagonals flips in triangulations on surfaces of various genera, in computational geometry investigations are limited mostly to diagonal flips in triangulations of point sets or of polygons in the plane.

A *triangulation* T is a partition of a geometric domain into simplices that meet only at shared faces.

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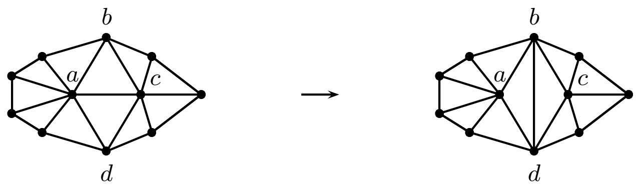

Figure 1: A diagonal flip in a triangulation

In topological graph theory therefore under a *triangulation* on a surface one understands a simple graph G embedded on the surface so that each face is triangular and any two faces share at most one edge [8]. A *diagonal flip* is a local transformation (deformation) of a triangulation G which replaces a diagonal edge ac with the other bd in a quadrilateral region obtained from two triangular faces abc and acd and which preserves the simplicity of the graph (as in Fig. 1).

Let us note that positions of vertices can be changed freely and edges can be bent. One of the research directions concerns the study of *equivalence* (connectedness) of one or another class of triangulations under diagonal flips, *i.e.,* if any two triangulations that belong to the same class can be transformed to each other via sequence of diagonal flips.

The starting point in this research direction belongs to Wagner and is known as Wagner's theorem [9]:

THEOREM 1. *Any two triangulations on the sphere with the same number of vertices are equivalent to each other under diagonal flips, up to homeomorphism.*

The concept of homeomorphism between two graphs can be found in ([8]):

There is a simple algorithm to transform any triangulation of n vertices to the standard form Δ_n, which is depicted in Fig. 2

All standard forms are homeomorphic to each other.

In the field of computational geometry a geometric domain can be a point set, a polygon or a polyhedron. In the pla-

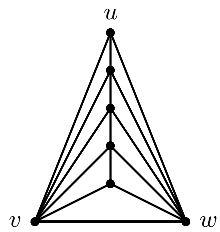

Figure 2: The standard form of spherical triangulations Δ_n

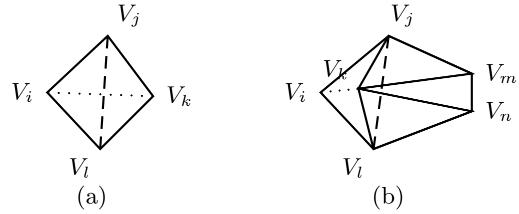

Figure 3: Swapping a diagonal in a quadrilateral: (a) Convex quadrilateral.(b) Non-convex quadrilateral. In both cases the dotted edge swapped to the dashed one.

nar case a triangulation represents a collection of triangles. The difference with respect to topological graph theory is that the points (vertices) have fixed positions and the edges are straight line segments. Therefore, we must distinguish between *convex* and *concave* quadrilaterals. Consequently, not all diagonal flips, which are possible topologically, might be allowed. We call diagonal flips which occur in topological graph theory *abstract diagonal flips.* In the field of computational geometry the diagonal flip (called then an exchange) was first introduced by Lawson [6]. The operation consists of swapping (flipping) a diagonal of the *convex* quadrilateral formed by two adjacent triangles, to the other diagonal, thus replacing one edge by a new one and obtaining another triangulation of the given data. He showed also that any two triangulation s T_1 and T_2 of point set V in domain Q (with the convex hull of Vas the boundary) are connected via a sequence of diagonal flips.

A diagonal flip is an example of a *local transformation*, *i.e.,* an operation that allows us to produce a new object in a class of given objects from previously generated objects by means of some small change [5]. In Surface Reconstruction the diagonal flip, as a local transformation, is often used to obtain an optimal triangulation [3],[2].

Let us note that if we have a given discrete point set in $3D$, we can, in general, construct triangulated surfaces of different genera with the given points as the vertices. In computational geometry such triangulated surfaces are called $2.5D$ triangulations. All $2.5D$ triangulations ST of the data set V can be separated into mutually disjoint classes of triangulations $ST_1, ...ST_l$ in such a way that all triangulations of the same class have the same genus and orientability (*i.e.,* topologically equivalent to the $2D$ sphere, the torus, and so on). It seems logical to assume that these characteristics should be preserved under the edge flip. Let us also note that every triangulation represents in general a new polyhedral surface (if the data are situated in general position). Every flip will produce a new triangulation and therefore a new polyhedral surface.

Traditionally the flip operation is only considered if it does not yield a self–intersection. Indeed, this is a logical restriction on this operation for plane triangulations, since a self–intersection results in producing no triangulation. If the quadrilateral is not convex, the second 'diagonal' will lie outside the 'body' of the quadrilateral and will intersect some other triangles or coincide with the already existing edge, and edge swapping will not produce a triangulation (see Fig. 3)

Unfortunately, the above–mentioned restriction in the case of $2.5D$ triangulations excludes the use of a traditional flip operation as a local transformation procedure. It has been shown that given two traingulations ST_1 and ST_2 with the same data set V, it is possible that there is no sequence of (traditional) flips transforming ST_1 into ST_2 (*i.e.,* at a certain step a triangulation with self–intersections is produced [1]).

2. MAIN RESULTS

The main contribution of the paper is a generalisation of the diagonal flip operation. We call this new operation the *extended diagonal flip*, or simply the *EDF*. A $2.5D$–triangulation (a triangulated surface) is a collection of triangles, which are, in contrast to the $2D$ case, situated not in plane but in space. The EDF can produce self–intersections, but a valid triangulation can still be determined.

The operation of *EDF* preserves the given genus and orientability of a triangulated surface. In this paper we restict ourselves to triangulations of genus 0.

Let us make the concept of a triangulation more accurate. The following definition is standard:

Definition 1. A triangulation T is a collection of triangles, that satisfies the following properties:

1. Two triangles are either disjoint, or have one vertex in common, or have two vertices and consequently the entire edge joining them in common.
2. T is connected.

In the case of compact surfaces a triangulation T consists of a finite number of triangles, and we can conclude that the two following conditions are valid [7]:

1. Each edge is an edge of exactly two triangles.
2. For every vertex V of a triangulation T, we may arrange the set of all triangles with V as a vertex in cyclic order, $T_0, T_1, ..., T_{n-1}, T_n = T_0$, such that T_i and T_{i+1} have an edge in common for $0 \le i \le n-1$.

The last condition means, that our triangulated surface is a manifold, *i.e.,* the neighbourhood of every point, as well as a vertex, is topologically the same as the open unit ball in $\mathbf{R^2}$. The first condition of Definition 1 excludes any intersection. Actually, the above–mentioned definition is the definition of an *abstract triangulation* of an abstract surface [4]. 'Real' surfaces can have self–intersections and/or have singularities, in which a surface is not a manifold. Therefore, we should distinguish between the abstract topological representation of a surface S (as a 2-dimensional *manifold*) and its realisation in 3-space (as an image of this 'canonical' manifold).

Quadrilaterals in the space. In space, if the data are situated in general position, any four vertices form a tetrahedron. We would like to stick to a surface ('plane') terminology, so we prefer to consider two adjacent triangles, formed by edges connecting the vertices, instead of a tetrahedron. The figure, built of two adjacent triangles, will be called a *spatial quadrilateral.* The edges of two triangles except the common edge form a closed polygonal line. We call it the *boundary* of the quadrilateral. If we keep the boundary edges fixed, then there are only two possible spatial quadrilaterals for every four vertices. If a triangulation is given then the boundary of a quadrilateral is always fixed. The common side of two adjacent triangles in a spatial quadrilateral will be called a *(spatial) diagonal.* The orientation of the triangles forming the spatial quadrilateral should be coherent. As we deal with oriented surfaces, we can determine two directions of normals to a surface (and therefore, to each triangle of our oriented surface). Usually, the direction of outwards pointing normals is said to be positive. We can say that an edge (diagonal) is *reflex*, if two lines, determined by the unit normals to the adjacent triangles, sharing this edge as the common side, intersect each other in the positive direction, otherwise, the edge is called *convex.* The edge is *flat* if the normals are parallel.

Then flipping in space means the exchange of a convex edge to a reflex edge and vice versa.

At each step we work only with two triangles, which together form a spatial quadrilateral. As the data are situated in general position and if we connect two vertices of the quadrilateral that are not the end–points of the common edge, the new edge will lie outside the surface of the quadrilateral and we can determine two new triangles. We will call a spatial quadrilateral *convex*, if its development on the plane is a convex quadrilateral; otherwise, we call it *reflex.* Therefore, we have four cases: convex quadrilateral with a convex diagonal, convex quadrilateral with a reflex diagonal, reflex quadrilateral with a convex diagonal, reflex quadrilateral with a reflex diagonal. We can denote the above–mentioned quadrilaterals by **CC, CR, RC, RR**. See Fig. 4, where we present the swapping of the concave (reflex) diagonal in CR and RR quadrilaterals.

We can encounter a degenerated situation when a quadrilateral collapses into two glued together triangles, as any three vertices lie always in a plane. The EDF is also valid for such a degenerated situation, but we need to introduce the definition of a generalised 2.5D triangulation.

Definition 2. (EXTENDED DEFINITION OF A TRIANGULATION) A triangulation T (triangulated surface,

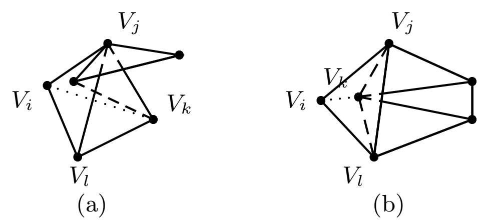

Figure 4: Swapping diagonal V_iV_k in a spatial quadrilateral: (a) CR quadrilateral (b) RR quadrilateral. Vertex V_j on the left picture is a so-called pinch point.

$2.5D$ triangulation) either corresponds to the properties of Definition 1 or, if not, then the following exceptions may occur:

1. For each triangle we can still single out three adjacent triangles, with which it shares a common edge. Two of three adjacent triangles may coincide.
2. Two triangles can have some points in common, besides vertices or edges. If these triangles are not adjacent then they can intersect each other along a line segment. Two adjacent triangles may have a triangular domain in common.

In other words we consider now not only 'pure' simplicial complexes and we allow some singularities. We call such triangulations generalised 2.5D triangulations, or shortly **2.5GD**. Let us note that the above definition is valid also if the data are situated not in general position.

THEOREM 2. *All generalised 2.5D triangulations of the data, which are in general position, and that are topologically equivalent to the 2D sphere are connected under the EDF.*

The proof is based on the following procedure: we put our 2.5D triangulation in one–to–one correspondence to some abstract triangulation (topological graph) on the 2D sphere and then by adapting in a proper way Wagner's theorem, we show that our generalised 2.5D triangulations are connected under the EDF.

COROLLARY 1. *Among 2.5GD (closed) triangulations of the given data there are always some with self–intersections.*

We refer the interested reader to the full version of the paper.

3. REFERENCES

[1] O. Aichholzer, L. Alboul, and F. Hurtado. On flips in polyhedral surfaces. *Report MA2-IR-00-00036, Universitat polytèchnica de Catalunya. (Accepted (conditionally) to Int. J. of Found. of Comp. Sc.)*, 2000.

[2] L. Alboul, G. Kloosterman, C. Traas, and R. van Damme. Best data–dependent triangulations. *J. of Comp. and Applied Math.*, 119:1–12, 2000.

[3] N. Dyn, D. Levin, and A. Rippa. Data dependent triangulations for piecewise linear interpolation. *IMA Journal of Numerical Analysis*, 10:137–154, 1990.

[4] P. Giblin. *Graphs, Surfaces and Homology.* Chapman and Hall, London, 1977.

[5] C. Hernando, F. Hurtado, and M. Houle. On local transformations of simple polygons. *Lecture Notes in Computer Science, Vol. 1858 (Proc. 6th International Computing and Combinatorics Conference (COCOON'00) Sydney)*, pages 54–63, 2000.

[6] C. Lawson. Transforming triangulations. *Discrete mathematics*, 3:365–372, 1972.

[7] W. Massey. *A Basic Course in Algebraic Topology.* Springer-Verlag, Berlin Heidelberg New York, 1991.

[8] S. Negami. Diagonal flips of triangulations on surfaces. *Yokohama Math.Journal (special issue)*, 47:1–40, 1999.

[9] K. Wagner. Bemekungen zum vierfarbenproblem. *J. der Deut. Math.*, 46(1):26–32, 1936.