The Diffiety Institute Preprint Series

Preprint DIPS–09 **November 4, 2002**

A simple method to prove locality of symmetry hierarchies

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A simple method to prove locality of symmetry hierarchies

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Abstract. A simple method to check locality of symmetry hierarchies generated by recursion operators is described.

Recursion operators for symmetries arising in studies of integrable systems usually contain terms of the type D_x^{-1}, [3]. For example, for the classical KdV equation

$$u_t = u_{xxx} + uu_x$$

the recursion operator is of the form

$$R = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1} \tag{1}$$

while for the modified KdV

$$u_t = u_{xxx} + u^2 u_x$$

one has

$$R = D_x^2 + \frac{2}{3}u^2 + \frac{2}{3}u_x D_x^{-1} \circ u. \tag{2}$$

Therefore, it is always a problem to prove whether the obtained hierarchies are local. We know no simple methods to prove this fact in reasonable generality. The proof given in [2] for the KdV, being undoubtedly quite beautiful, suggests no hints how to generalize it to other equations.

Below, we describe a simple method to prove locality applicable, at least, for polynomial (x,t)-independent evolution equations possessing a scaling symmetry. Let $\mathscr{E}$

$$u_t = F(u, u_1, \dots, u_k) \tag{3}$$

be our equation, where, as usual, $u_i = \partial^i u / \partial x^i$.

1. A general observation

Let a and b be two 'quantities' on $\mathscr{E}$ satisfying the equations

$$\ell_{\mathscr{E}}(a) = 0, \qquad \ell_{\mathscr{E}}^*(b) = 0$$

and assume that the pairing $\langle b, a\rangle$ makes sense taking its values in n-cochains of some horizontal de Rham complex. Then, by Green's formula

$$\begin{aligned} D_t\langle b, a\rangle &= \langle D_t b, a\rangle + \langle b, D_t a\rangle = \langle -\ell_F^*(b), a\rangle + \langle b, \ell_F(a)\rangle \\ &= \langle b, -\ell_F(a)\rangle + \langle b, \ell_F(a)\rangle + \bar{d}(\omega) = \bar{d}(\omega), \end{aligned} \tag{4}$$

i.e., $D_t\langle b, a\rangle$ is a cocycle.

2000 *Mathematics Subject Classification.* 35Q53.
Key words and phrases. Recursion operators.

2. A particular case

Let us apply (4) to the case when $a = S \in \varkappa$ is a symmetry and $b = G \in \hat{\varkappa}$ is a generating function. Then (4) reads

$$D_t\langle G, S\rangle = D_x(T_{G,F}), \qquad T_{G,F} \in \mathscr{F}(\mathscr{E}), \tag{5}$$

i.e., $\omega_{G,F} = \langle G, S\rangle\, dx + T_{G,F}\, dt$ is a conservation law[1]. From this fact we obtain

Proposition 1. *The operator $D_x^{-1} \circ G$, applied to S, produces a local quantity if and only if $\omega_{G,F}$ is a trivial conservation laws.*

Of course, in such a form the result looks rather tautological, but nevertheless, taking in account additional considerations, one may reach the desirable result in some cases.

Recall [1] that if $\omega = X\, dx + T\, dt$ is a conservation law, then $g_\omega = \mathcal{E}(X)$, where $\mathcal{E}$ is the Euler operator, is called its generating function and satisfies the equation $\ell_{\mathscr{E}}^*(g_\omega) = 0$. A conservation law is trivial if and only if $g_\omega = 0$.

3. Examples

Let us consider three examples. In all three cases we make use of the fact that the equations at hand are scaling-invariant and thus admit gradings. Consequently, all objects under consideration (symmetries, generating functions, etc.) may be considered homogeneous with respect to these gradings.

Example 1 (Burgers equation)**.** Consider the equation

$$u_t = u_{xx} + uu_x. \tag{6}$$

It is known to possess the recursion operator

$$R = D_x + \frac{1}{2}u + \frac{1}{2}u_1 D_x^{-1}. \tag{7}$$

This equation becomes homogeneous if we introduce the following gradings:

$$[x] = 1,\ [t] = 2,\ [u] = -1,\ [u_k] = -(k+1).$$

Equation (6) possesses the only generating function equal to 1. Hence, by (5), $\mathcal{E}(S)$ equals either 1 or 0 for any symmetry S of the Burgers equation, $\mathcal{E}$ being the Euler operator.

Symmetries belonging to the classical (i.e., (x,t)-independent) hierarchy of (6) are of the form

$$S_k = u_k + \text{ terms of lower order}, \quad k = 1, 2, \dots,$$

and hence $[S_k] = -(k+1)$. Consequently, $[\mathcal{E}(S_k)] = [S_k] - [u] = -k < 0$. Therefore, $\mathcal{E}(S_k) = 0$, i.e., all S_k are total derivatives. This means that the action of D_x^{-1} on the classical hierarchy is well defined.

Example 2 (KdV equation)**.** The KdV equation becomes homogeneous if we assign the following gradings:

$$[x] = 1,\ [t] = 3,\ [u] = -2,\ [u_k] = -(k+2).$$

[1] This observation belongs to P.H.M. Kersten.

The symmetries of the classical hierarchy are of the form

$$S_k = u_{2k-1} + \text{ terms of lower order}, \quad k = 1, 2, \dots,$$

and hence $[S_k] = -(2k+1)$. Consequently,

$$[\mathcal{E}(S_k)] = [S_k] - [u] = -(2k-1). \tag{8}$$

On the other hand, generating functions of the KdV equation are

$$G_0 = 1,\ G_k = u_{2k-2} + \text{ terms of lower order}, \quad k = 1, 2, \dots$$

Therefore, $[G_k] = -2k$, $k = 0, 1, \dots$ Comparing these gradings with (8), we deduce that, as in the previous example, $\mathcal{E}(S_k) = 0$ which means that the action of D_x^{-1} on the classical hierarchy of symmetries produces local quantities.

Example 3 (mKdV equation)**.** The mKdV equation becomes homogeneous if we assign the following gradings:

$$[x] = 1,\ [t] = 3,\ [u] = -1,\ [u_k] = -(k+1).$$

The symmetries of the classical hierarchy are of the form

$$S_k = u_{2k-1} + \text{ terms of lower order}, \quad k = 1, 2, \dots,$$

and hence $[uS_k] = -(2k+1)$. Consequently,

$$[\mathcal{E}(uS_k)] = [uS_k] - [u] = -2k. \tag{9}$$

Generating functions of the KdV equation are

$$G_0 = 1,\ G_k = u_{2k-2} + \text{ terms of lower order}, \quad k = 1, 2, \dots$$

and one has

$$[G_0] = 0,\ [G_k] = -(2k-1), \quad k > 0.$$

Comparing these gradings with (9), we again see that uS is always a total derivative and thus arrive to the final result: the operator $D_x^{-1} \circ u$ produces local quantities when acting on the classical mKdV hierarchy.

Remark 1. By yet unclear reasons all recursion operators (i.e., all operators found in our computations with Paul Kersten) can be presented in the form

$$R = \text{ local part } + \sum_\alpha S_\alpha\, D_x^{-1} \circ G_\alpha,$$

where G_α are generating functions and S_α are symmetries (unfortunately, they may be nonlocal, and in this case our method will not work).

Acknowledgments

The author is grateful to A. Verbovetsky and S. Igonin for discussions an finding a severe mistake in the draft version.

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