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Part I: Triangulations

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# Stars and bunches in planar graphs.

## Part I: Triangulations <sup>\*</sup>

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### Abstract

Given a plane graph, a  $k$ -star at  $u$  is a set of  $k$  vertices with a common neighbour  $u$ ; and a bunch is a maximal collection of paths of length at most two in the graph, such that all paths have the same end vertices and the edges of the paths form consecutive edges (in the natural order in the plane graph) around the two end vertices. We prove a theorem on the structure of plane triangulations in terms of stars and bunches. The result states that a plane triangulation contains a  $(d - 1)$ -star centred at a vertex of degree  $d \leq 5$  and the sum of the degrees of the vertices in the star is bounded, or there exists a large bunch.

*Keywords:* planar graph, discharging method, colouring

*AMS Subjects Classifications:* 05C15, 05C12

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## 1 Introduction

Throughout this paper,  $G$  is a plane graph (i.e., a representation in the plane of a planar graph), that is simple (i.e., without loops and multiple edges) and with vertex set  $V$  and edge set  $E$ . A  $k$ -star at  $u$  is a set of  $k$  vertices with a common neighbour  $u$ . A *bunch* is a maximal collection of paths of length at most two in the graph, such that all paths have the same end vertices and the edges of the paths form consecutive edges (in the natural order in the plane graph) around the two end vertices. The *weight* of a subgraph is the sum of the degrees of the vertices in that subgraph.

A significant amount of research has been done on the structure of plane triangulations, especially concerning bounds on the weights of small subgraphs. See for instance [1,2,4,5] and references in those. In [4] the conjecture of Kotzig (1978) that a plane triangulation with minimum degree 5 contains a cycle of length 4 of weight at most 25 is proved. Another result, more directly related to the main theorem in this paper, can be found in [2] and gives a best possible upper bound on the minimum weight of a face in a plane triangulation depending on the maximum length of a path of vertices of degree 4 in the graph.

The proof of Kotzig's Conjecture in [4] is based on the existence in plane triangulations with minimum degree 5 of a 4-star of weight at most 25 centred at some vertex of degree 5. On the other hand, for triangulations that contain vertices with degree less than 5 it is impossible to give a maximum value for the weight of a  $(d - 1)$ -star centred at a vertex of degree  $d \leq 5$  (a so-called *minor vertex*). For instance, the  $n$ -bipyramid shows that every minor vertex in a plane triangulation can be adjacent to at least two vertices of arbitrarily high degree.

In this paper we prove a theorem on the structure of plane triangulations in terms of stars centred at minor vertices and bunches. (Note that a large bunch in a plane triangulation always contains a long path of vertices of degree 4.) We prove that a plane triangulation contains a  $(d - 1)$ -star centred at a vertex of degree  $d \leq 5$  of bounded weight, if and only if there is no large bunch. The bound of the size of the bunch in the main theorem is best possible. In a sequel paper [3], this result is generalised to general planar graphs. That generalisation is used to prove a best possible upper bound on the minimum degree and on the minimum number of colours needed in a greedy colouring of the square of a planar graph.

## 2 Definitions and result

Throughout this paper,  $G$  is a plane graph (i.e., a representation in the plane of a planar graph), that is simple (i.e., without loops and multiple edges) and with vertex set  $V$  and edge set  $E$ . The *distance* between two vertices  $u$  and  $v$  is the length of a shortest path joining them. We are mainly interested in pairs at distance one or two, for which we also can define: a pair of vertices  $u, v$ ,  $u \neq v$ , have distance one if they are adjacent; and they have distance two if they are not adjacent but have a common neighbour.

In this paper we will prove a result on so-called unavoidable configurations in plane triangulations. In the sequel [3] this is generalised to plane graphs in general, and used to prove an

upper bound on the number of colours needed for a planar graph in which vertices at distance one or two have different colours (a so-called *distant-2-colouring*). Some background and earlier work on distant-2-colourings can also be found in [3].

Before we can state our main result, we need some more definitions.

We say that  $G$  has a *bunch* of length  $m \geq 3$  with *poles* at vertices  $p$  and  $q$ , where  $p \neq q$ , if  $G$  contains a sequence of paths  $P_1, P_2, \dots, P_m$  with the following properties. Each  $P_i$  has length 1 or 2 and joins  $p$  with  $q$ . Furthermore, for each  $i = 1, \dots, m-1$ , the cycle formed by  $P_i$  and  $P_{i+1}$  is not separating in  $G$  (i.e., has no vertex of  $G$  inside) (see Fig. 2.1). Moreover, this sequence of paths is maximal in the sense that there is no path  $P_0$  (or  $P_{m+1}$ ) that could be added to the bunch, preserving the above properties.

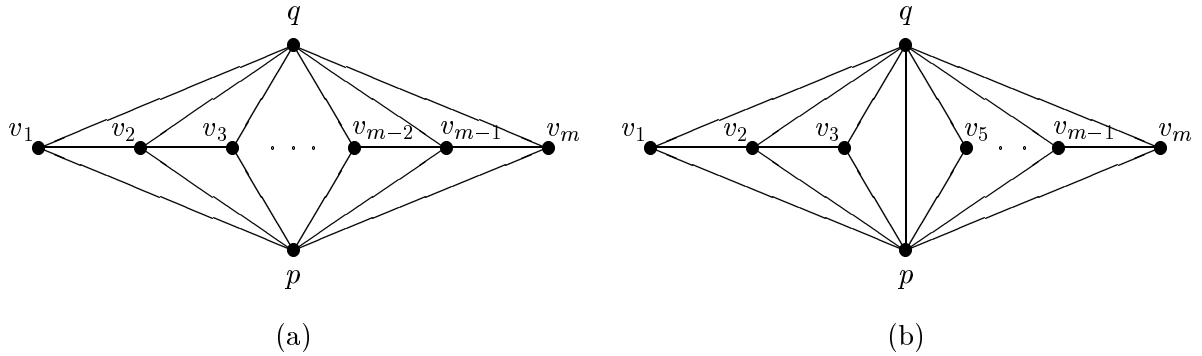


Fig. 2.1: A bunch without a parental edge (a) and with a parental edge (b)

If a path  $P_i$  in the bunch has length 2, i.e.,  $P_i = pv_iq$ , then the vertex  $v_i$  will be called a *brother* or a *bunch vertex*. A path  $P_i = pq$  of length 1 in the bunch will be referred to as a *parental edge* (Fig. 2.1(b)).

If the cycle bounded by  $P_1$  and  $P_m$  separates  $G$ , then the edges in  $P_1$  and  $P_m$  are called *boundary edges*, and the vertices  $v_1$  and  $v_m$  (if they exist) are the *end vertices* (or *ends*) of the bunch. The vertex  $v_i$  in the bunch is *interior* if  $2 \leq i \leq m-1$  and *strictly interior* if  $3 \leq i \leq m-2$ . Each edge  $v_i v_{i+1}$  joining consecutive bunch vertices is called *horizontal*, while the edges of the  $P_i$ 's are called *vertical* in the bunch. Observe that each interior vertex has degree 2, 3 or 4 and is adjacent only to the poles and possibly to one or two brothers.

A  $d$ -vertex in  $G$  is a vertex of degree  $d$ . The  $B$ -vertices in  $G$  are those of degree at least 26,  $L$ -vertices have degree at most 25, and *minor vertices* at most 5.

Let  $u$  be a  $d$ -vertex, and let  $v_1, \dots, v_k$  be adjacent to  $u$ . We say that the vertices  $v_1, \dots, v_k$  form a  $k$ -star at  $u$ , of *weight*  $\sum_{i=1}^k d(v_i)$ . Each  $(d-1)$ -star at  $u$  is called *precomplete*, and each  $d$ -star *complete*.

The following is the main result in this paper.

**Theorem 2.1**

For each plane triangulation  $G$  at least one of the following holds :

- (A)  $G$  has a precomplete star of weight at most 38 that does not contain B-vertices and is centred at a minor vertex.
- (B)  $G$  has a B-vertex  $b$  that satisfies at least one of the following conditions :
  - (i)  $b$  is a pole for a bunch of length greater than  $d(b)/5$ ;
  - (ii)  $b$  is a pole for a bunch of length precisely  $d(b)/5$  with a parental edge;
  - (iii)  $b$  is a pole for 5 bunches of length  $d(b)/5$  without parental edges and with pairwise different end vertices. Moreover, all but possibly one end vertices have degree 5, while the other end vertex has degree at most 11 (see Fig. 2.2).

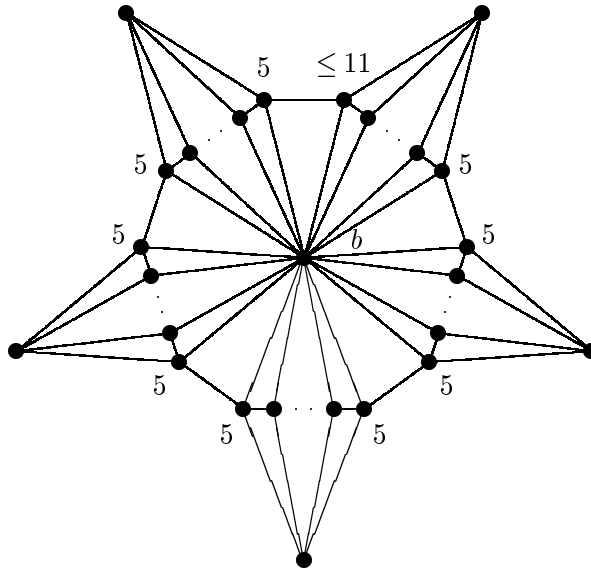


Fig. 2.2

**3 Proof of Theorem 2.1**

Let  $G$  be a counterexample, then the following facts are obvious :

- (A') Each precomplete star in  $G$  at a minor vertex either contains a B-vertex or has weight at least 39.
- (B') Each B-vertex  $b$  in  $G$  can be a pole for bunches of length at most  $d(b)/5$ .

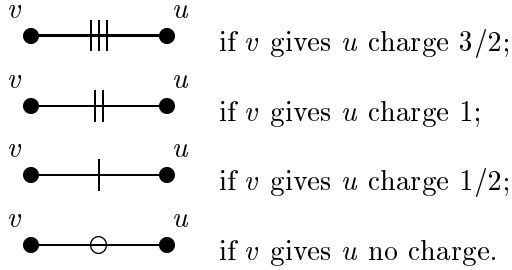
Euler's formula for  $G$  can be written as

$$\sum_{v \in V} (d(v) - 6) = \sum_{v \in V} \mu(v) = -12. \tag{3.1}$$

Here,  $\mu(v) = d(v) - 6$  is called a *charge* of  $v \in V$ . Observe that only minor vertices in  $G$  have a negative charge. We redistribute the charges among the vertices of  $G$  so that each  $v \in V$  gets a nonnegative *new charge*  $\mu^*(v)$ , while the sum of all charges in  $G$  remains the same. This will contradict (3.1):

$$0 \leq \sum_{v \in V} \mu^*(v) = \sum_{v \in V} \mu(v) = -12. \quad (3.2)$$

We will say that  $v$  gives  $u$  charge  $c$  if an amount  $c$  is subtracted from  $\mu(v)$ , transferred and added to  $\mu(u)$ . We use the following notation:



The edge  $vu$  will be called *sesquialteral* in the first case, *unitary* in the second, *half* in the third, and *zero* in the last case. The direction of transferring charge will always be clear from the context.

Next, we define the rules R1 – R3 of transferring charge from vertices of degree at least 9 to minor vertices so that the new charge of each minor vertex becomes nonnegative.

**R1:** Let  $u$  be a 4-vertex adjacent to vertices  $v_1, v_2, v_3, v_4$  in a cyclic order. By (A'), at least two of the  $v_i$ 's have degree at least 12.

- (a) If  $d(v_i) \geq 12$  for all  $i = 1, \dots, 4$ , then each  $v_i$  gives  $u$  charge 1 ( Fig. 3.1 (a) ).
- (b) If  $d(v_1) \leq 11$  and  $d(v_i) \geq 12$  for  $i = 2, 3, 4$ , then  $v_3$  gives  $u$  charge 1, while each of  $v_2$  and  $v_4$  gives  $1/2$  ( Fig. 3.1 (b) ).
- (c) If  $d(v_1) \leq 11$ ,  $d(v_3) \leq 11$ ,  $d(v_2) \geq 12$ , and  $d(v_4) \geq 12$ , then each of  $v_2$  and  $v_4$  gives  $u$  charge 1 ( Fig. 3.1 (c) ).
- (d) If  $9 \leq d(v_1) \leq 11$ ,  $9 \leq d(v_2) \leq 11$ ,  $d(v_3) \geq 12$ , and  $d(v_4) \geq 12$ , then each  $v_i$  gives  $u$  charge  $1/2$  ( Fig. 3.1 (d) ).
- (e) If  $d(v_1) \leq 8$ ,  $9 \leq d(v_2) \leq 11$ ,  $d(v_3) \geq 12$ , and  $d(v_4) \geq 12$ , then  $v_3$  gives  $u$  charge 1, while each of  $v_2$  and  $v_4$  gives  $1/2$  ( Fig. 3.1 (e) ).
- (f) Finally, if  $d(v_1) \leq 8$ ,  $d(v_2) \leq 8$ ,  $d(v_3) \geq 12$ , and  $d(v_4) \geq 12$ , then each of  $v_3$  and  $v_4$  gives  $u$  charge 1 ( Fig. 3.1 (f) ).

**R2:** Let  $u$  be a 3-vertex with neighbours  $v_1, v_2, v_3$ . By (A'), at least two of the  $v_i$ 's have degree at least 12.

- (a) If  $d(v_i) \geq 12$  for all  $i = 1, 2, 3$ , then each  $v_i$  gives  $u$  charge 1 ( Fig. 3.2 (a) ).
- (b) If  $d(v_1) \leq 11$ , then due to (A') both  $v_2$  and  $v_3$  are B-vertices. Then each of  $v_2$  and  $v_3$  gives  $u$  charge  $3/2$  ( Fig. 3.2 (b) ).

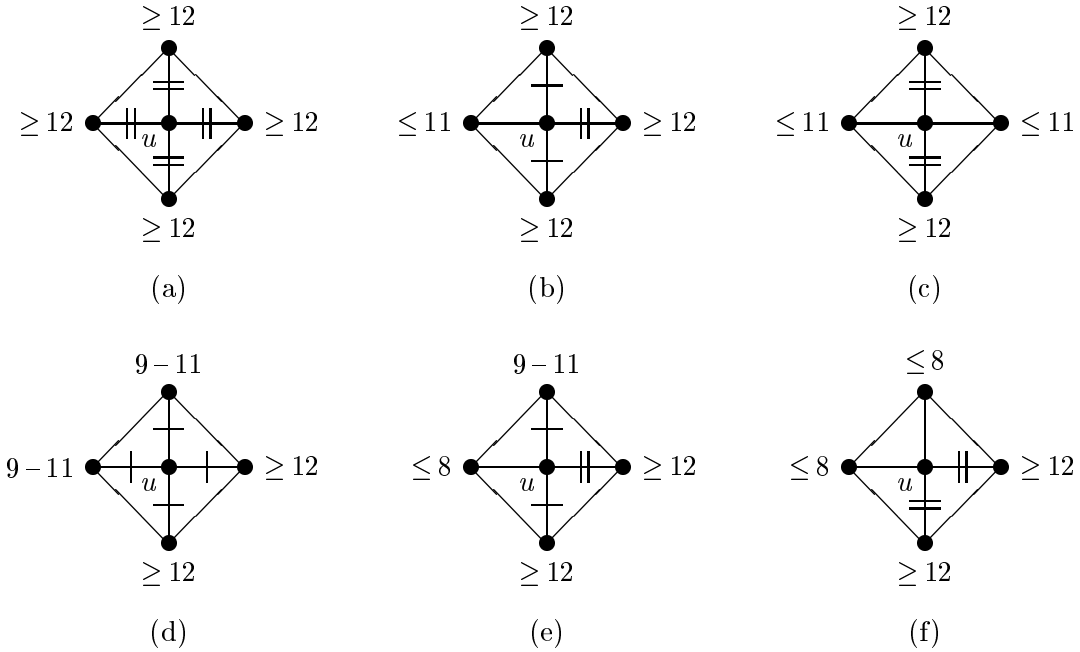


Fig. 3.1: The rules of R1

**R3:** Let  $u$  be a 5-vertex, and let  $v_1, \dots, v_5$  be its neighbours in a cyclic order. By (A'), at least two of the  $v_i$ 's have degree at least 9.

- (a) If each of  $v_{i-1}$ ,  $v_i$  and  $v_{i+1}$  is a B-vertex, then  $v_i$  gives  $u$  charge 1. Suppose this is not the case, but  $v_i$  has degree at least 9 and at least one of  $v_{i-1}$  and  $v_{i+1}$  also has degree at least 9, then  $v_i$  gives  $u$  charge  $1/2$  (Fig. 3.3 (a)).
- (b) If  $u$  has no two consecutive neighbours of degree at least 9, then each of its two neighbours of degree at least 9 gives  $u$  charge  $1/2$  (Fig. 3.3 (b)).

REMARK. Rule R3 implies that each neighbour  $v_1$  of degree at least 9 of a 5-vertex  $u$  always gives a positive charge to  $u$ , unless  $d(v_2) \leq 8$ ,  $d(v_3) \geq 9$ ,  $d(v_4) \geq 9$ , and  $d(v_5) \leq 8$ , in which case

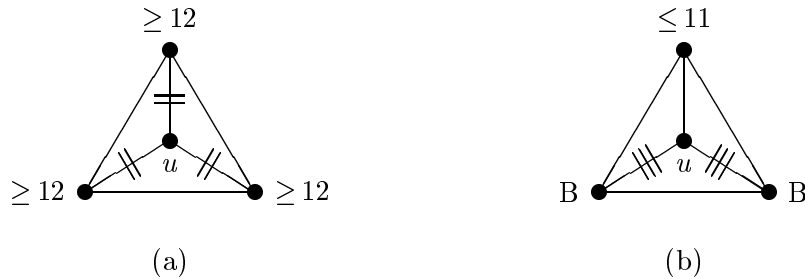


Fig. 3.2: The rules of R2

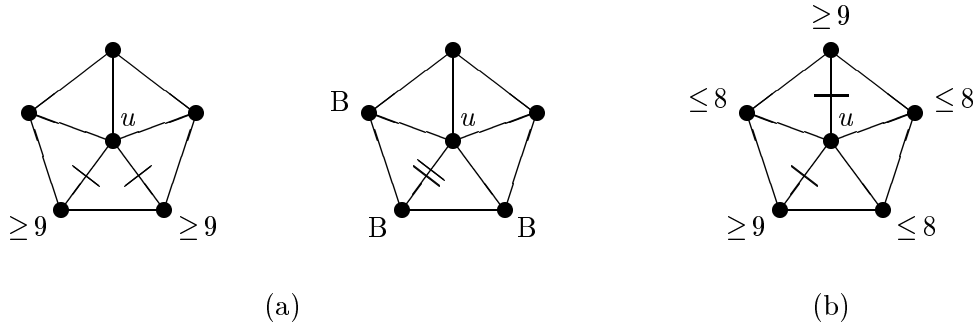


Fig. 3.3: The rules of R3

each of  $v_3$  and  $v_4$  gives  $1/2$  to  $u$  by a), while  $v_1$  is exempted from transferring charge.

The new charge of  $v \in V$  after applying R1–R3 is denoted by  $\mu^*(v)$ .

**Lemma 3.1** (on L-vertices)

Each L-vertex  $v \in V$  satisfies  $\mu^*(v) \geq 0$ .

**Proof** It follows directly from R1–R3 that  $\mu^*(v) \geq 0$  if  $d(v) \leq 8$ .

Our next goal is to prove  $\mu^*(v) \geq 0$  if  $9 \leq d(v) \leq 25$ . We estimate the total donation of  $v$  according to R1–R3 by means of a simple averaging argument. The *generous* donation of  $v$  to its minor neighbour  $u_i$  is defined as follows:

Let  $\lambda(v) = \mu(v)/d(v)$ . Then  $v$  (generously) gives  $u_i$  the following charge:

- $2\lambda(v)$  if none of  $u_{i-1}$ ,  $u_{i+1}$  is minor;
- $3\lambda(v)/2$  if precisely one of  $u_{i-1}$ ,  $u_{i+1}$  is minor, and
- $\lambda(v)$  if both  $u_{i-1}$  and  $u_{i+1}$  are minor.

Clearly,  $v$  gives to all its minor neighbours at most  $\mu(v)$  in total. To see this, imagine that  $v$  first sends  $\lambda(v)$  to each neighbour, i.e., precisely  $\mu(v)$  in total, and then the donation to a non-minor neighbour  $u_k$  is shared by  $\lambda(v)/2$  between  $u_{k-1}$  and  $u_{k+1}$ . As a result, each minor neighbour gets from  $v$  in this imaginary experiment exactly what is prescribed by the generous scheme.

It remains to show that in practice, i.e., according to R1–R3, each minor neighbour gets from  $v$  not more than under the generous scheme.

Observe that  $\lambda(v) \geq 1/3$  if  $d(v) \geq 9$ ,  $\lambda(v) \geq 1/2$  if  $d(v) \geq 12$ , and  $\lambda(v) \geq 2/3$  if  $d(v) \geq 18$ . This clearly implies that the generous donation of our  $v$  is not less than by R1–R3 everywhere except for possibly in R1 (c), R1 (f) and R3 (b).

Let us prove the same for the remaining cases. First consider R1 (c). If neither  $v_1$  nor  $v_3$  is minor, the statement is obvious. Suppose that precisely one of  $v_1$  and  $v_3$  is minor. By (A') (for  $u$ ), we have  $d(v_2) > 18$  and  $d(v_4) > 18$ , so that  $v_2$  and  $v_4$  give at least as much as required by R1–R3. Finally, if both  $v_1$  and  $v_3$  are minor, then both  $v_2$  and  $v_4$  are B-vertices by (A').

Now let us consider R1 (f). By (A'), we have  $d(v_3) > 18$  and  $d(v_4) > 18$ , and the same argument works. Finally, in the case R3 (b), using (A') again, we deduce that  $d(v_1) > 12$  and  $d(v_3) > 12$ , whence the statement follows. This completes the proof of Lemma 3.1.  $\square$



Now suppose that  $G$  has a B-vertex  $b$  such that  $\mu^*(b) < 0$ . Let  $b$  have degree  $d$ , and denote its neighbours in a cyclic order by  $v_1, v_2, \dots, v_d$ .

**Lemma 3.2** (structural)

- (a) If  $b$  is incident with a sesquialteral edge  $bv_2$ , then precisely one of  $v_1$  and  $v_3$  is a B-vertex (so that the corresponding edge  $bv_1$  or  $bv_3$  is zero).
- (b) If  $b$  is incident with a half edge  $bv_2$ , then at most one of  $v_1$  and  $v_3$  is a B-vertex.
- (c) Suppose  $b$  has a B-neighbour  $v_1$ , while the edge  $bv_2$  is unit. Then one of the following statements is true:
  - (i)  $bv_3$  is zero (Fig. 3.4 (a));
  - (ii)  $bv_3$  is a half edge (Fig. 3.4 (b));
  - (iii)  $bv_3$  is unit and  $v_4$  is a B-vertex (whence  $bv_4$  is zero) (Fig. 3.4 (c)).

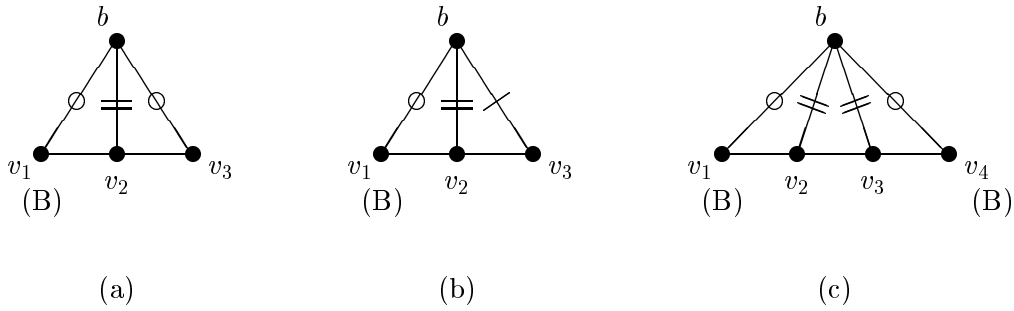


Fig. 3.4

**Proof** For (a), see R2 (b); for (b), apply R1 (b,d,e) and R3.

Let us prove (c). Assume that neither (i) nor (ii) hold. Then we show that (iii) should hold. Indeed,  $bv_2$  is unit, while  $bv_3$  is either unit or sesquialteral. It follows that both  $v_2$  and  $v_3$  are minor.

We now prove  $d(v_2) = 4$ . Observe that  $v_2$  has two B-neighbours,  $b$  and  $v_1$ , and a minor neighbour  $v_3$ . Hence if  $d(v_2) = 3$ , then  $bv_2$  is sesquialteral by R2 (b), contrary to the hypotheses of the lemma. Suppose  $d(v_2) = 5$ . Then due to R3,  $bv_2$  is unit only if each of  $v_1$  and  $v_3$  is a B-vertex. Since  $v_3$  is already known to be minor, it follows that  $d(v_2) = 4$ .

Let  $t$  be the neighbour of  $v_2$  other than  $b$ ,  $v_1$  or  $v_3$  (Fig. 3.5 (a)). Note that  $d(t) \leq 8$ , for otherwise  $bv_2$  would be a half edge by R1 (b,e). Hence,  $d(v_2) + d(t) \leq 12$ , and due to (A') for  $v_3$  we have  $d(v_3) \neq 3$ . It follows from R3 that  $d(v_3) \neq 5$ , because  $v_2$  is an L-vertex. Thus,  $d(v_3) = 4$ . Furthermore, the vertex  $v_4$  adjacent to  $v_3$  is a B-vertex, for otherwise  $v_3$  would have a precomplete star on  $v_2$ ,  $v_4$  and  $t$  of weight at most 37, contrary to (A') (Fig. 3.5 (b)). Finally, observe that  $bv_3$  is unit by R1 (f). This implies part (iii) of (c), and completes the proof of Lemma 3.2.  $\square$

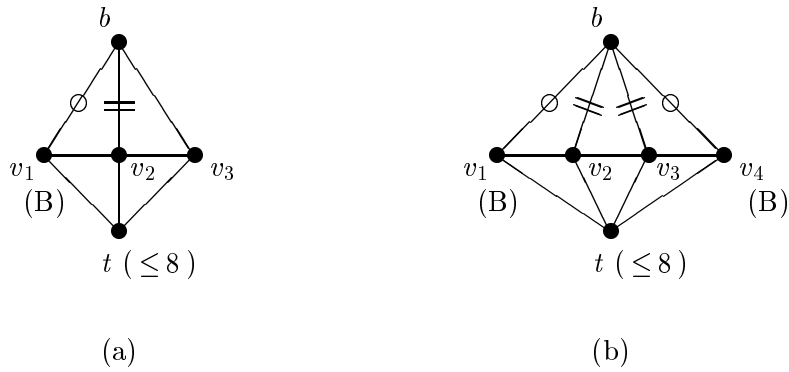


Fig. 3.5

To estimate the total donation of  $b$  along incident edges, we introduce the following averaging rule AR.

**AR** (averaging rule): Let  $bv_i$  transfer charge  $\lambda_i \neq 0$ , and let  $v_{i+1}$  be a B-vertex. Then  $b$  shifts charge  $1/2$  from  $bv_i$  to  $bv_{i+1}$  (Fig. 3.6 (a)). As a result,  $bv_{i+1}$  becomes at least a half edge (it becomes unit if it gets  $1/2$  also from  $bv_{i+2}$ ). If  $v_{i-1}$  is an L-vertex, then  $bv_i$  now takes away charge  $\lambda_i - 1/2$  (Fig. 3.6 (b)). However, if  $v_{i-1}$  is a B-vertex, then  $bv_i$  also shifts  $1/2$  to  $bv_{i-1}$ , so that  $bv_i$  finally transfers  $\lambda_i - 1$  (Fig. 3.6 (c)).

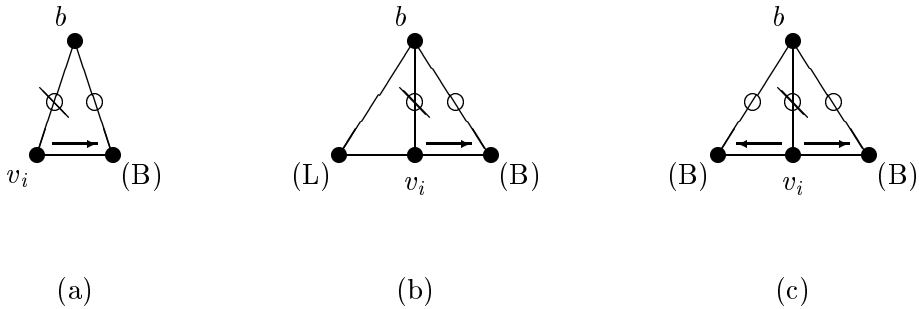


Fig. 3.6

To see that AR is well-defined if  $bv_i$  is a half edge, use Lemma 3.2 (b). It says that a half edge  $bv_i$  can shift  $1/2$  to at most one of the two immediate neighbour edges. From Lemma 3.2 (a) it follows that each sesquialteral edge incident with  $b$  is made by AR into unit. Hence,  $b$  has no more sesquialteral edges after applying AR. Also observe that the shift from unit edge  $bv_2$  to zero edge  $bv_1$ , results in one of the cases (i)–(iii) described in Lemma 3.2 (c). Finally, if  $b$  is incident with a zero edge leading to an L-vertex, then this edge remains zero after applying AR. As for zero edges leading to B-vertices, they clearly become either half or unit edges, depending on the number of  $1/2$ 's obtained. Let us formulate another useful consequence of AR.

**Claim 3.3** (on unit edges)

If the edge  $bv_i$  becomes unit after applying AR, then  $v_i$  is either a B-vertex or has degree 3 or 4.

**Proof** Observe that if  $d(v_i) = 5$  and  $bv_i$  was unit before averaging, then  $bv_i$  becomes zero by R3 (a). If  $d(v_i) = 5$  and  $bv_i$  was half or zero, then it cannot become unit. It follows that if  $v_i$  is minor, then its degree is 3 or 4.

Now suppose  $d(v_i) > 5$ . Then  $bv_i$  was zero initially, and gets  $1/2$  from each of  $bv_{i-1}$  and  $bv_{i+1}$ . This is only possible if  $v_i$  is a B-vertex. This completes the proof.  $\square$

From now on, by zero, half and unit edges in the vicinity of  $b$  we mean those AFTER averaging. Denote their numbers by  $e_0$ ,  $e_{1/2}$ , and  $e_1$ , respectively. From  $\mu^*(b) < 0$  it follows that

$$2e_0 + e_{1/2} \leq 11. \quad (3.3)$$

By a *prebunch* of length  $k$  we mean any maximal (non-extendable) sequence of  $k$  consecutive unit edges in the vicinity of  $b$ . (It cannot be extended either because its boundary edges are not unit, or if  $k = d$ .) A *separator* of length  $\ell$  is a sequence of  $\ell$  consecutive non-unit edges in the vicinity of  $b$  bounded from both sides by unit edges (here,  $\ell = d$  or  $\ell = d - 1$  is impossible due to (3.3)).

Thus, the set of edges in the vicinity of  $b$  is split into disjoint and alternating prebunches and separators. Clearly, their numbers are the same, unless all edges incident with  $b$  are unit. Sometimes we shall refer to a non-unit edge as *separating*. It follows from (3.3) that  $b$  sees at most 11 separating edges.

**Claim 3.4** (on the boundary of a separator)

*If  $bv_1$  shifts  $1/2$  to  $bv_2$  by AR, then either  $bv_2$  becomes unit after applying AR, or each of  $bv_2$  and  $bv_3$  becomes separating. In particular, if  $bv_1$  is separating, then  $bv_2$  cannot be a boundary edge in a separator that contains  $bv_1$ .*

**Proof** If  $bv_3$  was not zero before averaging, then it shifts  $1/2$  to  $bv_2$ , so that  $bv_2$  becomes unit. If  $bv_3$  was zero, it cannot become unit because it receives nothing from  $bv_2$ . In this case, both  $bv_2$  and  $bv_3$  become separating, as claimed.  $\square$

Our next lemma explains the role of prebunches and shows how helpful AR is.

**Lemma 3.5** (on prebunches)

*Each prebunch of length  $k \geq 3$  in the vicinity of  $b$  is a part of a bunch of length at least  $k + 2$  with one pole at  $b$  and the other at a B-vertex  $t$ . Moreover, all the edges of the prebunch are consecutive vertical edges of the bunch, and none of them is a boundary edge in the bunch.*

**Proof** Let our prebunch consist of the edges  $bv_1, bv_2, \dots, bv_k$ . Claim 3.3 implies that  $v_i$  is either a B-vertex or has degree 3 or 4 whenever  $i = 1, \dots, k$ . We next prove that if  $d(v_i) = 3$  (where  $i = 1, \dots, k$ ), then  $v_i$  has precisely two B-neighbours in  $G$ . It suffices to prove that  $bv_i$  transfers charge according to R2 (b). Assume otherwise that R2 (a) takes place. Then before averaging,  $bv_i$  was unit, and both  $bv_{i-1}$  and  $bv_{i+1}$  were zero. Note that none of the two latter

can become unit, because there is no shift from  $bv_i$ . This contradiction proves that  $v_i$  has two B-neighbours (one of which is  $b$ ).

Next, we define an “other pole” map  $\pi : \{v_1, \dots, v_k\} \rightarrow V$  as follows. If  $v_i$  is a B-vertex, we put  $\pi(v_i) = v_i$ . If  $d(v_i) = 3$ , then let  $\pi(v_i)$  be the only B-neighbour of  $v_i$  other than  $b$ . If  $d(v_i) = 4$ , then  $\pi(v_i)$  is defined to be the vertex opposite to  $b$  in the vicinity of  $v_i$ .

Let us prove that the image of all  $\pi$  consists of one point; i.e., that  $G$  has a B-vertex  $t$  such that  $\pi(v_i) = t$  for all  $i = 1, \dots, k$ . To this end, it suffices to prove  $\pi(v_i) = \pi(v_{i+1})$  whenever  $i = 1, \dots, k - 1$ .

If  $d(v_i) = 3$ , then one of  $v_{i-1}$  and  $v_{i+1}$  was proved to be a B-vertex. If this B-vertex is  $v_{i+1}$ , then the definition of  $\pi$  implies  $\pi(v_i) = \pi(v_{i+1}) = v_{i+1}$ . Suppose this B-vertex is  $v_{i-1}$ . Then as proved above,  $v_{i+1}$  has degree 3 or 4. If  $d(v_{i+1}) = 3$ , then  $G$  has two adjacent 3-vertices  $v_i$  and  $v_{i+1}$ , which is impossible in a triangulation without loops and multiple edges. Thus  $d(v_{i+1}) = 4$ , and by the definition of  $\pi$  we have  $\pi(v_i) = \pi(v_{i+1}) = v_{i-1}$  (Fig. 3.7).

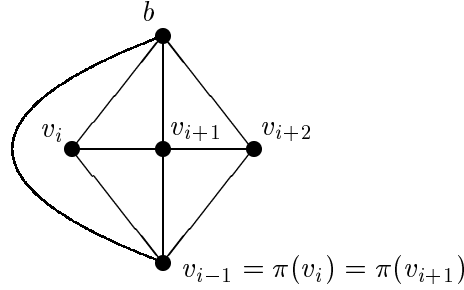


Fig. 3.7

Suppose  $d(v_i) = 4$ . If  $d(v_{i+1}) = 3$ , then the only B-neighbour of  $v_{i+1}$  other than  $b$  is  $v_{i+2}$ . Now the definition of  $\pi$  implies that  $\pi(v_i) = \pi(v_{i+1}) = v_{i+2}$ . If  $d(v_{i+1}) = 4$ , then since the faces of  $G$  are triangles, it follows that  $\pi(v_i) = \pi(v_{i+1})$ . Let us prove that  $v_{i+1}$  cannot be a B-vertex. Indeed, otherwise before averaging  $bv_{i+1}$  had to be zero and  $bv_i$  unit. However, then  $bv_i$  shifts  $1/2$  to  $bv_{i+1}$  by AR. As a result,  $bv_i$  could not become unit ultimately, contrary to the assumption of the lemma.

Let  $v_i$  be a B-vertex. If  $d(v_{i+1}) = 3$ , then the definition of  $\pi$  implies that  $\pi(v_i) = \pi(v_{i+1}) = v_i$ . As proved above,  $d(v_{i+1}) = 4$  is impossible. Furthermore,  $v_{i+1}$  cannot be a B-vertex, since in this case each of  $bv_i$  and  $bv_{i+1}$  was initially zero, and none of them can become unit in the end. So, we have proved that  $G$  has a vertex  $t$  such that  $\pi(v_i) = t$  for all  $i = 1, \dots, k$ .

Now it follows from the definition of  $\pi$  that  $bv_1, \dots, bv_k$  are consecutive vertical edges of a bunch of length at least  $k$  with poles  $b$  and  $t$ . Let us prove that  $t$  is a B-vertex. If for some  $i \in \{1, \dots, k\}$  the degree of  $v_i$  is not 4, then  $\pi(v_i) = t$  must be a B-vertex. If  $d(v_i) = 4$  for all  $i = 1, \dots, k$ , then  $v_2$  is adjacent to two 4-vertices  $v_1$  and  $v_3$  (and also to  $b$  and  $t$ ). By (A'), it follows that  $t$  is a B-vertex.

Since the length of the bunch found at the B-vertex  $b$  is at least  $k$ , it follows from property (B') that  $k < d - 1$ . The last inequality means that there are two different edges  $bv_d$  and  $bv_{k+1}$  not

belonging to the prebunch. Let us prove that both of them belong to the bunch considered. It suffices to prove this for  $bv_d$ . If  $v_1$  is a B-vertex, then  $t = v_1$  by the definition of  $\pi$ . In this case,  $bv_1$  is the parental edge of our bunch, and the path  $bv_dv_1$  extends the bunch. If  $d(v_1) = 4$ , then since  $G$  is triangular, it follows that the path  $bv_d t$  also extends the bunch. Suppose  $d(v_1) = 3$ . If  $t = \pi(v_1) = v_d$ , then the edge  $bv_d$  is parental in the bunch. Finally, if  $t = \pi(v_1) = v_2$ , then the bunch is extended by the path  $bv_d v_2 = bv_d t$ . This completes the proof of Lemma 3.5.  $\square$

**Lemma 3.6** (on half separators)

*After averaging, the vicinity of  $b$  does not contain separators consisting of one half edge.*

**Proof** Suppose there is a separator of one half edge  $bv_3$ . By definition, both  $bv_2$  and  $bv_4$  are unit edges.

CASE 1.  $bv_3$  was a half edge before averaging.

It follows from R1–R3 that  $d(v_3) \in \{4, 5\}$ . If  $d(v_3) = 4$ , then by R1 at least one of  $bv_2$  and  $bv_4$  must be zero initially. Moreover, this edge cannot become unit because it does not receive charge from  $bv_3$ .

Suppose  $d(v_3) = 5$ . Since  $bv_2$  and  $bv_4$  became unit, and did not receive  $1/2$  from  $bv_3$ , it follows that each of them was unit or sesquialteral before averaging.

First assume that both of them were unit. Then Claim 3.3, R1, and R2 imply that  $d(v_2) = d(v_4) = 4$  (see Fig. 3.8). Let us prove that precisely one of  $x, y$  in Fig. 3.8 has degree at least 9.

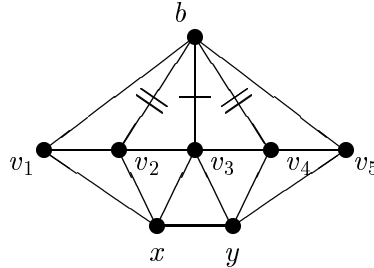


Fig. 3.8

Indeed, if  $d(x) \leq 8$  and  $d(y) \leq 8$ , then in the vicinity of the 5-vertex  $v_3$  there is a precomplete star defined by  $v_2, v_4, x$  and  $y$ , whose weight is at most 24. However, if  $d(x) \geq 9$  and  $d(y) \geq 9$ , then  $bv_3$  is zero by R3 (a), contrary to the assumption. Hence we may assume by symmetry, that  $d(x) \geq 9$  and  $d(y) \leq 8$ . Now by (A'), it follows that  $v_5$  is a B-vertex, and  $bv_5$  was initially zero, for otherwise we have a precomplete star at  $v_4$  of weight at most 38. So,  $bv_4$  shifts  $1/2$  to  $bv_5$ , and  $bv_4$  becomes unit, contrary to the assumption.

Now suppose  $bv_2$  was sesquialteral and  $bv_4$  unit. Then  $d(v_2) = 3$  and  $d(v_4) = 4$ . In the notation of Fig. 3.8, it means that  $x$  coincides with  $v_1$  and is a B-vertex (see R2 (b)). As above, we see that  $bv_3$  is zero if  $d(y) \geq 9$ , or  $bv_4$  becomes a half edge if  $d(y) \leq 8$ .

Finally, suppose that each of  $bv_2$  and  $bv_4$  was sesquialteral. Now both  $v_2$  and  $v_4$  are 3-vertices, and it follows from R2 (b) that each of  $x, y$  is a B-vertex. Again, we conclude that  $bv_3$  is zero, a contradiction.

CASE 2.  $bv_3$  becomes half after averaging.

First suppose that  $bv_3$  was initially zero and obtained  $1/2$  from  $bv_2$ . Then  $v_3$  is a B-vertex, and  $bv_2$  was sesquialteral. If  $bv_4$  was not zero, then it also shifted  $1/2$  to  $bv_3$ , so that  $bv_3$  becomes unit. But if  $bv_4$  was zero, then it cannot become unit because it does not receive  $1/2$  from  $bv_3$ .

Now suppose  $bv_3$  was unit before averaging and becomes half due to shifting  $1/2$  to (the initially zero edge)  $bv_2$ . Then Lemma 3.2 (c) implies that  $bv_4$  becomes zero or half, contrary to the assumption. This completes the proof of Lemma 3.6.  $\square$

**Corollary 3.7**

*The vicinity of  $b$  consists of at most 5 separators and at most 5 prebunches.*

**Proof** We can say that each zero edge saves a unit of charge for  $b$  (as compared to a unit edge), and a half edge saves  $1/2$ . Since  $\mu^*(b) < 0$ , the total saving at  $b$  is at most  $5\frac{1}{2}$ . Since by Lemma 3.6 each separator saves at least 1, the statement follows.  $\square$

**Lemma 3.8** (on triple separators)

*After averaging, there is no separator  $S$  at  $b$  consisting of three half edges.*

**Proof** Suppose the contrary, and consider two cases.

CASE 1. All three edges of  $S$  were half before averaging.

We first observe that each edge of  $S$  leads to a 5-vertex. Indeed, by R1 each half edge  $bu$  going to a 4-vertex lies in a 3-face  $buvb$  with an initially zero edge  $bv$ . Moreover, if  $bu \in S$ , then  $bv$  cannot become unit because it does not receive  $1/2$  from  $bu$ . But if  $bv$  becomes half after averaging, then it also belongs to  $S$ , contrary to the assumption.

Thus, we may assume that  $S$  consists of the edges  $bv_2$ ,  $bv_3$  and  $bv_4$  as shown in Fig. 3.9 (a).

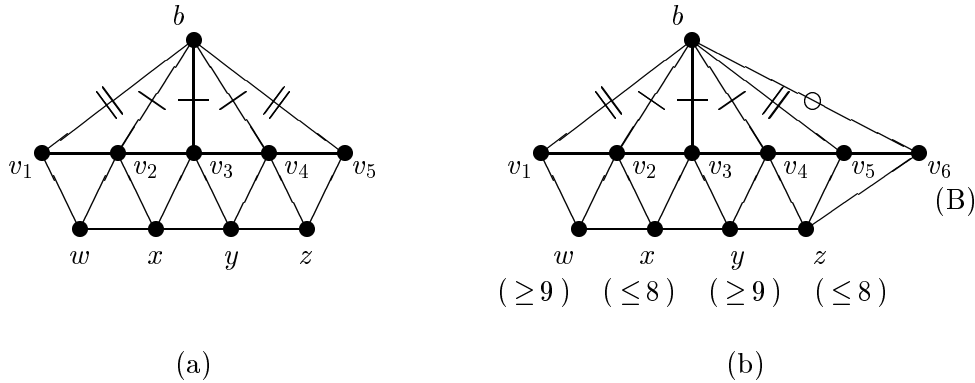


Fig. 3.9

The same argument as in the proof of Lemma 3.6 implies that precisely one of the vertices  $x, y$  in Fig. 3.9 (a) has degree at least 9 (for otherwise  $bv_3$  is not half). The same is true for  $y$  and  $z$ , and also for  $x$  and  $w$ . By symmetry, we may assume that  $d(w) \geq 9$ ,  $d(x) \leq 8$ ,  $d(y) \geq 9$  and  $d(z) \leq 8$  (Fig. 3.9 (b)).

Observe that the unit edge  $bv_5$  could not be zero before averaging because it did not receive  $1/2$  from  $bv_4$ . Now Claim 3.3 implies that  $d(v_5) \leq 4$ . Since  $d(v_4) + d(z) \leq 13$ , it follows from (A') that  $v_5$  has degree 4 and  $v_6$  adjacent to  $v_5$  is a B-vertex. By R1 (f), the edge  $bv_5$  was initially unit (Fig. 3.9 (b)). It means that  $bv_5$  gave  $1/2$  to the initially zero edge  $bv_6$ , so that  $bv_5$  became a half edge, contrary to the assumption.

CASE 2. An edge of  $S$  became half after averaging.

If  $bv_i$  was zero and became half due to the shift of  $1/2$  from the half edge  $bv_{i+1}$ , then  $bv_{i+1}$  becomes zero after averaging. Hence,  $bv_{i+1}$  cannot belong to  $S$ .

Let  $bv_2$  belong to  $S$  and become half due to the shift from the sesquialteral edge  $bv_1$ . Then  $v_2$  must be a B-vertex. Since  $bv_1$  became unit, it follows that  $S$  consists of  $bv_2$ ,  $bv_3$  and  $bv_4$ . If  $bv_3$  was not zero, then it also shifted  $1/2$  to  $bv_2$ , so that  $bv_2$  becomes unit. Suppose that  $bv_3$  was zero and becomes half due to the shift from  $bv_4$ . Since both  $bv_3$  and  $bv_4$  became half, the edge  $bv_4$  was initially unit and  $v_3$  was a B-vertex (Fig. 3.10). Note that the shift from the unit edge  $bv_4$

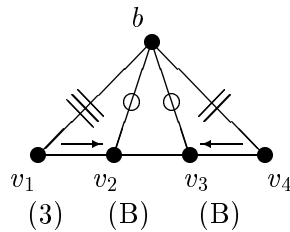


Fig. 3.10

to the zero edge  $bv_3$  can occur only in one of the cases (i)–(iii) described in Lemma 3.2 (c). It is not hard to see that in none of these cases  $bv_5$  can become unit nor  $bv_4$  can become zero. In all these cases,  $S$  cannot consist of the three half edges  $bv_2$ ,  $bv_3$  and  $bv_4$ .

It remains to consider the case that a certain edge of  $S$  became half by shifting  $1/2$  from the unit edge  $bv_3$  to the zero edge  $bv_2$ . If  $bv_2$  became half, then  $S$  must consist of  $bv_1$ ,  $bv_2$  and  $bv_3$ , or of  $bv_2$ ,  $bv_3$  and  $bv_4$ . If  $bv_2$  becomes unit after averaging, then  $S$  consists of  $bv_3$ ,  $bv_4$  and  $bv_5$ . Remark that  $S$  cannot consist of  $bv_2$ ,  $bv_3$  and  $bv_4$  due to Claim 3.4. From Lemma 3.2 (c) it follows that  $S$  also cannot consist of  $bv_1$ ,  $bv_2$  and  $bv_3$ .

Suppose that  $S$  consists of  $bv_3$ ,  $bv_4$  and  $bv_5$ , and that  $bv_2$  becomes unit. According to Lemma 3.2 (c), the edge  $bv_4$  was initially zero, half or unit. Moreover, in the last case  $bv_5$  was zero and  $v_5$  was a B-vertex.

If  $bv_4$  was zero, then  $v_4$  must be an L-vertex, for otherwise  $bv_3$  shifts  $1/2$  to  $bv_4$ , and  $bv_3$  becomes zero. This means that  $bv_4$  remains zero, contrary to the assumptions.

Suppose that  $bv_4$  was unit. By Lemma 3.2 (c),  $bv_4$  shifts  $1/2$  to the initially zero edge  $bv_5$ . Then by Claim 3.4,  $bv_5$  cannot be a boundary edge in  $S$ , contrary to the assumptions.

Now suppose  $bv_4$  was initially half. Since  $bv_3$  was unit and  $v_4$  is minor, we conclude by R1–R3 that  $d(v_3) = 4$ . By assumption,  $bv_4$  is a half edge both before and after averaging; hence  $d(v_4) \in \{4, 5\}$ . Assume  $d(v_4) = 4$ . Then R1 implies that  $d(v_5) \geq 12$  and that  $bv_5$  was zero

before averaging. Now if  $v_5$  is an L-vertex, then  $bv_5$  remains zero after averaging. But if  $v_5$  is a B-vertex, then  $bv_4$  shifts  $1/2$  to  $bv_5$ , so that  $bv_4$  becomes zero.

Now assume  $d(v_4) = 5$  (see Fig. 3.11 (a)). Since  $bv_3$  was unit, it follows from R1 that the

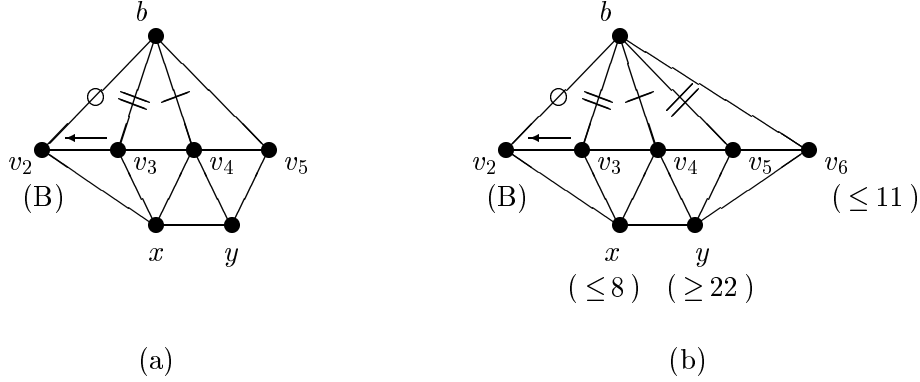


Fig. 3.11

degree of the vertex  $x$  in Fig. 3.11 (a) is at most 8. As  $bv_4$  does not shift  $1/2$  to  $bv_5$ ,  $v_5$  is an L-vertex. This implies that  $bv_5$  was half or unit before averaging, so that  $v_5$  is minor. As seen above,  $d(v_3) = 4$  and  $d(x) \leq 8$ , while (A') applied to  $v_4$  yields  $d(y) \geq 22$ .

As shown above,  $bv_5$  may be either half or unit. If it was unit, then it follows from R1–R3 that  $d(v_5) = 4$  (because the common neighbour  $v_4$  of  $v_5$  and  $b$  has degree 5). Moreover, the fact that  $bv_5$  is unit, by R1 implies that the degree of the vertex  $v_6$  adjacent to  $v_5$  is at most 11 (Fig. 3.11 (b)). In this case,  $bv_5$  does not shift  $1/2$  to  $bv_6$ , so that  $bv_5$  remains unit, a contradiction.

Now suppose that  $bv_5$  was initially half. If  $d(v_5) = 4$ , then  $bv_6$  was zero by R1. Moreover, it cannot become unit because of the lack of shift from  $bv_5$ . We must have  $d(v_5) = 5$ . Since  $bv_6$  becomes unit and there was no shift to it from  $bv_5$ , we see from Claim 3.3 that  $d(v_6) \leq 4$  (Fig. 3.12 (a)). Since  $bv_5$  was initially half, it follows from R3 that the degree of  $z$  in Fig. 3.12

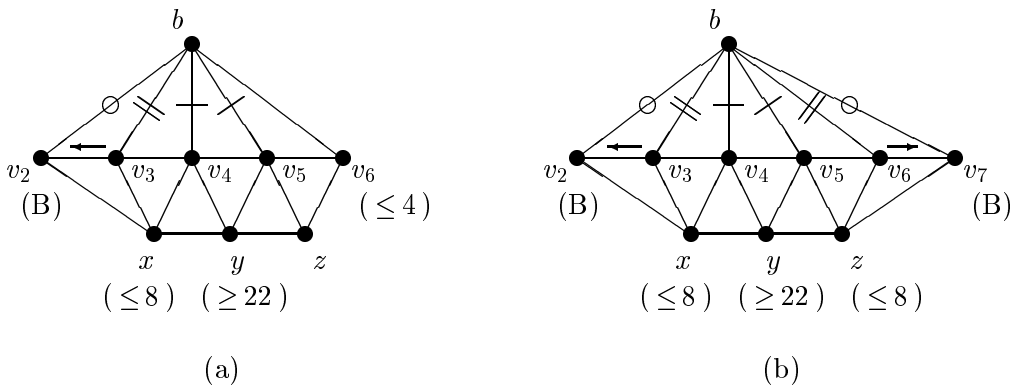


Fig. 3.12

is at most 8. Hence, by applying (A') to  $v_6$  we conclude that  $d(v_6) \neq 3$ , as it was proved above



that  $d(v_5) + d(z) \leq 13$ . Thus  $d(v_6) = 4$  and, moreover, the vertex  $v_7$  adjacent to  $v_6$  is different from  $z$  and is a B-vertex. Finally,  $bv_6$  was initially unit by R1 (f) ( Fig. 3.12 (b) ). It means that there was a shift from the unit edge  $bv_6$  to the zero edge  $bv_7$ , so that  $bv_6$  becomes half. This contradiction completes the proof of Lemma 3.8.  $\square$

**Lemma 3.9** (main)

*The vicinity of  $b$  consists of precisely 5 separators and precisely 5 bunches, bounded by these separators. Moreover, each separator consists of two edges, and at most one separating edge is zero.*

**Proof** By Corollary 3.7, the number of separators is at most 5. Due to (3.3), there are at most 11 separating edges, whence the number of unit edges is at least  $d - 11$ . Suppose that the number of separators, and hence that of prebunches, is at most 4. Then there is a prebunch of length at least  $\lceil (d - 11)/4 \rceil$ . By Lemma 3.5, this prebunch forms a part of a bunch of length at least  $\lceil (d - 11)/4 \rceil + 2$  with a pole at  $b$ . Since  $b$  is a B-vertex, it follows that  $\lceil (d - 11)/4 \rceil + 2 > d/5$ , which contradicts property (B'). Hence,  $b$  is incident with precisely 5 separators and 5 prebunches.

We now prove that each prebunch at  $b$  is a part of a bunch. By Lemma 3.5, it suffices to prove that each prebunch has length at least 3. If a shorter prebunch exists, then the other 4 prebunches contain at least  $d - 13$  unit edges in total, and there is a prebunch of length at least  $\lceil (d - 13)/4 \rceil$ . By Lemma 3.5, this prebunch is contained in a bunch of length at least  $\lceil (d - 13)/4 \rceil + 2$  with a pole at  $b$ . Since  $b$  is a B-vertex, it follows that  $\lceil (d - 13)/4 \rceil + 2 > d/5$ , contrary to (B'). Hence,  $b$  is incident with precisely 5 separators and 5 bunches.

Next, we prove that each separator has length two. Recall the notion of saving used in proving Corollary 3.7. Lemma 3.8 implies that each separator of length at least three saves for  $b$  at least two units of charge. If such a separator exists, then by Lemma 3.6 the total saving of all five separators is at least 6, which contradicts the assumption  $\mu^*(b) < 0$ . Hence, the length of each separator is at most two. If there were a separator of length one (i.e., consisting of just one edge), then by Lemma 3.5 the total length of all five bunches at  $b$  would be greater than  $d$ , and there is a bunch of length greater than  $d/5$ , contrary to (B').

It remains to observe that, by (3.3), among the 10 separating edges at  $b$  there may exist at most one zero edge. This completes the proof of Lemma 3.9.  $\square$

By Lemma 3.9, the vicinity of  $b$  consists of at most 5 bunches, whose total length is at least  $d$ . If not all the bunches have the same length, then there is a bunch of length greater than  $d/5$ , contrary to (B'). So, we can assume that each bunch has length precisely  $d/5$ , and no bunch has a parental edge (for otherwise the second part of statement (B) in Theorem 2.1 holds). Moreover, each parental edge in the vicinity of  $b$  is a boundary edge for precisely one bunch.

It follows from the proof of Lemma 3.5 that each unit edge at  $b$ , leading to a B-vertex, is a parental edge for a certain bunch. It also follows that if a bunch contains a bunch vertex of degree 3, then it contains a parental edge too. Hence, we may assume that each unit edge at  $b$  leads to a 4-vertex. Then R1 and AR imply that each unit edge at  $b$  was also unit before

averaging. Let us prove a stronger fact; namely, no shift by AR happens at  $b$ . Indeed, it was just proved for unit edges, and for separating edges it follows from Lemma 3.9 and the definition of AR.

Let us prove that each half edge at  $b$  leads to a 5-vertex. Suppose there is a separator consisting of edges  $bv_1$  and  $bv_2$ , where  $bv_2$  is half and  $d(v_2) = 4$ . Since  $G$  is a triangulation, it follows that  $v_2$  is an interior vertex in a bunch that contains  $v_3$  as a bunch vertex (see the proof of Lemma 3.5). Then  $bv_2$  is not a boundary edge of this bunch, and the length of this bunch is greater than  $d/5$ . This contradiction implies that each half edge at  $b$  leads to a 5-vertex and this vertex is an end vertex for one bunch. We have thus proved that if each separator edge at  $b$  is half, then the last part of statement (B) of Theorem 2.1 holds.

Finally, due to Lemma 3.9, we have to consider a separator at  $b$  that consists of a half edge  $bv_1$  and a zero edge  $bv_2$ . If  $d(v_2) \leq 11$ , then the last part of statement (B) of Theorem 2.1 holds. Therefore, suppose  $d(v_2) \geq 12$ . As seen above, we now have  $d(v_1) = 5$  and  $d(v_3) = d(v_4) = 4$ . Moreover,  $v_3$  and  $v_4$  are interior vertices and  $v_2$  is an end vertex in a bunch with one pole at  $b$  and the other at a B-vertex  $t$  (Fig. 3.13).

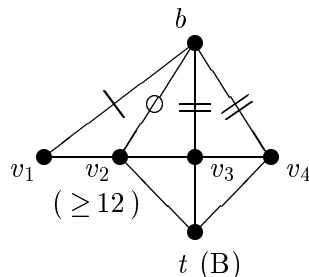


Fig. 3.13

It remains to observe that due to R1 (b), the edge  $bv_3$  must be half, contrary to the assumptions. This completes the proof of Theorem 2.1. ■

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