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A Stability Analysis Based on Dissipativity of Linear and Nonlinear Repetitive Control

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Abstract: This paper deals with repetitive control (RC). More specifically, a parametrised version of the repetitive compensator, i.e. of the infinite-dimensional controller employed in RC schemes, modelled as a boundary control system (BCS) in port-Hamiltonian form is presented. Well-posedness and stability of such control scheme are rigorously addressed thanks to novel tools based on dissipativity theory and originally developed for the stabilisation of BCS. Here, the linear and the nonlinear cases are tackled, and in both the cases the classes of plants for which RC schemes are exponentially stable are determined. Moreover, and explicit motivation of perfect asymptotic tracking and disturbance rejection for exponentially stable RC systems without relying on the internal model theory is provided. To show the validity of the analysis, simulations are reported.

1. INTRODUCTION

Repetitive control (RC) is a method for tracking periodic exogenous signals with period T, supposed known, see Hara et al. (1988). Its main properties depend on an element, denoted by $C(s)$ in Fig. 1 and called *repetitive compensator*. This is a neutral dynamical system able to generate any periodic signal of period T, store information about the tracking error, and adjust the control output to drive the error to zero. Due to the presence of the delay, $C(s)$ is an infinite-dimensional system and, in particular, it has infinite many poles on the imaginary axis at $\pm j\frac{2\pi}{T}k$, $k \in \mathbb{Z}$, independently from the value of the gains $\mathcal{F}$ and $\mathcal{K}$, see Fig. 2.

In this paper, the stability analysis of RC schemes is tackled within the port-Hamiltonian framework, by extending in a non-trivial manner the results presented in Califano et al. (2017) and in Califano et al. (2018). In these works, the repetitive compensator is modelled as a boundary control system (BCS) in port-Hamiltonian form, see e.g. Le Gorrec et al. (2005), and the results presented in Ramírez et al. (2014) and dealing with the synthesis of finite-dimensional boundary controller capable to guarantee an exponentially stable closed-loop system have been exploited. What has been obtained is the characterisation of classes of linear and nonlinear plants for which the corresponding closed-loop RC scheme is well-posed, exponentially stable and able to track / reject periodic reference signals.

The novelty of this paper lies in the parametrisation adopted for the repetitive compensator, shown in Fig. 2, and consequently in the associated dissipativity property. The control gains $\mathcal{K}$ and $\mathcal{F}$ are used to modify the dissipativity conditions associated to the admissible plants for which the RC scheme works, but without interfering with the nature of such systems. In this way, the class of

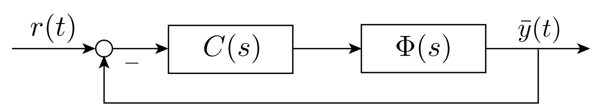

Fig. 1. Basic structure of the repetitive control, see Hara et al. (1988). The controller $C(s)$ is called *repetitive compensator*, and the plant is denoted by $\Phi(s)$

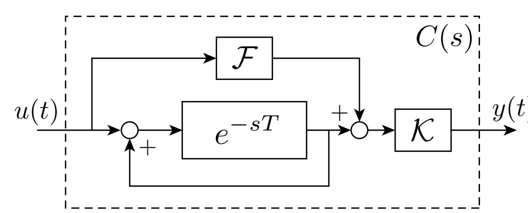

Fig. 2. The repetitive compensator with two control gains.

systems for which this control technique can be employed is enlarged with respect to the classical formulation of the RC illustrated in Hara et al. (1988) and that corresponds to have $\mathcal{F} = 0$ and $\mathcal{K} = I$ in the scheme of Fig. 2. The main contribution of this paper is that the stability analysis is carried out either in the linear and in the nonlinear case. As far as the linear case is concerned, thanks to the general control synthesis methodology presented in Macchelli and Califano (2018), the results presented in Califano et al. (2017) are extended and conditions for the exponential stability of RC scheme are given for *all* the possible parametrisation of the repetitive compensator. On the other hand, in the nonlinear case, such particular parametrisation of the repetitive compensator is exploited to well-posedness and stability for an important class of physical nonlinear and passive systems. A similar problem has been also investigated in Califano et al. (2018), where the repetitive compensator has been assumed in its classical formulation, i.e. $\mathcal{F} = 0$ and $\mathcal{K} = I$. As a result, the analysis has been quite involved, and the result

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is that the class of nonlinear systems for which the RC scheme works is not easily characterised. Instead, in this paper, the family of nonlinear systems for which RC can be successfully applied is shown to be made of nonlinear passive systems such as mechanical systems with actuator saturation, nonlinear stiffness and damping, and electrical systems with nonlinear capacitances and / or inductances.

As a final contribution, in this work an explicit motivation of the asymptotic tracking capabilities of RC is given, under the main requirement that closed-loop system is exponentially stable. In this way, periodic output regulation is proved without relying on the internal model property, and a novel perspective for interpreting the Internal Model principle for periodic signals in the linear case is given.

The paper is organised as follows. In Section 2, the required mathematical background is briefly presented, and the formulation of the repetitive compensator as a BCS in port-Hamiltonian form is provided. In Sections 3 and 4, the stability analysis of RC schemes in case the plant is a linear or a nonlinear system is developed. In Section 5, the asymptotic tracking of exponentially stable RC systems is addressed, while conclusions are reported in Section 6.

2. THE REPETITIVE COMPENSATOR AS DISTRIBUTED PORT-HAMILTONIAN SYSTEM

The aim of this section is to present the mathematical background that allows to formulate the repetitive compensator as a BCS in port-Hamiltonian form. This will be instrumental to study RC schemes within the port-Hamiltonian framework.

2.1 BCS in port-Hamiltonian form

We refer to the class of port-Hamiltonian systems studied in Le Gorrec et al. (2005), i.e. to systems described by

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\big(\mathcal{L}(z)x(t,z)\big) + (P_0 - G_0)\mathcal{L}(z)x(t,z) \quad (1)$$

with $x \in \mathbb{R}^n$, and $z \in [a, b]$. Moreover, $P_1 = P_1^{\mathrm{T}}$ and invertible, $P_0 = -P_0^{\mathrm{T}}$, $G_0 = G_0^{\mathrm{T}}$, and $\mathcal{L}(\cdot)$ is a bounded and Lipschitz continuous matrix-valued function such that $\mathcal{L}(z) = \mathcal{L}^{\mathrm{T}}(z) \geq \kappa I$, with $\kappa > 0$, for all $z \in [a, b]$. For the sake of clearness, $(\mathcal{L}x)(t, z) := \mathcal{L}(z)x(t, z)$. The state space is $X = L^2(a, b; \mathbb{R}^n)$, and is endowed with the inner product $\langle x_1 \mid x_2\rangle_{\mathcal{L}} = \langle x_1 \mid \mathcal{L}x_2\rangle$ and norm $\|x_1\|^2_{\mathcal{L}} = \langle x_1 \mid x_1\rangle_{\mathcal{L}}$, where $\langle\cdot \mid \cdot\rangle$ denotes the natural L^2-inner product. The selection of this space for the state variable is motivated by the fact that $\|\cdot\|^2_{\mathcal{L}}$ is linked to the energy function of (1). As a consequence, X is also called the space of energy variables, and $\mathcal{L}x$ denotes the co-energy variables.

Remark 2.1. Here $|\cdot|$ denotes the norm of a vector or of a matrix, to distinguish it from $\|\cdot\|$, used for the norm of an element of an infinite-dimensional space.

Remark 2.2. The PDE (1) can be also written as $\dot{x} = \mathcal{J}x$, where $\mathcal{J}x := P_1\frac{\partial}{\partial z}(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ is a linear operator with domain $D(\mathcal{J}) = \{\mathcal{L}x \in H^1(a, b; \mathbb{R}^n)\}$, being $H^1(a, b; \mathbb{R}^n)$ the Sobolev space of order one.

For the PDE (1), the boundary port (f_∂, e_∂) is now defined, where f_∂, $e_\partial \in \mathbb{R}^n$ are two vectors such that

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}\begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix} \quad (2)$$

Theorem 2.1. (Le Gorrec et al. (2005)). Let W be a full rank $n\times 2n$ matrix, and define the map $\mathcal{B} : H^1(a, b; \mathbb{R}^n) \to \mathbb{R}^n$, with $D(\mathcal{B}) = D(\mathcal{J})$, and the input $u(t)$ as

$$u(t) = \mathcal{B}x(t) := W\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \quad (3)$$

The operator $\bar{\mathcal{J}}x := P_1\frac{\partial}{\partial z}(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain $D(\bar{\mathcal{J}}) = \{\mathcal{L}x \in H^1(a, b; \mathbb{R}^n) \mid \mathcal{B}x = 0\}$ generates a contraction semigroup on X if and only if $W\Sigma W^{\mathrm{T}} \geq 0$, with $\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, and the system (1) with input (2) is a BCS on X, see (Curtain and Zwart, 1995, Definition 3.3.2), provided that $u \in C^2(0, \infty; \mathbb{R}^n)$. Moreover, let $\tilde{W}$ be a full rank $n \times 2n$ matrix such that $\begin{pmatrix} W^{\mathrm{T}} & \tilde{W}^{\mathrm{T}} \end{pmatrix}$ is invertible, and define the output as

$$y(t) = \mathcal{C}x(t) := \tilde{W}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \quad (4)$$

with $\mathcal{C} : H^1(a, b; \mathbb{R}^n) \to \mathbb{R}^n$. For $(\mathcal{L}x)(0) \in H^1(a, b; \mathbb{R}^n)$, and $u(0) = \mathcal{B}x(0)$, the energy-balance inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{L}} \leq \frac{1}{2}\begin{pmatrix} u(t) \\ y(t) \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} W \\ \tilde{W} \end{pmatrix}^{-\mathrm{T}}\Sigma\begin{pmatrix} W \\ \tilde{W} \end{pmatrix}^{-1}\begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \quad (5)$$

is satisfied.

2.2 The repetitive compensator in port-Hamiltonian form

The repetitive compensator of Fig. 2 is a BCS in the sense of Theorem 2.1. This fact is summarised in the next proposition that generalises a similar result presented in Califano et al. (2017), where it is shown that also the "classical" repetitive compensator of Hara et al. (1988), that corresponds to the choice $\mathcal{F} = 0$ and $\mathcal{K} = I$, is in fact a BCS in port-Hamiltonian form.

Proposition 2.1. The repetitive compensator of Fig. 2 admits the port-Hamiltonian representation (1) with $P_1 = -I$, $P_0 = G_0 = 0$, $\mathcal{L}(z) = I$, and $z \in [0, T]$. Moreover, with Theorem 2.1 in mind and under the condition that $\mathcal{K} \neq 0$, we have that

$$W = \sqrt{2}\,(I \;\; 0) \qquad \tilde{W} = \frac{\sqrt{2}}{2}\,(2\mathcal{F} - \mathcal{K} \;\; \mathcal{K}) \quad (6)$$

the repetitive compensator is a BCS, and the following energy-balance relation holds true:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_2 = \frac{1}{2}\begin{pmatrix} u(t) \\ y(t) \end{pmatrix}^{\mathrm{T}}\underbrace{\begin{pmatrix} 0 & \mathcal{K} \\ \mathcal{K}^{\mathrm{T}} & \mathcal{K}(2\mathcal{F} - \mathcal{K}) \end{pmatrix}^{-1}}_{=:P_{rc}}\begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \quad (7)$$

Proof. The delay is described by the PDE

$$\frac{\partial x}{\partial t}(t,z) = -\frac{\partial x}{\partial z}(t,z) \quad (8)$$

which is in the form (1). Since the spatial domain is $[0, T]$, it is easy to see that $x(t, T) = x(t - T, 0)$, which means that $x(t, T)$ is the delayed copy of $x(t, 0)$, for all $t \geq T$. With Fig. 2 in mind, we can write that $y(t) = \mathcal{K}x(t, T) + \mathcal{F}u(t)$ and that $x(t, 0) = x(t, T) + u(t)$, which implies that $u(t) = x(t, 0) - x(t, T)$ and $y(t) = (\mathcal{K} - \mathcal{F})x(t, T) + \mathcal{F}x(t, 0)$. By applying (2) to system (8), we have that

$$f_\partial = \frac{1}{\sqrt{2}}[-x(T) + x(0)] \quad e_\partial = \frac{1}{\sqrt{2}}[x(T) + x(0)]$$

so $u(t)$ and $y(t)$ are obtained from (3) and (4) if the matrices W and $\tilde{W}$ are selected as in (6). Note that $W\Sigma W^{\mathrm{T}} = 0$, so (6) with input $u(t)$ is a BCS in the sense of Theorem 2.1. Finally, (7) follows by substituting (6) in (5).

Relation (7) shows that the BCS of Proposition 2.1, i.e. the repetitive compensator, is dissipative with storage function $\frac{1}{2}\|x\|_2^2$ and quadratic supply rate. Note that $W\Sigma W^{\mathrm{T}} = 0$ and that the matrix $\begin{pmatrix} W^{\mathrm{T}} & \tilde{W}^{\mathrm{T}} \end{pmatrix}$ is invertible independently from the choice of $\mathcal{K} \neq 0$ and $\mathcal{F}$. This means that $\mathcal{K}$ and $\mathcal{F}$ have an effect on the dissipativity properties of repetitive compensator, i.e. of the associated BCS, but not on its well-posedness.

Remark 2.3. The modulation of the repetitive compensator with the feedforward gain $\mathcal{F}$ has been developed in the frequency domain in Hara et al. (1988), where the stability regions of the Nyquist plot for a linear plant have been determined in the SISO case. Here, the analysis is carried out in the time domain, and the resulting dissipativity properties are directly used to characterise the admissible plants, either in the linear and in the nonlinear scenario.

Remark 2.4. For the "classical" repetitive compensator, i.e. when $\mathcal{K} = I$ and $\mathcal{F} = 0$, the energy-balance relation (7) becomes $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x\|_2^2 = \frac{1}{2}|u|^2 + u^{\mathrm{T}}y$. An intersting case and at the best of our knowledges not explored in the RC literature corresponds to have $\mathcal{K} = I$ and $\mathcal{F} = \frac{1}{2}$. As a result, the repetitive compensator becomes an *impedance passive* BCS, i.e. the energy balance is $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x\|_2^2 = u^{\mathrm{T}}y$. This is one of the key points in the development of the main contribution of this work.

3. STABILITY ANALYSIS OF RC SYSTEMS: THE LINEAR CASE

Let us consider the minimal realisation of the LTI system Σ_Φ in Fig. 1 and with transfer function $\Phi(s)$:

$$\Sigma_\Phi : \begin{cases} \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \\ \bar{y}(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t) \end{cases} \tag{9}$$

where $\bar{x} \in \mathbb{R}^{\bar{n}}$, $\bar{u} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^n$ are the state variable, input and output, respectively. The matrices $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ have the appropriate dimensions. The RC scheme of Fig. 1 is the result of the negative feedback interconnection

$$\begin{pmatrix} u(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} \bar{u}(t) \\ \bar{y}(t) \end{pmatrix} + \begin{pmatrix} r(t) \\ 0 \end{pmatrix} \tag{10}$$

between the repetitive compensator $C(s)$, written as BCS as in Proposition 2.1, and the Σ_Φ, as shown in Fig. 3. This is a typical control design scenario, as discussed e.g. in Ramírez et al. (2014) and in Macchelli and Califano (2018). However, in case of RC systems, the BCS in port-Hamiltonian form is not the plant that has to be stabilised in some equilibrium by a finite dimensional controller. Indeed, it represents the infinite-dimensional regulator based on the Internal Model Principle that let the (finite-dimensional) plant to track exogenous reference trajectories.

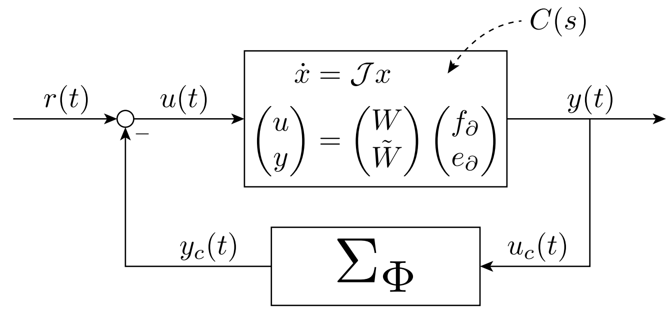

Fig. 3. An equivalent representation of the RC scheme in Fig. 1.

To determine the family of linear plant Σ_Φ in the form (9) for which periodic output regulation by means of RC laws can be achieved, the first step consists in defining the conditions for which the closed-loop system is exponentially stable for *all* the possible parametrisation of the repetitive compensator. This point is discussed in this section, and the first problem to be solved is to determine the dissipativity properties for Σ_Φ. More specifically, we have that (9) is dissipative with storage function

$$E_c(\bar{x}) = \frac{1}{2}\bar{x}^{\mathrm{T}}\bar{Q}\bar{x}, \qquad \bar{Q} = \bar{Q}^{\mathrm{T}} > 0 \tag{11}$$

and supply rate

$$s_c(\bar{u}, \bar{y}) = \frac{1}{2}\begin{pmatrix} \bar{u} \\ \bar{y} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} U_c & S_c \\ S_c^{\mathrm{T}} & Y_c \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{y} \end{pmatrix} \tag{12}$$

with $U_c = U_c^{\mathrm{T}}$, and $Y_c = Y_c^{\mathrm{T}}$, if and only if

$$E_c(\bar{x}) - E_c(\bar{x}(0)) \leq \int_0^t s_c(\bar{u}, \bar{y})\, \mathrm{d}\tau$$

along system trajectories and for all $t \geq 0$. The next result is an extension of the Kalman-Yakubovic-Popov Lemma and it gives necessary and sufficient conditions for the dissipativity of (9).

Proposition 3.1. The linear system (9) is dissipative with storage function (11) and supply rate (12) if and only if:

$$\mathcal{M}_c - \mathcal{N}_c \leq 0 \tag{13}$$

in which

$$\begin{aligned} \mathcal{M}_c &= \begin{pmatrix} \bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q} & \bar{Q}\bar{B} \\ \bar{B}^{\mathrm{T}}\bar{Q} & 0 \end{pmatrix}, \\ \mathcal{N}_c &= \begin{pmatrix} \bar{C}^{\mathrm{T}}Y_c\bar{C} & \bar{C}^{\mathrm{T}}Y_c\bar{D} \\ \bar{D}^{\mathrm{T}}Y_c\bar{C} & U_c + \bar{D}^{\mathrm{T}}Y_c\bar{D} \end{pmatrix} + \\ &\quad + \begin{pmatrix} 0 & \bar{C}^{\mathrm{T}}S_c \\ S_c^{\mathrm{T}}\bar{C} & \bar{D}^{\mathrm{T}}S_c + S_c^{\mathrm{T}}\bar{D} \end{pmatrix} \end{aligned} \tag{14}$$

The exponential stability of the system that results from the feedback interconnection (9) of the linear system Σ_Φ for which the dissipativity property of Proposition 3.1 holds, and the parametrised repetitive compensator of Proposition 2.1 is discussed in the next proposition. Since P_{rc} in (7) is symmetric and it depends on $\mathcal{K}$ and $\mathcal{F}$, we can write that

$$P_{rc}(\mathcal{K}, \mathcal{F}) = \begin{pmatrix} U(\mathcal{K}, \mathcal{F}) & S(\mathcal{K}, \mathcal{F}) \\ S^{\mathrm{T}}(\mathcal{K}, \mathcal{F}) & Y(\mathcal{K}, \mathcal{F}) \end{pmatrix}$$

Note that U, Y, and S are known. Moreover, to simplify the notation, the dependence on $\mathcal{K}$ and $\mathcal{F}$ is not reported.

Proposition 3.2. In (9), assume that $\bar{A}$ is Hurwitz and that the pair $(\bar{A}, \bar{B})$ is controllable. Suppose that for sufficiently small $\delta_x > 0$ and $\delta_u > 0$, we have that

$$\frac{1}{2}\left(\mathcal{M}_c - \mathcal{N}_c\right) \leq -\begin{pmatrix} -\delta_x(\bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q}) & 0 \\ 0 & \delta_u I \end{pmatrix} \qquad (15)$$

under the condition that

$$\begin{pmatrix} Y & -S^{\mathrm{T}} \\ -S & U \end{pmatrix} + \begin{pmatrix} U_c & S_c \\ S_c^{\mathrm{T}} & Y_c \end{pmatrix} \leq 0 \qquad (16)$$

Then, the autonomous RC scheme obtained from the interconnection of Σ_Φ and of the repetitive compensator via (10) is exponentially stable.

Proof. This result is a simple application of Proposition 12 in Macchelli and Califano (2018).

Remark 3.1. The inequalities (15) and (16) combine the dissipativity properties of either the plant and the controller to end up with sufficient conditions for the exponential stability of the closed-loop system and, consequently, for perfect tracking at steady state. Condition (16) together with (13) assures the well-posedness of the closed-loop system and the existence of differentiable trajectories for smooth enough inputs. More details on this point can be found in Proposition 7 in Macchelli and Califano (2018). During the design phase, it is possible to “play” with the gains $\mathcal{K}$ and $\mathcal{F}$ in the controller to change the admissible plants in the RC scheme. Condition (15) implies and it is clearly stronger than (13). More precisely, (15) implies that the finite dimensional system is characterised by the presence of damping able to attenuate both the low and high frequency dynamic. The first requirement is associated to the existence of a $\delta_x > 0$, the second one of a $\delta_u > 0$.

Example 3.1. Let us consider the repetitive compensator introduced in Hara et al. (1988), i.e. let us assume that $\mathcal{K} = I$ and $\mathcal{F} = 0$ in Proposition 2.1. In this case, we have that

$$P_{rc} = \begin{pmatrix} I & I \\ I & 0 \end{pmatrix}$$

and condition (16) is fulfilled if the plant Σ_Φ is dissipative with respect to the supply rate

$$\bar{s}(\bar{u}, \bar{y}) = \frac{1}{2}\begin{pmatrix} \bar{u} \\ \bar{y} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 & I \\ I & -\sigma I \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{y} \end{pmatrix} \qquad (17)$$

with $\sigma \geq 1$. This means that the closed-loop system is well-posed for all the linear systems Σ_Φ that are α-output strictly passive, with $\alpha > \frac{1}{2}$. This situation has been studied in detail in Macchelli and Califano (2018).

Example 3.2. If $\mathcal{F} = \frac{1}{2}$ and $\mathcal{K} = I$, then we have that

$$P_{rc} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and (17) holds when $\sigma \geq 0$. This allows to have passive plants Σ_Φ to get a well-posed RC scheme.

In all of these examples, by adding dissipation in order to satisfy (15), the resulting closed-loop system turns out to be exponentially stable, thus guaranteeing perfect asymptotic tracking for periodic reference signals $r(t)$.

4. NONLINEAR RC APPLIED TO PASSIVE SYSTEMS

The aim of this section is to extend the previous result to nonlinear RC schemes, i.e. to systems represented as in Fig. 1 where now Σ_Φ is nonlinear and whose output has to track a periodic reference $r(t)$. The infinite-dimensional and nonlinear nature of the system makes a mathematical rigorous analysis quite complex because of the lack of general mathematical tools to support the study of properties like well-posedness, stabilisability and tracking in this framework. Nevertheless, by relying on a proper choice of the repetitive compensator and by exploiting some recent results on the nonlinear control design for port-Hamiltonian BCS presented in Ramírez et al. (2017), it is shown that for an important class of passive nonlinear systems Σ_Φ, RC laws can be successfully applied.

In particular the key choice for the repetitive compensator is $\mathcal{K} = I$ and $\mathcal{F} = \frac{1}{2}$. As pointed out in Remark 2.4, in this case BCS that represents the controller is impedance passive, i.e. it is dissipative with storage function $\frac{1}{2}\|x\|_2^2$ and supply rate $s(u, y) = u^{\mathrm{T}} y$. These are the conditions under which most of the theory on the stabilisation of BCS in port-Hamiltonian form has been developed so far in the linear case (see e.g. Villegas et al. (2009); Macchelli (2013); Ramírez et al. (2014); Macchelli et al. (2017)), and the only situation in which nonlinear boundary controllers have been employed, as illustrated in Ramírez et al. (2017).

Let us consider the nonlinear system

$$\Sigma_\Phi : \begin{cases} \dot{v}_1(t) = K v_2(t) \\ \dot{v}_2(t) = -\dfrac{\partial \mathcal{P}}{\partial v_1}(v_1(t)) - \mathcal{R}(K v_2(t)) + B\bar{u}(t) \\ \bar{y}(t) = B^{\mathrm{T}} K v_2(t) + S\bar{u}(t) \end{cases} \qquad (18)$$

where $v_1, v_2 \in \mathbb{R}^{n_c}$, $\mathcal{P} : \mathbb{R}^{n_c} \to \mathbb{R}^+$ is a Fréchet differentiable function, B an $n_c \times n$ real matrix, $\mathcal{R}(\cdot)$ a locally Lipschitz-continuous matrix-valued function taking values in $\mathbb{R}^{n_c \times n_c}$, $S = S^{\mathrm{T}} \geq 0$ an $n \times n$ matrix, $K = K^{\mathrm{T}} > 0$ a $n_c \times n_c$ real matrix. Furthermore $\frac{\partial \mathcal{P}}{\partial v_1}(v_1)$ is locally Lipschitz-continuous. It is easy to check that such a system is passive with storage function

$$E_c(v_1, v_2) = \mathcal{P}(v_1) + \frac{1}{2} v_2^{\mathrm{T}} K v_2$$

Theorem 7 in Ramírez et al. (2017) shows that under the assumptions

- $\mathcal{P}(v_1) > \mathcal{P}(0) = 0$, $\forall v_1 \neq 0$,
- $\mathcal{P}(v_1)$ is radially unbounded, i.e. $\mathcal{P}(v_1) \to \infty$ as long as $|v_1| \to \infty$,
- $v_2^{\mathrm{T}} \mathcal{R}(v_2) \geq 0$, $\forall v_2$,

the closed-loop system in Fig. 3 possesses mild solutions which are uniformly bounded for any initial condition. On the other hand, the next propositions characterises the exponential stability of a class of nonlinear RC systems.

Proposition 4.1. The nonlinear RC system obtained from the feedback interconnection (10) between the repetitive compensator of Proposition 2.1 and the nonlinear system Σ_Φ defined in (18) is exponentially stable when $r(t) = 0$ if the following conditions hold:

- The gains of the repetitive compensator are $\mathcal{F} = \frac{1}{2} I$ and $\mathcal{K} = I$;
- For some δ_1, $\delta_2 > 0$, and $\forall v_1 \in \mathbb{R}^{n_c}$, we have that

$$v_1^{\mathrm{T}} \frac{\partial \mathcal{P}}{\partial v_1}(v_1) \geq \delta_1 \mathcal{P}(v_1) \geq \delta_2 |v_1|^2$$

- For some ϵ_1, $\epsilon_2 > 0$, and $\forall v_2 \in \mathbb{R}^{n_c}$, we have that

$$v_2^{\mathrm{T}}\mathcal{R}(v_2) \geq \epsilon_1 |v_2|^2 \geq \epsilon_2 |\mathcal{R}(v_2)|^2$$

- The controller is strictly input passive, i.e. $S > 0$.

Proof. Since $\mathcal{F} = \frac{1}{2}I$ and $\mathcal{K} = I$, the repetitive compensator, and consequently its associated port-Hamiltonian BCS, is impedance passive. Thanks to (6), from (2), (3), and (4), it is easy to see that there exists $\epsilon > 0$ such that $|u(t)|^2 + |y(t)|^2 \geq \epsilon |x(t, 0)|^2$, which corresponds to Assumption 14 in Ramírez et al. (2017). This completes the set of assumptions necessary to invoke Theorem 20 in Ramírez et al. (2017), and then to finally prove the exponential stability of the RC scheme.

The previous result is powerful because it assures well-posedness and exponential stability for a class of nonlinear RC systems by relying on a particular and original formulation of the repetitive compensator. Furthermore, Σ_Φ is a physical system, i.e. it is passive and it has to satisfy some specific requirements reported in Proposition 4.1.

Remark 4.1. The main difference with respect to Ramírez et al. (2017) and the literature dealing with the stabilisation of BCS in port-Hamiltonian form is once again related to the interpretation of the repetitive compensator. In fact it does not represent a distributed parameter system to be stabilised but the controller itself. On the contrary Σ_Φ represents a class of nonlinear systems for which RC can be successfully applied.

5. ASYMPTOTIC TRACKING IN RC SCHEMES

This section is devoted to the analysis of perfect asymptotic tracking in RC schemes. In the linear case discussed e.g. in Hara et al. (1988), this property has been assumed to be satisfied once the closed-loop system is exponentially stable. In fact, according to internal model-based arguments that "classically" state that if the model of the exogenous signal generator is properly included in the loop, the repetitive compensator $C(s)$ in this case, and if the closed-loop system is exponentially stable, then asymptotic tracking of the exogenous signals is assured. Even if this interpretation is somehow wrong because this yields for LTI finite-dimensional systems, the nontrivial extension to continuous-time RC systems is present in Yamamoto (1993). Nevertheless, in the nonlinear case it is not possible to invoke such arguments to conclude perfect tracking from exponential stability of the closed-loop system and other methodologies need to be used.

Here, a new way of checking asymptotic tracking in RC schemes is investigated without relying on internal-model arguments. At first, it is easy to see that asymptotic tracking in the RC scheme of Fig. 1 is achieved if, for any periodic reference signal $r(t)$ of period T, the error $e(t) = r(t) - \bar{y}(t)$ vanishes asymptotically in an L^2-sense over a period, i.e. if for $n \in \mathbb{N}$, we have that

$$\lim_{n\to\infty} \int_{nT}^{(n+1)T} |r(t) - \bar{y}(t)|^2 \mathrm{d}t = 0$$

The idea is to give sufficient conditions for which such property holds.

Definition 5.1. A signal $s(t) : \mathbb{R} \to \mathbb{R}^n$ is called asymptotically T-periodic if and only if for $n \in \mathbb{N}$ and $\forall k \in \mathbb{N}$

$$\lim_{n\to\infty} \int_{nT}^{(n+1)T} |s(t) - s(t + kT)|^2 \mathrm{d}t = 0$$

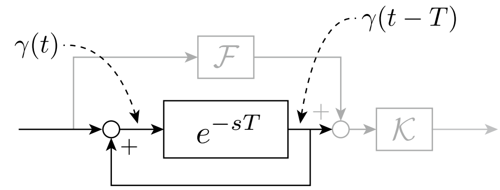

Fig. 4. A detail of the RC scheme of Fig. 1, used in the proof of Proposition 5.1.

The next proposition provides a way to check perfect tracking in RC schemes.

Proposition 5.1. Let us refer to the RC scheme of Fig. 1, and consider the formulation of the delay as a BCS in port-Hamiltonian form of Prop. 2.1. Asymptotic tracking of periodic reference signals $r(t)$ of period T is achieved if and only if the state of the delay equation in the closed-loop system is asymptotically T-periodic for all $z \in [0, T]$, $x(t, z)$.

Proof. The result is immediately checked by referring to Fig. 4, where a detail of the RC scheme is reported. In this respect, asymptotic tracking is achieved if, at steady state

$$\lim_{n\to\infty} \int_{nT}^{(n+1)T} |e(t)|^2 \mathrm{d}t = 0$$

being $e(t) = \gamma(t) - \gamma(t - T)$ where $\gamma(t)$ is the input of the delay. This is possible if and only if $\gamma(t)$ is an asymptotic T-periodic signal, which implies that $x(t, z)$ is asymptotically T-periodic for all $z \in [0, T]$.

Proposition 5.2. Let us consider the linear RC of Fig. 1, in which the plant Σ_Φ is given in (9). Let us assume that the closed-loop system is exponentially stable, i.e. that Proposition 3.2 holds true. Then, asymptotic tracking for any periodic signal $r(t)$ of period T is achieved.

Proof. Exponentially stable linear systems produce at steady state asymptotically T-periodic solutions if they are excited by a T-periodic input. Then, the proof is an immediate consequence of Proposition 5.1. It is worth noting that this fact can be interpreted as the internal model property for periodic reference signals.

In a similar manner, asymptotic tracking in the nonlinear case, i.e. for the RC systems characterised in Proposition 4.1, can be equivalently analysed by determining the conditions under which asymptotic T-periodicity of the state solution in the delay is obtained. In this respect, similar arguments to those presented in Califano et al. (2018) can be used. The major requirement is that the origin of closed-loop system is exponentially stable.

To illustrate the validity of the approach, some simulative results are reported in Fig. 5. The implemented RC scheme satisfies assumptions of Prop. 4.1, making the repetitive compensator impedence passive. The graphs show that RC system is able to let a nonlinear plant to track a reference signal and reject an additive disturbance on the output, both periodic of period $T = 1\,\mathrm{s}$. The plant is a 2^{nd}-order passive system Σ_Φ in the form (18) that could model, for example, a mechanical actuator with nonlinear damping.

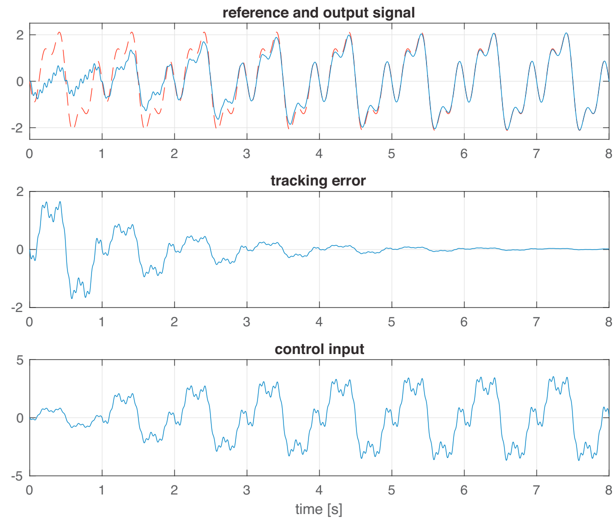

Fig. 5. Asymptotic tracking and disturbance rejection of the RC system in which the plant Σ_Φ takes the form (18). In the simulation $S = 0.6$, $B = 1$, $K = 0.15$, $\mathcal{R}(Kv_2) = K^3 v_2^3$ and $\mathcal{P} = 0.25v_1^2$.

6. CONCLUSIONS

In this paper, continuous-time RC has been studied by relying on tools originally developed for the design of regulators for (linear) BCS in port-Hamiltonian form. The main contribution is the characterisation of a class of systems for which the resulting RC scheme is well-posed and exponentially stable, either in case the plant is linear and nonlinear. The problem has been tackled by decomposing the closed-loop system into the feedback interconnection of a BCS in port-Hamiltonian form, associated with the repetitive compensator, and the plant, while well-posedness and stability have been proved by exploiting their dissipativity properties, that can be properly "tuned" by acting on two control gains in the repetitive compensator. Finally, a novel perspective on the asymptotic tracking in RC schemes has been provided under the requirement that the closed-loop system is exponentially stable and without relying on internal model arguments.

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