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Generalized projection dynamics in evolutionary game theory*

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Abstract

We introduce a new kind of projection dynamics by employing a ray-projection both locally and globally. By global (local) we mean a projection of a vector (close to the unit simplex) unto the unit simplex along a ray through the origin. Using a correspondence between local and global ray-projection dynamics we prove that every interior evolutionarily stable strategy is an asymptotically stable fixed point. We also show that every strict equilibrium is an evolutionarily stable state and an evolutionarily stable equilibrium.

Then, we employ several projections on a wider set of functions derived from the payoff structure. This yields an interesting class of so-called generalized projection dynamics which contains best-response, logit, replicator, and Brown-Von-Neumann dynamics among others.
Key words: evolutionary game theory, projection dynamics, orthogonal projection, ray projection, asymptotical and evolutionary stability.
JEL-Codes: A12; C62; C72; C73; D83

1 Introduction

We introduce a class of dynamics to model evolutionary changes in game theory. We were inspired by rather early literature on price-adjustment processes as introduced by Samuelson [1941, 1947] and subsequent results by Arrow & Hurwicz [1958, 1960a,b] and Arrow, Block & Hurwicz [1959].[1] A second source of inspiration was recent work featuring projection dynamics, e.g., Lahkar & Sandholm [2008], Hofbauer & Sandholm [2008].

In the latter papers it is shown that if a stable game possesses an interior evolutionarily stable state (*ESS*, Maynard Smith & Price [1973]), the

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[1] For a surveys on Walrasian processes, see e.g., Uzawa [1961], Negishi [1962].

projection dynamics converge to it from any starting point. In fact, the proofs imply that for projection dynamics every interior evolutionarily stable state is an evolutionarily stable equilibrium (Joosten [1996]), i.e., trajectories converge to the equilibrium and along any such trajectory the Euclidean distance to it decreases strictly in time.

In the literature on price-adjustment processes, a similar result was established about half a century ago. If the Weak Axiom of Revealed Preferences (*WARP,* Samuelson [1938]) holds, the price-adjustment process of Samuelson [1947] given by

$$\dot{x} = \frac{dx}{dt} = f(x) \text{ for all } x \in \mathbb{R}^{n+1}_{+} \backslash \{0^{n+1}\},$$

converges to an economic equilibrium. Here, x denotes a vector of prices for $n+1$ commodities, 0^{n+1} denotes the $n+1$-vector of zeros, and the (vector) function $f : \mathbb{R}^{n+1}_{+} \backslash \{0^{n+1}\} \to \mathbb{R}^{n+1}$ is an excess demand function. An excess demand function gives for each commodity the difference between its demand and supply given a price for each commodity. An equilibrium is a price vector for which there exists no positive excess demand for any commodity, i.e., y is an equilibrium iff $f(y) \leq 0^{n+1}$.

Under *WARP*, any trajectory converges to an equilibrium and the Euclidean distance to it decreases strictly over time. This inspired the concept of the evolutionarily stable equilibrium (*ESE*) in Joosten [1996], a notion defined on the dynamics instead of on the underlying system, guaranteeing that trajectories converge to the equilibrium as described.

As shown in Joosten [2006], an implication of *WARP* in economics is very similar to an implication of *ESS* in mathematical biology. Hence, alternatives to the many dynamics highlighted in the literature[2] may exist such that each *ESS* is an asymptotically stable fixed point. Samuelson's tâtonnement process however, does not induce dynamics on the unit simplex, it induces dynamics on a sphere with the origin as center, and radius equal to the length of the starting vector.

Our basic idea is to project a(ny) trajectory of Samuelson's tâtonnement process on the unit simplex such that every point of the original is projected on the unit simplex along the ray through this point and the origin. By the convergence result of the unrestricted dynamics under *WARP* mentioned, it follows that the projected dynamics also converge to an equilibrium. Which means that for these dynamics applied to a game theoretical model, each interior *ESS* is an asymptotically stable fixed point. We show that the ray-projection dynamics of Samuelson tâtonnement process on the unit simplex are for every $y = \lambda x \in int\ \mathbb{R}^{n+1}_{+} \backslash \{0^{n+1}\}$ given by

$$\dot{x} = \frac{1}{\lambda} \left[f(x) - x \left(\sum_{i=1}^{n+1} f_i(x) \right) \right],$$

[2] See e.g., Hofbauer [1995,2000], Hofbauer & Sandholm [2008].

where $\lambda = \sum_{i=1}^{n+1} y_i$ and $x \in S^n = \{z \in \mathbb{R}^{n+1} | z_j \geq 0$ for all $j \in \{1, 2, ..., n+1\}$ and $\sum_{j=1}^{n+1} z_j = 1\}$.

One might think that the dynamics obtained in that manner, are equivalent to the projection dynamics of Lahkar & Sandholm [2008] on the interior of the unit simplex, and if not globally then at least locally. By a global projection, we mean a projection of an arbitrary trajectory unto the unit simplex. By local projection, we mean that the trajectory is started on the unit simplex and then continuously be forced back on the unit simplex by projection, i.e., $\lambda = 1$ always. This intuition is false, as the mathematical forms of local and global ray-projection dynamics differ crucially from their orthogonal-projection relatives. The dynamics of Lahkar & Sandholm [2008] are for $x \in int\ S^n$ given (in our notations) by

$$\dot{x} = f(x) - \frac{1}{n+1} \left(\sum_{i=1}^{n+1} f_i(x) \right) i,$$

where $i = (1, ..., 1) \in \mathbb{R}^{n+1}$.

We demonstrate that under the ray-projection dynamics every interior evolutionarily stable state is an asymptotically stable fixed point. An elegant geometric interpretation of this fact is the following. It is well-established that Samuelson's process moves on a sphere with the origin as its center and with a fixed radius. Points having equal Euclidean distance to the equilibrium form a circle on this sphere.[3] Connecting this circle to the origin yields a cone. This cone is intersected by the unit simplex, a subset of a plane. Hence, the projection of the circle unto the unit simplex is an ellipse. Since the unrestricted process always moves inwards relative to the circle around the equilibrium on which the process happens to be, the process projected unto unit simplex moves inwards relative to the ellipse it happens to be on.

We also show that the concept of a strict equilibrium unifies two notions of evolutionary stability, namely static evolutionary stability as embodied by the *ESS* and dynamic evolutionary stability as embodied by *ESE*.

Our next idea was to generalize the approach with ray-projections by employing modifications of the relative fitness function. Many well-known evolutionary dynamics can be represented as projection dynamics by choosing appropriate variants of the relative fitness function. These include e.g., the best-response dynamics of Matsui [1991], the Brown-Von Neumann dynamics (Brown & Von Neumann [1950]) and generalizations implied by Björnerstedt & Weibull [1996] and Hofbauer [2000], the logit dynamics (Fudenberg & Levine [1998]), but also the replicator dynamics of Taylor & Jonker [1978].

The next section gives an exposé on ideas leading to our new concept, the ray-projection dynamics. In Section 3 we generalize both ray-projection

[3] For all of these objects in $\mathbb{R}^3$ a proper higher-dimensional parallel exists.

and orthogonal-projection dynamics. Well-known dynamics are presented as special cases of generalized projection dynamics. Section 4 deals with conditions guaranteeing that the dynamics do not cross the boundary of the unit simplex. This yields a small list of desiderata for generalized projection dynamics. Section 5 concludes, all proofs are to be found in the Appendix.

2 Comparing the old and the new

In Joosten [2006] connections were highlighted between models formalizing evolutionary dynamics and price-adjustment processes. One of the correspondences found was that a condition resulting from the Weak Axiom of Revealed Preferences (*WARP*) can be translated almost one-to-one to a condition resulting from the evolutionarily stable strategy (*ESS*). We first discuss the result on price-adjustment dynamics.

2.1 On price-adjustment dynamics

The condition implied by *WARP*, cf., e.g., Uzawa [1961], is the following

$$(y-x)\cdot f(x) > 0,$$

for all $x, y \in \mathbb{P} = \mathbb{R}^{n+1}_{+} \backslash \{0^{n+1}\}$ such that $y \in E = \left\{z \in \mathbb{P}\} |\ f(z) \leq 0^{n+1}\right\}$, $x \notin E$. Here, $f : \mathbb{P} \to \mathbb{R}^{n+1}$ satisfies continuity, **homogeneity (of degree zero in prices)**, i.e., $f(\lambda x) = f(x)$ for all $\lambda > 0$, and **complementarity**, i.e., $x \cdot f(x) = 0$ for all $x \in \mathbb{P}$. Often, since the function f satisfies homogeneity of degree zero, analysis is restricted to a normalized subspace of $\mathbb{R}^{n+1}$, for instance to the n-dimensional S^n, i.e.,

$$S^n = \left\{ x \in \mathbb{R}^{n+1} |\ x_j \geq 0 \text{ for all } j \in I^{n+1} \text{ and } \sum_{j \in I^{n+1}} x_j = 1 \right\},$$

where $I^{n+1} = \{1, ..., n+1\}$.

In economics, $x \in S^n$ represents a vector of relative prices adding up to unity; the function f represents a so called **generalized excess demand function**. A price vector $y \in S^n$ satisfying $f(y) \leq 0^{n+1}$ is called an **equilibrium** or a **Walrasian equilibrium**. At an equilibrium no commodity has positive excess demand. *Existence of an equilibrium (ray)* is readily shown by using homogeneity in order to restrict analysis to the unit simplex, constructing an adequate continuous function from this unit simplex unto itself, and then using Brouwer's fixed point theorem.

The work of Sonnenschein [1972, 1973], Mantel [1974] and Debreu [1974] shows that any function satisfying continuity, complementarity and desirability[4], can be approximated by an excess demand function on an arbi-

[4] Desirability of all goods means that if the price of a commodity equals zero, then the supply of that good can not exceed its demand, i.e., $x_j = 0$ implies $f_j(x) \geq 0$.

trarily large subset of the interior of the unit simplex resulting from a pure exchange economy with as many agents as commodities in which each of the agents has well-behaved preferences and positive initial endowments of all commodities. If the property of desirability is dropped one obtains a generalized excess demand function, if one furthermore restricts attention to the unit simplex, homogeneity of degree zero in prices becomes void. So, a generalized excess demand function on the unit simplex is characterized by continuity and complementarity.

A well-known result by Arrow & Hurwicz [1958,1960a,b], Arrow *et al.* [1959] is that the tâtonnement process of Samuelson [1947]:

$$\dot{x} = \frac{dx}{dt} = f(x), \tag{1}$$

converges to an equilibrium if $(y - x) \cdot f(x) > 0$ for all $y \in E$, and $x \notin E$ and if desirability holds. Here, $E = \left\{ x \in \mathbb{R}^{n+1} | \ f(x) \leq 0^{n+1} \right\}$ denotes the set of (economic) equilibria, and if the condition mentioned holds, it can be shown that E is convex (cf., Arrow & Hurwicz [1960b]).

The sketch of the proof is simple. Complementarity of f implies

$$\frac{d||x||^2}{dt} = \sum_{i \in I^{n+1}} 2x_i \frac{dx_i}{dt} = 2 \sum_{i \in I^{n+1}} x_i f_i(x) = 2x \cdot f(x) = 0.$$

Continuity and desirability of all commodities imply that if the process starts in the non-negative orthant it remains on the sphere in this orthant having the origin as its center and containing the starting point. Furthermore, let $y \in E$ and let $x \notin E$ satisfy $||x|| = ||y||$, $x \neq y$, then

$$||y - x||^2 > 0, \text{ moreover } \frac{d||y - x||^2}{dt} < 0.$$

So, under the dynamics the Euclidean distance to y decreases monotonically in time. The actual proof uses Lyapunov's second method, and the Euclidean distance can be interpreted as a so-called Lyapunov function. Recall that by homogeneity of degree zero of f, a ray $\{\lambda y\}_{\lambda > 0}$ exists satisfying $f(x) = 0^{n+1}$ for all $x \in \{\lambda y\}_{\lambda > 0}$.

2.2 Ray-projection of Samuelson's tatonnement process

Now, we derive the dynamics being the projection of Samuelson's tatonnement process on the unit simplex. Note that the trajectory $\{y_t\}_{t \geq 0}$ with $y_0 \in \mathbb{P}$ under (1) may be approximated at $y \in \{y_t\}_{t \geq 0}$ by $y + \Delta t f(y)$. The projection of $y + \Delta t f(y)$ unto the unit simplex is given by

$$\frac{y + \Delta t f(y)}{\sum_{i=1}^{n+1} y_i + \Delta t \sum_{i=1}^{n+1} f_i(y)}.$$

Here, Δt is the length of the time interval elapsed, $\sum_{i=1}^{n+1} y_i + \Delta t \sum_{i=1}^{n+1} f_i(y)$ is a number, whereas y and $f(y)$ are vectors. Then, this implies a move from $x = \frac{y}{\sum_{i=1}^{n+1} y_i} \in S^n$ to $\frac{y+\Delta t f(y)}{\sum_{i=1}^{n+1} y_i + \Delta t \sum_{i=1}^{n+1} f_i(y)} \in S^n$ and therefore

$$\begin{aligned}
\Delta x &= \frac{y + \Delta t f(y)}{\sum_{i=1}^{n+1} y_i + \Delta t \sum_{i=1}^{n+1} f_i(y)} - \frac{y}{\sum_{i=1}^{n+1} y_i} \\
&\overset{y=\lambda x}{=} \frac{\lambda x + \Delta t f(\lambda x)}{\sum_{i=1}^{n+1} \lambda x_i + \Delta t \sum_{i=1}^{n+1} f_i(\lambda x)} - \frac{\lambda x}{\sum_{i=1}^{n+1} \lambda x_i} \\
&\overset{\sum_{i=1}^{n+1} \lambda x_i = \lambda}{=} \frac{\lambda x + \Delta t f(\lambda x)}{\lambda + \Delta t \sum_{i=1}^{n+1} f_i(\lambda x)} - \frac{\lambda x}{\lambda} \\
&= \frac{\lambda x + \Delta t f(\lambda x)}{\lambda + \Delta t \sum_{i=1}^{n+1} f_i(\lambda x)} - x \\
&= \frac{\lambda x + \Delta t f(\lambda x) - x\left(\lambda + \Delta t \sum_{i=1}^{n+1} f_i(\lambda x)\right)}{\lambda + \Delta t \sum_{i=1}^{n+1} f_i(\lambda x)} \\
&\overset{f(\lambda x)=f(x)}{=} \frac{\lambda x + \Delta t f(x) - x\left(\lambda + \Delta t \sum_{i=1}^{n+1} f_i(x)\right)}{\lambda + \Delta t \sum_{i=1}^{n+1} f_i(x)} \\
&= \Delta t \frac{f(x) - x\left(\sum_{i=1}^{n+1} f_i(x)\right)}{\lambda + \Delta t \sum_{i=1}^{n+1} f_i(x)}.
\end{aligned}$$

So, this means that

$$\dot{x} = \lim_{\Delta t \downarrow 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \downarrow 0} \frac{\Delta t}{\Delta t} \frac{f(x) - x\left(\sum_{i=1}^{n+1} f_i(x)\right)}{\lambda + \Delta t \sum_{i=1}^{n+1} f_i(x)} = \frac{1}{\lambda}\left[f(x) - x\left(\sum_{i=1}^{n+1} f_i(x)\right)\right].$$

Note that the term $\frac{1}{\lambda}$ has no influence on the direction of the dynamics, merely on the speed of the dynamics. As $\lambda = \sum_{i=1}^{n+1} y_i$, the speed of the projected dynamics decreases, roughly speaking, the distance of the unrestricted trajectory to the origin, increases. Furthermore, if $y \in S^n$, then $\lambda = 1$. So, if the ray-projection dynamics are local, we may dispense with this speed parameter. This leads to the following definition.

Definition 1 *Let $f : \mathbb{P} \to \mathbb{R}^{n+1}$ satisfying continuity, complementarity, and (positive) homogeneity of degree zero. Let for all $y \in \mathbb{P}$, $\dot{y} = \frac{dy}{dt} = f(y)$. Then, the ray-projection dynamics on the unit simplex are for every $x = \frac{1}{\sum_{i=1}^{n+1} y_i} y \in S^n$ given by*

$$\dot{x} = \frac{1}{\lambda}\left[f(x) - x\left(\sum_{i=1}^{n+1} f_i(x)\right)\right],$$

where $\lambda = \sum_{i=1}^{n+1} y_i$.

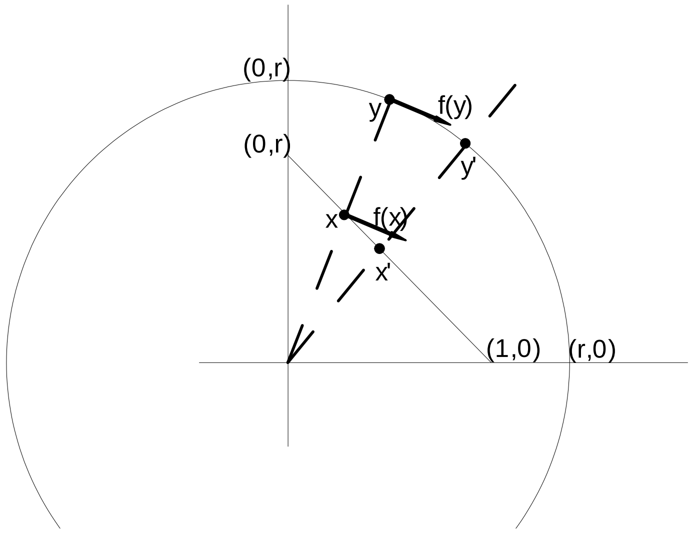

Figure 1: The price-adjustment process induces a trajectory from y to y' in $\mathbb{R}^2$ on the sphere with radius $r = ||y||$ and the origin as center. The projection of this trajectory unto S^1 is the one from x towards x'. We have depicted vectors $f(x) = f(y)$.

Remark 1 *If $\lambda = 1$, i.e., $x = y \in S^n$, we call the ray-projection dynamics local, and global otherwise. Local and global ray-projection dynamics can be transformed one into the other by a transformation of time.*

Here, we are not concerned for the behavior of these dynamics on the boundary of the unit simplex, as price-adjustment processes tend to stay away from the boundary of $\mathbb{P}$.

2.3 On dynamics and equilibria in evolutionary game theory

In evolutionary game theory, for a population[5] having $n+1$ distinguishable subgroups, $x \in S^n$ is a vector of population shares for each subgroup, i.e., x_i is the population share of subgroup $i \in I^{n+1}$. Let $F : S^n \to \mathbb{R}^{n+1}$ be a fitness function, i.e., a function attributing to each subgroup in the population its fitness. The fitness of a subgroup may be interpreted as the subgroup's potential to reproduce or alternatively the average number of offspring. Fitness of a subgroup depends on the composition of the population, i.e., $x \in S^n$.

The **relative fitness function** $f : S^n \to \mathbb{R}^{n+1}$ is given by

$$f_i(x) = F_i(x) - x \cdot F(x) \text{ for all } x \in S^n \text{ and all } i \in I^{n+1}.$$

[5] We remain faithful to the terminology from mathematical biology.

So, a relative fitness function (Joosten [1996]) attributes to each subgroup the difference of its fitness and the population share weighted average fitness of the population. If the fitness function F is continuous, the same property follows immediately for the relative fitness function f. Observe furthermore that for all $x \in S^n$, it holds that $x \cdot f(x) = 0$.

The evolution of the composition of the population is usually represented by a system of $n+1$ autonomous differential equations:

$$\dot{x} = \frac{dx}{dt} = h(x).$$

Here, the function $h : S^n \to \mathbb{R}^{n+1}$ is connected to the relative fitness function f in one of the ways proposed, cf., e.g., Nachbar [1990], Friedman [1991], Swinkels [1993], Joosten [1996], Ritzberger & Weibull [1995]. (Lipschitz) continuity of h implies existence (and uniqueness) of a solution to the differential equation for every starting point $x_0 \in S^n$; differentiability of h implies both existence and uniqueness (cf., e.g., Perko [1991]). We are we are reluctant to impose conditions on the function h at this point since many interesting evolutionary dynamics are neither differentiable, nor continuous.

For **sign-compatible dynamics**, we have

$$sign\ h_i(x) = sign\ f_i(x) \text{ whenever } x_i > 0.$$

i.e., the change in population share of each subgroup with positive population share corresponds in sign with its relative fitness; for **weakly sign-compatible dynamics**, at least one subgroup with positive relative fitness grows in population share. A more general alternative than sign compatibility is provided by Friedman [1991], evolutionary dynamics are **weakly compatible** if $f(x) \cdot h(x) \geq 0$ for all $x \in S^n$.

The state $y \in S^n$ is a **saturated equilibrium** if $f(y) \leq \mathbf{0}^{n+1}$, a **fixed point** if $h(y) = \mathbf{0}^{n+1}$; a fixed point y is **(asymptotically) stable** if, for any neighborhood $U \subset S^n$ of y, there exists an open neighborhood $V \subset U$ of y such that any trajectory starting in V remains in U (and converges to y). A **limit point** is a point $y \in S^n$ satisfying $\lim_{t\to\infty} x_t = y$ for at least one solution $\{x_t\}_{t\geq 0}$ to $x_0 \in S^n$ and the differential equation above.

At a saturated equilibrium all subgroups with below average fitness have population share equal to zero. So, rather than 'survival of the fittest', we have 'extinction of the less fit'. If the fitness function is given by $F(x) = Ax$ for some square matrix A, every saturated equilibrium corresponds to a Nash equilibrium of the evolutionary game at hand. The term is due to Hofbauer & Sigmund [1988], in the sequel we may omit the term 'saturated'.

The fixed point $y \in S^n$ is a **generalized evolutionarily stable state** (cf., Joosten [1996]) if and only if there exists an open neighborhood $U \subset S^n$ of y satisfying

$$(y - x) \cdot f(x) > 0 \text{ for all } x \in U \backslash \{y\}. \tag{2}$$

A geometric interpretation of a generalized evolutionarily stable state (*GESS*) is that near such an equilibrium the angle between the vector pointing from x towards the equilibrium, i.e., $(y - x)$, and the vector $f(x)$ is always acute. The concept of a *GESS* generalizes the concept of an *ESS* of Maynard Smith & Price [1973] in order to deal with *arbitrary* (relative) fitness functions. For the more standard fitness functions, the two notions coincide.

Taylor & Jonker [1978] introduced the replicator dynamics into mathematical biology and gave conditions guaranteeing that an *ESS* is an asymptotically stable fixed point of these dynamics. Zeeman [1981] extended this result and pointed out that the conditions formulated by Taylor and Jonker [1978] are almost always satisfied. The most general result on asymptotic stability regarding the replicator dynamics for the *ESS* is probably Hofbauer *et al.* [1979] as it stipulates an equivalence of the *ESS* and existence of a Lyapunov function of which the time derivative is similar to Eq. (2).

Friedman [1991] has an elegant way of coping with evolutionary stability as he defines any asymptotically stable fixed point of given evolutionary dynamics as an evolutionary equilibrium. Most approaches however, deal with conditions on the underlying system in order to come up with a viable evolutionary equilibrium concept, or deal with refinements of the asymptotically stable fixed point concept (e.g., Weissing [1990]).

In Joosten [1996] we defined an evolutionary equilibrium concept on the dynamic system, wishing to rule out some asymptotically stable fixed points. Namely, the ones which induce trajectories starting nearby, but going far away from the equilibrium before converging to it in the end. The fixed point $y \in S^n$ is an **evolutionarily stable equilibrium** if and only if there exists an open neighborhood $U \subset S^n$ of y satisfying

$$(y - x) \cdot h(x) > 0 \text{ for all } x \in U \backslash \{y\}. \tag{3}$$

A geometric interpretation of (3) is that sufficiently close to the equilibrium the angle between $(y - x)$ and the vector representing the direction of the dynamics is always acute. The concept was inspired by the Euclidean distance approach of early contributions in economics as mentioned, since (3) implies that the Euclidean distance is a Lyapunov function for U.

2.4 Projection dynamics in evolutionary games

Lahkar & Sandholm [2008] introduced dynamics into evolutionary game theory which converge to an interior evolutionarily stable equilibrium, because for the dynamics at hand Eq. (2) and (3) are equivalent. The authors quote Nagurney & Zhang [1996] as a main source of inspiration.

Definition 2 *(Lahkar & Sandholm [2008]) Given relative fitness function* $f : S^n \to \mathbb{R}^{n+1}$*, the orthogonal-projection dynamics are for every* $x \in int\ S^n$

given by

$$\dot{x} = f(x) - \left(\frac{1}{n+1} \sum_{i=1}^{n+1} f_i(x) \right) i$$

Here, i is the $n+1$-dimensional vector of ones, i.e., $i = (1, ..., 1) \in \mathbb{R}^{n+1}$. For the time being, we are only interested in the behavior of the dynamics of Lahkar & Sandholm [2008] in the interior of the unit simplex. The definition of orthogonal-projection dynamics takes due care of boundary behavior.

Remark 2 *Lahkar & Sandholm [2008] actually define their dynamics on the fitness function but for (the interior of the unit simplex) we have*

$$\begin{aligned}
\dot{x} &= f(x) - \left(\frac{1}{n+1} \sum_{i=1}^{n+1} f_i(x) \right) i \\
&= E(x) - (x \cdot E(x))\, i - \left(\frac{1}{n+1} \sum_{i=1}^{n+1} E_i(x) - x \cdot E(x) \right) i \\
&= E(x) - \left(\frac{1}{n+1} \sum_{i=1}^{n+1} E_i(x) \right) i. \qquad \blacksquare
\end{aligned}$$

Below, we present the ray-projection dynamics, corresponding to the local variant of the definition given in the economic framework.

Definition 3 *Let $f : S^n \to \mathbb{R}^{n+1}$ be a relative fitness function. Then, the ray-projection dynamics are for every $x \in int\ S^n$ given by*

$$\dot{x} = f(x) - x \left(\sum_{i=1}^{n+1} f_i(x) \right).$$

Lemma 4 *Every interior equilibrium is a fixed point of the both types of projection dynamics and every interior fixed point of both types of projection dynamics is an equilibrium.*

2.5 On stability of interior equilibria

Hofbauer & Sandholm [2008] introduce the class of stable games. A stable game is a game in which the following property holds:

$$(y - x) \cdot (F(y) - F(x)) \leq 0 \text{ for all } x, y \in S^n.$$

Here, F is a *fitness function*, but it follows easily that in our notations using the *relative fitness function* f we get

$$(y - x) \cdot (f(y) - f(x)) \leq 0 \text{ for all } x, y \in S^n.$$

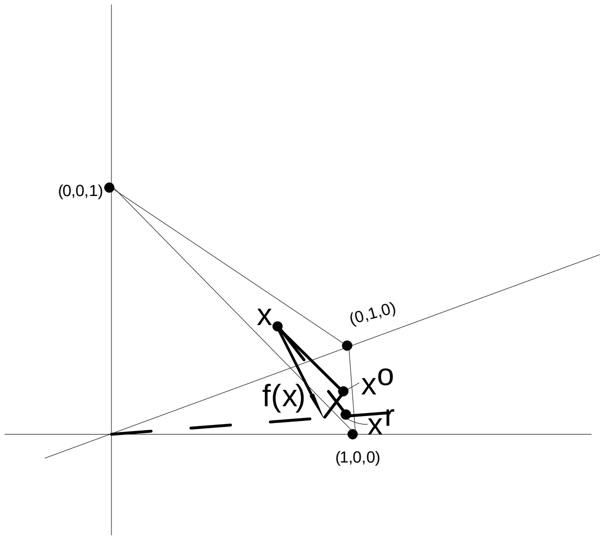

Figure 2: The point x^o is the orthogonal projection of $x + f(x)$ on the S^2; x^r is the ray-projection of $x + f(x)$ on S^n.

The property which defines a stable game is called monotonicity (MON) elsewhere and is connected to a multitude of important results guaranteeing uniqueness and dynamic stability of equilibria and fixed points (see Joosten [2006], Harker & Pang [1990]). MON is a weaker version of strict monotonicity ($SMON$) which can be written as

$$(y - x) \cdot (f(y) - f(x)) < 0 \text{ for all } x, y \in S^n, x \neq y.$$

A game in which $SMON$ holds for all states $x, y \in S^n, x \neq y$, is called a strictly stable game by Hofbauer & Sandholm [2008]. It can be shown that $SMON$ implies that there is a unique saturated equilibrium, and that MON implies that the set of equilibria is compact and convex.

Joosten [2006] showed that if the relative fitness function is given by $f(x) = Ax - (xAx)\, i$ for all $x \in S^n$, then strict monotonicity is equivalent to Haigh's criterion (Haigh [1975]) which can be written as

$$\xi A \xi < 0 \text{ for all } \xi \in \mathbb{R}^{n+1} \text{ satisfying } \sum_{j=1}^{n+1} \xi_j = 0.$$

The version where $\xi A \xi \leq 0$ replaces $\xi A \xi < 0$, is equivalent to MON.

For an interior equilibrium $y \in S^n$, $(S)MON$ implies

$$(y - x) \cdot f(x) \geq (>) 0 \text{ for all } x \in S^n \backslash \{y\}.$$

So, every interior equilibrium of a strictly stable game is a *generalized evolutionarily stable state* ($GESS$, Joosten [1996]) for which the neighborhood U in Eq. (2) can be expanded to include the entire unit simplex. For every stable game, every interior equilibrium is a *neutrally stable state* following Joosten [2006] and Maynard Smith [1982]. Under the replicator dynamics every (generalized) evolutionarily stable state is an asymptotically stable fixed point and every neutrally stable state is stable (cf., e.g., Hofbauer & Sigmund [1998]).

For the orthogonal-projection dynamics it can be seen that every *interior evolutionarily stable equilibrium* is a *generalized evolutionarily stable state* and every *interior evolutionarily stable state* is a *generalized evolutionarily stable state*, as for $y \in int\ S^n$ we have

$$(y-x)\cdot h(x) > 0 \Longleftrightarrow$$

$$\sum_{i\in I^{n+1}} (y_i - x_i)\cdot \left[f_i(x) - \frac{1}{n+1} \sum_{h\in I^{n+1}} f_h(x) \right] > 0 \Longleftrightarrow$$

$$(y-x)\cdot f(x) - \left(\frac{1}{n+1} \sum_{h\in I^{n+1}} f_h(x) \right) \sum_{i\in I^{n+1}} (y_i - x_i) > 0 \Longleftrightarrow$$

$$(y-x)\cdot f(x) > 0.$$

This means that we have shown the validity of the following.

Proposition 5 *(Hofbauer & Sandholm [2008]) Every interior evolutionarily stable state is an interior evolutionarily stable equilibrium under the orthogonal-projection dynamics and vice versa.*

We now prove a corresponding result for ray-projection dynamics. Our strategy of proof is the following. From a given relative fitness function we construct a function on the relevant positive orthant, connect dynamics to that function and construct a trajectory under the dynamics converging to an equilibrium corresponding to a full-dimensional expansion of the interior evolutionarily stable state. Then we project this trajectory unto the unit simplex using the ray-projection. This projected trajectory converges then to the projected equilibrium point. The corresponding dynamics on the unit simplex are the ray-projection dynamics.

Proposition 6 *Under the ray-projection dynamics, every interior generalized evolutionarily stable state is an asymptotically stable equilibrium.*

3 Generalizations of projection dynamics

Here, we pursue the idea of generalizing both projection dynamics presented. For this purpose we define some $g : S^n \to \mathbb{R}^{n+1}$. We intend to examine dynamics induced by g in two variants:

$$\begin{aligned}
\overset{.}{x}{}_g^r &= \left[g(x) - x \left(\sum_{i=1}^{n+1} g_i(x) \right) \right], \\
\overset{.}{x}{}_g^o &= \left[g(x) - \left(\frac{1}{n+1} \sum_{i=1}^{n+1} g_i(x) \right) i \right].
\end{aligned}$$

Here, the superscript r (o) refers to the ray-projection (orthogonal-projection) dynamics and subscript g refers to the function g. Again, we will only consider points yielding projections in the interior of the unit simplex. However, in several cases the projected dynamics happen to be well-defined on the boundary of the unit simplex. There are various approaches tackling the boundary behavior of dynamics (e.g., Friedman [1991], Lahkar & Sandholm [2008]). In order to be relevant in an evolutionary framework it is of utmost importance to link the function g to the relative fitness function.

The following result is straightforward, its proof is left to the reader.

Lemma 7 *Let* $g : S^n \to \mathbb{R}^{n+1}$.

- *If* g *satisfies* $\sum_{i=1}^{n+1} g_i(x) = 0$, *then the local and global ray-projection dynamics, and the orthogonal-projection dynamics concur.*
- *If* g *is weak compatible with* f, *i.e.,* $g(x) \cdot f(x) \geq 0$ *for all* $x \in int\ S^n$, *then the ray-projection dynamics associated with* g *are weak compatible.*
- *If* g *is non-negative, i.e.,* $g : S^n \to \mathbb{R}_+^{n+1}$, *then the ray-projection dynamics remain on the unit simplex.*

Note that (trivially) all evolutionary dynamics on the unit simplex are projected 'unto themselves', hence in that case by the first statement of the lemma, ray-projection and orthogonal projection dynamics concur. For instance, the replicator dynamics of Taylor & Jonker [1978] are given by

$$\dot{x}_i = x_i f_i(x) \text{ for all } x \in S^n.$$

Hence, setting $g_i(x) = x_i f_i(x)$ for all $x \in S^n$ yields the replicator dynamics as both the ray-projection dynamics and the orthogonal-projection dynamics. The second statement of the lemma gives an easy-to-check criterion in order to determine the status of the ensuing ray-projection dynamics. Recall that evolutionary dynamics should be connected with the relative fitness function and weak compatibility of Friedman [1991] is one of the ways to accomplish this. The final statement deals with an equally easy-to-check criterion to guarantee that ray-projection dynamics do not cross the boundary of the unit simplex (or in the global case, the boundary of $\mathbb{P}$.

We now give several ways to obtain the replicator dynamics as ray-projection dynamics or orthogonal-projection dynamics, and the corresponding relatives are also of some interest. We continue with a set of examples of dynamics which can be regarded as projection dynamics.

Example 8 *We can have the function driving both projection dynamics depend on the fitness function $F : S^n \to \mathbb{R}^{n+1}$. For instance, let $\widetilde{g} : S^n \to \mathbb{R}^{n+1}$ be given by $\widetilde{g}_i(x) = x_i F_i(x)$ for all $x \in$ int S^n, $i \in I^{n+1}$. Then for all $i \in I^{n+1}$:*

$$\begin{aligned} \left(\overset{\cdot r}{x}_{\widetilde{g}}\right)_i &= x_i F_i(x) - x_i \left(\sum_{j=1}^{n+1} x_j F_j(x)\right) = x_i \left[F_i(x) - x \cdot F(x)\right] = x_i f_i(x), \\ \left(\overset{\cdot o}{x}_{\widetilde{g}}\right)_i &= x_i F_i(x) - \frac{1}{n+1} \left(\sum_{j=1}^{n+1} x_j F_j(x)\right). \end{aligned}$$

So, the generalized ray-projection dynamics connected to the function $\widetilde{g}$ as defined yield the ***replicator dynamics****.*
Another way of obtaining similar dynamics is particularly interesting in case the fitness function is given by $F(x) = Ax$ for a symmetric matrix A. Let $\underline{a} \leq \min(0, \min_{ij} a_{ij})$. Then, let $\widehat{g} : S^n \to \mathbb{R}^{n+1}$ be given by $\widehat{g}_i(x) = x_i f_i(x) - \underline{a}$ for all $x \in$ int S^n, $i \in I^{n+1}$. Then,

$$\begin{aligned} \left(\overset{\cdot o}{x}_{\widehat{g}}\right)_i &= x_i f_i(x) - \underline{a} - \frac{1}{n+1} \left(\sum_{j=1}^{n+1} \left[x_j f_j(x) - \underline{a}\right]\right) \\ &= x_i f_i(x) \text{ for all } i \in I^{n+1}. \end{aligned}$$

The ray-projection dynamics are given by

$$\left(\overset{\cdot r}{x}_{\widehat{g}}\right)_i = x_i f_i(x) - \underline{a}(1 - x_i(n+1)) \text{ for all } i \in I^{n+1}.$$

An advantage of this function is that $\widehat{g}_i(x) = x_i f_i(x) - a \geq 0$ for all $x \in$ int S^n, $i \in I^{n+1}$. So, the dynamics can not cross on the boundary of S^n. Here, orthogonal-projection dynamics yield the replicator dynamics. ■

Example 9 ***Best-response dynamics*** *(Matsui [1992]) are given by*

$$\dot{x} = BR(x) - x$$

where $BR : S^n \to S^n$ is given by

$$BR_i(x) = \begin{cases} x_i & \text{if } \max_{k \in I^{n+1}} f_k(x) = 0, \\ 1 & \text{if } i = \min\left\{h \in I^{n+1} \mid f_h(x) = \max_{k \in I^{n+1}} f_k(x) > 0\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, these dynamics are weakly sign-compatible. We introduced two slight changes to the original, one implying that $f(y) \leq \mathbf{0}^{n+1}$ implies $h(y) = \mathbf{0}^{n+1}$, and a tie-breaker for the case that multiple best-responses exist. Let

$$g_i(x) = \begin{cases} 1 & \text{if } i = \min\left\{h \in I^{n+1} \middle| \ f_h(x) = \max_{k \in I^{n+1}} f_k(x) > 0\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let for given $x \in S^n$, $j^* = \min\left\{h \in I^{n+1} \middle| \ f_h(x) = \max_{k \in I^{n+1}} f_k(x) > 0\right\}$ and let $e_k \in \mathbb{R}^{n+1}$ denote the k-th unit vector. Then, we obtain

$$\left(\dot{x}^r_g\right)_i = \begin{cases} 0 & \text{if } x \in E, \\ (e_{j^*})_i - x_i & \text{otherwise.} \end{cases} \quad \text{and}$$

$$\left(\dot{x}^o_g\right)_i = \begin{cases} 0 & \text{if } x \in E, \\ (e_{j^*})_i - \frac{1}{n+1} & \text{otherwise.} \end{cases}$$

So, every equilibrium is a fixed point of the ray-projection dynamics; both ray-projection dynamics and orthogonal-projection dynamics are well-defined for the entire unit simplex. ■

BR-dynamics have a predecessor in the continuous fictitious-play dynamics of Rosenmüller [1971], a continuous-time version of fictitious play (Brown [1951]). Brown formulated this process in order to compute a solution (i.e., a Nash equilibrium) of a zero-sum game. Brown has conceived several other ideas on dynamics to compute equilibria. The following example deals with one of them and variations thereof.

Example 10 ***(Generalized "Brownian motions")*** *The term including the quotation marks is due to Hofbauer [2000] after G.W. Brown (not botanist Robert Brown, the (re)discoverer of Brownian motion). The* ***Brown-von Neumann dynamics*** *(Brown & Von Neumann [1950]) given by*

$$\dot{x}_i = \max\{0, f_i(x)\} - x_i \sum_{j \in I^{n+1}} \max\{0, f_j(x)\},$$

are weakly compatible dynamics on the unit simplex.
It can be seen readily that for $g_i(x) = \max\{0, f_i(x)\}$ for all $i \in I^{n+1}$ we have

$$\left(\dot{x}^r_g\right)_i = \max\{0, f_i(x)\} - x_i \sum_{j \in I^{n+1}} \max\{0, f_j(x)\},$$

$$\left(\dot{x}^o_g\right)_i = \max\{0, f_i(x)\} - \frac{1}{n+1} \sum_{j \in I^{n+1}} \max\{0, f_j(x)\}.$$

The ray-projection dynamics coincide with those of Brown and Von Neumann on the interior of the unit simplex; the alternative orthogonal-projection dynamics have not studied as far as we know. For both types of dynamics, *each equilibrium is a fixed point, and each limit point is an equilibrium. More generally, let* $z : \mathbb{R}_+^{n+1} \to \mathbb{R}_+^{n+1}$ *be given by* $z(0) = 0$ *and* $z(x) > 0$ *for all* $x > 0$*. Then, defining* $g^z : S^n \to \mathbb{R}^{n+1}$ *by* $g_i^z(x) = z(\max\{0, f_i(x)\})$ *for all* $i \in I^{n+1}$*, we obtain*

$$\begin{aligned}\left(\overset{\cdot}{x}{}^{r}_{g^z}\right)_i &= z(\max\{0, f_i(x)\}) - x_i \sum_{j \in I^{n+1}} z(\max\{0, f_j(x)\}) \ \textit{and} \\ \left(\overset{\cdot}{x}{}^{o}_{g^z}\right)_i &= z(\max\{0, f_i(x)\}) - \frac{1}{n+1} \sum_{j \in I^{n+1}} z(\max\{0, f_j(x)\}).\end{aligned}$$

The orthogonal-projection variant is not studied as far as we know. Note that if $z(x) = x^\alpha$ *for* $\alpha > 0, x \geq 0$*, then clearly* $\alpha = 1$ *yields the BN-dynamics. An interesting case is then to let* $\alpha \to \infty$*, where the dynamics are very similar to the best-response dynamics.* ■

BN-dynamics converge to a Nash equilibrium, if the relative fitness function $f(x) = Ax - (x \cdot Ax)\, i$ is such that for matrix A it holds that $a_{ij} = -a_{ji}$ for all $i, j \in I^{n+1}$. Moreover, BN-dynamics are globally stable under strict monotonicity ($SMON$) of the generalized excess demand function (or relative fitness function) (cf., Nikaidô [1959]). Hofbauer [2000] treats families of dynamics including (smoothed) BN-dynamics, BR-dynamics and replicator dynamics. His convergence results on the *ESS* complement Nikaidô's. The majority of results in Hofbauer [2000] rely on the weak version of Haigh's criterion, for the stronger on Hofbauer [1995] already parallels.

Example 11 ***(Logit type dynamics)*** *Now, let* $\beta > 0$ *and let* $g^\beta : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *be given by* $g_i^\beta(x) = e^{\beta f_i(x)}$*. Then, we obtain projection dynamics given by*

$$\begin{aligned}\left(\overset{\cdot}{x}{}^{r}_{g^\beta}\right)_i &= e^{\beta f_i(x)} - x_i \left(\sum_{j=1}^{n+1} e^{\beta f_j(x)}\right) \\ \left(\overset{\cdot}{x}{}^{o}_{g^\beta}\right)_i &= e^{\beta f_i(x)} - \left(\frac{1}{n+1} \sum_{j=1}^{n+1} e^{\beta f_j(x)}\right).\end{aligned}$$

Clearly, the ray-projection dynamics do not cross the boundary of S^n*, as* $x_i = 0$ *implies* $\dot{x}_i = e^{\beta f_i(x)} \geq 0$*. Furthermore, for very large values of* β *only best-responses increase in population share under both variants. The former dynamics are known as the logit dynamics (Fudenberg & Levine [1998]), where* $\frac{1}{\beta}$ *is interpreted as an error-term. For error terms going to zero, i.e.,* β*'s going to infinity, the dynamics become more and more similar to the best response dynamics, but remain continuous.*

Fudenberg & Levine [1998] actually write

$$\dot{x}_i = \frac{e^{\beta F_i(x)}}{\sum_{j=1}^{n+1} e^{\beta F_j(x)}} - x_i \text{ for all } x \in S^n,\ i \in I^{n+1}.$$

However, notice that

$$\left(\dot{x}^r_{g^\beta}\right)_i = e^{\beta f_i(x)} - x_i \left(\sum_{j=1}^{n+1} e^{\beta f_j(x)}\right) = \xi(x)\left[\frac{e^{\beta F_i(x)}}{\sum_{j=1}^{n+1} e^{\beta F_j(x)}} - x_i\right].$$

Since, $\xi(x) = \frac{\sum_{j=1}^{n+1} e^{\beta F_j(x)}}{e^{\beta x \cdot F(x)}}$ *does not depend on the subgroup at hand, it follows that both dynamics have the same direction, but may differ in speed.*
A glaring shortcoming of the logit dynamics is that an interior equilibrium need not be a fixed point of the dynamics. In this sense, the orthogonal-projection dynamics are perhaps more interesting than the ray-projection variant, as $f(y) = 0^{n+1}$ *implies* $\dot{x}^o_{g^\beta} = 0^{n+1}$.
Logit-type dynamics which happen to be well-defined on the boundary of the unit simplex and which possess the property that an interior equilibrium is a fixed point of the dynamics are generated by

$$g_i^\beta(x) = \frac{x_i e^{\beta f_i(x)}}{\sum_{j=1}^{n+1} x_j e^{\beta f_j(x)}} \text{ for all } i \in I^{n+1},$$

which yields

$$\begin{aligned}\left(\dot{x}^r_{g^\beta}\right)_i &= x_i\left(\frac{e^{\beta f_i(x)}}{\sum_{j=1}^{n+1} x_j e^{\beta f_j(x)}} - 1\right),\\ \left(\dot{x}^o_{g^\beta}\right)_i &= \frac{x_i e^{\beta f_i(x)}}{\sum_{j=1}^{n+1} x_j e^{\beta f_j(x)}} - \frac{1}{n+1}.\end{aligned}$$

The ray-projection dynamics feature in e.g., Björnerstedt & Weibull [1996], and in Cabrales & Sobel [1992] in a discrete-time version . ■

We refer to Hopkins [1999] and Hofbauer [2000] for stability results of the *ESS* for the ray-projection variant of the logit dynamics. Sandholm [2007] provides a microfoundation for these dynamics (see also Fudenberg & Levine [1998], Hopkins [2002]).

Example 12 ***(Inflow dynamics)*** *We now formulate classes of dynamics which we envision as originating from inflows to the different subgroups (from others). The dynamics of Smith [1984] and Sethi [1998] are examples of a similar idea. Let us start with* ***Sethi-type inflow dynamics****. Let* $a = \mathbb{R}^{n+1}_{++} = \left\{x \in \mathbb{R}^{n+1} | x_j > 0\right\}$ *and let* $g^a : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *be given by*

$g_i^a(x) = a_i x_i \sum_{j \in I^{n+1}} x_j \max\{0, f_i(x) - f_j(x)\}.$
Then, $g_i(x)$ can be interpreted as an inflow from all other subgroups. It should be recalled that Sethi [1998] deals with learning, and in that context subjects observe another member of the population and may switch to the action that the observed member plays. All subgroups with (relative) fitness less than subgroup i are assumed to lose a fraction to subgroup i, the higher the differences in (relative) fitness, the stronger the inflow to i. No subgroup with fitness higher than subgroup i loses members to subgroup i. The number a_i is an indicator how easy it is to switch to subgroup i. A relatively low number indicates that it is difficult to switch to this subgroup. The term x_i can be motivated by probabilistic arguments, that it is easier to observe (more likely to draw) a member of a large subgroup than a member of a small subgroup. Sethi [1998] calls such numbers strategy-specific barriers to learning. Then,

$$\begin{aligned}\left(\dot{x}^r_{g^a}\right)_i &= a_i x_i \sum_{i \in I^{n+1}} x_j \max\{0, f_i(x) - f_j(x)\} - x_i C(x) \text{ and} \\ \left(\dot{x}^o_{g^a}\right)_i &= a_i x_i \sum_{i \in I^{n+1}} x_j \max\{0, f_i(x) - f_j(x)\} - \frac{1}{n+1} C(x),\end{aligned}$$

where $C(x) = \sum_{k \in I^{n+1}} a_k x_k \left(\sum_{j \in I^{n+1}} x_j \max\{0, f_k(x) - f_j(x)\}\right).$
*We now define **Sethi-Smith-type inflow dynamics**. Let $g^a : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be given by $g_i^a(x) = a_i \sum_{j \in I^{n+1}} x_j \max\{0, f_i(x) - f_j(x)\}$. Then,*

$$\begin{aligned}\left(\dot{x}^r_{g^a}\right)_i &= a_i \sum_{i \in I^{n+1}} x_j \max\{0, f_i(x) - f_j(x)\} - x_i C(x) \text{ and} \\ \left(\dot{x}^o_{g^a}\right)_i &= a_i \sum_{i \in I^{n+1}} x_j \max\{0, f_i(x) - f_j(x)\} - \frac{1}{n+1} C(x),\end{aligned}$$

where $C(x) = \sum_{k \in I^{n+1}} a_k \left(\sum_{j \in I^{n+1}} x_j \max\{0, f_k(x) - f_j(x)\}\right).$ ■

4 Boundary conditions

The standard way of dealing with Samuelson's dynamics on the boundary of $\mathbb{P}$ is to define them as being zero for every zero component of the state variable, see e.g., Arrow & Hurwicz [1958, 1960a,b], Arrow *et al.* 1959]. In our notations the extension to include the boundary of $\mathbb{P}$ would be given by

$$\dot{x}_i = \begin{cases} 0 & \text{if } x_i = 0, \\ f_i(x) & \text{otherwise.} \end{cases}$$

So, the dynamics extended to the boundary may be **discontinuous**. For the ray-projection dynamics this extension to the boundary does not pose great problems as we may (re)define

$$\dot{x}_i = \begin{cases} 0 & \text{if } x_i = 0, \\ f_i(x) - \left(\sum_{j:x_j>0} f_j(x)\right) & \text{otherwise.} \end{cases} \tag{a}$$

This definition is identical to our previous definition for the interior of the unit simplex. Under (a), a trajectory might in finite time reach the boundary of the unit simplex, and then remain on it while the relative fitness of a subgroup with population share zero becomes positive again.

An alternative is to define the dynamics extended as

$$\dot{x}_i = \begin{cases} 0 & \text{if } x_i = 0 \text{ and } f_i(x) < 0, \\ f_i(x) - x_i \left(\sum_{\substack{j:x_j>0 \text{ or} \\ f_j(x)\geq 0}} f_j(x)\right) & \text{otherwise.} \end{cases} \tag{b}$$

This way, the dynamics escape the boundary of S^n as soon as $f_i(x) > 0$. So, at a limit point $y \in bd\ S^n$, we can never have $y_i = 0$ and $f_i(y) > 0$.

The following small result has interesting implications. Let, $ZP = \{x \in S^n |\ f(x) = 0^{n+1}\}$ and $FP = \{x \in S^n |\ \dot{x} = 0^{n+1}\}$.

Lemma 13 *Let $\{x_t\}_{t\geq 0}$ be a trajectory under the ray-projection dynamics and let $y = \lim_{t\to\infty} x_t$. If t^* exists such that $\{x_t\}_{t\geq t^*} \subset int\ S^n$, then $y \in ZP$; otherwise, $y \in bd\ S^n$ and under (a) $y \in FP$, under (b) $y \in E$.*

Boundary conditions are obviously of high relevance for boundary equilibria, fixed points and limit points. A refinement of the saturated equilibrium concept is the **strict saturated equilibrium** (cf., Joosten [1996]) which is a saturated equilibrium satisfying $f_j(y) = 0$ for precisely one $j \in I^{n+1}$. For this type of equilibrium we have the following result.

Proposition 14 *Every strict saturated equilibrium is an asymptotically stable fixed point of the ray-projection dynamics.*

Let $SSAT$, $ASFP$, and LP denote the sets of strict saturated equilibria, asymptotically stable fixed points, and limit points respectively; let LP_{int*} denote the set of limit points satisfying there is at least one $\{x_t\}_{t\geq 0}$ with $y = \lim_{t\to\infty} x_t$ satisfying that some t^* exists such that $\{x_t\}_{t\geq t^*} \subset int\ S^n$. Note that in Joosten [1996] it was shown that $SSAT \subseteq GESS \subseteq E$, then the following summarizes results.

Corollary 15 *For arbitrary dynamics, $SSAT \subseteq GESS \subseteq E$. For the ray-projection dynamics: $LP_{int*} \subseteq ZP \subseteq E \subseteq FP$; (a) implies $SSAT \subseteq ESE \subseteq ASFP \subseteq LP \subseteq FP$; (b) implies $SSAT \subseteq ESE \subseteq ASFP \subseteq LP \subseteq E \subseteq FP$.*

4.1 Desiderata for generalized ray-projection dynamics

It is remarkable that several generalizations of the ray-projection dynamics presented in the examples of the previous section happen to be well-defined on the boundary of the unit simplex, as obviously

$$\dot{x}_i = g_i(x) \geq 0 \text{ whenever } x_i = 0.$$

However, projection dynamics do not necessarily become non-negative for boundary states. So, convergence results may depend crucially on how the boundary dynamics are specified.

For future work the results of the preceding subsection show that it may be useful to formulate desiderata:

$$g \text{ satisfies continuity and sign compatibility with } f, \tag{A}$$

$$g \text{ satisfies continuity and weak compatibility with } f. \tag{B}$$

Note that (A) implies (B), and that a sign compatible function g need not yield sign compatible ray-projection dynamics. The proper generalizations of (a) and (b) for g are immediate, i.e.,

$$\dot{x}_i = \begin{cases} 0 & \text{if } x_i = 0, \\ f_i(x) - \left(\sum_{j:x_j>0} f_j(x)\right) & \text{otherwise.} \end{cases} \tag{a'}$$

$$\dot{x}_i = \begin{cases} 0 & \text{if } x_i = 0 \text{ and } f_i(x) < 0, \\ f_i(x) - x_i \left(\sum_{\substack{j:x_j>0 \text{ or} \\ f_j(x)\geq 0}} f_j(x)\right) & \text{otherwise.} \end{cases} \tag{b'}$$

We have a preference for the combination of (b') and (A).

5 Conclusions

We introduced new dynamics on the unit simplex, the ray-projection dynamics. These dynamics form a useful alternative to the orthogonal-projection dynamics of Lahkar & Sandholm [2008]. As the names already indicate, the orthogonal-projection dynamics project the relative fitness function at every point of the unit simplex orthogonally unto it, whereas the ray-projection variant does the same along a ray through the origin.

Under orthogonal-projection dynamics every evolutionarily stable strategy is an evolutionarily stable equilibrium and vice versa (cf., Hofbauer & Sandholm [2008]). This implies that along every trajectory approaching the evolutionarily stable strategy under these dynamics the Euclidean distance to it decreases strictly over time. In this paper, we showed that each strict equilibrium is both an *ESS* and an *ESE* for ray-projection dynamics.

We have proven a convergence (stability) result with the same flavor for ray-projection dynamics. We have shown that every interior evolutionarily stable strategy is an asymptotically stable fixed point of the ray-projection dynamics. The result is immediate if one is familiar with the early economic literature on price-adjustment processes, but we provided a new proof for evolutionary dynamics. For this we transformed a dynamic process on the unit simplex to a dynamic process in the positive orthant, and then projected the latter unto the unit simplex again. Similar tools are used in economics to prove existence of a competitive equilibrium justified by the fact that excess demand functions are homogeneous of degree zero in prices.

We generalized our approach applying both ray-projection dynamics and orthogonal-projection dynamics to more general functions connected to the relative fitness function. It turns out that well-known dynamics in evolutionary game theory can be represented as projection dynamics for appropriately chosen functions. To facilitate future research and applicability of these generalized projection dynamics a natural set of desiderata was presented.

Tsakas & Voorneveld [2008] show that target projection dynamics (Sandholm [2005]), closely related to orthogonal-projection dynamics, can be associated to rational choice behavior if control costs (as in e.g., Van Damme [1991]) can be assumed (see also, Hofbauer & Sandholm [2002], Mattson & Weibull [2002] and Voorneveld [2006]). Further research must reveal which, if any, generalized ray- or orthogonal-projection dynamics can be motivated with similar microeconomic foundations.

Further research must reveal to which extent additional convergence results for price adjustment dynamics of the late fifties and early sixties can be recovered for evolutionary games while remaining within the class of these generalized projection dynamics.

6 Appendix

Proof of Lemma 4. Let $y \in E \cap int\ S^n$, then $f(y) = 0^{n+1}$. Hence, y is a fixed point of both ray-projection and orthogonal-projection dynamics. Conversely, let $y \in int\ S^n$ be a fixed point of the ray-projection dynamics. Then, $f_i(y) - y_i\left(\sum_{j=1}^{n+1} f_j(y)\right) = 0$ for all $i \in I^{n+1}$. This in turn implies $y_i f_i(y) = y_i^2\left(\sum_{j=1}^{n+1} f_j(y)\right)$ for all $i \in I^{n+1}$. Then, summing over all $i \in I^{n+1}$ and complementarity of f lead to $0 = \sum_{i=1}^{n+1} y_i f_i(y) = \sum_{i=1}^{n+1} y_i^2\left(\sum_{j=1}^{n+1} f_j(y)\right)$. This can only hold if $\sum_{j=1}^{n+1} f_j(y) = 0$, hence $f(y) = 0^{n+1}$. For orthogonal-projection dynamics, the reasoning is similar. ■

Proof of Proposition 6. Let $f : S^n \to \mathbb{R}^{n+1}$ be a continuous relative fitness function. Define $\widetilde{f} : \mathbb{P} \to \mathbb{R}^{n+1}$ by $\widetilde{f}(\lambda x) = f(x)$ for all $\lambda > 0$. Then, $\widetilde{f}$ is continuous, homogeneous of degree zero, and satisfies complementarity.

Define for all $x \in \mathbb{P}$:

$$\dot{x} = \widetilde{f}(x). \tag{4}$$

Clearly, this implies that $\frac{d||x||^2}{dt} = 2\sum_{j=1}^{n+1} x_j \dot{x}_j = 2\sum_{j=1}^{n+1} x_j \widetilde{f}_j(x) = 0$. Let $\{x_t\}_{t\geq 0}$ denote a solution to $x_0 \in \mathbb{P}$ and Eq. (5). Then, $\{x_t\}_{t\geq 0}$ remains on the sphere with the origin as center and with radius $r = ||x_0||$.

Let $y \in S^n$ be an interior generalized evolutionarily stable state, i.e., an open neighborhood $U \subseteq int\ S^n$ containing y exists such that

$$(y - x) \cdot f(x) > 0 \text{ for all } x \in U \backslash \{y\}.$$

Let $E = \{x \in \mathbb{P} |\ x = \lambda y,\ \lambda > 0\}$. Define for $z \in \mathbb{P}$, $\lambda_z = \sum_{k=1}^{n+1} z_k$. Then, let $x \in \mathbb{P}$ satisfy $\frac{1}{\lambda_x} x \in U \backslash \{y\}$ and let $y^* \in E$ such that $||x|| = ||y^*||$. Then, obviously $d(x, y^*)^2 > 0$, $d(y^*, y^*)^2 = 0$ and under the dynamics we have

$$\begin{aligned} \dot{d}(x, y^*)^2 &= \dot{\left(\sum_{j=1}^{n+1} (y_j^* - x_j)^2\right)} = -2\sum_{j=1}^{n+1} (y_j^* - x_j)\dot{x}_j \\ &= -2\sum_{j=1}^{n+1} (y_j^* - x_j)\widetilde{f}_j(x) = -2\sum_{j=1}^{n+1} \left(\lambda_{y^*}\frac{y_j^*}{\lambda_{y^*}} - \lambda_x \frac{x_j}{\lambda_x}\right) f_j\left(\frac{x}{\lambda_x}\right) \\ &= -2\sum_{j=1}^{n+1} \left(\lambda_{y^*} y - \lambda_x \frac{x_j}{\lambda_x}\right) f_j\left(\frac{x}{\lambda_x}\right) = -2\lambda_{y^*} \sum_{j=1}^{n+1} \left(y - \frac{x_j}{\lambda_x}\right) f_j\left(\frac{x}{\lambda_x}\right) < 0. \end{aligned}$$

This means that the squared (Euclidean) distance is a strict Lyapunov function for $U' = \left\{x \in \mathbb{P} |\ \frac{1}{\lambda_x} x \in U\right\}$. Hence, an open neighborhood U'' of y^* exists such that every trajectory $\{x_t\}_{t\geq 0}$ with $x_0 \in U'' \backslash \{y^*\}$ such that $||x_0|| = ||y^*||$, converges to y^*, i.e., $\lim_{t\to\infty} x_t = y^*$.

The ray-projection $\{x_t'\}_{t\geq 0}$ of such a trajectory $\{x_t\}_{t\geq 0}$ with $x_0 \in U'' \backslash \{y^*\}$ such that $||x_0|| = ||y^*||$, and $\lim_{t\to\infty} x_t = y^*$ is given by $x' = \frac{x_0}{\sum_{j=1}^{n+1} (x_0)_j}$ and

$$\dot{x}' = \frac{1}{\lambda_x}\left[f(x) - x\left(\sum_{i=1}^{n+1} f_i(x)\right)\right] \text{ for every } x \in \{x_t\}_{t\geq 0}.$$

Clearly, $\lim_{t\to\infty} x_t' = y$. As the factor $\frac{1}{\lambda_x}$ only influences the speed of the dynamics but not the direction, it follows that any trajectory $\{x_t\}_{t\geq 0}$with $x_0 \in U'''$ converges to y under the local ray-projection dynamics given by

$$\dot{x} = f(x) - x\left(\sum_{i=1}^{n+1} f_i(x)\right). \tag{5}$$

So, y is an asymptotically stable fixed point of the dynamics given by (6). ■

Proof of Lemma 13. Let $h : S^n \to \mathbb{R}^{n+1}$ be given by $h(x) = f(x) - \left(\sum_{j=1}^{n+1} f_j(x)\right)$ for all $x \in S^n$. Clearly, h is continuous because f is continuous on the unit simplex. Let $\{x_t\}_{t\geq 0}$ satisfy that some t^* exists such that $\{x_t\}_{t\geq t^*} \subset int\ S^n$ and $\lim_{t\to\infty} x_t = y$. If $y \in int\ S^n$, then by continuity of h it follows that $h(y) = 0^{n+1}$. So, y is an interior fixed point of the dynamics and our earlier result applies, i.e., $y \in E$.

If $y \in bd\ S^n$, then assume $y_j = 0$ and $f_j(y) > 0$. By continuity of h we have $h_j(y) > 0$, and an open neighborhood $U \ni y$ exists such that $h_j(x) > 0$ for all $x \in U$. However, since $y_j = 0$ and $x_j > 0$ for all $x \in \{x_t\}_{t\geq t^*}$ a subsequence $\{x_{t_k}\}_{k\in\mathbb{N}} \subseteq \{x_t\}_{t\geq t^*}$ must exist such that $\left(\dot{x}_{t_k}\right)_j = h_j(x_{t_k}) < 0$ for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty} x_{t_k} = y$, $\{x_{t_k}\}_{k\in\mathbb{N}} \cap U \neq \varnothing$. This yields a contradiction. Hence, $y_j = 0$ implies $f_j(y) \leq 0$. Furthermore, for $y_j > 0$ we have $h_j(y) = 0 = f_j(y) - y_j\left(\sum_{k=1}^{n+1} f_k(y)\right)$ by continuity which implies $f_j(y) = y_j\left(\sum_{k=1}^{n+1} f_k(x)\right)$. However, then $0 = \sum_{j:y_j>0} y_j f_j(y) = \sum_{j:y_j>0} y_j^2\left(\sum_{k=1}^{n+1} f_k(x)\right)$ and therefore $\sum_{k=1}^{n+1} f_k(x) = 0$ which in turn implies $f_j(y) = 0$ whenever $y_j > 0$, hence $f(y) = 0^{n+1}$.

Suppose $\{x_t\}_{t\geq 0} \overset{t\to\infty}{\to} y$ and it does not hold that t^* exists such that $\{x_t\}_{t\geq t^*} \subset int\ S^n$. Let $T = \{k \in I^{n+1} |\ y_k > 0 \text{ or } [y_k = 0 \text{ and } (x_t)_k > 0 \text{ for all } t > t' \text{ for some } t' \geq 0]\}$. If follows from the above that for $k \in T$ it must hold that $f_k(y) = 0$. Now, let $h \in I^{n+1}\backslash T$ then $y_h = (x_t)_h = 0$. If (a) holds, then $\dot{x}_h = 0$ regardless whether $f_h(x) > 0$ or $f_h(x) \leq 0$, hence $y \in FP$. Under (b), $\dot{x}_h > 0$ whenever $f_h(x) > 0$ and therefore $f_h(y) \leq 0$ and $y \in E$. ■

Proof of Proposition 14. Let y be a strict saturated equilibrium, then $m = \max_{h\neq j} f_h(y) < 0$ and continuity implies that a neighborhood $U \ni y$ exists such that $\max_{h\neq j} f_h(x) = \frac{m}{2}$ for all $x \in U$. Complementarity implies $y = e_j$. Let $C_S(x) = \sum_{h\in S\cup\{j\}} f_h(x)$ for $\varnothing \neq S \subseteq I^{n+1}\backslash\{j\}$. Then, clearly $C_S(y) \leq m < 0$ for all nonempty $S \subseteq I^{n+1}\backslash\{j\}$ and a neighborhood $U' \ni y$ exists such that $\max_{h\neq j} f_h(x) = \frac{m}{2}$ for all $x \in U$. Then, let $x \in U \cap U'$

$$
\begin{aligned}
(y - x)\cdot\dot{x} &= (e_j - x)\cdot f(x) - C_{S'}(x)(e_j - x)\cdot x \geq f_j(x) - (x_j - x\cdot x)\, m \\
&= -\frac{\left(\sum_{h\neq j} x_h f_h(x)\right)}{x_j} - (x_j - x\cdot x)\,\frac{m}{2} \\
&\geq -\frac{1 - x_j}{x_j}\frac{m}{2} - (1 - x_j)\,\frac{m}{2}\left(x_j - \max_{h\neq j} x_h\right) \\
&= -(1 - x_j)\,\frac{m}{2}\left(\frac{1}{x_j} + \left(x_j - \max_{h\neq j} x_h\right)\right) \geq -(1 - x_j)\,\frac{m}{2} \geq 0.
\end{aligned}
$$

Here, we have a strict inequality whenever $x_j \neq 1$. So, $y \in ESE$. ■

7 References

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