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On sufficient spectral radius conditions for hamiltonicity of k-connected graphs ☆

Qiannan Zhou [a,b], Hajo Broersma [b,*], Ligong Wang [a], Yong Lu [c]

[a] School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, People's Republic of China
[b] Faculty of EEMCS, University of Twente, P.O. Box 217, 7500 AE Enschede, the Netherlands
[c] School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, Jiangsu 221116, People's Republic of China

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ABSTRACT

In this paper, we present two new sufficient conditions on the spectral radius $\rho(G)$ that guarantee the hamiltonicity and traceability of a k-connected graph G of sufficiently large order, respectively, unless G is a specified exceptional graph. In particular, if $k \geq 2$, $n \geq k^3+k+2$, and $\rho(G) > n-k-1-\frac{1}{n}$, then G is hamiltonian, unless G is a specified exceptional graph. If $k \geq 1$, $n \geq k^3+k^2+k+3$, and $\rho(G) > n-k-2-\frac{1}{n}$, then G is traceable, unless G is a specified exceptional graph.

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* Corresponding author.
E-mail addresses: qnzhoumath@163.com (Q. Zhou), h.j.broersma@utwente.nl (H. Broersma), lgwangmath@163.com (L. Wang), luyong@jsnu.edu.cn (Y. Lu).

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1. Introduction

We use the textbook of Bondy and Murty [3] for any terminology and notation not defined here. Throughout this paper, we consider only finite undirected simple graphs.

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We use $n(G) = |V(G)|$ to denote the order of G, and $e(G) = |E(G)|$ to denote the size of G. For $X \subseteq V(G)$, the induced subgraph $G[X]$ is the subgraph of G whose vertex set is X and whose edge set consists of all edges of G which have both ends in X. For $v \in V(G)$ and subgraphs H and R of G, we let $N_H(v)$ and $N_H(R)$ denote the neighbors of the vertex v and the subgraph R in H, that is, $N_H(v) = \{u \in V(H) \mid uv \in E(G)\}$ and $N_H(R) = (\bigcup_{u \in V(R)} N_H(u)) \setminus V(R)$. When $H = G$, $|N_H(v)|$ is called the degree of the vertex v, simply written as $d(v)$. Denote by $\delta(G)$ the minimum degree of G. For two subsets S and T of $V(G)$, we say S is adjacent to T if every vertex of S is adjacent to every vertex of T. Let P_{uv} and P_{wz} be two disjoint paths. Denote by $P_{uv} \bigsqcup P_{wz}$ a path obtained from P_{uv} and P_{wz} by joining v and w with an edge. A connected graph G is said to be k-connected (or k-vertex connected) if it has more than k vertices and remains connected whenever fewer than k vertices are removed. The connectivity $\kappa(G)$ of G is the maximum value of k for which G is k-connected. The independence number of G, denoted by $\alpha(G)$, is the cardinality of a largest independent (mutually nonadjacent) set of vertices. Denote by $\omega(G)$ the clique number of G, that is, the cardinality of a largest clique, i.e., a set of mutually adjacent vertices. For two graphs G_1 and G_2, we use $G_1 + G_2$ to denote the disjoint union of G_1 and G_2, and $G_1 \vee G_2$ to denote the join of G_1 and G_2.

For a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$, the adjacency matrix $A(G)$ is the symmetric $n \times n$ matrix with entries $A(i, j) = 1$ if and only if $v_iv_j \in E(G)$ and zeros elsewhere. The largest eigenvalue of $A(G)$ is called the spectral radius of G, denoted by $\rho(G)$.

A graph is hamiltonian (traceable) if it contains a Hamilton cycle (Hamilton path), i.e., a cycle (path) containing all vertices of G. Deciding whether a given graph is hamiltonian (traceable) is an NP-complete problem. Many graph theorists have focused on finding sufficient conditions for the existence of a Hamilton cycle (or path). In 2010, Fiedler and Nikiforov [6] gave some bounds on $\rho(G)$ that imply the existence of Hamilton cycles and paths in G. This work motivated further research, as in [1,10,12,13,15,16,18].

For graphs with minimum degree $\delta(G) \geq k$, Nikiforov [15] presented the following result on the spectral radius guaranteeing the hamiltonicity of G.

Theorem 1.1 *([15]). Let G be a graph of order n with $\delta(G) \geq k$. If $k \geq 2$, $n \geq k^3 + k + 4$ and $\rho(G) \geq n - k - 1$, then G is hamiltonian unless $G = K_1 \vee (K_{n-k-1} + K_k)$ or $G = K_k \vee (K_{n-2k} + kK_1)$.*

Recently, Ge and Ning [7] showed that the statement in the above theorem due to Nikiforov also holds for $k \geq 1$ and $n \geq \max\{\frac{1}{2}k^3 + k + \frac{5}{2}, 6k + 5\}$.

For the traceability of G, Li and Ning [10] obtained the following result involving the spectral radius.

Theorem 1.2 *([10]). Let G be a graph of order n with $\delta(G) \geq k$. If $k \geq 0$, $n \geq \max\{6k + 10, (k^2 + 7k + 8)/2\}$ and $\rho(G) \geq \rho(K_k \vee (K_{n-2k-1} + (k+1)K_1))$, then G is traceable unless $G = K_k \vee (K_{n-2k-1} + (k+1)K_1)$.*

Recently, for graphs with connectivity κ and minimum degree δ, Li [11] presented the following sufficient conditions on the spectral radius implying the hamiltonicity and traceability of G, respectively.

Theorem 1.3 *([11]). Let G be a graph of order $n \geq 3$ with connectivity $\kappa \geq 2$. If $\rho(G) \leq \delta\sqrt{\frac{\kappa+1}{n-\kappa-1}}$, then G is hamiltonian or G is $K_{\kappa,\kappa+1}$.*

Theorem 1.4 *([11]). Let G be a graph of order $n \geq 12$ with connectivity $\kappa \geq 1$. If $\rho(G) \leq \delta\sqrt{\frac{\kappa+2}{n-\kappa-2}}$, then G is traceable or G is $K_{\kappa,\kappa+2}$.*

In this paper, we present new sufficient conditions based on the spectral radius for the hamiltonicity and traceability of k-connected graphs. Since being k-connected is a stronger condition than having minimum degree at least k, our motivation was to improve on the bounds for $\rho(G)$ in Theorems 1.1 and 1.2. Obviously, we still have to exclude $K_k \vee (K_{n-2k} + kK_1)$ and $K_k \vee (K_{n-2k-1} + (k+1)K_1)$, respectively, since these graphs are k-connected.

Before stating our results, we introduce some families of graphs based on the two exceptional graphs $K_k \vee (K_{n-2k} + kK_1)$ and $K_k \vee (K_{n-2k-1} + (k+1)K_1)$. For $n \geq 2k+1$, we define

$$G^1_{n,k} = K_k \vee (K_{n-2k} + kK_1).$$

For $n \geq 2k+2$, we start with a graph consisting of vertex-disjoint graphs kK_1 and K_{n-k-1}, and an additional new vertex v, and we let $V(kK_1) = X$ and $V(K_{n-k-1}) = Y$. We choose $Y_2 \subseteq Y$ with $|Y_2| = k$. Now, we let $G^2_{n,k}$ denote the graph obtained from this $kK_1 + K_{n-k-1} + \{v\}$ by joining X to Y_2, and v to Y_2 and an arbitrary vertex in X (see the graph sketched in the left part of Fig. 1). We also define $G^3_{n,k} = K_k \vee (K_{n-2k-1} + (k+1)K_1)$.

For $n \geq 2k+3$, we start with a graph consisting of vertex-disjoint graphs $(k+1)K_1$ and K_{n-k-1}, and we let $V((k+1)K_1) = X$ and $V(K_{n-k-1}) = Y$. We choose $X_1 \subseteq X$ with $|X_1| = k$, and $X_2 = X \setminus X_1$, so with $|X_2| = 1$, and $Y_1 \subseteq Y$ with $|Y_1| = k$, and $Y_2 \subseteq Y \setminus Y_1$ with $|Y_2| = 1$. Now, we let $G^4_{n,k}$ denote the graph obtained from $(k+1)K_1 + K_{n-k-1}$ by adding edges from X to Y_1 and X_2 to Y_2 (see the graph sketched in the right part of Fig. 1). We also define $G^5_{n,k} = K_{k+1} \vee (K_{n-2k-2} + (k+1)K_1)$.

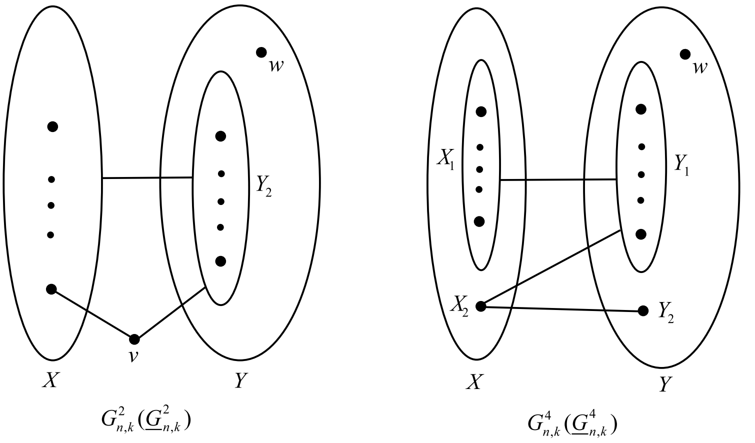

Fig. 1. Some of the graphs appearing in our proofs.

Additionally, we denote by $\underline{G}^1_{n,k}, \underline{G}^2_{n,k}, \underline{G}^3_{n,k}, \underline{G}^4_{n,k}, \underline{G}^5_{n,k}$ the graphs obtained from $G^1_{n+1,k+1}$, $G^2_{n+1,k+1}, G^3_{n+1,k+1}, G^4_{n+1,k+1}, G^5_{n+1,k+1}$ by deleting one vertex of degree n, respectively, i.e.,

$$\underline{G}^1_{n,k} = K_k \vee (K_{n-2k-1} + (k+1)K_1),$$
$$\underline{G}^3_{n,k} = K_k \vee (K_{n-2k-2} + (k+2)K_1),$$
$$\underline{G}^5_{n,k} = K_{k+1} \vee (K_{n-2k-3} + (k+2)K_1).$$

One easily checks that $\underline{G}^2_{n,k}$ is (again) the left graph in Fig. 1 with $|X| = k+1$, $|Y| = n-k-2$, and $|Y_2| = k$. Similarly, $\underline{G}^4_{n,k}$ is (again) the right graph in Fig. 1 with $|X| = k+2$, $|X_1| = k+1$, $|Y| = n-k-2$, and $|Y_1| = k$.

In Section 3, we prove the following two main results.

Theorem 1.5. *Let G be a k-connected graph of order $n \geq k^3 + k + 2$, where $k \geq 2$. If $\rho(G) > n - k - 1 - \frac{1}{n}$, then G is hamiltonian, unless $G = G^1_{n,k}$.*

Note that the exceptional graph $G^1_{n,k}$ in the above theorem is precisely the k-connected graph $K_k \vee (K_{n-2k-1} + (k+1)K_1)$ that was excluded in the conclusion of Theorem 1.1. The condition $\rho(G) > n-k-1-\frac{1}{n}$ in the above theorem looks better than the condition $\rho(G) > n - k - 1$ in Theorem 1.1, but one should note the different meaning of k: Theorem 1.1 involves graphs with minimum degree at least k, whereas the above result deals with k-connected graphs. In this sense, the two results are incomparable.

Theorem 1.6. *Let G be a k-connected graph of order $n \geq k^3 + k^2 + k + 3$, where $k \geq 1$. If $\rho(G) > n - k - 2 - \frac{1}{n}$, then G is traceable, unless $G = \underline{G}^1_{n,k}$.*

Similar remarks pertain to the comparison of the above theorem and Theorem 1.2. The exceptional graph $\underline{G}^1_{n,k}$ in the above theorem is precisely the k-connected graph

$K_k \vee (K_{n-2k-1} + (k+1)K_1)$ that was excluded in the conclusion of Theorem 1.2. The conditions on $\rho(G)$ in the two results are incomparable, in the sense that k in our result refers to k-connected graphs, whereas k in Theorem 1.2 refers to graphs with minimum degree at least k.

The rest of the paper is organized as follows. In Section 2, we will give some useful techniques and lemmas that will be used in our proofs. In Section 3, we present several other necessary lemmas, together with our proofs of Theorems 1.5 and 1.6.

2. Preliminaries

We start this section by introducing a technique based on the concept of equitable partitions. Let M be a symmetric real $n \times n$ matrix. The rows and columns of M are indexed by $X = \{1, \dots, n\}$. Suppose $\pi = \{X_1, \dots, X_m\}$ is a partition of X. Let M be partitioned according to $\{X_1, \dots, X_m\}$, i.e.,

$$M = \begin{pmatrix} M_{11} & \dots & M_{1m} \\ \vdots & & \vdots \\ M_{m1} & \dots & M_{mm} \end{pmatrix},$$

where M_{ij} denotes the block of M formed by the rows in X_i and the columns in X_j. Let $b_{ij} = \frac{\mathbf{1}^T M_{ij} \mathbf{1}}{|X_i|}$, i.e., the average row sum of M_{ij}, where $\mathbf{1}$ is the column vector (of the correct dimension) with all entries equal to 1. Then the matrix $M/\pi = (b_{ij})_{m \times m}$ is called the quotient matrix of M. If the row sum of each block M_{ij} is a constant, then the partition is called equitable.

The following lemma shows that equitable partitions can provide a simple way to calculate eigenvalues.

Lemma 2.1 *([8]). Let G be a graph, and let π be an equitable partition of G. Then $\rho(G) = \rho(A(G)) = \rho(A(G)/\pi)$.*

The concept of the closure of a graph was introduced by Bondy and Chvátal [2]. The k-closure of a graph G is denoted by $cl_k(G)$, and is the unique graph that can be obtained from G by recursively joining nonadjacent vertices with degree sum at least k until no such pair remains in the current graph. When $k = n$, we use $cl(G)$ instead of $cl_n(G)$. A graph is called closed if $G = cl(G)$.

Lemma 2.2 *([2]). A graph G is hamiltonian if and only if $cl(G)$ is hamiltonian.*

The next lemma is one of the generalizations of Dirac's theorem due to Chvátal and Erdős.

Lemma 2.3 *([5]). A graph G with at least three vertices is hamiltonian if $\kappa(G) \geq \alpha(G)$.*

Since we consider hamiltonicity (and traceability) of k-connected graphs, we can make use of Lemma 2.3 (and its counterpart for traceable graphs) if we are able to use the conditions on $\rho(G)$ to obtain upper bounds on the independence number of G in our proofs. This is the main difference between our approach and the applied methods in [15] and [10].

We will also frequently use the following known lemmas involving $\rho(G)$.

Lemma 2.4 *([4,8]). Let G be a connected graph. If H is a subgraph of G, then $\rho(H) \leq \rho(G)$, with strict inequality in case H is a proper subgraph of G.*

Lemma 2.5 *([14]). Let G be a graph on n vertices and m edges with minimum degree δ. Then $\rho(G) \leq \frac{\delta-1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}$.*

In conjunction with Lemma 2.5, we also use the following property.

Lemma 2.6 *([9,14]). For nonnegative integers p and q with $2q \leq p(p-1)$ and $0 \leq x \leq p-1$, the function $f(x) = \frac{x-1}{2} + \sqrt{2q - px + \frac{(x+1)^2}{4}}$ is decreasing with respect to x.*

3. The proofs of our results

In this section, we first give two results involving a lower bound on the number of edges, the first of which is known and implies a lower bound on the clique number. This is then used in the second result to characterize exceptional, i.e., nonhamiltonian k-connected graphs meeting the conditions, which is one of the key ingredients of our proof of Theorem 1.5. The proof is completed by determining upper bounds on $\rho(G)$ for (subgraphs of) the exceptional graphs, the content of Lemma 3.3 below. For our proof of Theorem 1.6, we take a similar approach and need a counterpart of Theorem 3.2 below on traceability.

From Lemma 1 in [10], we have the following lemma.

Lemma 3.1. *Let G be a closed graph of order $n \geq 6k+11$, where $k \geq 0$. If $e(G) > \binom{n-k-2}{2} + (k+2)^2$, then $\omega(G) \geq n-k-1$.*

Using the above, we can prove the following structure theorem.

Theorem 3.2. *Let G be a k-connected graph of order $n \geq 6k+11$, where $k \geq 2$. If $e(G) > \binom{n-k-2}{2} + (k+2)^2$, then G is hamiltonian unless $cl(G) \in \{G^1_{n,k}, G^2_{n,k}, G^3_{n,k}, G^4_{n,k}, G^5_{n,k}\}$.*

Proof. Let $H = cl(G)$. If H is hamiltonian, then so is G by Lemma 2.2. Now, we assume that H is not hamiltonian. Note that H is k-connected and $e(H) \geq e(G)$. By Lemma 3.1, we have $\omega(H) \geq n-k-1$. We claim that $\omega(H) \leq n-k$. In fact, if $\omega(H) \geq n-k+1$, then $\alpha(H) \leq k$. Then, by Lemma 2.3, H is hamiltonian, which contradicts our assumption. Note that $\delta(G) \geq \kappa(G) \geq k$. Next we distinguish two cases.

Case 1. $\omega(H) = n-k$.
In this case, we have $\alpha(H) = k+1$. Set $V(H) = X \cup Y$, where $X \cap Y = \emptyset$, $|X| = k$, $|Y| = n-k$, $H[Y] = K_{n-k}$ and X together with one vertex in Y, say w, is a maximum independent set in H. Note that $\delta(H) \geq \kappa(H) \geq k$. From the fact that H is closed and $d_H(w) = n-k-1$, we have $d_H(x) = k$ for every $x \in X$. Let $F \subseteq Y$ be the neighbors of X in Y. Obviously $w \notin F$. Then every vertex in F is adjacent to every vertex in X, and hence $|F| = k$. So $H = K_k \vee (K_{n-2k} + kK_1) = G^1_{n,k}$.

Case 2. $\omega(H) = n-k-1$.
In this case, we have $\alpha(H) = k+1$ or $k+2$. We discuss these two subcases separately.

Subcase 2.1 $\alpha(H) = k+1$.
We first consider the situation that $V(H)$ can be divided into two subsets, i.e., $V(H) = X \cup Y$, such that $X \cap Y = \emptyset$, $|X| = k+1$, $|Y| = n-k-1$, $H[X] = (k+1)K_1$ and $H[Y] = K_{n-k-1}$. In this situation, each vertex of Y must be adjacent to some vertex in X; otherwise $\alpha(H) = k+2$. Set $Y = Y_1 \cup Y_2$ such that each vertex of Y_1 has only one neighbor in X and each vertex of Y_2 has at least two neighbors in X. Recall that H is closed. If $Y_1 = \emptyset$, then $H = K_{n-k-1} \vee (k+1)K_1$, which is hamiltonian, a contradiction. If $Y_2 = \emptyset$, then obviously H has a Hamilton cycle, a contradiction. Next, let $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$. Then $d_H(y) = n-k-1$ for $y \in Y_1$, and Y_2 is adjacent to X since $d_H(y) \geq n-k$ for $y \in Y_2$. Denote by X_1 the neighbor set of Y_1 in X, and let $X_2 = X \setminus X_1$. If $|X_1| \geq 2$, then $d_H(x) = k$ for $x \in X_1$; otherwise Y_1 will be adjacent to X_1, and each vertex of Y_1 has more than one neighbor in X. Then we have $|Y_2| \leq k-1$ and hence $d_H(x) \leq k-1$ for $x \in X_2$, a contradiction. If $|X_1| = 1$, then the vertex $x \in X_1$ is adjacent to Y, which means $\omega(H) = n-k$, a contradiction.

The other situation is that $V(H) = X \cup Y \cup \{v\}$, such that $X \cap Y \cap \{v\} = \emptyset$, $|X| = k$, $|Y| = n-k-1$, $H[Y] = K_{n-k-1}$, and X together with one vertex of Y, say w, is a maximum independent set in H. We use the same notations as in the first situation. We deduce that v is adjacent to Y_2, and that v must be adjacent to at least one vertex in X; otherwise $\alpha(H) = k+2$.

We first assume v is also adjacent to $Y \setminus \{Y_1 \cup Y_2\}$, that is to say, all possible w have degree $n-k-1$. Then $d_H(x) = k$ for all $x \in X$. In this case, we have $Y_1 \neq \emptyset$; otherwise v is adjacent to Y and $\omega(H) = n-k$, and hence $X_1 \neq \emptyset$. If $X_2 \neq \emptyset$, then $|Y_2| = k-1$, v is adjacent to X_2, and each vertex of X_1 has exactly one neighbor in Y_1. Hence $|Y_1| = |X_1| \leq k-1$. Suppose $|X_1| = a \geq 1$ and $|X_2| = b \geq 1$, where $a+b = k$. We use G_1 to denote the graph when v has no neighbors in $Y \setminus (Y_1 \cup Y_2)$ (see the left graph of Fig. 2). Obviously, we have $G_1 \subseteq H$. Label the vertices of G_1 as $x_{11}, \ldots, x_{1a}$; $x_{21}, \ldots, x_{2b}$; $y_{11}, \ldots, y_{1a}$; $y_{21}, \ldots, y_{2a}, y_{31}, \ldots, y_{3,b-1}$; $y_{41}, \ldots, y_{4c}$ (see the left graph of Fig. 2), where $c = n-2k-a$. Let $Q_1 = y_{11}x_{11}y_{21}x_{12}y_{12}y_{13}x_{13}$, $Q_2 = y_{22}x_{14}y_{23}x_{15} \cdots y_{2,a-2}x_{1a}$, $Q_3 =$

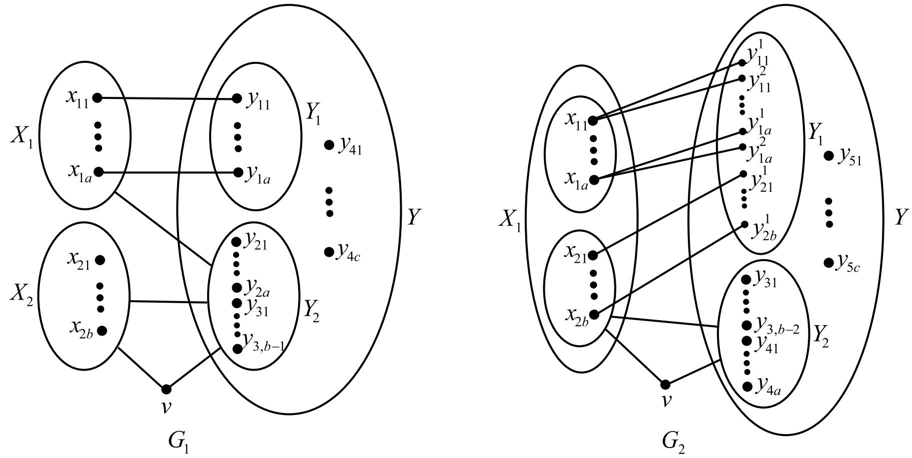

Fig. 2. The graphs G_1 and G_2 from the proof.

$y_{31}x_{21}\cdots y_{3,b-1}x_{2,b-1}vx_{2b}$ and $Q_4 = y_{2,a-1}y_{2a}y_{14}\cdots y_{1a}y_{41}\cdots y_{4c}y_{11}$. Then $\bigsqcup_{i=1}^{4} Q_i$ is a Hamilton cycle. Hence H is also hamiltonian, a contradiction. If $X_2 = \emptyset$, then $|Y_2| \leq k-2$. Indeed, if $|Y_2| \geq k-1$, then either v has no neighbors in X_1 or the vertices in X_1 have degree more than k, both leading to a contradiction. Suppose $|Y_2| = t$. Then each $x \in X_1 \setminus N_{H[X_1]}(v)$ has $k-t$ neighbors in Y_1, and each $x \in N_{H[X_1]}(v)$ has $k-t-1$ neighbors in Y_1. When $t = k-2$, we use G_2 to denote the graph if v has no neighbors in $Y \setminus (Y_1 \cup Y_2)$ (see the right graph of Fig. 2). Obviously, $G_2 \subseteq H$. Label the vertices of G_2 as $x_{11}, \ldots, x_{1a}$; $x_{21}, \ldots, x_{2b}$; $y_{11}^1, y_{11}^2, \ldots, y_{1a}^1, y_{1a}^2$; $y_{21}^1, \ldots, y_{2b}^1$; $y_{31}, \ldots, y_{3,b-2}, y_{41}, \ldots, y_{4a}$; $y_{51}, \ldots, y_{5c}$, where $a \geq 1$, $b \geq 1$, $a+b = k$ and $c = n-3k-a+1$. Let $Q_1 = y_{11}^1x_{11}y_{11}^2\cdots y_{1a}^1x_{1a}y_{1a}^2$, $Q_2 = y_{21}^1x_{21}y_{31}\cdots x_{2,b-2}y_{3,b-2}x_{2,b-1}vx_{2b}y_{2b}^1$ and $Q_3 = y_{22}^1\cdots y_{2,b-1}^1y_{41}\cdots y_{4a}y_{51}\cdots y_{5c}y_{11}^1$ (see the right graph of Fig. 2). Then $\bigsqcup_{i=1}^{3} Q_i$ is a Hamilton cycle of G_2. Hence H is also hamiltonian, a contradiction. When $t \leq k-3$, we have that each vertex of X_1 has at least two neighbors in Y_1. Obviously, H has a Hamilton cycle, a contradiction.

Next, we assume that there exists a vertex w such that its degree is $n-k-2$, i.e., v is not adjacent to w. In this case, $d_H(x) = k$ or $k+1$ for $x \in X$.

If $Y_1 = \emptyset$, then $X_1 = \emptyset$. If $d_H(x) = k+1$ for all $x \in X_2$, then $|Y_2| = k$, and v is adjacent to X_2. Then we can easily find a Hamilton cycle in H, a contradiction. If $d_H(x) = k$ for all $x \in X_2$, then $|Y_2| = k-1$ ($k \geq 2$) and again v is adjacent to X. Hence $d_H(v) \geq 2k-1$. We see that v must have at least one neighbor in $Y \setminus Y_2$; otherwise $\kappa(H) = k-1$. Now $d_H(v) \geq 2k$ and $d_H(v) + d_H(w) \geq 2k + n - k - 2 = n + k - 2 \geq n$, which means v is adjacent to $Y \setminus Y_2$, a contradiction. If some vertices in X_2 have degree k and some vertices in X_2 have degree $k+1$, then $|Y_2| = k$, and the vertices in X_2 with degree $k+1$ are adjacent to v. In this case, if v has a neighbor in $Y \setminus Y_2$ or v has at least two neighbors in X, then H is obviously hamiltonian, a contradiction. If v has no neighbor in $Y \setminus Y_2$ and has only one neighbor in X, then $H = G_{n,k}^2$.

If $Y_2 = \emptyset$, it is easy to see that H is hamiltonian, a contradiction.

If $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$, when $X_2 \neq \emptyset$, then $d_H(x) = k$ for $x \in X_2$ since the vertices in X_2 have no neighbors in Y_1. If $|Y_2| = k$, then on the one hand, there is no neighbor of v in X_2; otherwise its neighbor will be adjacent to Y_1. On the other hand, v cannot be adjacent to any $x \in X_1$; otherwise $d_H(x) = k+2$, a contradiction. Therefore, $|Y_2| = k-1$ and hence, X_2 is adjacent to v. In this case, every vertex in X_1 has a one-to-one neighbor in Y_1. Also, v must be adjacent to at least one vertex in $Y \setminus (Y_1 \cup Y_2)$; otherwise Y_2 will be a cut set of H. Now we conclude that $G_1 \subseteq H$. By the former proof, we obtain that H is hamiltonian, a contradiction. When $X_2 = \emptyset$, then for $x \in X_1$, we have $d_H(x) = k$. We can obtain that $|Y_2| \leq k-2$. Let $|Y_2| = t$. Denote by $X_{11} = N_{H[X_1]}(v)$ and $X_{12} = X_1 \setminus X_{11}$. Then each vertex of X_{11} has $k-t-1$ neighbors in Y_1, and each vertex of X_{12} has $k-t$ neighbors in Y_1. Obviously, when $t \leq k-3$, every vertex in X_1 has more than two neighbors in Y_1, and it is easy to find a Hamilton cycle in H, a contradiction. When $t = k-2$, we obtain that $G_2 \subseteq H$. Then, by the former proof, H is also hamiltonian, a contradiction.

Subcase 2.2 $\alpha(H) = k+2$.

Set $V(H) = X \cup Y$, where $X \cap Y = \emptyset$, $|X| = k+1$, $|Y| = n-k-1$, $H[Y] = K_{n-k-1}$, and X together with one vertex in Y, say w, is a maximum independent set in H. Since $d_H(w) = n-k-2$, we know that $d_H(x) = k$ or $k+1$ for $x \in X$. Let $X_1 \subseteq X$ be the set, each vertex of which has degree k, and let $X_2 = X \setminus X_1$ be the set, each vertex of which has degree $k+1$. Denote by Y_1 the neighbors of X_1 in Y, and $Y_2 = N_{H[Y]}(X_2) \setminus Y_1$.

The first situation we consider is that $X_1 = \emptyset$. Then X is adjacent to Y_2, and $|Y_2| = k+1$. Hence $H = K_{k+1} \vee (K_{n-2k-2} + (k+1)K_1) = G^5_{n,k}$.

The second situation we consider is that $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$. Firstly, Y_1 is adjacent to X_2 since $d_H(y) \geq n-k-1$ for $y \in Y_1$. So, now every vertex of Y_1 has at least two neighbors in X, and hence $d_H(y) \geq n-k$ for $y \in Y_1$. Then every vertex in X_1 is adjacent to every vertex in Y_1, and hence $|Y_1| = k$. Then every $x \in X_2$ has a one-to-one neighbor in Y_2, hence $|X_2| = |Y_2|$. When $|X_2| = |Y_2| = 2$, label the vertices of H as $x_1, x_2, \ldots, x_{k+1}$; $y_0, y_1, \ldots, y_{k+1}$; $y_{11}, y_{12}, \ldots, y_{1,n-2k-3}$ (see Fig. 3 (1)). Let $Q_1 = y_0 x_k y_k y_{11} y_{12} \cdots y_{1,n-2k-3} y_{k+1} x_{k+1}$, $Q_2 = y_1 x_1 y_2 x_2 \cdots y_{k-1} x_{k-1} y_0$. Obviously $P = Q_1 \bigsqcup Q_2$ is a Hamilton cycle, a contradiction. If $|X_2| = |Y_2| \geq 3$, label the vertices of H as $x_{11}, \ldots, x_{1a}$; $x_{21}, \ldots, x_{2b}$; $y_{11}, \ldots, y_{1a}$, $y_{1,a+1}, \ldots, y_{1,a+b-1}$; $y_{21}, \ldots, y_{2b}$; $y_{31}, \ldots, y_{3,n-2k-b-1}$ (see Fig. 3 (2)), where $|X_1| = a$, $|X_2| = b \geq 3$, $a+b = k+1$. Let $Q_1 = y_{11} x_{11} \cdots y_{1a} x_{1a}$. If b is even, let $Q_{2i} = y_{1,a+i} x_{2,2i-1} y_{2,2i-1} y_{2,2i} x_{2,2i}$ $(1 \leq i \leq \frac{b}{2})$ and $Q_3 = y_{1,\frac{b}{2}+1} \cdots y_{1,a+b-1} y_{31} \cdots y_{3,n-2k-b-1}$. Then we obtain a Hamilton cycle $Q_1 \bigsqcup (\bigsqcup_{i=1}^{b/2} Q_{2i}) \bigsqcup Q_3 y_{11}$, a contradiction. If b is odd, let $Q'_3 = y_{1,\frac{b-1}{2}+1} x_{2b} y_{2b} y_{1,\frac{b-1}{2}+2} \cdots y_{1,a+b-1} y_{31} \cdots y_{3,n-2k-b-1}$. Then we obtain a Hamilton cycle $Q_1 \bigsqcup (\bigsqcup_{i=1}^{(b-1)/2} Q_{2i}) \bigsqcup Q'_3 y_{11}$, a contradiction. When $|X_2| = |Y_2| = 1$, $H = G^4_{n,k}$.

The third situation we consider is that $X_2 = \emptyset$. Then $Y_2 = \emptyset$. Set $Y_1 = Y_{11} \cup Y_{12}$, where Y_{11} is the set of vertices with only one neighbor in X, and Y_{12} is the set of vertices with at least two neighbors in X. Let $Y_3 = Y \setminus Y_1$, i.e., Y_3 is the set of vertices with no

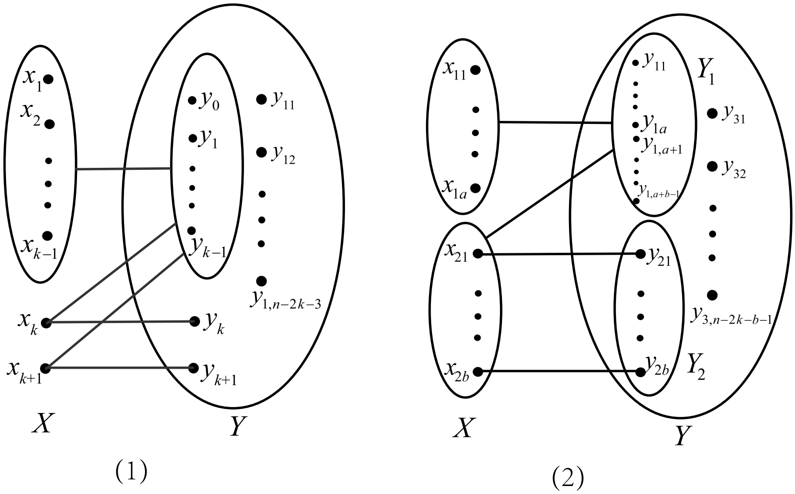

Fig. 3. The graph H from the proof.

neighbor in X. Obviously $w \in Y_3$, $d_H(y) = n-k-1$ for $y \in Y_{11}$ and $d_H(y) \geq n-k$ for $y \in Y_{12}$.

If $Y_{11} = \emptyset$, then it is easy to see that X is adjacent to Y_{12}, $|Y_{12}| = k$, and hence $H = K_k \vee (K_{n-2k-1} + (k+1)K_1) = G^3_{n,k}$.

If $Y_{12} = \emptyset$, then obviously H is hamiltonian, a contradiction.

If $Y_{11} \neq \emptyset$ and $Y_{12} \neq \emptyset$, then $|Y_{11}| + |Y_{12}| \geq k+1$, and X is adjacent to Y_{12}. Let $|Y_{11}| = a$. Then $1 \leq a \leq k+1$. If $1 \leq a \leq k$, then $|Y_{12}| = k$ and each vertex in $N_{H[X]}(Y_{11})$ has degree $k+1$, a contradiction. So $a = k+1$, i.e., every vertex in X has a one-to-one neighbor in Y_{11}, which leads to $|Y_{12}| = k-1$. If $k+1$ is even, then label the vertices of H as $x_1, x_2, \ldots, x_{k+1}$; $y_{11}, y_{12}, \ldots, y_{1,k+1}$; $y_{21}, \ldots, y_{2,\frac{k+1}{2}}, y_{2,\frac{k+3}{2}}, \ldots, y_{2,k-1}$; $y_{31}, \ldots, y_{3b}$ (see Fig. 4 (3)), where $b = n-3k-1$. Let $Q_i = y_{1,2i-1}x_{2i-1}y_{2,i}x_{2i}y_{1,2i}$ ($i = 1, 2, \ldots, \frac{k+1}{2}$) and let $Q = y_{2,\frac{k+3}{2}} \cdots y_{2,k-1}y_{31}y_{32}\cdots y_{3b}$. Then $(\bigsqcup_{i=1}^{\frac{k+1}{2}} Q_i) \bigsqcup Qy_{11}$ is a Hamilton cycle in H, a contradiction.

If $k+1$ is odd, label the vertices of H as $x_1, x_2, \ldots, x_{k+1}$; $y_{11}, y_{12}, \ldots, y_{1,k+1}$; $y_{21}, \ldots, y_{2,\frac{k}{2}}, y_{2,\frac{k}{2}+1}, \ldots, y_{2,k-1}$; $y_{31}, \ldots, y_{3b}$ (see Fig. 4 (4)), where $b = n-3k-1$. Let $Q' = y_{1,k+1}x_{k+1}y_{2,\frac{k}{2}+1}y_{2,\frac{k}{2}+2}\cdots y_{2,k-1}y_{31}y_{32}\cdots y_{3b}$. Then $(\bigsqcup_{i=1}^{\frac{k}{2}} Q_i) \bigsqcup Q'y_{11}$ is a Hamilton cycle in H, a contradiction. □

To complete our proof of Theorem 1.5, we also need the following result on the spectral radius.

Lemma 3.3. *Let G be a graph of order n with minimum degree $\delta(G) \geq k$, where $n \geq k^3 + k + 2$ and $k \geq 2$.*

(i) If G is a proper subgraph of $G^1_{n,k}$, then $\rho(G) < n-k-1-\frac{1}{n}$.

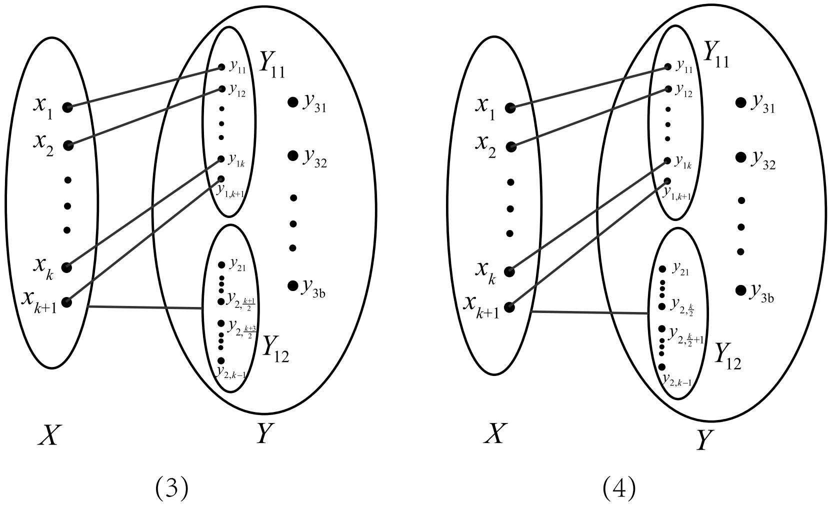

Fig. 4. The graph H from the proof.

(ii) If $G \in \{G^2_{n,k}, G^3_{n,k}, G^4_{n,k}, G^5_{n,k}\}$, *then* $\rho(G) < n - k - 1 - \frac{1}{n}$.

Proof. (i) For $G^1_{n,k} = K_k \vee (K_{n-2k} + kK_1)$, let X be the set of vertices with degree k, let Y be the set of neighbors of X, and let Z be the remaining set of $n-2k$ vertices. Assume that G is a proper subgraph obtained from $G^1_{n,k}$ by deleting one edge $uv \in E(G^1_{n,k})$. According to the proof of Theorem 1.6 in [15], we know that there are three possible cases, and that the case with $u, v \in Z$ yields the largest spectral radius. It is sufficient to prove $\rho(G) < n - k - 1 - \frac{1}{n}$ for this case.

Let us consider the following partition π of $V(G)$ in $X_1 = X$, $X_2 = Y$, $X_3 = Z \setminus \{u, v\}$, and $X_4 = \{u, v\}$. It is easy to check that this partition is equitable, and that the corresponding adjacency matrix of the quotient matrix of G is as follows:

$$A(G/\pi) = \begin{pmatrix} 0 & k & 0 & 0 \\ k & k-1 & n-2k-2 & 2 \\ 0 & k & n-2k-3 & 2 \\ 0 & k & n-2k-2 & 0 \end{pmatrix}.$$

By using the Laplace expansion for the calculation of the determinant, we obtain the following characteristic polynomial of $A(G/\pi)$.

$$f_1(x) = x^4 + (k-n+4)x^3 + (-k^2 + 3k - 3n + 7)x^2 + (2k - 2n + k^2n - 3k^2 - 2k^3 + 4)x + 2k^2n - 4k^2 - 4k^3.$$

Then we can obtain the following derivatives of $f_1(x)$ with respect to x, by standard analysis:

$$
\begin{aligned}
f_1'(x) &= 4x^3 + 3(k-n+4)x^2 + 2(-k^2+3k-3n+7)x + 2k - 2n \\
&\quad + k^2 n - 3k^2 - 2k^3 + 4, \\
f_1^{(2)}(x) &= 12x^2 + 6(k-n+4)x + 2(-k^2+3k-3n+7), \\
f_1^{(3)}(x) &= 24x + 6(k-n+4), \\
f_1^{(4)}(x) &= 24.
\end{aligned}
$$

By substitution and simple but tedious calculations, we obtain

$$
\begin{aligned}
f_1(n-k-1-\frac{1}{n}) &= n^2 - (k^3+k+2)n + \frac{k^3+k^2-3k+2}{n} + \frac{2k^2-2}{n^2} + \frac{3k}{n^3} + \frac{1}{n^4} \\
&\quad + k^4 - k^3 - 2k^2 + 2k \\
&> n^2 - (k^3+k+2)n + k^4 - k^3 - 2k^2 + 2k \\
&= g_1(n) \geq 0,
\end{aligned}
$$

where $g_1(x) = x^2 - (k^3+k+2)x + k^4 - k^3 - 2k^2 + 2k$. For the last inequality, obviously, the function $g_1(x)$ is strictly increasing when $x \geq k^3+k+2$. Using $n \geq k^3+k+2$, we obtain $g_1(n) \geq g_1(k^3+k+2) = k^4 - k^3 - 2k^2 + 2k \geq 0$.

$$
\begin{aligned}
f_1'(n-k-1-\frac{1}{n}) &= n^3 - 3kn^2 + (2k^2-3)n - \frac{4k^2-7}{n} - \frac{9k}{n^2} - \frac{4}{n^3} - k^3 - k^2 + 9k - 2 \\
&= g_2(n) \geq g_2(k^3+k+2) \\
&= k^9 + 6k^6 - k^5 + 8k^3 - 3k^2 + 6k - \frac{4k^2-7}{k^3+k+2} - \frac{9k}{(k^3+k+2)^2} \\
&\quad - \frac{4}{(k^3+k+2)^3} \\
&\geq k^9 + (6k^6 - k^5) + (8k^3 - 3k^2) + 6k - \frac{3}{4} - \frac{1}{8} - \frac{1}{432} \\
&> 0,
\end{aligned}
$$

where $g_2(x) = x^3 - 3kx^2 + (2k^2-3)x - \frac{4k^2-7}{x} - \frac{9k}{x^2} - \frac{4}{x^3} - k^3 - k^2 + 9k - 2$. For the first inequality, the first derivative of $g_2(x)$ is $g_2'(x) = 3x^2 - 6kx + 2k^2 - 3 + \frac{4k^2-7}{x^2} + \frac{18k}{x^3} + \frac{12}{x^4}$. Since $g_2'(x) \geq g_2'(k^3+k+2) = 3k^6 + 12k^3 - k^2 + 9 + \frac{4k^2-7}{(k^3+k+2)^2} + \frac{18k}{(k^3+k+2)^3} + \frac{12}{(k^3+k+2)^4} > 0$ for $x \geq k^3+k+2$, we obtain that $g_2(x)$ is strictly increasing when $x \geq k^3+k+2$. Using $n \geq k^3+k+2$, we have $g_2(n) \geq g_2(k^3+k+2)$.

$$
\begin{aligned}
f_1^{(2)}(n-k-1-\frac{1}{n}) &= 6n^2 - 12kn + \frac{18k}{n} + \frac{12}{n^2} + 4k^2 - 16 \\
&> 6n^2 - 12kn + 4k^2 - 16 \\
&= g_3(n) > 0,
\end{aligned}
$$

where $g_3(x) = 6x^2 - 12kx + 4k^2 - 16$. For the last inequality, it is easy to see that $g_3(x)$ is strictly increasing when $x \geq k^3 + k + 2$. Using $n \geq k^3 + k + 2$, we obtain $g_3(n) \geq g_3(k^3 + k + 2) = 6k^6 + 24k^3 - 2k^2 + 8 > 0$.

$$f_1^{(3)}(n - k - 1 - \frac{1}{n}) = 6(3n - \frac{4}{n} - 3k) > 0.$$
$$f_1^{(4)}(n - k - 1 - \frac{1}{n}) = 24 > 0.$$

Hence, by the Fourier-Budan Theorem (see, e.g., [17]), there is no root of $f_1(x)$ in the interval $[n - k - 1 - \frac{1}{n}, +\infty)$. Then by Lemma 2.4, all subgraphs of $G^1_{n,k}$ have spectral radius less than $n - k - 1 - \frac{1}{n}$.

(ii) Since the proofs are very similar, we omit the details. □

Proof of Theorem 1.5. Note that $\delta(G) \geq \kappa(G) \geq k$. By Lemmas 2.5 and 2.6, we have

$$n - k - 1 - \frac{1}{n} < \rho(G) \leq \frac{k-1}{2} + \sqrt{2e(G) - nk + \frac{(k+1)^2}{4}}.$$

Therefore, when $n \geq k^3 + k + 2$, we obtain

$$\begin{aligned} e(G) &> \frac{1}{2}[n^2 - (2k+1)n + \frac{3k+1}{n} + \frac{1}{n^2} + 2k^2 + k - 2] \\ &> \frac{n^2 - (2k+1)n + 2k^2 + k - 2}{2} \\ &> \binom{n-k-2}{2} + (k+2)^2. \end{aligned}$$

By Theorem 3.2, G is hamiltonian or $cl(G) \in \{G^1_{n,k}, G^2_{n,k}, G^3_{n,k}, G^4_{n,k}, G^5_{n,k}\}$. Since $K_{n-k} \subseteq G^1_{n,k}$, by Lemma 2.4, we have $\rho(G^1_{n,k}) > \rho(K_{n-k}) = n - k - 1$. Combining the above with Lemmas 2.4 and 3.3, we conclude that $G = G^1_{n,k}$. □

Next, we turn to traceable graphs and the counterparts of the above results that we need for our proof of Theorem 1.6.

Theorem 3.4. *Let G be a k-connected graph of order $n \geq 6k + 16$, where $k \geq 1$. If $e(G) > \binom{n-k-3}{2} + (k+2)(k+3)$, then G is traceable unless $cl_{n-1}(G) \in \{\underline{G}^1_{n,k}, \underline{G}^2_{n,k}, \underline{G}^3_{n,k}, \underline{G}^4_{n,k}, \underline{G}^5_{n,k}\}$.*

Proof. Let $G' = G \vee K_1$. Then we have $\kappa(G') = \kappa(G) + 1 \geq k + 1$, $n(G') = n + 1 = 6(k + 1) + 11$ and $e(G') = e(G) + n > \binom{n-k-3}{2} + (k + 2)(k + 3) + n = \binom{n-k-2}{2} + (k + 3)^2$. By Theorem 3.2, we know G' is hamiltonian unless

$cl_{n+1}(G') \in \{G^1_{n+1,k+1}, G^2_{n+1,k+1}, G^3_{n+1,k+1}, G^4_{n+1,k+1}, G^5_{n+1,k+1}\}$. Hence, G is traceable unless $cl_{n-1}(G) \in \{\underline{G}^1_{n,k}, \underline{G}^2_{n,k}, \underline{G}^3_{n,k}, \underline{G}^4_{n,k}, \underline{G}^5_{n,k}\}$. $\square$

Lemma 3.5. *Let G be a graph of order n with minimum degree $\delta(G) \geq k$, where $n \geq k^3 + k^2 + k + 3$ and $k \geq 1$.*

(i) If G is a proper subgraph of $\underline{G}^1_{n,k}$, then $\rho(G) < n - k - 2 - \frac{1}{n}$.
(ii) If $G \in \{\underline{G}^2_{n,k}, \underline{G}^3_{n,k}, \underline{G}^4_{n,k}, \underline{G}^5_{n,k}\}$, then $\rho(G) < n - k - 2 - \frac{1}{n}$.

Proof. (i) For $\underline{G}^1_{n,k} = K_k \vee (K_{n-2k-1} + (k+1)K_1)$, let X be the set of degree k, Y be the set of neighbors of X, and Z be the set of remaining $n - 2k - 1$ vertices. Assume that G is a proper subgraph obtained from $\underline{G}^1_{n,k}$ by deleting one edge $uv \in E(\underline{G}^1_{n,k})$. Similarly as in the proof of Lemma 3.3, the case $u, v \in Z$ yields the largest spectral radius. It is sufficient to prove $\rho(G) < n - k - 2 - \frac{1}{n}$ for this case.

Let us consider the following partition π of $V(G)$ in $X_1 = X$, $X_2 = Y$, $X_3 = Z \setminus \{u, v\}$, and $X_4 = \{u, v\}$. It is easy to check that this partition is equitable, and that the corresponding adjacency matrix of G/π is as follows:

$$A(G/\pi) = \begin{pmatrix} 0 & k & 0 & 0 \\ k+1 & k-1 & n-2k-3 & 2 \\ 0 & k & n-2k-4 & 2 \\ 0 & k & n-2k-3 & 0 \end{pmatrix}.$$

By using the Laplace expansion for the calculation of the determinants, we obtain the characteristic polynomial of $A(G/\pi)$:

$$\begin{aligned} f_2(x) &= x^4 + (k - n + 5)x^3 + (-k^2 + 2k - 3n + 10)x^2 \\ &\quad + (kn - 2n - 2k + k^2 n - 6k^2 - 2k^3 + 6)x + 2kn - 6k + 2k^2 n - 10k^2 - 4k^3. \end{aligned}$$

Then we can obtain the following derivatives of $f_2(x)$ with respect to x, by standard analysis:

$$\begin{aligned} f_2'(x) &= 4x^3 + 3(k - n + 5)x^2 + 2(-k^2 + 2k - 3n + 10)x + kn - 2n - 2k + k^2 n \\ &\quad - 6k^2 - 2k^3 + 6, \\ f_2^{(2)}(x) &= 12x^2 + 6(k - n + 5)x + 2(-k^2 + 2k - 3n + 10), \\ f_2^{(3)}(x) &= 24x + 6(k - n + 5), \\ f_2^{(4)}(x) &= 24. \end{aligned}$$

By substitution and simple but tedious calculations, we obtain

$$f_2(n - k - 2 - \frac{1}{n}) = n^2 - (k^3 + k^2 + k + 3)n + \frac{k^3 + 3k^2}{n} + \frac{2k^2 + 5k + 1}{n^2} + \frac{3k + 3}{n^3} + \frac{1}{n^4}$$

$$\begin{aligned}&+k^4+k^3-2k^2-k+1\\&>n^2-(k^3+k^2+k+3)n+k^4+k^3-2k^2-k+1\\&=h_1(n)\geq 0,\end{aligned}$$

where $h_1(x)=x^2-(k^3+k^2+k+3)x+k^4+k^3-2k^2-k+1$. For the last inequality, obviously, the function $h_1(x)$ is strictly increasing when $x\geq k^3+k^2+k+3$. Since $n\geq k^3+k^2+k+3$, we obtain $f_2(n-k-2-\frac{1}{n})>h_1(n)\geq h_1(k^3+k^2+k+3)=k^4+k^3-2k^2-k+1\geq 0$.

$$\begin{aligned}f_2'(n-k-2-\frac{1}{n})&=n^3-3(k+1)n^2+(2k^2+5k)n-\frac{4k^2+10k-1}{n}-\frac{9(k+1)}{n^2}-\frac{4}{n^3}\\&\quad-k^3-3k^2+6k+6\\&=h_2(n)\geq h_2(k^3+k^2+k+3)\\&=k^9+3k^8+3k^7+7k^6+11k^5+4k^4+7k^3+5k^2+3k+6\\&\quad-\frac{4k^2+10k-1}{k^3+k^2+k+3}-\frac{9k+9}{(k^3+k^2+k+3)^2}-\frac{4}{(k^3+k^2+k+3)^3}\\&\geq k^9+3k^8+3k^7+7k^6+11k^5+4k^4+7k^3+5k^2+3k+6\\&\quad-\frac{13}{6}-\frac{1}{2}-\frac{1}{54}\\&>0,\end{aligned}$$

where $h_2(x)=x^3-3(k+1)x^2+(2k^2+5k)x-\frac{4k^2+10k-1}{x}-\frac{9(k+1)}{x^2}-\frac{4}{x^3}-k^3-3k^2+6k+6$. For the first inequality, the first derivative of $h_2(x)$ is $h_2'(x)=3x^2-6(k+1)x+(2k+5)k+\frac{4k^2+10k-1}{x^2}+\frac{18(k+1)}{x^3}+\frac{12}{x^4}$. Since

$$\begin{aligned}h_2'(x)&\geq h_2'(k^3+k^2+k+3)\\&=3k^6+6k^5+3k^4+12k^3+11k^2-k+9\\&\quad+\frac{4k^2+10k-1}{(k^3+k^2+k+3)^2}+\frac{18k+18}{(k^3+k^2+k+3)^3}+\frac{12}{(k^3+k^2+k+3)^4}\\&>0,\end{aligned}$$

we have $h_2(x)$ is strictly increasing when $x\geq k^3+k^2+k+3$. Since $n\geq k^3+k^2+k+3$, we obtain $h_2(n)\geq h_2(k^3+k^2+k+3)$.

$$\begin{aligned}f_2^{(2)}(n-k-2-\frac{1}{n})&=6n^2-(12k+12)n+\frac{18k+18}{n}+\frac{12}{n^2}+4k^2+10k-10\\&>6n^2-(12k+12)n+4k^2+10k-10\\&=h_3(n)>0.\end{aligned}$$

For the last inequality, let $h_3(x) = 6x^2 - (12k+12)x + 4k^2 + 10k - 10$. It is easy to see that $h_3(x)$ is strictly increasing when $x \geq k^3 + k^2 + k + 3$. Since $n \geq k^3 + k^2 + k + 3$, we obtain $h_3(n) \geq h_3(k^3 + k^2 + k + 3) = 6k^6 + 12k^5 + 6k^4 + 24k^3 + 22k^2 - 2k + 8 > 0$.

$$f_2^{(3)}(n-k-2-\frac{1}{n}) = 18n - \frac{24}{n} - 18k - 18 > 0.$$
$$f_2^{(4)}(n-k-2-\frac{1}{n}) = 24 > 0.$$

Hence, by the Fourier-Budan Theorem (see, e.g., [17]), there is no root of $f_2(x)$ in the interval $[n-k-2-\frac{1}{n}, +\infty)$. Then by Lemma 2.4, all subgraphs of $\underline{G}^1_{n,k}$ have spectral radius less than $n-k-2-\frac{1}{n}$.

(ii) Since the proofs are similar, we omit the details. □

Proof of Theorem 1.6. Note that $\delta(G) \geq \kappa(G) \geq k$. By Lemmas 2.5 and 2.6, we have

$$n-k-2-\frac{1}{n} < \rho(G) \leq \frac{k-1}{2} + \sqrt{2e(G) - nk + \frac{(k+1)^2}{4}}.$$

Therefore, when $n \geq k^3 + k^2 + k + 3$, we obtain

$$\begin{aligned} e(G) &> \frac{1}{2}[n^2 - (2k+3)n + \frac{3k+3}{n} + \frac{1}{n^2} + 2k^2 + 4k] \\ &> \frac{n^2 - (2k+3)n + 2k^2 + 4k}{2} \\ &> \binom{n-k-3}{2} + (k+2)(k+3). \end{aligned}$$

By Theorem 3.4, G is traceable or $cl_{n-1}(G) \in \{\underline{G}^1_{n,k}, \underline{G}^2_{n,k}, \underline{G}^3_{n,k}, \underline{G}^4_{n,k}, \underline{G}^5_{n,k}\}$. Since $K_{n-k-1} \subseteq \underline{G}^1_{n,k}$, by Lemma 2.4, we have $\rho(\underline{G}^1_{n,k}) > \rho(K_{n-k-1}) = n-k-2$. Combining this with Lemmas 2.4 and 3.5, we get $G = \underline{G}^1_{n,k}$. □

Declaration of competing interest

The authors declare that they have no conflict of interest.

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