

Rank-based change-point analysis for long-range dependent time series*

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Abstract: We consider change-point tests based on rank statistics to test for structural changes in long-range dependent observations. Under the hypothesis of stationary time series and under the assumption of a change with decreasing change-point height, the asymptotic distributions of corresponding test statistics are derived. For this, a uniform reduction principle for the sequential empirical process in a two-parameter Skorohod space equipped with a weighted supremum norm is proved. Moreover, we compare the efficiency of rank tests resulting from the consideration of different score functions. Under Gaussianity, the asymptotic relative efficiency of rank-based tests with respect to the CuSum test is 1, irrespective of the score function. Regarding the practical implementation of rank-based change-point tests, we suggest to combine self-normalized rank statistics with subsampling. The theoretical results are accompanied by simulation studies that, in particular, allow for a comparison of rank tests resulting from different score functions. With respect to the finite sample performance of rank-based change-point tests, the Van der Waerden rank test proves to be favorable in a broad range of situations. Finally, we analyze data sets from economy, hydrology, and network traffic monitoring in view of structural changes and compare our results to previous analysis of the data.

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1. Introduction

Let X_1, \dots, X_n be random variables and let F_i , $i = 1, \dots, n$, denote the marginal distribution functions of X_i , $i = 1, \dots, n$. If $F_k \neq F_{k+1}$ for some $k \in \{1, \dots, n-1\}$, we say that there is a *change-point* in k and we refer to k as the *time of change*. The testing problem

$$H : F_1 = F_2 = \dots = F_n$$

against

$$A : F_1 = F_2 = \dots = F_k \neq F_{k+1} = F_{k+2} = \dots = F_n \\ \text{for some } k \in \{1, \dots, n-1\}$$

is called *change-point problem*.

The most frequently considered change-point problems relate to the identification of shifts in the mean value of time series. Writing

$$X_n = \mu_n + Y_n$$

for a sequence of unknown constants μ_n , $n \in \mathbb{N}$, and a mean-zero stochastic process Y_n , $n \in \mathbb{N}$, a *change-point in the location* of the time series X_n , $n \in \mathbb{N}$, is characterized by the sequence μ_n , $n \in \mathbb{N}$, satisfying

$$\mu_i = \begin{cases} \mu & \text{for } i = 1, \dots, k, \\ \mu + h_n & \text{for } i = k+1, \dots, n \end{cases}$$

for some $k = \lfloor n\tau \rfloor$, $0 < \tau < 1$, denoting the time of change, and a deterministic sequence of *shift heights* h_n , $n \in \mathbb{N}$, with $h_n \neq 0$ for all $n \in \mathbb{N}$. If the sequence of shift-heights converges to 0, i.e., $\lim_{n \rightarrow \infty} h_n = 0$, we refer to *local changes* and *local alternatives*, respectively.

Motivated by change-point tests for the change-in-location problem based on the consideration of the partial sums

$$\sum_{i=1}^k (X_i - \bar{X}_n), \quad \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i,$$

i.e., *CuSum-tests*, we consider a class of change-point tests based on rank statistics

$$S_{k,n}(a) := \sum_{i=1}^k (a(R_i) - \bar{a}_n), \quad \bar{a}_n = \frac{1}{n} \sum_{i=1}^n a(i),$$

where $a = (a(1), \dots, a(n))$ is a vector of scores, and $R_i = \sum_{j=1}^n 1_{\{X_j \leq X_i\}}$ denotes the rank of X_i among X_1, \dots, X_n .

Rank statistics for change-point detection have been studied for over 50 years, starting with [Bhattacharyya and Johnson \(1968\)](#), [Sen \(1978\)](#) and [Lombard \(1987\)](#). Given independent data-generating random variables with a change in location, the statistical properties of rank-based statistics have been investigated by [Praagman \(1988\)](#), [Gombay \(1994\)](#) and [Gombay and Hušková \(1998\)](#).

Under the assumption that the time of change is unknown under the alternative, it seems natural to consider the statistics $|S_{k,n}(a)|$ for every possible time of change k and to decide in favor of the alternative hypothesis A if the maximum exceeds a predefined critical value. As a result, change-point tests base test decisions on the values of the statistics

$$S_n(a) := \max_{1 \leq k < n} |S_{k,n}(a)|. \quad (1)$$

Choosing $a(i) = i$, a short calculation yields

$$\sum_{i=1}^k (a(R_i) - \bar{a}_n) = \sum_{i=1}^k \sum_{j=k+1}^n \left(1_{\{X_j \leq X_i\}} - \frac{1}{2} \right),$$

i.e., this score results in the Wilcoxon-two-sample statistic. [Darkhovskh \(1976\)](#), [Pettitt \(1979\)](#) and [Wolfe and Schechtman \(1984\)](#) study Wilcoxon-type change-point statistics under the assumption of independent data-generating variables. For long-range dependent time series, [Wang \(2008\)](#), [Dehling, Rooch and Taqqu \(2013\)](#) and [Dehling, Rooch and Taqqu \(2017\)](#) characterize the asymptotic behavior of change-point tests that are based on the two-sample Wilcoxon statistic. A self-normalized version of the Wilcoxon-type change-point test is proposed by [Betken \(2016\)](#).

To the best of our knowledge, for dependent data there do not yet exist results for rank-based change-point tests stemming from general score functions. The aim of this paper is to study the (asymptotic and finite sample) behavior of general rank statistics under long-range dependence. This allows for an application of other score functions, including the Median test (choosing $a(i) = \text{sign}(i - \frac{n+1}{2})$) and the Van der Waerden test (choosing $a(i) = \phi^{-1}(\frac{i}{n+1})$). We will use weighted empirical processes to determine the limit distribution of rank statistics following an approach in [Pyke and Shorack \(1968\)](#). For independent data, this techniques are considered in the context of change-point detection by [Szyszkowicz \(1994\)](#).

Section 2 introduces the mathematical framework of weighted Skorohod spaces and subordinated Gaussian processes. The main results on the asymptotic be-

havior of rank statistics under the hypothesis and under local alternatives follow in Section 3. We discuss self-normalization and subsampling as means of a practical implementation of change-point tests in Section 4. Section 5 contains simulation studies that give insight into the finite sample behavior of rank-based change-point tests. Real life data sets are discussed in Section 6. The proofs of our theoretical results and additional simulation results can be found in the appendix.

2. Preliminaries

Given dependent data, the exact distribution of the statistic $S_n(a)$ is unknown and, in general, hard to obtain. For this reason, test decisions are based on a comparison of the value of the test statistic with quantiles of its limit distribution. For the determination of the asymptotic distribution of the statistic $S_n(a)$, it is useful to note that for any function $h : (0, 1) \rightarrow \mathbb{R}$ satisfying $h\left(\frac{i}{n+1}\right) = a(i)$ we have

$$\begin{aligned} S_{k,n}(a) &= \sum_{i=1}^k a(R_i) - \frac{k}{n} \sum_{i=1}^n a(i) \\ &= \sum_{i=1}^k \left(h\left(\frac{1}{n+1}R_i\right) - \frac{1}{n} \sum_{i=1}^n h\left(\frac{1}{n+1}R_i\right) \right) \\ &= \int_0^1 h(x) d\left(\hat{G}_k(x) - \frac{k}{n}\hat{G}_n(x)\right), \end{aligned}$$

where $\hat{G}_k(x) := \sum_{i=1}^k 1_{\{\frac{1}{n+1}R_i \leq x\}}$ is the empirical distribution function of the (rescaled) ranks. Under an additional assumption, introduced in Section 2.1, we can use integration by parts (see Lemma B.1 in [Beutner et al. \(2012\)](#)) to further simplify the above representation, so that

$$S_{k,n}(a) = \int_0^1 h(x) d\left(\hat{G}_k(x) - \frac{k}{n}\hat{G}_n(x)\right) = - \int_0^1 \left(\hat{G}_k(x-) - \frac{k}{n}\hat{G}_n(x-)\right) dh(x).$$

2.1. Weighted Skorohod space

In order to derive the asymptotic distribution of the test statistic $S_n(a)$ defined by (1), we consider the process

$$\left(\hat{G}_k(x-) - \frac{k}{n}\hat{G}_n(x-)\right), \quad x \in [0, 1],$$

as an element of the space $D[0, 1]$, i.e., the set of all functions on $[0, 1]$ which are right-continuous and have left limits, and the statistic $S_{k,n}(a)$ as the image of this process under the mapping $g : D[0, 1] \rightarrow \mathbb{R}$, $f \mapsto \int_0^1 f(x) dh(x)$. It is

important to note that this function is not necessarily continuous with respect to the supremum norm on $D[0, 1]$. In particular, the function g is unbounded for $h = \Phi^{-1}$, i.e., when considering the Van der Waerden test statistic, and, as a linear functional, consequently nowhere continuous. As a result, we must not apply the continuous mapping theorem without further discussion. For this reason, we introduce the weighted supremum norm $\|\cdot\|_\lambda$ on $D[0, 1]$, defined by

$$\|f\|_\lambda := \sup_{x \in [0, 1]} |(\min\{x, 1-x\})^{-\lambda} f(x)|,$$

and we consider the space

$$D_\lambda[0, 1] := \{f \in D[0, 1] : \|f\|_\lambda < \infty\}.$$

Note that

$$\begin{aligned} \left| \int_0^1 f(x) dh(x) - \int_0^1 g(x) dh(x) \right| &\leq \int_0^1 |f(x) - g(x)| d\bar{h}(x) \\ &\leq \|f - g\|_\lambda \int_0^1 (\min\{x, 1-x\})^\lambda d\bar{h}(x), \end{aligned}$$

where we define the function $\bar{h} : [0, 1] \rightarrow \mathbb{R}$ by

$$\bar{h}(x) := \begin{cases} V_{1/2}^x(h) & \text{for } x \geq 1/2 \\ -V_x^{1/2}(h) & \text{for } x < 1/2 \end{cases} \quad (2)$$

with $V_a^b(f)$ denoting the total variation of a function f over the interval $[a, b]$. For this reason, we impose the following assumption:

Assumption 1. We assume that for $\bar{h} : [0, 1] \rightarrow \mathbb{R}$ defined by (2) and some $\lambda \in (0, \frac{1}{3})$

$$\int_0^1 (\min\{x, 1-x\})^\lambda d\bar{h}(x) < \infty.$$

Given Assumption 1, the mapping $f \mapsto \int_0^1 f(x) dh(x)$ is continuous. Moreover, the process $\hat{G}_k(x-) - \frac{k}{n} \hat{G}_n(x-)$, $x \in [0, 1]$, takes values in $D_\lambda[0, 1]$ almost surely. Due to continuity of g with respect to $\|\cdot\|_\lambda$, convergence in distribution will follow from the continuous mapping theorem and (after rescaling) convergence of $\hat{G}_k(x-) - \frac{k}{n} \hat{G}_n(x-)$, $x \in [0, 1]$, in $D_\lambda[0, 1]$.

The following example shows that the Van der Waerden score function satisfies Assumption 1:

Example. Assumption 1 holds for the score function Φ^{-1} and for any $\lambda > 0$,

since Φ^{-1} is of bounded variation on compact intervals and since

$$\begin{aligned}
& \int_0^1 (\min\{x, 1-x\})^\lambda d\bar{h}(x) \\
&= \int_0^{\frac{1}{2}} x^\lambda d\left(\Phi^{-1}(x) - \Phi^{-1}\left(\frac{1}{2}\right)\right) + \int_{\frac{1}{2}}^1 (1-x)^\lambda d\left(\Phi^{-1}(x) - \Phi^{-1}\left(\frac{1}{2}\right)\right) \\
&= \int_0^{\frac{1}{2}} x^\lambda d\Phi^{-1}(x) + \int_{\frac{1}{2}}^1 (1-x)^\lambda d\Phi^{-1}(x) \\
&= \int_{-\infty}^0 \Phi(x)^\lambda dx + \int_0^\infty (1-\Phi(x))^\lambda dx = \int_{-\infty}^0 \Phi(x)^\lambda dx + \int_0^\infty \Phi(-x)^\lambda dx \\
&= 2 \int_{-\infty}^0 (\Phi(x))^\lambda dx < \infty.
\end{aligned}$$

2.2. Long-range dependence

In time series analysis, the rate of decay of the autocovariance function is crucial to the characterization of a statistic's limit distribution. A relatively slow decay of the autocovariances characterizes long-range dependent time series, while a relatively fast decay characterizes short-range dependent processes; see [Pipiras and Taqqu \(2017\)](#), p. 17. We will focus on the consideration of long-range dependent subordinated Gaussian time series, i.e., on random observations generated by transformations of Gaussian processes:

Model. Let $Y_n = G(\xi_n)$, where $G : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function and let $\xi_n, n \in \mathbb{N}$, be a stationary, long-range dependent Gaussian time series with long-range dependence (LRD) parameter D , i.e, $E \xi_1 = 0$, $\text{Var} \xi_1 = 1$, and

$$\gamma(k) := \text{Cov}(\xi_1, \xi_{k+1}) \sim k^{-D} L(k), \quad \text{as } k \rightarrow \infty,$$

for some $D \in (0, 1)$ and a slowly-varying function L .

Remark 2.1. For any particular distribution function F , an appropriate choice of the transformation G yields subordinated Gaussian processes with marginal distribution F . Moreover, there exist algorithms for generating Gaussian processes that, after suitable transformation, yield subordinated Gaussian processes with marginal distribution F and a predefined covariance structure; see [Pipiras and Taqqu \(2017\)](#). As a result, subordinated Gaussian processes provide a flexible model for long-range dependent time series.

A very useful tool for studying subordinated Gaussian processes are Hermite polynomials. For $n \geq 0$, the *Hermite polynomial* of order n is defined by

$$H_n(x) := (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2}, \quad x \in \mathbb{R}.$$

For any function G with $E[G^2(\xi_1)] < \infty$, the r -th Hermite-coefficient is defined by

$$J_r(G) := E[G(\xi_1)H_r(\xi_1)]. \quad (3)$$

Every such G has an expansion in Hermite polynomials, i.e., we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left(G(\xi_1) - \sum_{r=0}^n \frac{J_r(G)}{r!} H_r(\xi_1) \right)^2 \right] = 0.$$

Given the Hermite expansion, it is possible to characterize the dependence structure of subordinated Gaussian time series $G(\xi_n)$, $n \in \mathbb{N}$: The behavior of the autocorrelations of the transformed process is completely determined by the dependence structure of the underlying process. In fact, it holds that

$$\text{Cov}(G(\xi_1), G(\xi_{k+1})) = \sum_{r=1}^{\infty} \frac{J_r^2(G)}{r!} (\gamma(k))^r.$$

Under the assumption that, as k tends to ∞ , $\gamma(k)$ converges to 0 with a certain rate, the asymptotically dominating term in this series is the summand corresponding to the smallest integer r for which the Hermite coefficient $J_r(G)$ is non-zero. This index, which decisively depends on G , is called *Hermite rank*.

As, in the following, we will study empirical processes, we do not only consider a single transformation G , but the class of transformations $1_{\{G(\xi_1) \leq x\}} - F(x)$, $x \in \mathbb{R}$. For this, we need to define the Hermite rank of this class.

Definition 2.1. For $G : \mathbb{R} \rightarrow \mathbb{R}$, let $J_r(G; x)$ denote the r -th Hermite coefficient in the Hermite expansion of $1_{\{G(\xi_i) \leq x\}} - F(x)$, i.e.,

$$J_r(G; x) := \mathbb{E} \left(1_{\{G(\xi_i) \leq x\}} H_r(\xi_i) \right),$$

and let r denote the Hermite rank of the class of functions $1_{\{G(\xi_1) \leq x\}} - F(x)$, $x \in \mathbb{R}$, defined by

$$r := \min_{x \in \mathbb{R}} r(x), \quad r(x) := \min \{q \geq 1 : J_q(G; x) \neq 0\}.$$

An appropriate scaling for partial sums of a subordinated Gaussian sequence $Y_n = G(\xi_n)$, $n \in \mathbb{N}$, depends on the Hermite rank r of G and the long-range dependence parameter D of ξ_n , $n \in \mathbb{N}$. More precisely, a corresponding scaling sequence $d_{n,r}$, $n \in \mathbb{N}$, is defined by

$$d_{n,r}^2 := \text{Var} \left(\sum_{i=1}^n H_r(\xi_i) \right). \quad (4)$$

Given the previous definitions and notations, we are now in a position to formulate a general assumption on the data-generating process needed for our theoretical results in the following section:

Assumption 2. Let $Y_n = G(\xi_n)$, where ξ_n , $n \in \mathbb{N}$, is a stationary Gaussian time series with mean 0, variance 1, and autocovariance function γ satisfying

$$\gamma(k) := \text{Cov}(\xi_1, \xi_{k+1}) \sim k^{-D} L(k),$$

as $k \rightarrow \infty$. We assume that $Dr < 1$, where r denotes the Hermite rank of the class of functions $1_{\{G(\xi_1) \leq x\}} - F(x)$, $x \in \mathbb{R}$. Moreover, we assume that the marginal distribution function F of Y_n , $n \in \mathbb{N}$, is continuous.

Remark 2.2. Without loss of generality, we may assume that $F(x) = x$, because by continuity of F , the generalized inverse F^- is strictly increasing, $F(X_i)$ is uniformly distributed on $[0, 1]$ and rank statistics are therefore not affected by a corresponding transformation.

3. Main Results

Recall that

$$S_{k,n}(a) = - \int_0^1 \left(\hat{G}_k(x-) - \frac{k}{n} \hat{G}_n(x-) \right) dh(x),$$

where $\hat{G}_k(x) := \sum_{i=1}^k 1_{\{\frac{i}{n+1} R_i \leq x\}}$ with $R_i = \sum_{j=1}^n 1_{\{X_j \leq X_i\}}$ denoting the rank of X_i among observations X_1, \dots, X_n . Given the parametrization

$$S_n(a) = \max_{1 \leq k < n} |S_{k,n}(a)| = \sup_{t \in [0,1]} \left| \int_0^1 \left(\hat{G}_{\lfloor nt \rfloor}(x-) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x-) \right) dh(x) \right|, \quad (5)$$

the asymptotic distribution of $S_n(a)$ can be derived from an application of the continuous mapping theorem and a limit theorem for the two-parameter process

$$\hat{G}_{\lfloor nt \rfloor}(x-) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x-), \quad t \in [0, 1], \quad x \in [0, 1].$$

For proofs of corresponding limit theorems, we initially derive reduction principles for the sequential empirical process $F_{\lfloor nt \rfloor}(x) - x$, $t \in [0, 1], x \in [0, 1]$, where F_n refers to the empirical distribution function of X_1, \dots, X_n , i.e.,

$$F_n(x) := \frac{1}{n} \sum_{i=1}^n 1_{\{X_i \leq x\}}.$$

3.1. Asymptotic behavior under stationarity

The following proposition can be considered as a reduction principle for the empirical process $F_{\lfloor nt \rfloor}(x) - x$, $t \in [0, 1], x \in [0, 1]$, with respect to the weighted supremum norm and under the assumption of a stationary data-generating process. It makes way for establishing a reduction principle for the two-parameter empirical process of the ranks under the hypothesis of no change; see Theorem 3.1.

Proposition 3.1. *Let $X_n = G(\xi_n)$, $n \in \mathbb{N}$, be a subordinated Gaussian sequence satisfying Assumption 2 with marginal distribution $F(x) = x$, $x \in [0, 1]$. Moreover, let $d_{n,r}$, $n \in \mathbb{N}$, be the deterministic sequence defined by (4) with r denoting the Hermite rank of the class of functions $1_{\{G(\xi_1) \leq x\}} - x$, $x \in [0, 1]$. Then, there exists a $\vartheta > 0$ such that, as $n \rightarrow \infty$,*

$$\begin{aligned} \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} \left| \lfloor nt \rfloor (F_{\lfloor nt \rfloor}(x) - x) - \frac{1}{r!} J_r(F^-(x)) \sum_{j=1}^{\lfloor nt \rfloor} H_r(\xi_j) \right| \\ = \mathcal{O}_P(n^{-\vartheta}). \quad (6) \end{aligned}$$

Remark 3.1. Proposition 3.1 is closely related to Theorem 2 in Buchsteiner (2015) that establishes a reduction principle for the sequential empirical process with respect to another class of weighted norms.

On the basis of Proposition 3.1, we derive a reduction principle for the two-parameter empirical process of the ranks, i.e., for

$$\hat{G}_{[nt]}(x-) - \frac{[nt]}{n} \hat{G}_n(x-), \quad t \in [0, 1], \quad x \in [0, 1],$$

with $\hat{G}_k(x) := \sum_{i=1}^k \mathbf{1}_{\{\frac{i}{k+1} R_i \leq x\}}$.

Theorem 3.1. *Let $X_n = G(\xi_n)$, $n \in \mathbb{N}$, be a subordinated Gaussian sequence satisfying Assumption 2 with marginal distribution $F(x) = x$, $x \in [0, 1]$. Moreover, let $d_{n,r}$, $n \in \mathbb{N}$, be the deterministic sequence defined by (4) with r denoting the Hermite rank of the class of functions $\mathbf{1}_{\{G(\xi_1) \leq x\}} - x$, $x \in [0, 1]$, and consider $\vartheta > 0$ such that (6) holds. For any $\lambda < 1/3$ such that $n^\lambda = o(d_{n,r}^{1-\lambda})$, $n^{2\lambda} d_{n,r} = o(n)$ and $d_{n,r}^\lambda = o(n^\vartheta)$, we have*

$$\begin{aligned} \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} & \left| \left(\hat{G}_{[nt]}(x-) - \frac{[nt]}{n} \hat{G}_n(x-) \right) \right. \\ & \left. - \frac{1}{r!} J_r(x) \left(\sum_{i=1}^{[nt]} H_r(\xi_i) - \frac{[nt]}{n} \sum_{i=1}^n H_r(\xi_i) \right) \right| = o_P(1). \end{aligned}$$

According to Theorem 3.1 and (5), it suffices to know the limit of the sequential partial sum process $\sum_{i=1}^{[n\cdot]} H_r(\xi_i) \in D[0, 1]$, in order to derive the asymptotic distribution of the statistics $S_n(a)$ under the hypothesis of stationarity. In fact, it follows by Theorem 5.6 in Taqqu (1979) that

$$\frac{1}{d_{n,r}} \sum_{i=1}^{[nt]} H_r(\xi_i) \xrightarrow{\mathcal{D}} Z_{r,H}(t), \quad t \in [0, 1],$$

where $Z_{r,H}$ is an r -th order Hermite process, $H = 1 - \frac{rD}{2}$, and $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution with respect to the σ -field generated by the open balls in $D[0, 1]$, equipped with the supremum norm. As a result, using the representation (5) and applying the continuous mapping theorem yields the asymptotic distribution of the test statistic $S_n(a)$:

Corollary 3.1. *Let the assumptions of Theorem 3.1 hold and let $h : (0, 1) \rightarrow \mathbb{R}$ satisfy Assumption 1. Then, we have*

$$d_{n,r}^{-1} S_n(a) \xrightarrow{\mathcal{D}} \sup_{t \in [0,1]} |Z_{r,H}(t) - tZ_{r,H}(1)| \int_0^1 J_r(F^-(x)) dh(x).$$

In practical applications, the sequence $d_{n,r}$, the parameters r , H , and the value of the integral on the right-hand side are typically unknown. For this reason, it is difficult to use Corollary 3.1 directly to obtain critical values. In Section 4, we will discuss nonparametric methods to derive critical values.

3.2. Asymptotic behavior under local alternatives

In the following, we assume that the considered observations are generated by a triangular array $X_{n,i}$, $1 \leq i \leq n$, $n \in \mathbb{N}$, with

$$X_{n,i} = \begin{cases} Y_i & \text{if } i \leq \lfloor n\tau \rfloor, \\ Y_i + h_n & \text{if } i > \lfloor n\tau \rfloor, \end{cases} \quad (7)$$

where $0 < \tau < 1$, h_n , $n \in \mathbb{N}$, is a non-negative deterministic sequence and $Y_n = G(\xi_n)$, $n \in \mathbb{N}$, is a subordinated Gaussian sequence according to Model 2 with continuous marginal distribution F and density f . For convergence of the test statistic $S_n(a)$ to a non-degenerate limit, we have to assume that $h_n \rightarrow 0$ (as $n \rightarrow \infty$) with a certain rate that will be specified later.

In analogy to the asymptotic results in Section 3.1 under the assumption of stationary time series, we first establish a reduction principle for the sequential empirical process with respect to the weighted supremum norm under the assumption of local alternatives:

Proposition 3.2. *Let $X_{n,i}$, $1 \leq i \leq n$, $n \in \mathbb{N}$, be a triangular according to (7) with $h_n = cn^{-1}d_{n,r}$ for some constant $c > 0$ and with $d_{n,r}$ defined by (4), where r is the Hermite rank of the class of functions $1_{\{G(\xi_1) \leq x\}} - F(x)$, $x \in [0, 1]$. Assume that F is strictly monotone,*

$$\sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} |h^{-1}(x - F(F^-(x) - h)) - f(F^-(x))| = \mathcal{O}(h^\rho), \quad (8)$$

as $h \rightarrow 0$, for some ρ , $0 < \rho < \min(1, (1 - 2\lambda - \vartheta)^{-1}\vartheta)$, with ϑ and λ as in Proposition 3.1, and

$$\sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} |f(F^-(x))| < \infty. \quad (9)$$

Then, if $n^{\lambda+\rho-1} = \mathcal{O}(d_{n,r}^{\rho-1})$, as $n \rightarrow \infty$, and $2\lambda + \vartheta < \frac{1}{2}$, we have

$$\begin{aligned} & \sup_{t \in [0,1], x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} \left| d_{n,r}^{-1} \lfloor nt \rfloor (F_{\lfloor nt \rfloor}(x) - x) - \frac{J_r(F^-(x))}{r! d_{n,r}} \sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) \right. \\ & \left. + 1_{\{t > \tau\}} \frac{\lfloor nt \rfloor - \lfloor n\tau \rfloor}{d_{n,r}} (x - F(F^-(x) - h_n)) \right| = \mathcal{O}_P(h_n^{\min\{\rho, \lambda\}}), \quad (10) \end{aligned}$$

where $J_r(F^-(x)) = \mathbb{E}(1_{\{G(\xi_1) \leq F^-(x)\}} H_r(\xi_1))$.

Note that, in comparison to Proposition 3.1, an additional deterministic term is needed to characterize the asymptotic behavior of the empirical process under the alternative.

On the basis of Proposition 3.1, we derive a reduction principle for the two-parameter empirical process of the ranks:

Theorem 3.2. Let $X_{n,i}$, $1 \leq i \leq n$, $n \in \mathbb{N}$, be a triangular according to (7) with $h_n = cn^{-1}d_{n,r}$ for some constant $c > 0$ and with $d_{n,r}$ defined by (4), where r is the Hermite rank of the class of functions $1_{\{G(\xi_1) \leq x\}} - F(x)$, $x \in [0, 1]$. Assume that for F and f the conditions of Proposition 3.2 hold and that, additionally, there is a constant C , such that for h small enough, there exists an $\epsilon_1 > 0$, such that

$$\left| 1 - \frac{f(F^-(x) + h)}{f(F^-(x))} \right| \leq C (\min\{|x|, |1-x|\})^{-\epsilon_1} |h|, \quad \text{for } x \in [0, 1]. \quad (11)$$

Then, for any $\lambda < 1/3$ such that $n^\lambda = o(d_{n,r}^{1-\lambda})$, $n^\lambda d_{n,r}^{1+\epsilon_1} = o(n)$ and $d_{n,r}^{\rho+\lambda} = o(n^\rho)$, where ρ as in Proposition 3.2, we have

$$\begin{aligned} & \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-(\lambda-\epsilon_1)} \left| \left(\hat{G}_{\lfloor nt \rfloor}(x-) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x-) \right) \right. \\ & \quad \left. - \frac{J_r(F^-(x))}{r!} \left(\sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) - \frac{\lfloor nt \rfloor}{n} \sum_{i=1}^n H_r(\xi_i) \right) \right. \\ & \left. + \left(1_{\{t > \tau\}} \frac{\lfloor nt \rfloor - \lfloor n\tau \rfloor}{n} - \frac{\lfloor nt \rfloor}{n} \frac{n - \lfloor n\tau \rfloor}{n} \right) d_{n,r}^{-1} n (x - F(F^-(x) - h_n)) \right| = o_P(1), \end{aligned}$$

where $J_r(F^-(x)) = \mathbb{E}(1_{\{G(\xi_1) \leq F^-(x)\}} H_r(\xi_1))$.

Example. It may not be obvious that the conditions (9) and (11) hold for specific distribution functions. Therefore, we discuss the standard normal distribution as an example. Assume that $G = id$ such that $f = \varphi$ and $F = \Phi$, where φ denotes the standard normal density and Φ the standard normal distribution function. It is well-known that for $x \rightarrow -\infty$

$$\Phi(x) \approx \frac{1}{|x|} \varphi(x);$$

see Feller (1968). Consequently $\varphi(x) \approx |x| \Phi(x)$. As a result, we have

$$\varphi(\Phi^{-1}(x)) \approx |\Phi^{-1}(x)| \Phi(\Phi^{-1}(x)) = x |\Phi^{-1}(x)| \quad \text{for } x \rightarrow 0.$$

As $\Phi(x) \leq e^x$ for $x \leq 0$, it holds that $0 \geq \Phi^{-1}(x) \geq \log(x)$ for $x \leq \frac{1}{2}$ and therefore $|\Phi^{-1}(x)| \leq |\log(x)|$. With similar arguments for $x \rightarrow 1$, it follows that (9) holds for any $\lambda < \frac{1}{2}$.

In order to show that (11) holds, one needs a tighter upper bound. For any $K > 0$, there exists a constant C , such that $\Phi(x) \leq Ce^{Kx}$ for $x \leq 0$, and we conclude that $|\Phi^{-1}(x)| \leq \frac{1}{K} |\log(x/C)|$ for $x \leq \frac{1}{2}$. We focus on the case $h > 0$ because for $h < 0$, the quotient of densities in (11) is smaller than 1 and thus the difference is bounded. For $x \leq \frac{1}{2}$ and $h > 0$, we have

$$\begin{aligned} \left| 1 - \frac{\varphi(\Phi^{-1}(x) + h)}{\varphi(\Phi^{-1}(x))} \right| &= \left| \frac{\varphi(\Phi^{-1}(x)) - \varphi(\Phi^{-1}(x) + h)}{\varphi(\Phi^{-1}(x))} \right| \\ &\leq \frac{h |\varphi'(\Phi^{-1}(x) + h)|}{\varphi(\Phi^{-1}(x))} = h |\Phi^{-1}(x) + h| \frac{\varphi(\Phi^{-1}(x) + h)}{\varphi(\Phi^{-1}(x))} \\ &= h |\Phi^{-1}(x) + h| e^{-h\Phi^{-1}(x) - \frac{h^2}{2}} \leq h |\Phi^{-1}(x) + h| e^{-h\Phi^{-1}(x)} \end{aligned}$$

as $\varphi'(x) = -x\varphi(x)$. Because $|\Phi^{-1}(x)| \leq \frac{1}{K}|\log(x/C)|$, we arrive at

$$\left| 1 - \frac{\varphi(\Phi^{-1}(x) + h)}{\varphi(\Phi^{-1}(x))} \right| \leq \tilde{C}|\log(x/C)|e^{-h\log(x/C)/K} \leq \tilde{C} \left(\frac{x}{C}\right)^{1/K}$$

for any $h \in (0, 1]$ and some constant \tilde{C} . As K can be chosen arbitrarily large, we conclude that (11) holds for any $\epsilon_1 > 0$.

Based on Theorem 3.2, under local alternatives, the asymptotic distribution of the statistic $S_n(a)$ can be derived by the same arguments as under the assumption of stationarity, i.e., by the representation (5) and the continuous mapping theorem.

Corollary 3.2. *Let the assumptions of Theorem 3.2 hold and let $h : (0, 1) \rightarrow \mathbb{R}$ satisfy Assumption 1. Then, we have*

$$d_{n,r}^{-1}S_n(a) \xrightarrow{\mathcal{D}} \sup_{t \in [0,1]} \left| (Z_{r,H}(t) - tZ_{r,H}(1)) \int_0^1 J_r(F^-(x))dh(x) + c\delta_\tau(t) \int_0^1 f(F^-(x))dh(x) \right|,$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution in $D_\lambda[0, 1]$ and

$$\delta_\tau(t) = \begin{cases} t(1-\tau) & \text{if } t \leq \tau, \\ \tau(1-t) & \text{if } t > \tau. \end{cases}$$

3.3. Asymptotic Relative Efficiency for level shifts

The goal of this section is to calculate the asymptotic relative efficiency of rank tests that are based on two different score functions a_1 and a_2 . For this, we calculate the number of observations needed to detect a level shift of height h at time τ with a test of predefined asymptotic level α and asymptotic power β . With $n_1(h)$ and $n_2(h)$ corresponding to these numbers for a_1 and a_2 , we define the asymptotic relative efficiency of the tests by

$$\lim_{h \rightarrow 0} \frac{n_1(h)}{n_2(h)}$$

assuming that this limit exists.

An asymptotic relative efficiency that is smaller than 1 indicates that the change-point test that corresponds to the score function a_2 needs on large scale more observations than the change-point test that corresponds to the score function a_1 in order to detect a given jump on the same level with the same power. It is therefore called *less efficient*.

The above definition of asymptotic relative efficiency has as well been considered in Dehling, Rooch and Taqqu (2017) for a comparison of change-point tests.

Dehling, Rooch and Taqqu (2017) show that, when considering the asymptotic relative efficiency of CuSum and Wilcoxon test, the above limit exists and does not depend on the choice of τ , α , or β .

In order to determine the asymptotic relative efficiency of two rank-based testing procedures, we proceed in the same way. For this, we calculate a quantity that is related to the asymptotic relative efficiency, namely the ratio of the sizes of level shifts that can be detected by the two tests, based on the same number of observations n , for given values of τ , α , and β . We denote the corresponding level shifts by $\Delta_1(n)$ and $\Delta_2(n)$, respectively, assuming that these numbers depend on n in the following way:

$$\Delta_1(n) \sim c_1 \frac{d_{n,r}}{n} \quad \text{and} \quad \Delta_2(n) \sim c_2 \frac{d_{n,r}}{n}.$$

In order to simplify the succeeding argument, we consider a one-sided change-point test, thus rejecting the hypothesis of no change-point for large values of

$$\max_{1 \leq k < n} S_{k,n}(a_1) \quad \text{and} \quad \max_{1 \leq k < n} S_{k,n}(a_2).$$

The rank tests reject the null hypothesis when the statistics

$$\left(\frac{1}{r!} \int_0^1 J_r(F^-(x)) dh_1(x) \right)^{-1} \max_{1 \leq k < n} S_{k,n}(a_1),$$

$$\left(\frac{1}{r!} \int_0^1 J_r(F^-(x)) dh_2(x) \right)^{-1} \max_{1 \leq k < n} S_{k,n}(a_2)$$

exceed the upper α quantile q_α of the distribution of

$$\sup_{0 \leq t \leq 1} BB_{r,H}(t), \quad BB_{r,H}(t) := Z_{r,H}(t) - tZ_{r,H}(1).$$

Thus, if we want the two tests to have identical power, we have to choose c_1 and c_2 that

$$\left(\frac{1}{r!} \int_0^1 J_r(F^-(x)) dh_1(x) \right)^{-1} c_1 \psi_\tau(t) \int_0^1 f_G(F^-(x)) dh_1(x)$$

$$= \left(\frac{1}{r!} \int_0^1 J_r(F^-(x)) dh_2(x) \right)^{-1} c_2 \psi_\tau(t) \int_0^1 f_G(F^-(x)) dh_2(x)$$

yielding

$$\frac{\Delta_1(n)}{\Delta_2(n)} = \frac{c_1}{c_2} = \frac{\int_0^1 J_r(F^-(x)) dh_1(x) \int_0^1 f_G(F^-(x)) dh_2(x)}{\int_0^1 J_r(F^-(x)) dh_2(x) \int_0^1 f_G(F^-(x)) dh_1(x)}.$$

Example. Assume that $G = id$ such that $f = \varphi$ and $F = \Phi$, where φ denotes the standard normal density and Φ the standard normal distribution function.

In this case, we have

$$\begin{aligned} \int_0^1 J_1(F^-(x)) dh_i(x) &= \int_0^1 J_1(\Phi^{-1}(x)) dh_i(x) = \int_0^1 \mathbb{E}(1_{\{\xi_1 \leq \Phi^{-1}(x)\}} \xi_1) dh_i(x) \\ &= \int_0^1 \int_{-\infty}^{\Phi^{-1}(x)} y \varphi(y) dy dh_i(x) = - \int_0^1 \varphi(\Phi^{-1}(x)) dh_i(x) = - \int_0^1 f(F^{-1}(x)) dh_i(x) \end{aligned}$$

for $i = 1, 2$.

As a result, the asymptotic relative efficiency of rank-based change-point tests is always 1 when considering Gaussian time series. Since [Dehling, Rooch and Taqqu \(2017\)](#) have shown that the Wilcoxon-type change-point test has a relative efficiency of 1 with respect to the CuSum change-point test, we may conclude that the asymptotic efficiency of all rank-based change-point tests corresponds to the asymptotic efficiency of the CuSum test under the assumption of Gaussian data. However, for other marginal distributions, this might be different. In particular, the simulation studies considered in [Section 5](#) indicate that rank-based change-point tests have a higher empirical power for heavy-tailed marginal distributions.

4. Practical implementation

In this section, we describe how to meet challenges that go along with an implementation of the established rank-based change-point tests in practice. For this, note that an application of rank tests to a given data set presupposes determination of the scaling factor $d_{n,r}$, which satisfies $d_{n,r}^2 \sim c_{r,D} n^{2-rD} L^r(n)$, where $c_{r,D}$ is a constant depending on r and D ; see [Dehling, Rooch and Taqqu \(2013\)](#). In statistical practice, the parameters D , r and the function L are usually unknown. With regard to the practical implementation of rank-based change-point tests, we therefore propose to replace the deterministic scaling of rank-based statistics by a data-driven standardization, i.e., by a normalizing sequence that depends on the given realizations only and which is therefore referred to as *self-normalization*.

Although, by the consideration of self-normalized statistics, we dispose of unknown quantities in the computation of test statistics, we will show that the resulting, self-normalized test statistics converge in distribution to limits that depend on unknown parameters (the Hurst index H and the Hermite rank r), as well. To overcome this problem in practice, a subsampling procedure is considered as an alternative to basing test decisions on the limit distributions of test statistics.

Taken by itself, both methods, i.e., self-normalization and subsampling, make applications of change-point tests more feasible. Nonetheless, the particular charm of their practical implementation lies in the combination of the two methods.

4.1. Self-normalized rank tests

The concept of self-normalization has recently been applied to several testing procedures in change-point analysis. Originally established by [Lobato \(2001\)](#) in another testing context, it has been adapted to the change-point problem in [Shao and Zhang \(2010\)](#) by definition of a self-normalized Kolmogorov-Smirnov test statistic. In these papers, short-range dependent processes are considered. An extension to possibly long-range dependent processes was introduced by Shao, who established a self-normalized change-point test based on the CuSum statistic; see [Shao \(2011\)](#). Several inference problems, including a self-normalized cumulative sum test for the change-point problem and a self-normalization-based wild bootstrap adjusting for time-dependent variances are considered in [Zhao and Li \(2013\)](#). CuSum-based procedures for sequential monitoring of time series with respect to structural changes are proposed by [Dette and Gösmann \(2019\)](#) and [Chan, Ng and Yau \(2018\)](#), among others. [Dette, Kocot and Volgushev \(2020\)](#) extend the concept of self-normalization to develop a methodology for testing the null hypothesis of no relevant deviation in functional time series data against the alternative of relevant changes. [Zhang and Lavitas \(2018\)](#) propose a self-normalized change-point test that does not require a priori information on the number of change-points and can thus be considered as unsupervised. [Pešta and Wendler \(2018\)](#) combine self-normalized CuSum-type statistics and the wild bootstrap, thereby establishing completely data-driven change-point tests. A self-normalized version of the Wilcoxon change-point test is considered in [Betken \(2016\)](#) and [Betken and Kulik \(2019\)](#). For further, recent discussions and references on self-normalization in different contexts, we refer to [Shao \(2015\)](#).

The definition of self-normalized rank statistics is motivated with reference to an application of a self-normalization procedure to the CuSum statistic. For this, it is crucial to note that rank statistics arise from an application of the CuSum statistic to the scores $a(R_1), \dots, a(R_n)$: Given observations X_1, \dots, X_n , the CuSum test bases test decisions on the statistic

$$C_n := \max_{1 \leq k < n} \left| \sum_{i=1}^k X_i - \frac{k}{n} \sum_{j=1}^n X_j \right|,$$

while rank-based change-point tests decide in favor of a change-point in the data for large values of the statistic

$$S_n(a) := \max_{1 \leq k < n} \left| \sum_{i=1}^k a(R_i) - \frac{k}{n} \sum_{j=1}^n a(R_j) \right|.$$

Therefore, it seems natural to choose a data-driven normalization for rank statistics by evaluation of the self-normalized CuSum statistic, defined in [Shao \(2011\)](#), in $a(R_1), \dots, a(R_n)$. For this reason, we define the self-normalized rank statistic

for the change-point problem by

$$T_n(a) := \max_{1 \leq k < n} |T_{k,n}(a)|, \quad (12)$$

$$T_{k,n}(a) := \frac{S_{k;1,n}(a)}{\left\{ \frac{1}{n} \sum_{t=1}^k S_{t;1,k}^2(a) + \frac{1}{n} \sum_{t=k+1}^n S_{t;k+1,n}^2(a) \right\}^{1/2}}, \quad (13)$$

where

$$S_{t;j,k}(a) := \sum_{h=j}^t (a(R_h) - \bar{a}_{j,k}) \quad \text{with} \quad \bar{a}_{j,k} := \frac{1}{k-j+1} \sum_{t=j}^k a(R_t).$$

In order to derive the asymptotic distribution of $T_n(a)$, recall that

$$S_{k;1,n}(a) = \sum_{i=1}^k a(R_i) - \frac{k}{n} \sum_{i=1}^n a(i) = - \int_0^1 \left(\hat{G}_k(x-) - \frac{k}{n} \hat{G}_n(x-) \right) dh(x),$$

where $\hat{G}_k(x) := \sum_{i=1}^k 1_{\{\frac{i}{n+1} R_i \leq x\}}$, and note that

$$T_{[nt],n}(a) := G_{S_{[nt];1,n}(a)} + \mathcal{O}_P(1), \quad t \in [0, 1],$$

where for $f \in D[0, 1]$ the function $G_f \in D[0, 1]$ is defined by

$$G_f(t) := \frac{f(t)}{V_f(t)},$$

$$V_f(t) := \left\{ \int_0^t \left(f(s) - \frac{s}{t} f(t) \right)^2 ds + \int_t^1 \left(f(s) - f(t) - \frac{s-t}{1-t} (f(1) - f(t)) \right)^2 ds \right\}^{\frac{1}{2}}.$$

As a result, the limit of the self-normalized process $T_{[nt],n}(a)$, $t \in [0, 1]$, can be derived from the limit distribution of the process $S_{[nt];1,n}(a)$, $t \in [0, 1]$.

Under the hypothesis, i.e., under the assumption of a stationary data-generating process, Theorem 3.1 and the argument that proves Theorem 1 in Betken (2016) establish the convergence of the self-normalized rank statistics $T_n(a)$:

Corollary 4.1. *Let the assumptions of Theorem 3.1 hold and let $h : (0, 1) \rightarrow \mathbb{R}$ satisfy Assumption 1. Then, we have*

$$T_n(a) \xrightarrow{\mathcal{D}} \sup_{t \in [0, 1]} \frac{|Z_{r,H}(t) - tZ_{r,H}(1)|}{\left\{ \int_0^t V_{r,H}^2(s; 0, t) ds + \int_t^1 V_{r,H}^2(s; t, 1) ds \right\}^{\frac{1}{2}}}$$

with

$$V_{r,H}(s; s_1, s_2) = Z_{r,H}(s) - Z_{r,H}(s_1) - \frac{s-s_1}{s_2-s_1} \{Z_{r,H}(s_2) - Z_{r,H}(s_1)\}$$

for $s \in [s_1, s_2]$, $0 < s_1 < s_2 < 1$.

4.2. Subsampling

The basic idea of resampling procedures is to approximate the distribution function F_{T_n} of a considered statistic $T_n := T_n(X_1, \dots, X_n)$ by the empirical distribution of values of the statistic computed over subsets of the original sample. The so-called *sampling-window method*, studied by [Politis and Romano \(1994\)](#), [Hall and Jing \(1996\)](#), and [Sherman and Carlstein \(1996\)](#), utilizes evaluations of the test statistic in subsamples of successive observations, i.e., for some blocklength $l_n < n$, the realizations $T_{l_n, k} := T_{l_n}(X_k, \dots, X_{k+l_n-1})$, $k = 1, \dots, m_n$, where $m_n := n - l_n + 1$, are considered. As a result, multiple (though dependent) realizations of the test statistic T_{l_n} are obtained. Due to the fact that consecutive observations are chosen, the subsamples retain the dependence structure of the original sample, so that the empirical distribution function of $T_{l_n, 1}, \dots, T_{l_n, m_n}$, defined by

$$\widehat{F}_{m_n, l_n}(t) := \frac{1}{m_n} \sum_{k=1}^{m_n} 1_{\{T_{l_n, k} \leq t\}}, \quad (14)$$

can be considered as an appropriate estimator for F_{T_n} .

The validity of the subsampling procedure is typically established by proving that the distance between \widehat{F}_{m_n, l_n} and F_{T_n} vanishes as the number of observations tends to ∞ , i.e., by showing that

$$\left| \widehat{F}_{m_n, l_n}(t) - F_{T_n}(t) \right| \xrightarrow{\mathcal{P}} 0, \quad \text{as } n \rightarrow \infty,$$

for all points of continuity t of F_T .

It is shown in [Sherman and Carlstein \(1996\)](#) that the sampling-window method is consistent for any time series satisfying an α -mixing condition and for any measurable statistic converging in distribution to a non-degenerate limit variable. Thereby, consistency of the sampling-window method can be derived for an extensive class of short-range dependent processes under the mildest possible assumptions on the blocklength l_n and the considered statistic T_n . In the long-range dependent case, the validity of subsampling has been shown to hold for specific statistics under various model assumptions. [Hall, Jing and Lahiri \(1998\)](#) prove consistency of the sampling-window method for the sample mean as well as a studentized version of the sample mean under the assumption of subordinated Gaussian processes. [Nordman and Lahiri \(2005\)](#) attained consistency results with respect to the same statistics for long-range dependent linear processes with possibly non-Gaussian innovations. For this model, an alternative proof for consistency can be found in [Beran et al. \(2013\)](#). [Zhang et al. \(2013\)](#) generalize these results by proving consistency with respect to the sample mean under the assumption of subordinated long-range dependent linear processes with possibly non-Gaussian innovations. [Jach, McElroy and Politis \(2012\)](#) provide a result on the validity of subsampling for a general class of statistics T_n and certain heavy-tailed long-range dependent time series that follow a long memory stochastic volatility model. However, their results are restricted by assumptions

that are difficult to check in practice. Moreover, although not explicitly stated in [Jach, McElroy and Politis \(2012\)](#), the proof of consistency only holds for statistics T_n that are Lipschitz-continuous (uniformly in n); see [Jach, McElroy and Politis \(2016\)](#). Many robust statistics do not satisfy this assumption. In fact, rank-based change-point test statistics can be taken as examples for non-Lipschitz-continuous statistics.

General results on the validity of subsampling for long-range dependent time series are established in [Bai and Taqqu \(2017\)](#) and, independently in [Betken and Wendler \(2018\)](#). Neither of both consistency results makes any restrictive demands concerning the statistic T_n or the considered time series, such as finite moments of the data-generating variables, or continuity of the considered statistics. As a result, both results are applicable to heavy-tailed random variables and rank-based test statistics. For this reason, in the following sections, we can formally justify an application of the sampling-window method in simulations and data analysis by referring to the aforementioned results.

5. Simulations

In the following, the finite sample performance of self-normalized, rank-based testing procedures is analyzed in the context of testing for changes in the mean of a given set of observations X_1, \dots, X_n . More precisely, we will consider two different rank-based testing procedures, the self-normalized Wilcoxon change-point test and the self-normalized Van der Waerden change-point test, and compare their finite sample performance to that of the self-normalized CuSum change-point test. For this purpose, the rejection rates of all three testing procedures are computed for simulated subordinated Gaussian time series X_n , $n \in \mathbb{N}$, $X_n = G(\xi_n)$, where ξ_n , $n \in \mathbb{N}$, is a fractional Gaussian noise sequence generated by the function `fgnSim` from the `fArma` package in R.

We consider the following choices of marginal distributions that, for subordinated Gaussian time series, are determined by the function G :

1. Normal margins: We choose $G(t) = t$. In this case, the Hermite coefficient $J_1(G; x)$ is not equal to 0 for all $x \in \mathbb{R}$ (see [Dehling, Rooch and Taqqu \(2013\)](#)), so that $r = 1$, where r denotes the Hermite rank of $1_{\{G(\xi_i) \leq x\}} - F(x)$, $x \in \mathbb{R}$.
2. Pareto margins: In order to get standardized Pareto-distributed data which has a representation as a functional of a Gaussian process, we consider the transformation

$$G(t) = \left(\frac{\alpha k^2}{(\alpha - 1)^2(\alpha - 2)} \right)^{-\frac{1}{2}} \left(k(\Phi(t))^{-\frac{1}{\alpha}} - \frac{\alpha k}{\alpha - 1} \right)$$

with parameters $k, \alpha > 0$ and with Φ denoting the standard normal distribution function. Since G is a strictly decreasing function, it follows by Theorem 2 in [Dehling, Rooch and Taqqu \(2013\)](#) that $r = 1$, where r denotes the Hermite rank of $1_{\{G(\xi_i) \leq x\}} - F(x)$, $x \in \mathbb{R}$.

3. Cauchy margins: In order to get standardized Pareto-distributed data which has a representation as a functional of a Gaussian process, we consider the transformation

$$G(t) = \tan \left(\pi \left(\Phi(t) - \frac{1}{2} \right) \right)$$

with Φ denoting the standard normal distribution function. Since G is a strictly increasing function, it follows by Theorem 2 in [Dehling, Rooch and Taqqu \(2013\)](#) that $r = 1$, where r denotes the Hermite rank of $1_{\{G(\xi_i) \leq x\}} - F(x)$, $x \in \mathbb{R}$.

4. $\chi^2(1)$ margins: In order to get standardized $\chi^2(1)$ -distributed data which has a representation as a functional of a Gaussian process, we consider the transformation

$$G(t) = \frac{1}{2} (t^2 - 1).$$

In this case, the Hermite coefficient $J_1(G; x)$ equals 0 for all $x \in \mathbb{R}$, while $J_2(G; x)$ is not equal to 0. It follows that $r = 2$, where r denotes the Hermite rank of $1_{\{G(\xi_i) \leq x\}} - F(x)$, $x \in \mathbb{R}$.

All calculations are based on 5,000 realizations of time series and test decisions are based on an application of the sampling-window method for a significance level of 5%, meaning that the values of the test statistics are compared to the 95%-quantile of the empirical distribution function \hat{F}_{m_n, l_n} defined by (14). The empirical rejection frequencies for all three testing procedures and a sample size of $n = 500$ can be found in [Figure 1](#).

As expected, the empirical power increases for greater values of the level shift h . Furthermore, the empirical power is higher for breakpoints located in the middle of the sample ($\tau = 0.5$) than for change-point locations that lie close to the boundary of the testing region ($\tau = 0.25$). In accordance with the asymptotic considerations in [Example 3.3](#), the three tests behave very similar under normality (upper panels). For the heavy-tailed Pareto and Cauchy distribution (middle panels), the rank-based test clearly outperform the CuSum test. In particular, the empirical power of the CuSum test is not much greater than 5% when considering Cauchy distributed data. Moreover, the rank-based testing procedures yield a better power than the CuSum test when considering χ^2 -distributed time series (lower panel), i.e., subordinated Gaussian time series with an Hermite rank $r = 2$.

Detailed simulation results can be found in [Tables 1, 2, 3, and 4](#) in the appendix. These display results for sample sizes $n = 300$ and $n = 500$ and for different block lengths ($l_n = \lfloor n^\gamma \rfloor$ with $\gamma \in \{0.4, 0.5, 0.6\}$). Not surprisingly, an increasing sample size goes along with an improvement of the finite sample performance, i.e., the empirical size approaches the level of significance and the empirical power increases. All three testing procedures have an empirical size that is relatively close to the level of significance, an observation that seems to be typical of self-normalized testing procedures as it corresponds to the so-called

better size but less power phenomenon for self-normalized tests, which has also been observed in Shao (2011), Shao and Zhang (2010), and Betken (2016). The block length $l_n = \sqrt{n}$ seems to give the best overall performance, although it does not yield the best results for each and every scenario.

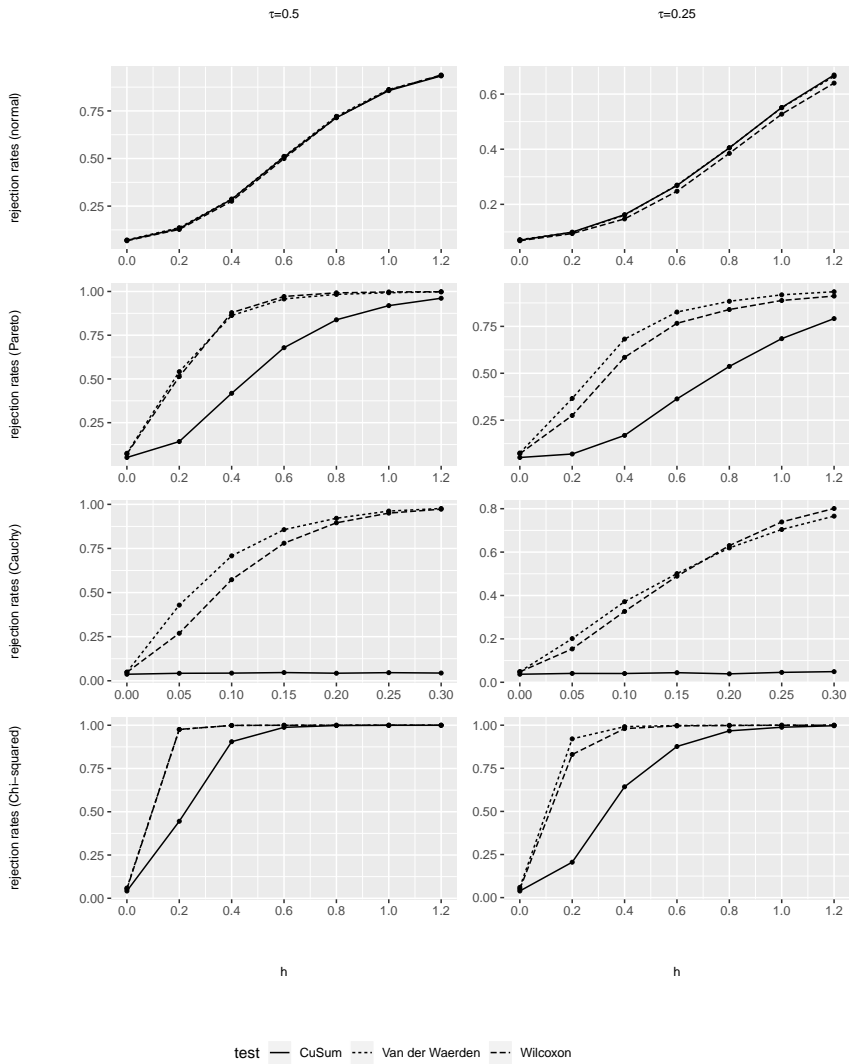


FIG 1. Rejection rates of the self-normalized CuSum, the self-normalized Wilcoxon and the self-normalized Van der Waerden change-point tests obtained by subsampling with block length $l_n = 22$ for transformed fractional Gaussian noise time series of length $n = 500$ with Hurst parameter $H = 0.7$, marginal standard normal, Pareto, and Cauchy distribution and a change in location of height h after a proportion τ of the simulated data.

Given normally distributed observations, there do not seem to be large deviations in the rejection rates of the three testing procedures. Nonetheless, the Van der Waerden test tends to be less conservative, but more efficient than the other testing procedures. At least for independent data, this observations corresponds to the fact that considering normal scores (yielding the Van der Waerden statistic) is known to result in a more efficient testing procedure; see [Hodges Jr. and Lehmann \(1961\)](#). For Pareto(3, 1)-distributed observations and Cauchy-distributed observations the two rank-based testing procedures clearly outperform the self-normalized CuSum test in that they yield considerably higher empirical rejection rates under the alternative. This observation specifically applies to the Cauchy-distributed time series as for these the self-normalized CuSum test has an extremely low power (which seems to be independent of the height and the location of the change-point). When considering subordinated Gaussian time series with Hermite rank $r = 2$, that is χ^2 -distributed observations, the rank-based testing procedures also perform better than the CuSum test. Comparing Wilcoxon and Van der Waerden test with respect to these time series, again the Van der Waerden test shows a slight tendency of being more efficient than the Wilcoxon test.

6. Data examples

In the following, three different data sets are analyzed with regard to structural changes by an application of rank-based testing procedures and the methodologies described in Section 4. The observations stem from hydrology, finance, and network traffic monitoring, areas that typically give rise to long-range dependent time series.

The data sets from hydrology and network traffic monitoring have been well-studied in the literature, such that we will embed our analysis into the context of existing results. As a relatively recent data set, the considered financial time series, which describes the performance of the British stock market against the background of the United Kingdom European Union membership referendum in 2016, has not yet been considered in the context of change-point analysis. With our consideration of this data, we hope to pave the way for new discussions in applied change-point analysis. Additionally, we aim at a comparison of rank-based change-point tests resulting from different score functions, as well as a comparison of rank-based change-point tests to CuSum-based testing procedures in practice.

6.1. *Argentina rainfall*

The first data set consists of 113 measurements of yearly rainfall volume in the Argentinian province of Tucumán from 1884 to 1996; see Figure 2. The data was monitored by the Agricultural Experiment Station Obispo Colombres (EEAOC). It was provided by Dr. César M. Lamelas, a meteorologist at EEAOC, and reported in [Wu, Woodroffe and Mentz \(2001\)](#). The construction

of a dam on the Salí river, one of the main running waters in the province of Tucumán, between 1952 and 1962 may serve as an explanation for an abrupt change in the data.

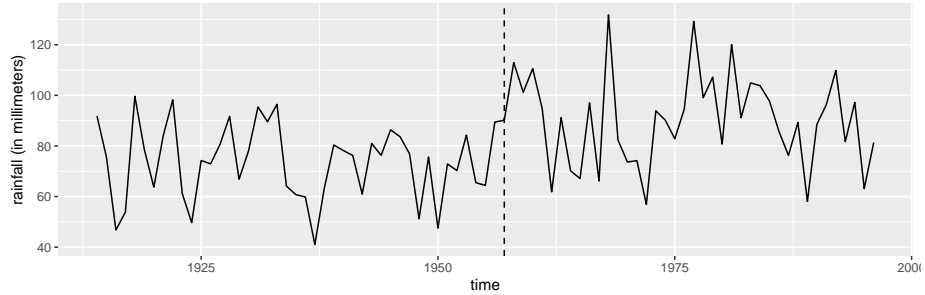


FIG 2. Annual rainfall volume (in millimeters) in the Argentinian province of Tucumán from 1884 to 1996.

We base our analysis of the data on the 83 measurements of yearly rainfall from 1914 to 1996. As examples of rank-based change-point tests we choose the self-normalized Wilcoxon and the self-normalized Van der Waerden test and compare the performances of these tests to that resulting from the change-point test based on the self-normalized CuSum statistic defined in Shao (2011).

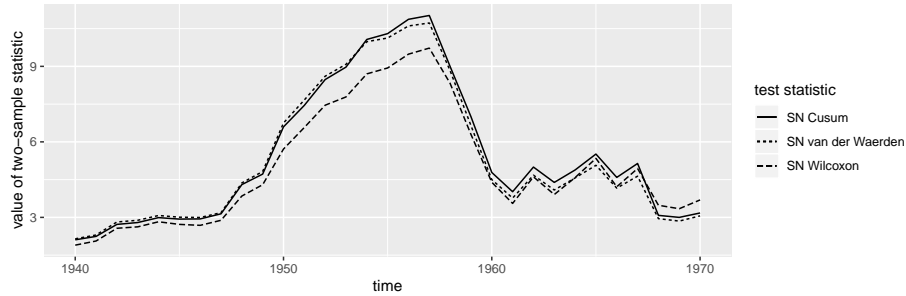


FIG 3. Values of the self-normalized two-sample CuSum, Wilcoxon, and Van der Waerden statistics between 1940 and 1970 computed on the basis of the annual rainfall volume in the Argentinian province of Tucumán from 1914 to 1996.

Figure 3 depicts the values of the two-sample statistics $T_{k,n}(a_i)$, $i = 1, 2$, as defined in formula (13), with $a_1(i) = (n+1)^{-1}i$ (the score function that yields the self-normalized two-sample Wilcoxon statistic) and $a_2(i) = \Phi^{-1}((n+1)^{-1}i)$ (the score function that yields the self-normalized two-sample Van der Waerden statistic) and the self-normalized two-sample CuSum statistic as defined in Shao (2011), between 1940 and 1970. All three line plots achieve their maximum in the year 1957, thereby indicating a potential change-point location that

corresponds to this year. In this regard, our findings agree with the results of previous analysis and the expectation of a change occurring as a consequence of the construction of a dam on the Salí river between 1952 and 1962.

However, approximating the distribution of the self-normalized test statistics by the sampling window method with block size $l = \lfloor \sqrt{n} \rfloor = 9$ yields p -values of 0 for the self-normalized CuSum test, 0.0225 for the self-normalized Van der Waerden test and 0.0858 for the self-normalized Wilcoxon test, i.e., at a level of significance of 5% the self-normalized Van der Waerden and the self-normalized CuSum test reject the hypothesis, while the self-normalized Wilcoxon test decides in favor of the hypothesis of no change. In order to decide which test decision is more plausible, we compare our findings with previous analysis on that same data set:

Wu, Woodroffe and Mentz (2001) base a change-point test on isotonic regression and consider additionally a trend detection test for stationary time series proposed in Brillinger (1989). The isotonic regression method of Wu, Woodroffe and Mentz (2001) strongly favors a location shift in the data between 1955 and 1956, whereas Brillinger's test does not show any evidence of a change. Chen, Gupta and Pan (2006) examine the possibility of changes in mean and variance of the observations. For this, they apply two different information criteria, both suggesting a change-point occurring around 1954. Since, according to their analysis, there is no sufficient indication of a change in the variability of the time series, the change-point is attributed to an increase in the mean precipitation.

Alvarez and Dey (2009) provide statistically significant evidence for a change-point by carrying out Bayesian inference. Jandhyala, Fotopoulos and You (2010) argue that assuming independence of the data-generating random variables cannot be justified with regard to the precipitation data, indicating that valid change-point analysis has to account for serial correlations among the observations. By adjusting for dependence, they base their analysis of the data on a Bayes-type statistic developed in Jandhyala and MacNeill (1991) and a likelihood ratio statistic studied in Csörgő and Horváth (1997). Both procedures provide statistical evidence for a change in the mean of the data around 1956. Incorporating the prior knowledge about a potential change-point location (construction of the dam on the Salí river) by restricting the search area for the change-point accordingly, Shao and Zhang (2010) identify a change in the data on the basis of a self-normalized CuSum statistic.

Since CuSum-type hypothesis tests for a change in mean may be susceptible to outliers in the data, Shao and Zhang (2010) additionally note that the self-normalized median test, as a robust alternative to CuSum-based testing procedures, rejects the hypothesis of stationarity at the 5% significance level, as well. Vogel and Wendler (2017) provide further evidence for a change in location by pointing out that Hodges-Lehmann and CuSum-type tests, resulting as special cases from the consideration of U-quantile-based change-point tests, both reject the null hypothesis of no change at the 5% level of significance.

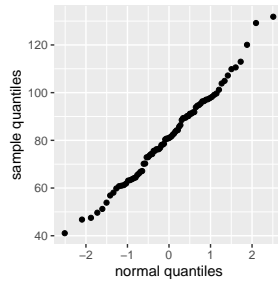


FIG 4. Normal quantile plot of the annual rainfall volume in the Argentinian province of Tucumán from 1914 to 1996.

As noted by [Vogel and Wendler \(2017\)](#), the normal quantile plot (see Figure 4) supports the assumption of normally distributed data. The Van der Waerden test is known to be more efficient when the normality assumption is satisfied, yielding an explanation for contradicting outcomes of the two rank-based testing procedures and a justification for choosing the Van der Waerden test over the Wilcoxon test when normally distributed data is considered.

6.2. FTSE 100 Index

The second data set corresponds to the closing values of the Financial Times Stock Exchange 100 Index (FTSE 100), a share index of the 100 companies with the highest market capitalisation listed on the London Stock Exchange, recorded daily over a time period of one year from March 2016 to March 2017.

Since in general stock prices do not follow a stationary process, whereas their log-returns display features of stationarity, we analyze the log-returns instead of considering the closing index itself; see Figure 5. Formally, the log-returns are defined by

$$L_t := \log R_t, \quad R_t := \frac{P_t}{P_{t-1}},$$

where P_t denotes the value of the index on day t .

The plot in Figure 5 does not indicate a change in the location of the time series, but rather a change in its volatility. For this reason, we intend to apply the change-point tests to the absolute log-returns, i.e., the absolute values of the log-returns. Again, we base our analysis of the data on the self-normalized Wilcoxon and the self-normalized Van der Waerden test and compare their performances to that resulting from the change-point test based on the self-normalized CuSum statistic as defined in [Shao \(2011\)](#).

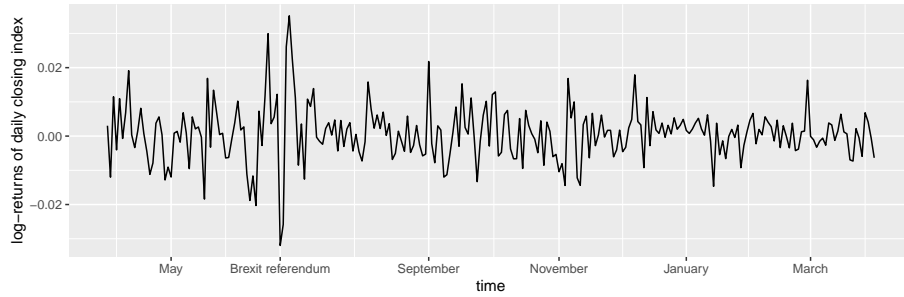


FIG 5. *Log-returns of the daily closing index of the FTSE 100 from March 2016 to March 2017.*

Figure 6 depicts the values of the two-sample statistics $T_{k,n}(a_i)$, $i = 1, 2$, as defined in formula (13), with $a_1(i) = (n+1)^{-1}i$ (the score function that yields the self-normalized two-sample Wilcoxon statistic) and $a_2(i) = \Phi^{-1}((n+1)^{-1}i)$ (the score function that yields the self-normalized two-sample Van der Waerden statistic) and the self-normalized two-sample CuSum statistic as defined in Shao (2011).

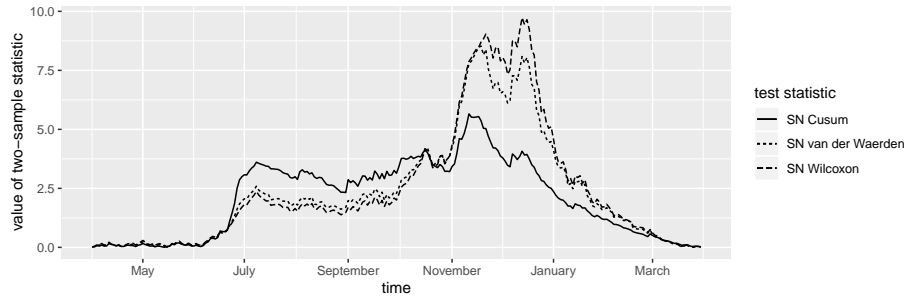


FIG 6. *Values of the self-normalized two-sample CuSum, Wilcoxon, and Van der Waerden statistics computed on the basis of the absolute log-returns of the daily closing index of the FTSE 100 from March 2016 to March 2017.*

All three line plots achieve their maximum in the autumn of 2016, thereby indicating that, if there is a change in the volatility of the time series, it is most likely located around that point in time. The occurrence of a structural change in financial time series in the year 2016 seems highly plausible due to the outcome of the United Kingdom European Union membership referendum on 23 June 2016. An explanation for a decrease of the volatility around November may refer to the Autumn Statement of the same year, a financial report on the state of the economy, published by the British government on 23 November 2016, possibly soothing markets and thereby resulting in a change of the FTSE 100's volatility.

Approximating the distribution of the self-normalized test statistics by the sampling window method with block size $l = \lfloor \sqrt{n} \rfloor = 15$ yields p -values of 0.0118 for the self-normalized Wilcoxon test, 0.0216 for the self-normalized Van der Waerden test and 0.2071 for the self-normalized CuSum test, i.e., at a level of significance of 5% the self-normalized Wilcoxon and the self-normalized Van der Waerden test reject the hypothesis, while the self-normalized CuSum test decides in favor of the hypothesis of no change. This seems plausible insofar the normal quantile plot (see Figure 7) does not substantiate the assumption of a normal distribution, as the tails of the empirical distribution are too heavy. In this case, the more robust Wilcoxon test should be more reliable.

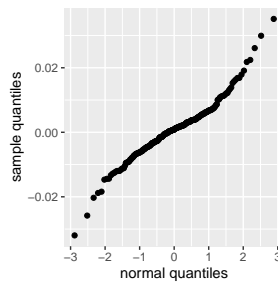


FIG 7. Normal quantile plot of the absolute log-returns of the daily closing index of the FTSE 100 from January 2015 to December 2017.

6.3. Ethernet traffic

The third data set consists of the arrival rate of Ethernet data (bytes per 10 milliseconds) from a local area network (LAN) measured at Bellcore Research and Engineering Center in 1989. The data has been taken from the `longmemo` package in R. For more information on the LAN traffic monitoring see [Leland and Wilson \(1991\)](#) and [Beran \(1994\)](#).

Figure 8 reveals that the observations are strongly right-skewed. As the Wilcoxon and the Van der Waerden statistics are computed from ranks, this is not expected to affect tests and estimators that are based on these statistics. Moreover, estimation of the Hurst parameter by the local Whittle procedure with bandwidth parameter $b_n = \lfloor n^{2/3} \rfloor$ yields an estimate $\hat{H} = 0.845$ indicating long-range dependence. This is consistent with the results of [Leland et al. \(1994\)](#) and [Taqqu and Teverovsky \(1997\)](#).

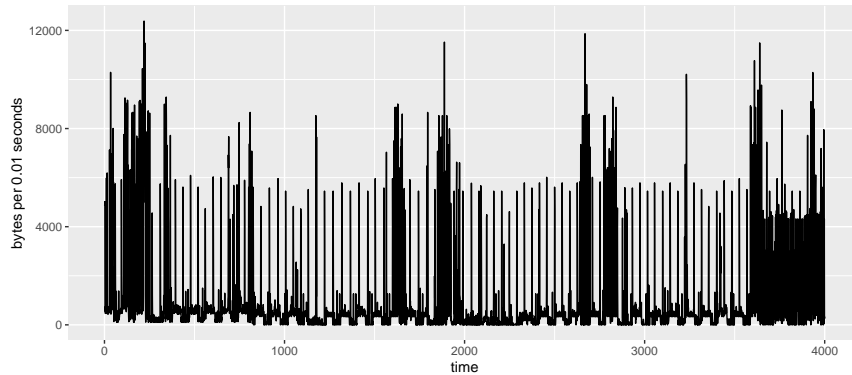


FIG 8. Ethernet traffic in bytes per 10 milliseconds from a LAN measured at Bellcore Research Engineering Center.

Again, we base our analysis of the data on the two self-normalized rank-based test statistics $T_n(a_i)$, $i = 1, 2$, as defined in formula (12), and the self-normalized CuSum statistic as defined in Shao (2011). If compared to their asymptotic critical values, none of the test statistics values can be interpreted as an indication of a structural change in the data for any value of $H \in (\frac{1}{2}, 1)$. Furthermore, approximating the distribution of the self-normalized test statistics by the sampling window method with block size $l = 40$ yields p -values of 0.7159 for the self-normalized Wilcoxon test, 0.7164 for the self-normalized Van der Waerden test and 0.7972 for the self-normalized CuSum test, i.e., at any common choice of a level of significance all three change-point tests reject the hypothesis. In addition, as shown in Betken and Wendler (2018), even when accounting for ties or multiple changes in the data, an application of the self-normalized Wilcoxon change-point test does not provide evidence for a change-point in the mean.

Let us compare our findings to previous analysis by other authors: Coulon, Chabert and Swami (2009) examined this data set in view of change-points under the assumption that a FARIMA model holds for segments of the data. The number of different sections and the location of potential change-points are chosen by a model selection criterion. The algorithm proposed by Coulon, Chabert and Swami (2009) detects multiple changes in the parameters of the corresponding FARIMA time series. However, the change-point estimation algorithm proposed in that paper is not robust to skewness or heavy-tailed distributions and decisively relies on the assumption of FARIMA time series. This seems to contradict observations made by Bhansali and Kokoszka (2001) as well as Taqqu and Teverovsky (1997) who stress that the Ethernet traffic data is very unlikely to be generated by FARIMA processes.

All in all, we analyzed three data sets from different domains of application in change-point analysis. Even though the self-normalized CuSum and Wilcoxon change-point tests have been studied before, our theoretical results facilitate the consideration of other rank-based statistics. In particular, we find that test

decisions that are based on the self-normalized Van der Waerden test in some cases concur with those of the self-normalized CuSum change-point test while in others they coincide with conclusions drawn from an application of the self-normalized Wilcoxon change-point test.

Proofs

Proofs under stationarity

Proof of Proposition 3.1. Our goal is to prove a reduction principle for the sequential empirical process $F_{\lfloor nt \rfloor}(x) - x$, $t \in [0, 1], x \in [0, 1]$, where

$$F_n(x) := \frac{1}{n} \sum_{i=1}^n 1_{\{X_i \leq x\}}$$

with respect to the weighted supremum norm. For this, we consider the transformed random variables Z_n , $n \in \mathbb{N}$, with

$$Z_n := \begin{cases} \frac{1}{X_n} & \text{if } X_n \leq \frac{1}{2}, \\ \frac{1}{X_n - 1} & \text{if } X_n > \frac{1}{2}. \end{cases}$$

It follows that for $x > 2$

$$P(Z_n > x) = \frac{1}{x} \quad \text{and} \quad P(Z_n < -x) = \frac{1}{x}.$$

Hence, Z_n has finite 3λ -moment for $\lambda \in (0, 1/3)$. According to Theorem 2 in [Buchsteiner \(2015\)](#), there exists a constant $\kappa > 0$ such that

$$\begin{aligned} \sup_{t \in [0, 1], x \in [-\infty, \infty]} d_{n,r}^{-1} (1 + |x|)^\lambda \left| \sum_{j=1}^{\lfloor nt \rfloor} \left(1_{\{Z_j \leq x\}} - F_Z(x) - \frac{1}{r!} J_{Z,r}(x) H_r(\xi_j) \right) \right| \\ = O_p \left(n^{-\kappa/3} \right), \end{aligned}$$

where $J_{Z,r}(x) := E(1_{\{Z_1 \leq x\}} H_r(\xi_1))$ and $F_Z(x) := P(Z_1 \leq x)$. For $x \leq 1/2$, we have $X_n \leq x$ if and only if $Z_n \geq 1/x$, i.e., for $x \leq 1/2$, we have

$$\begin{aligned} & 1_{\{X_j \leq x\}} - x - \frac{1}{r!} J_r(x) H_r(\xi_j) \\ &= 1_{\{Z_j \geq \frac{1}{x}\}} - (1 - F_Z(x^{-1})) - \frac{1}{r!} E(1_{\{Z_1 \geq x^{-1}\}} H_r(\xi_j)) H_r(\xi_j) \\ &= - \left(1_{\{Z_j \leq x^{-1}\}} - F_Z(x^{-1}) - \frac{1}{r!} J_{Z,r}(x^{-1}) \right). \end{aligned}$$

Using analogous arguments, we arrive at the same estimation in the case $x > 1/2$. Moreover, we have $x^{-\lambda} \leq (1 + |1/x|)^\lambda$. As a result, we obtain

$$\begin{aligned} & \sup_{t \in [0, 1], x \in [0, 1]} d_{n,r}^{-1} (\min\{x, 1 - x\})^{-\lambda} \left| \lfloor nt \rfloor (F_{\lfloor nt \rfloor}(x) - x) - \frac{1}{r!} J_r(x) \sum_{j=1}^{\lfloor nt \rfloor} H_r(\xi_j) \right| \\ & \leq \sup_{t \in [0, 1], x \in [-\infty, \infty]} d_{n,r}^{-1} (1 + |x|)^\lambda \left| \sum_{j=1}^{\lfloor nt \rfloor} \left(1_{\{Z_j \leq x\}} - F_Z(x) - \frac{1}{r!} J_{Z,r}(x) H_r(\xi_j) \right) \right|. \end{aligned}$$

This completes the proof. \square

Proof of Theorem 3.1. Our goal is to derive a reduction principle for the two-parameter empirical process of the ranks, i.e., for

$$\hat{G}_{\lfloor nt \rfloor}(x-) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x-), \quad t \in [0, 1], \quad x \in [0, 1],$$

with $\hat{G}_k(x) := \sum_{i=1}^k \mathbb{1}_{\{\frac{i}{n+1} R_i \leq x\}}$ and $R_i = \sum_{j=1}^n \mathbb{1}_{\{X_j \leq X_i\}}$.

Recall that F_n^- denotes the generalized inverse of F_n . It then follows that

$$\begin{aligned} \hat{G}_{\lfloor nt \rfloor}(x-) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x-) &= \sum_{i=1}^{\lfloor nt \rfloor} \left(\mathbb{1}_{\{\frac{i}{n+1} R_i \leq x\}} - \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{\frac{i}{n+1} R_i \leq x\}} \right) \\ &= \sum_{i=1}^{\lfloor nt \rfloor} \left(\mathbb{1}_{\{F_n(X_i) \leq \frac{n+1}{n} x\}} - \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{F_n(X_i) \leq \frac{n+1}{n} x\}} \right) \\ &= \sum_{i=1}^{\lfloor nt \rfloor} \left(\mathbb{1}_{\{X_i < F_n^-(\frac{n+1}{n} x)\}} - \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i < F_n^-(\frac{n+1}{n} x)\}} \right). \end{aligned}$$

Unfortunately, the generalized inverse F_n^- is not continuous, which may cause difficulties when considering the weighted supremum norm of the above expression. Therefore, we will consider a continuous modification of F_n^- . For this, let $X_{(1)} \leq X_{(2)} \leq \dots, X_{(n)}$ denote the order statistic of X_1, X_2, \dots, X_n and define a continuous modification of F_n^- by $\tilde{F}_n^- : [0, 1] \rightarrow [0, 1]$ by

$$\tilde{F}_n^-(x) = \begin{cases} 0 & \text{for } x = 0 \\ X_{(i)} & \text{for } x = \frac{i}{n+1} \\ 1 & \text{for } x = 1 \\ \text{linear interpolated in between.} & \end{cases} \quad (15)$$

Because F_n and $F_{\lfloor nt \rfloor}$ are constant on the intervals $[X_{(i)}, X_{(i+1)})$, we have

$$\hat{G}_{\lfloor nt \rfloor}(x-) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x-) = \lfloor nt \rfloor F_{\lfloor nt \rfloor}(\tilde{F}_n^-(x) -) - \lfloor nt \rfloor F_n(\tilde{F}_n^-(x) -).$$

For the proof of Theorem 3.1, we eliminate the expression $\tilde{F}_n^-(x)$ on the right-hand side of the equality and then apply Proposition 3.1. For this, we have to replace $J_r(x)$ by $J_r(\tilde{F}_n^-(x))$, i.e., we have to show that

$$\sup_{x \in [0, 1]} (\min\{x, 1-x\})^{-\lambda} \left(J_r(\tilde{F}_n^-(x)) - J_r(x) \right) = o_P(1).$$

Observe that for $x < y$

$$|J_r(x) - J_r(y)| \leq C\sqrt{y-x}$$

for some constant C . Therefore, it suffices to show that

$$\begin{aligned} \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} \sqrt{|\tilde{F}_n^-(x) - x|} \\ = \sqrt{\sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} |\tilde{F}_n^-(x) - x|} = o_P(1). \end{aligned}$$

Because \tilde{F}_n^- is piecewise linear, we have

$$\begin{aligned} \frac{\tilde{F}_n^-(x)}{x} &= (n+1)\tilde{F}_n^-\left(\frac{1}{n+1}\right) && \text{for } x \leq \frac{1}{n+1}, \\ \frac{1 - \tilde{F}_n^-(x)}{1-x} &= (n+1)\left(1 - \tilde{F}_n^-\left(\frac{n}{n+1}\right)\right) && \text{for } x \geq \frac{n}{n+1}. \end{aligned}$$

The function $\tilde{F}_n^-(x) - x$, $x \in [0, 1]$, takes its maximum for some $x \in \{\frac{1}{n+1}, \frac{2}{n+1}, \dots, \frac{n}{n+1}\}$, as it is linear between these points. As a result, we may conclude that

$$\begin{aligned} \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} |\tilde{F}_n^-(x) - x| \\ = \sup_{x \in [1/(n+1), 1/2]} x^{-2\lambda} |\tilde{F}_n^-(x) - x| + \sup_{x \in [1/2, n/(n+1)]} (1-x)^{-2\lambda} |\tilde{F}_n^-(x) - x| \\ \leq (n+1)^{2\lambda} \sup_{x \in [0,1]} |\tilde{F}_n^-(x) - x|. \end{aligned}$$

Letting $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ denote the order statistics of X_1, X_2, \dots, X_n , such that $nF_n(X_{(i)}) = i$ by definition of $X_{(i)}$, it follows that

$$\begin{aligned} \sup_{x \in [0,1]} |\tilde{F}_n^-(x) - x| &= \max_{i=1, \dots, n} \left| \tilde{F}_n^-\left(\frac{i}{n+1}\right) - \frac{i}{n+1} \right| = \max_{i=1, \dots, n} \left| X_{(i)} - \frac{i}{n+1} \right| \\ &\leq \max_{i=1, \dots, n} \left| \frac{i}{n} - X_{(i)} \right| + \frac{1}{n+1} \leq \sup_{t \in [0,1]} |F_n(x) - x| + \frac{1}{n+1}. \end{aligned}$$

By Proposition 3.1, it therefore holds that

$$\sup_{x \in [0,1]} |\tilde{F}_n^-(x) - x| = \mathcal{O}_P(n^{-1}d_{n,r}).$$

Hence, we finally arrive at

$$\sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} |\tilde{F}_n^-(x) - x| = \mathcal{O}_P(n^{2\lambda-1}d_{n,r}) = o_P(1). \quad (16)$$

Since $J_r(\tilde{F}_n^-(x))$ converges in probability to $J_r(x)$ with respect to the weighted

supremum norm, it remains to show that

$$\begin{aligned} & \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} \left| \left(\hat{G}_{\lfloor nt \rfloor}(x-) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x-) \right) \right. \\ & \quad \left. - \frac{1}{r!} J_r(\tilde{F}_n^-(x)) \left(\sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) - \frac{\lfloor nt \rfloor}{n} \sum_{i=1}^n H_r(\xi_i) \right) \right| = o_P(1). \end{aligned}$$

For this, note that

$$\begin{aligned} & \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} \left| \left(\hat{G}_{\lfloor nt \rfloor}(x-) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x-) \right) \right. \\ & \quad \left. - \frac{1}{r!} J_r(\tilde{F}_n^-(x)) \left(\sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) - \frac{\lfloor nt \rfloor}{n} \sum_{i=1}^n H_r(\xi_i) \right) \right| \\ & \leq \sup_{x \in [0,1]} \frac{(\min\{\tilde{F}_n^-(x), 1 - \tilde{F}_n^-(x)\})^\lambda}{(\min\{x, 1-x\})^\lambda} \\ & \quad \times \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} \left| (\lfloor nt \rfloor F_{\lfloor nt \rfloor}(x-) - \lfloor nt \rfloor F_n(x-)) \right. \\ & \quad \left. - \frac{1}{r!} J_r(x) \left(\sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) - \frac{\lfloor nt \rfloor}{n} \sum_{i=1}^n H_r(\xi_i) \right) \right|. \end{aligned}$$

We will treat the two factors on the right-hand side of the above formula separately. For the first factor, we have

$$\begin{aligned} & \sup_{x \in [0,1]} \left| (\min\{x, 1-x\})^{-\lambda} (\min\{\tilde{F}_n^-(x), 1 - \tilde{F}_n^-(x)\})^\lambda \right| \\ & \leq \sup_{x \in [0, \frac{1}{2}]} \left| \frac{\tilde{F}_n^-(x)}{x} \right|^\lambda + \sup_{x \in [\frac{1}{2}, 1]} \left| \frac{1 - \tilde{F}_n^-(x)}{1-x} \right|^\lambda \\ & \leq \sup_{x \in [0, \frac{1}{2}]} \left| \frac{\tilde{F}_n^-(x) - x}{x} \right|^\lambda + \sup_{x \in [\frac{1}{2}, 1]} \left| \frac{\tilde{F}_n^-(x) - x}{1-x} \right|^\lambda + 2. \end{aligned}$$

Because \tilde{F}_n^- is piecewise linear, it holds that

$$\begin{aligned} \frac{\tilde{F}_n^-(x)}{x} &= (n+1) \tilde{F}_n^-\left(\frac{1}{n+1}\right) & \text{for } x \leq \frac{1}{n+1}, \\ \frac{1 - \tilde{F}_n^-(x)}{1-x} &= (n+1) \left(1 - \tilde{F}_n^-\left(\frac{n}{n+1}\right) \right) & \text{for } x \geq \frac{n}{n+1}, \end{aligned}$$

such that

$$\begin{aligned} & \sup_{x \in [0,1]} \left| (\min\{x, 1-x\})^{-\lambda} (\min\{\tilde{F}_n^-(x), 1 - \tilde{F}_n^-(x)\})^\lambda \right| \\ & \leq 2(n+1)^\lambda \left(\sup_{x \in [0,1]} \left| \tilde{F}_n^-(x) - x \right| \right)^\lambda + 2. \end{aligned}$$

By (16), it therefore follows that

$$\sup_{x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} (\min\{\tilde{F}_n^-(x), 1 - \tilde{F}_n^-(x)\})^\lambda = \mathcal{O}_P(d_{n,r}^\lambda).$$

So as to determine the order of the second factor, we split it into two summands:

$$\begin{aligned} & \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} \left| ([nt]F_{[nt]}(x-) - [nt]F_n(x-)) \right. \\ & \quad \left. - \frac{J_r(x)}{r!} \left(\sum_{i=1}^{[nt]} H_r(\xi_i) - \frac{[nt]}{n} \sum_{i=1}^n H_r(\xi_i) \right) \right| \\ & \leq \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} \left| ([nt]F_{[nt]}(x-) - F(x-)) - \frac{J_r(x)}{r!} \sum_{i=1}^{[nt]} H_r(\xi_i) \right| \\ & \quad + \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} \frac{[nt]}{n} \left| (nF_n(x-) - F(x-)) - \frac{J_r(x)}{r!} \sum_{i=1}^n H_r(\xi_i) \right|. \end{aligned}$$

The second summand is smaller than the first summand. Therefore, it suffices to deal with the first summand. Due to continuity of J_r and F , we have

$$\begin{aligned} & \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} \left| ([nt]F_{[nt]}(x-) - F(x-)) - \frac{1}{r!} J_r(x) \sum_{i=1}^{[nt]} H_r(\xi_i) \right| \\ & = \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} \left| ([nt]F_{[nt]}(x) - F(x)) - \frac{1}{r!} J_r(x) \sum_{i=1}^{[nt]} H_r(\xi_i) \right|. \end{aligned}$$

The right-hand side of the above equation is $\mathcal{O}_P(n^{-\vartheta})$ due to Proposition 3.1.

All in all, we arrive at

$$\begin{aligned} & \sup_{x \in [0,1]} \frac{(\min\{\tilde{F}_n^-(x), 1 - \tilde{F}_n^-(x)\})^\lambda}{(\min\{x, 1-x\})^\lambda} \\ & \quad \times \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} \left| ([nt]F_{[nt]}(x-) - [nt]F_n(x-)) \right. \\ & \quad \left. - \frac{J_r(x)}{r!} \left(\sum_{i=1}^{[nt]} H_r(\xi_i) - \frac{[nt]}{n} \sum_{i=1}^n H_r(\xi_i) \right) \right| \\ & \quad = \mathcal{O}_P(d_{n,r}^\lambda (n^{-\vartheta}) + d_{n,r}^{-1} n^\lambda) = o_P(1), \end{aligned}$$

since by assumption $n^\lambda = o(d_{n,r}^{1-\lambda})$, $n^\lambda d_{n,r} = o(n)$ and $d_{n,r}^\lambda = o(n^\vartheta)$ for any $\lambda < 1/3$. This completes the proof of Theorem 3.1. \square

Proofs under local alternatives

Recall that, under the alternative of a change in the mean, the observations are generated by a triangular array $X_{n,i}$, $1 \leq i \leq n$, $n \in \mathbb{N}$, defined by

$$X_{n,i} = \begin{cases} Y_i & \text{if } i \leq \lfloor n\tau \rfloor, \\ Y_i + h_n & \text{if } i > \lfloor n\tau \rfloor, \end{cases}$$

where $0 < \tau < 1$, h_n , $n \in \mathbb{N}$, is a non-negative deterministic sequence and $Y_n = G(\xi_n)$, $n \in \mathbb{N}$, is a subordinated Gaussian sequence.

Let F denote the marginal distribution function of Y_n , $n \in \mathbb{N}$, and let $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ denote the order statistics of Y_1, Y_2, \dots, Y_n .

Consider the following (stochastic) transformation:

$$H_n(x) := \begin{cases} F(x) & \text{if } x < Y_{(n)} - h_n, \\ F(x - h_n) & \text{if } x > Y_{(n)} + h_n, \\ \text{linear interpolated in between.} \end{cases}$$

Its inverse is given by

$$H_n^-(x) = \begin{cases} F^-(x) & \text{if } x < F(Y_{(n)} - h_n), \\ F^-(x) + h_n & \text{if } x > F(Y_{(n)}), \\ \text{linear interpolated in between.} \end{cases}$$

Let $Y_{n,i} := H_n(X_{n,i})$, $1 \leq i \leq n$, $n \in \mathbb{N}$, denote the transformed observations and note that H_n is a strictly monotone function. As a consequence, we have

$$R_i = \sum_{j=1}^n 1_{\{X_{n,j} \leq X_{n,i}\}} = \sum_{j=1}^n 1_{\{Y_{n,j} \leq Y_{n,i}\}},$$

i.e., the rank statistics are not affected by the transformation H_n . Instead of considering the triangular array $X_{n,i}$, $1 \leq i \leq n$, $n \in \mathbb{N}$, we may therefore as well consider the transformed observations $Y_{n,i}$, $1 \leq i \leq n$, $n \in \mathbb{N}$.

In the following, $F_{k,l}$ refers to the empirical distribution function of $Y_{n,k}, \dots, Y_{n,l}$, i.e.,

$$F_{k,l}(x) := \frac{1}{l - k + 1} \sum_{i=k}^l 1_{\{Y_{n,i} \leq x\}}.$$

For notational convenience, we write F_l instead of $F_{1,l}$.

Proof of Proposition 3.2. Our goal is to prove a reduction principle for the sequential empirical process $F_{\lfloor nt \rfloor}(x) - x$, $t \in [0, 1]$, $x \in [0, 1]$, where

$$F_n(x) := \frac{1}{n} \sum_{i=1}^n 1_{\{Y_{n,i} \leq x\}}, \quad Y_{n,i} := H_n(X_{n,i}),$$

with respect to the weighted supremum norm.

For this, we split the considered expression in (10) into two parts:

$$\begin{aligned}
& \sup_{t \in [0,1], x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} \left| d_{n,r}^{-1} \lfloor nt \rfloor (F_{\lfloor nt \rfloor}(x) - x) \right. \\
& \quad \left. - \frac{1}{r!} J_r(F^-(x)) d_{n,r}^{-1} \sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) + 1_{\{t > \tau\}} \frac{\lfloor nt \rfloor - \lfloor n\tau \rfloor}{d_{n,r}} (x - F(F^-(x) - h_n)) \right| \\
\leq & \sup_{t \in [0,\tau], x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} \left| d_{n,r}^{-1} \lfloor nt \rfloor (F_{\lfloor nt \rfloor}(x) - x) \right. \\
& \quad \left. - \frac{1}{r!} J_r(F^-(x)) d_{n,r}^{-1} \sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) \right| \\
+ & \sup_{t \in [\tau,1], x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} \left| d_{n,r}^{-1} \lfloor nt \rfloor (F_{\lfloor nt \rfloor}(x) - x) \right. \\
& \quad \left. - \frac{1}{r!} J_r(F^-(x)) d_{n,r}^{-1} \sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) + \frac{\lfloor nt \rfloor - \lfloor n\tau \rfloor}{d_{n,r}} (x - F(F^-(x) - h_n)) \right|.
\end{aligned}$$

Convergence of the first summand follows directly from Proposition 3.1. Therefore, it remains to show that the second summand is $\mathcal{O}_P(h_n^\rho)$.

Noting that $\lfloor nt \rfloor F_{\lfloor nt \rfloor}(x) = \lfloor n\tau \rfloor F_{\lfloor n\tau \rfloor}(x) + (\lfloor nt \rfloor - \lfloor n\tau \rfloor) F_{\lfloor n\tau \rfloor + 1, \lfloor nt \rfloor}(x)$, we split the summand as follows:

$$\begin{aligned}
& \sup_{t \in [\tau,1], x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} \left| d_{n,r}^{-1} \lfloor nt \rfloor (F_{\lfloor nt \rfloor}(x) - x) \right. \\
& \quad \left. - \frac{1}{r!} J_r(F^-(x)) d_{n,r}^{-1} \sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) + \left(\frac{\lfloor nt \rfloor}{n} - \frac{\lfloor n\tau \rfloor}{n} \right) d_{n,r}^{-1} n (x - F(F^-(x) - h_n)) \right| \\
= & \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} \left| d_{n,r}^{-1} \lfloor n\tau \rfloor (F_{\lfloor n\tau \rfloor}(x) - x) - \frac{1}{r!} J_r(F^-(x)) d_{n,r}^{-1} \sum_{i=1}^{\lfloor n\tau \rfloor} H_r(\xi_i) \right| \\
& + \sup_{t \in [\tau,1], x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} \left| d_{n,r}^{-1} (\lfloor nt \rfloor - \lfloor n\tau \rfloor) (F_{\lfloor n\tau \rfloor + 1, \lfloor nt \rfloor}(x) - x) \right. \\
& \quad \left. - \frac{1}{r!} J_r(F^-(x)) d_{n,r}^{-1} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} H_r(\xi_i) + \left(\frac{\lfloor nt \rfloor}{n} - \frac{\lfloor n\tau \rfloor}{n} \right) d_{n,r}^{-1} n (x - F(F^-(x) - h_n)) \right|.
\end{aligned}$$

The first summand on the right-hand side of the above inequality is of order $\mathcal{O}_P(h_n^\rho)$ according to Proposition 3.1. Therefore, we restrict our considerations

to the second summand, which can be written as

$$\begin{aligned}
& \sup_{t \in [\tau, 1], x \in [0, 1]} (\min\{x, 1 - x\})^{-\lambda} |d_{n,r}^{-1} (\lfloor nt \rfloor - \lfloor n\tau \rfloor) (F_{\lfloor n\tau \rfloor + 1, \lfloor nt \rfloor}(x) - x) \\
& - \frac{J_r(F^-(x))}{r!} d_{n,r}^{-1} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} H_r(\xi_i) + \left(\frac{\lfloor nt \rfloor}{n} - \frac{\lfloor n\tau \rfloor}{n} \right) d_{n,r}^{-1} n (x - F(F^-(x) - h_n)) \Big| \\
= & \sup_{t \in [\tau, 1], x \in [0, 1]} (\min\{x, 1 - x\})^{-\lambda} |d_{n,r}^{-1} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} \left(\mathbf{1}_{\{Y_i \leq H_n^-(x) - h_n\}} - x \right) \\
& - \frac{J_r(F^-(x))}{r!} d_{n,r}^{-1} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} H_r(\xi_i) + \left(\frac{\lfloor nt \rfloor}{n} - \frac{\lfloor n\tau \rfloor}{n} \right) d_{n,r}^{-1} n (x - F(F^-(x) - h_n)) \Big|.
\end{aligned}$$

Repeated application of the triangular inequality yields

$$\begin{aligned}
& \sup_{t \in [\tau, 1], x \in [0, 1]} (\min\{x, 1 - x\})^{-\lambda} |d_{n,r}^{-1} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} \left(\mathbf{1}_{\{Y_i \leq H_n^-(x) - h_n\}} - x \right) \\
& - \frac{J_r(F^-(x))}{r!} d_{n,r}^{-1} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} H_r(\xi_i) + \left(\frac{\lfloor nt \rfloor}{n} - \frac{\lfloor n\tau \rfloor}{n} \right) d_{n,r}^{-1} n (x - F(F^-(x) - h_n)) \Big| \\
\leq & \sup_{t \in [\tau, 1], x \in [0, 1]} (\min\{x, 1 - x\})^{-\lambda} |d_{n,r}^{-1} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} \left(\mathbf{1}_{\{Y_i \leq H_n^-(x) - h_n\}} - F(H_n^-(x) - h_n) \right) \\
& - \frac{1}{r!} J_r(H_n^-(x) - h_n) d_{n,r}^{-1} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} H_r(\xi_i) \Big| \tag{17}
\end{aligned}$$

$$\begin{aligned}
& + \sup_{t \in [\tau, 1], x \in [0, 1]} (\min\{x, 1 - x\})^{-\lambda} |d_{n,r}^{-1} (\lfloor nt \rfloor - \lfloor n\tau \rfloor) (F(H_n^-(x) - h_n) - x) \\
& + \left(\frac{\lfloor nt \rfloor}{n} - \frac{\lfloor n\tau \rfloor}{n} \right) d_{n,r}^{-1} n (x - F(F^-(x) - h_n)) \Big| \tag{18} \\
& + \frac{1}{r!} \sup_{t \in [\tau, 1], x \in [0, 1]} (\min\{x, 1 - x\})^{-\lambda} |J_r(H_n^-(x) - h_n) - J_r(F^-(x))|
\end{aligned}$$

$$\times \sup_{t \in [\tau, 1]} \left| d_{n,r}^{-1} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} H_r(\xi_i) \right|.$$

For the first summand (17) on the right-hand side of the above inequality, we

have

$$\begin{aligned}
& \sup_{\substack{t \in [\tau, 1], \\ x \in [0, 1]}} (\min\{x, 1-x\})^{-\lambda} |d_{n,r}^{-1} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} \left(1_{\{Y_i \leq H_n^-(x) - h_n\}} - F(H_n^-(x) - h_n) \right) \\
& \quad - \frac{1}{r!} J_r(H_n^-(x) - h_n) d_{n,r}^{-1} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} H_r(\xi_i) | \\
&= \sup_{\substack{t \in [\tau, 1], \\ x \in [0, 1]}} (\min\{x, 1-x\})^{-\lambda} |d_{n,r}^{-1} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} \left(1_{\{F(Y_i) \leq F(H_n^-(x) - h_n)\}} - F(H_n^-(x) - h_n) \right) \\
& \quad - \frac{1}{r!} J_r(H_n^-(x) - h_n) d_{n,r}^{-1} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} H_r(\xi_i) | \\
&\leq \sup_{x \in [0, 1]} \left(\frac{\min\{F(H_n^-(x) - h_n), 1 - F(H_n^-(x) - h_n)\}}{\min\{x, 1-x\}} \right)^\lambda \\
& \quad \sup_{\substack{t \in [\tau, 1], \\ x \in [0, 1]}} (\min\{x, 1-x\})^{-\lambda} d_{n,r}^{-1} \left| \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} (1_{\{F(Y_i) \leq x\}} - x) - \frac{J_r(F^-(x))}{r!} \sum_{i=\lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} H_r(\xi_i) \right|.
\end{aligned}$$

The second factor on the right-hand side of the above inequality is $\mathcal{O}_P(n^{-\vartheta})$ according to Proposition 3.1.

For the second summand (18), we have

$$\begin{aligned}
& \sup_{t \in [\tau, 1], x \in [0, 1]} (\min\{x, 1-x\})^{-\lambda} |d_{n,r}^{-1} (\lfloor nt \rfloor - \lfloor n\tau \rfloor) (F(H_n^-(x) - h_n) - x) \\
& \quad + \left(\frac{\lfloor nt \rfloor}{n} - \frac{\lfloor n\tau \rfloor}{n} \right) d_{n,r}^{-1} n (x - F(F^-(x) - h_n)) | \\
&\leq \sup_{x \in [0, 1]} (\min\{x, 1-x\})^{-\lambda} \frac{n}{d_{n,r}} |(F(H_n^-(x) - h_n) - x) - (F(F^-(x) - h_n) - x)|.
\end{aligned}$$

All in all, it therefore remains to show that

$$\sup_{x \in [0, 1]} \left(\frac{\min\{F(H_n^-(x) - h_n), 1 - F(H_n^-(x) - h_n)\}}{\min\{x, 1-x\}} \right)^\lambda = \mathcal{O}_P(1), \quad (19)$$

$$\sup_{x \in [0, 1]} (\min\{x, 1-x\})^{-\lambda} \frac{n}{d_{n,r}} |F(F^-(x) - h_n) - F(H_n^-(x) - h_n)| = \mathcal{O}_P(h_n^{\min\{\rho, \lambda\}}), \quad (20)$$

and

$$\sup_{x \in [0, 1]} (\min\{x, 1-x\})^{-\lambda} |J_r(H_n^-(x) - h_n) - J_r(F^-(x))| = \mathcal{O}_P(h_n^{\min\{\rho, \lambda\}}). \quad (21)$$

In order to show (19), note that

$$\begin{aligned}
& \sup_{x \in [0,1]} \left(\frac{\min\{F(H_n^-(x) - h_n), 1 - F(H_n^-(x) - h_n)\}}{\min\{x, 1 - x\}} \right)^\lambda \\
& \leq \sup_{x \in [0,1]} \left(\frac{F(H_n^-(x) - h_n)}{x} \right)^\lambda + \sup_{x \in [0,1]} \left(\frac{1 - F(H_n^-(x) - h_n)}{1 - x} \right)^\lambda \\
& \leq \sup_{x \in [0,1]} \left(\frac{F(F^-(x))}{x} \right)^\lambda + \sup_{x \in [0,1]} \left(\frac{1 - F(H_n^-(x) - h_n)}{1 - x} \right)^\lambda \\
& = 1 + \sup_{x \in [0,1]} \left(\frac{1 - F(H_n^-(x) - h_n)}{1 - x} \right)^\lambda.
\end{aligned}$$

For the second summand on the right-hand side, we have

$$\sup_{x \in [F(Y_n), 1]} \left(\frac{1 - F(H_n^-(x) - h_n)}{1 - x} \right)^\lambda = 1.$$

Moreover, using the fact that $H_n^-(x) \geq F^-(x)$, it follows that

$$\begin{aligned}
\sup_{x \in [0, F(Y_n)]} \left(\frac{1 - F(H_n^-(x) - h_n)}{1 - x} \right)^\lambda & \leq \sup_{x \in [0, F(Y_n)]} \left(\frac{1 - F(F^-(x) - h_n)}{1 - x} \right)^\lambda \\
& = 1 + \sup_{x \in [0, F(Y_n)]} \left(\frac{x - F(F^-(x) - h_n)}{1 - x} \right)^\lambda.
\end{aligned}$$

Note that, since $\sup_{x \in [0, F(Y_n)]} (1 - x)^{2\lambda - 1} = (1 - F(Y_n))^{2\lambda - 1} = \mathcal{O}_P(n^{2\lambda - 1})$,

$$\begin{aligned}
& \sup_{x \in [0, F(Y_n)]} \left(\frac{x - F(F^-(x) - h_n)}{1 - x} \right)^\lambda \\
& \leq h_n n^{\lambda(1-2\lambda)} \left[\sup_{x \in [0,1]} (\min\{x, 1 - x\})^{-2\lambda} h_n^{-1} (x - F(F^-(x) - h_n)) \right]^\lambda \\
& = \mathcal{O}(h_n n^{\lambda(1-2\lambda)}).
\end{aligned}$$

The right-hand side of the above inequality is $\mathcal{O}(1)$ due to the fact that $n^{\lambda + \rho - 1} = \mathcal{O}(d_{n,r}^{\rho - 1})$ by assumption. All in all, (19) follows.

In order to show (21), recall that for $x < y$

$$|J_r(x) - J_r(y)| \leq C \sqrt{F(y) - F(x)}.$$

As a result, we have

$$\begin{aligned}
& \sup_{x \in [0,1]} (\min\{x, 1 - x\})^{-\lambda} |J_r(H_n^-(x) - h_n) - J_r(F^-(x))| \\
& \leq \sqrt{\sup_{x \in [0,1]} (\min\{x, 1 - x\})^{-2\lambda} (x - F(H_n^-(x) - h_n))}.
\end{aligned}$$

Note that

$$\begin{aligned}
& \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} (x - F(H_n^-(x) - h_n)) \\
& \leq h_n \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} d_{n,r}^{-1} n \left(x - F(F^-(x) - h_n) \right) \\
& \leq h_n \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} |d_{n,r}^{-1} n (x - F(F^-(x) - h_n)) - f(F^-(x))| \\
& \quad + h_n \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} f(F^-(x)).
\end{aligned}$$

Due to Assumptions (8) and (9), the right-hand side of the above inequality is $\mathcal{O}_P(h_n)$ and (21) follows, so that it remains to show (20). For this, it is enough to consider the interval $[F(Y_{(n)} - h_n), 1]$. To see this, note that $H_n^-(x) = F^-(x)$ for $x \leq F(Y_{(n)} - h_n)$, so that

$$\sup_{x \in [0, F(Y_{(n)} - h_n)]} (\min\{x, 1-x\})^{-\lambda} |d_{n,r}^{-1} n (F(F^-(x) - h_n) - F(H_n^-(x) - h_n))| = 0.$$

On the other hand, we have $|H_n^-(x) - F^-(x)| \leq h_n$ for $x \geq F(Y_{(n)} - h_n)$. As $1 = F_n(Y_{(n)})$ and consequently

$$1 - F(Y_{(n)}) = F_n(Y_{(n)}) - F(Y_{(n)}) \leq \frac{d_n}{n} \sup_x \frac{n}{d_n} |F_n(x) - x| = \mathcal{O}_P(h_n),$$

it follows that for $x \geq Y_{(n)} - h_n$

$$(\min\{x, 1-x\})^{-\lambda} \leq \mathcal{O}_P(h_n^\lambda) (\min\{x, 1-x\})^{-2\lambda}$$

for $x \geq F(Y_{(n)} - h_n)$. As a result, we obtain

$$\begin{aligned}
& \sup_{x \in [F(Y_{(n)} - h_n), 1]} (\min\{x, 1-x\})^{-\lambda} |d_{n,r}^{-1} n (F(F^-(x) - h_n) - F(H_n^-(x) - h_n))| \\
& = \sup_{x \in [F(Y_{(n)} - h_n), 1]} (\min\{x, 1-x\})^{-\lambda} |d_{n,r}^{-1} n (x - F(F^-(x) - h_n)) - f(F^-(x))| \\
& \quad + \sup_{x \in [F(Y_{(n)} - h_n), 1]} (\min\{x, 1-x\})^{-\lambda} f(F^-(x)) \\
& \leq \mathcal{O}_P(h_n^\rho) + \mathcal{O}_P(h_n^\lambda) \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} f(F^-(x)) \\
& = \mathcal{O}_P\left(h_n^{\min\{\rho, \lambda\}}\right),
\end{aligned}$$

using Assumptions (8) and (9). Therefore, (20) holds and the proof is complete. \square

Before proving Theorem 3.2, we establish a number of auxiliary results:

Lemma A.1. *Given the assumptions of Theorem 3.2, it holds that*

$$\sup_{x \in [0,1]} \left| \tilde{F}_n^-(x) - x \right| = \mathcal{O}_P(h_n)$$

with \tilde{F}_n^- defined by

$$\tilde{F}_n^-(x) = \begin{cases} 0 & \text{if } x = 0, \\ Y_{n,(i)} & \text{if } x = \frac{i}{n+1}, \\ 1 & \text{if } x = 1, \\ & \text{linear interpolated in between,} \end{cases}$$

and

$$\sup_{x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} \left| \tilde{F}_n^-(x) - x \right| = \mathcal{O}_P(n^\lambda h_n).$$

Proof. It is clear that the function $\tilde{F}_n^-(x) - x$, $x \in [0,1]$, takes its maximum for some $x \in \{1/(n+1), 2/(n+1), \dots, n/(n+1)\}$, because it is linear between these points. As a result, we may conclude that

$$\begin{aligned} & \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} \left| \tilde{F}_n^-(x) - x \right| \\ &= \sup_{x \in [1/(n+1), 1/2]} x^{-\lambda} \left| \tilde{F}_n^-(x) - x \right| + \sup_{x \in [1/2, n/(n+1)]} (1-x)^{-\lambda} \left| \tilde{F}_n^-(x) - x \right| \\ &\leq (n+1)^\lambda \sup_{x \in [0,1]} \left| \tilde{F}_n^-(x) - x \right|. \end{aligned}$$

By definition, $F_n(Y_{n,(i)}) = i/n$. It then follows that

$$\begin{aligned} \sup_{x \in [0,1]} \left| \tilde{F}_n^-(x) - x \right| &= \max_{i=1, \dots, n} \left| \tilde{F}_n^-\left(\frac{i}{n+1}\right) - \frac{i}{n+1} \right| = \max_{i=1, \dots, n} \left| Y_{n,(i)} - \frac{i}{n+1} \right| \\ &\leq \max_{i=1, \dots, n} \left| \frac{i}{n} - Y_{n,(i)} \right| + \frac{1}{n+1} = \sup_{x \in [0,1]} |F_n(x) - x| + \frac{1}{n+1}. \end{aligned}$$

Proposition 3.2 yields

$$\sup_{x \in [0,1]} |F_n(x) - x| = \mathcal{O}_P\left(\frac{d_{n,r}}{n}\right).$$

This completes the proof. \square

Lemma A.2. *Given the assumptions of Theorem 3.2, it holds that*

$$\sup_{x \in [0,1]} \left(\frac{\min\{\tilde{F}_n^-(x), 1 - \tilde{F}_n^-(x)\}}{\min\{x, 1-x\}} \right)^\lambda = \mathcal{O}_P(n^\lambda h_n^\lambda)$$

with \tilde{F}_n^- defined in (15) and

$$\sup_{x \in [0,1]} \left(\frac{\min\{x, 1-x\}}{\min\{\tilde{F}_n^-(x), 1-\tilde{F}_n^-(x)\}} \right)^{\epsilon_1} = O_P(n^{\epsilon_1} h_n^{\epsilon_1}).$$

Proof. Because \tilde{F}_n^- is piecewise linear, we have

$$\begin{aligned} \frac{\tilde{F}_n^-(x)}{x} &= (n+1)\tilde{F}_n^-\left(\frac{1}{n+1}\right) && \text{for } x \leq \frac{1}{n+1}, \\ \frac{1-\tilde{F}_n^-(x)}{1-x} &= (n+1)\left(1-\tilde{F}_n^-\left(\frac{n}{n+1}\right)\right) && \text{for } x \geq \frac{n}{n+1}. \end{aligned}$$

Using this and the inequality $x^\lambda \leq 1 + |x-1|^\lambda$, we get

$$\begin{aligned} &\sup_{x \in [0,1]} \left(\frac{\min\{\tilde{F}_n^-(x), 1-\tilde{F}_n^-(x)\}}{\min\{x, 1-x\}} \right)^\lambda \\ &\leq \sup_{x \in [1/(n+1), 1/2]} \left(\frac{\tilde{F}_n^-(x)}{x} \right)^\lambda + \sup_{x \in [1/2, n/(n+1)]} \left(\frac{1-\tilde{F}_n^-(x)}{1-x} \right)^\lambda \\ &\leq \sup_{x \in [1/(n+1), 1/2]} \left(\frac{|\tilde{F}_n^-(x) - x|}{x} \right)^\lambda + \sup_{x \in [1/2, n/(n+1)]} \left(\frac{|1-\tilde{F}_n^-(x) - (1-x)|}{(1-x)^\lambda} \right)^\lambda + 2 \\ &\leq (n+1)^\lambda \sup_{x \in [0,1]} |\tilde{F}_n^-(x) - x|^\lambda + 2 = O_P(d_{n,r}^\lambda). \end{aligned}$$

Similar arguments lead to

$$\begin{aligned} &\sup_{x \in [0,1]} \left(\frac{\min\{x, 1-x\}}{\min\{\tilde{F}_n^-(x), 1-\tilde{F}_n^-(x)\}} \right)^{\epsilon_1} \\ &\leq \left(\max \left\{ \frac{1}{\tilde{F}_n^-(1/(n+1))}, \frac{1}{1-\tilde{F}_n^-(n/(n+1))} \right\} \right)^{\epsilon_1} \sup_{x \in [0,1]} |\tilde{F}_n^-(x) - x|^{\epsilon_1} + 2. \end{aligned}$$

It remains to show that the first factor on the right-hand side of the above inequality is of order $O_P(n^{\epsilon_1})$. For any constant $C > 0$, it holds that

$$P\left(\frac{1}{\tilde{F}_n^-(1/(n+1))} \geq Cn\right) = P\left(Y_{n,(1)} \leq \frac{1}{Cn}\right) \leq \sum_{i=1}^n P\left(Y_i \leq \frac{1}{Cn}\right) \leq n \frac{1}{Cn} = \frac{1}{C}$$

and consequently $\left(\tilde{F}_n^-(1/(n+1))\right)^{-1} = O_P(n)$. The same holds for $\left(1-\tilde{F}_n^-(n/(n+1))\right)^{-1}$.

This completes the proof. \square

Lemma A.3. *Given the assumptions of Theorem 3.2, it holds that*

$$\sup_{x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} \left| J_r \left(F^- \left(\tilde{F}_n^-(x) \right) \right) - J_r \left(F^-(x) \right) \right| = O_P \left(n^\lambda \sqrt{h_n} \right).$$

Proof. Recall that for $x < y$

$$|J_r(x) - J_r(y)| \leq C\sqrt{F(y) - F(x)}.$$

With Lemma A.1 it then follows that

$$\begin{aligned} & \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} \left| J_r \left(F^{-} \left(\tilde{F}_n^{-}(x) \right) \right) - J_r \left(F^{-}(x) \right) \right| \\ & \leq \sqrt{\sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} \left| \tilde{F}_n^{-}(x) - x \right|} \\ & = \mathcal{O}_P \left(n^\lambda \sqrt{h_n} \right). \end{aligned}$$

□

Lemma A.4. *Given the assumptions of Theorem 3.2, it holds that*

$$\begin{aligned} & \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-(\lambda-\epsilon_1)} \left| d_{n,r}^{-1} n \left(\tilde{F}_n^{-}(x) - F \left(F^{-} \left(\tilde{F}_n^{-}(x) \right) - h_n \right) \right) \right. \\ & \quad \left. - d_{n,r}^{-1} n \left(x - F \left(F^{-}(x) - h_n \right) \right) \right| = o_P(1). \end{aligned}$$

Proof. We rewrite the difference as

$$\left(\tilde{F}_n^{-}(x) - F \left(F^{-} \left(\tilde{F}_n^{-}(x) \right) - h_n \right) \right) - \left(x - F \left(F^{-}(x) - h_n \right) \right) = g(\tilde{F}_n^{-}(x)) - g(x)$$

with $g(x) := x - F(F^{-}(x) - h_n)$. The function g has the derivative

$$g'(x) = 1 - \frac{f(F^{-}(x) - h_n)}{f(F^{-}(x))},$$

which yields

$$g(\tilde{F}_n^{-}(x)) - g(x) = \left(1 - \frac{f(F^{-}(\zeta_x) - h_n)}{f(F^{-}(\zeta_x))} \right) (\tilde{F}_n^{-}(x) - x)$$

for some $\zeta_x \in (\min\{\tilde{F}_n^{-}(x), x\}, \max\{\tilde{F}_n^{-}(x), x\})$. We conclude that

$$\begin{aligned} & \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-(\lambda-\epsilon_1)} \left| d_{n,r}^{-1} n \left(\tilde{F}_n^{-}(x) - F \left(F^{-} \left(\tilde{F}_n^{-}(x) \right) - h_n \right) \right) \right. \\ & \quad \left. - d_{n,r}^{-1} n \left(x - F \left(F^{-}(x) - h_n \right) \right) \right| \\ & \leq d_{n,r}^{-1} n \sup_{x \in [0,1]} \left(\frac{\min\{\zeta_x, 1-\zeta_x\}}{\min\{x, 1-x\}} \right)^{-\epsilon_1} \\ & \quad \sup_{x \in [0,1]} (\min\{\zeta_x, 1-\zeta_x\})^{\epsilon_1} \left(1 - \frac{f(F^{-}(\zeta_x) - h_n)}{f(F^{-}(\zeta_x))} \right) \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} \left| \tilde{F}_n^{-}(x) - x \right|. \end{aligned}$$

By condition (11), Lemma A.1, and Lemma A.2 it follows that the right-hand side of the above inequality is of the order $O_P(nd_n^{-1}n^{\epsilon_1}h_n^{\epsilon_1}h_n n^\lambda h_n) = O_P(n^{\lambda-1}d_n^{1+\epsilon_1}) = o_P(1)$. \square

Proof of Theorem 3.2. Our goal is to derive a reduction principle for the two-parameter empirical process of the ranks, i.e., for

$$\hat{G}_{\lfloor nt \rfloor}(x-) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x-), \quad t \in [0, 1], \quad x \in [0, 1]$$

with $\hat{G}_k(x) := \sum_{i=1}^k \mathbf{1}_{\{\frac{i}{n+1} R_i \leq x\}}$ and $R_i = \sum_{j=1}^n \mathbf{1}_{\{Y_{n,j} \leq Y_{n,i}\}}$. Recall that

$$\hat{G}_{\lfloor nt \rfloor}(x-) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x-) = \lfloor nt \rfloor F_{\lfloor nt \rfloor}(\tilde{F}_n^-(x) -) - \lfloor nt \rfloor F_n(\tilde{F}_n^-(x) -),$$

i.e., we have to show that

$$\begin{aligned} & \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} \left| \lfloor nt \rfloor F_{\lfloor nt \rfloor}(\tilde{F}_n^-(x) -) - \lfloor nt \rfloor F_n(\tilde{F}_n^-(x) -) \right. \\ & \quad \left. - \frac{1}{r!} J_r(F^-(x)) \left(\sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) - \frac{\lfloor nt \rfloor}{n} \sum_{i=1}^n H_r(\xi_i) \right) \right. \\ & \left. + \left(\mathbf{1}_{\{t > \tau\}} \frac{\lfloor nt \rfloor - \lfloor n\tau \rfloor}{n} - \frac{\lfloor nt \rfloor}{n} \left(1 - \frac{\lfloor n\tau \rfloor}{n} \right) \right) \frac{n}{d_{n,r}} (x - F(F^-(x) - h_n)) \right| = o_P(1). \end{aligned}$$

According to Lemma A.3 and Lemma A.4, we have

$$\begin{aligned} & \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} \left| \lfloor nt \rfloor F_{\lfloor nt \rfloor}(\tilde{F}_n^-(x) -) - \lfloor nt \rfloor F_n(\tilde{F}_n^-(x) -) \right. \\ & \quad \left. - \frac{1}{r!} J_r(F^-(x)) \left(\sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) - \frac{\lfloor nt \rfloor}{n} \sum_{i=1}^n H_r(\xi_i) \right) \right. \\ & \quad \left. + \left(\mathbf{1}_{\{t > \tau\}} \left(\frac{\lfloor nt \rfloor}{n} - \frac{\lfloor n\tau \rfloor}{n} \right) - \frac{\lfloor nt \rfloor}{n} \left(1 - \frac{\lfloor n\tau \rfloor}{n} \right) \right) \frac{n}{d_{n,r}} (x - F(F^-(x) - h_n)) \right| \\ = & \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1-x\})^{-\lambda} \left| \lfloor nt \rfloor F_{\lfloor nt \rfloor}(\tilde{F}_n^-(x) -) - \lfloor nt \rfloor F_n(\tilde{F}_n^-(x) -) \right. \\ & \quad \left. - \frac{1}{r!} J_r(F^-(\tilde{F}_n^-(x))) \left(\sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) - \frac{\lfloor nt \rfloor}{n} \sum_{i=1}^n H_r(\xi_i) \right) \right. \\ & \quad \left. + \left(\mathbf{1}_{\{t > \tau\}} \frac{\lfloor nt \rfloor - \lfloor n\tau \rfloor}{n} - \frac{\lfloor nt \rfloor}{n} \left(1 - \frac{\lfloor n\tau \rfloor}{n} \right) \right) \frac{n}{d_{n,r}} (\tilde{F}_n^-(x) - F(F^-(\tilde{F}_n^-(x)) - h_n)) \right| \\ & + o_P(1). \end{aligned}$$

Moreover, Lemma A.2 yields

$$\begin{aligned}
& \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1}(\min\{x, 1-x\})^{-\lambda} \left| [nt]F_{[nt]}(\tilde{F}_n^-(x) -) - [nt]F_n(\tilde{F}_n^-(x) -) \right. \\
& - \frac{1}{r!} J_r(F^-(\tilde{F}_n^-(x))) \left(\sum_{i=1}^{[nt]} H_r(\xi_i) - \frac{[nt]}{n} \sum_{i=1}^n H_r(\xi_i) \right) \\
& + \left(\mathbf{1}_{\{t > \tau\}} \left(\frac{[nt]}{n} - \frac{[n\tau]}{n} \right) - \frac{[nt]}{n} \left(1 - \frac{[n\tau]}{n} \right) \right) d_{n,r}^{-1} n (x - F(F^-(x) - h_n)) \left. \right| \\
= & \mathcal{O}_P(n^\lambda h_n^\lambda) \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1}(\min\{x, 1-x\})^{-\lambda} \left| [nt]F_{[nt]}(x -) - [nt]F_n(x -) \right. \\
& - \frac{J_r(F^-(x))}{r!} \left(\sum_{i=1}^{[nt]} H_r(\xi_i) - \frac{[nt]}{n} \sum_{i=1}^n H_r(\xi_i) \right) \\
& + \left(\mathbf{1}_{\{t > \tau\}} \frac{[nt] - [n\tau]}{n} - \frac{[nt]}{n} \left(1 - \frac{[n\tau]}{n} \right) \right) \frac{n}{d_{n,r}} \left(\tilde{F}_n^-(x) - F(F^-(\tilde{F}_n^-(x) -) - h_n) \right) \left. \right|.
\end{aligned}$$

Due to continuity of J_r and F , it remains to show that

$$\begin{aligned}
& \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1}(\min\{x, 1-x\})^{-\lambda} \left| [nt]F_{[nt]}(x) - [nt]F_n(x) \right. \\
& - \frac{1}{r!} J_r(F^-(x)) \left(\sum_{i=1}^{[nt]} H_r(\xi_i) - \frac{[nt]}{n} \sum_{i=1}^n H_r(\xi_i) \right) \\
& + \left(\mathbf{1}_{\{t > \tau\}} \frac{[nt] - [n\tau]}{n} - \frac{[nt]}{n} \left(1 - \frac{[n\tau]}{n} \right) \right) \frac{n}{d_{n,r}} (x - F(F^-(x) - h_n)) \left. \right| = o_P(d_{n,r}^{-\lambda}).
\end{aligned}$$

It follows by Proposition 3.2 that the above expression is $\mathcal{O}_P(h_n^\rho)$ with ρ as in that proposition. As $d_{n,r}^{\rho+\lambda} = o(n^\rho)$, this completes the proof. \square

Additional simulation results

This section provides a detailed description of the finite sample performance of rank-based testing procedures. More precisely, Tables 1, 2, and 3 report the frequencies of rejections of the self-normalized Wilcoxon change-point test, the self-normalized Van der Waerden change-point test, and the self-normalized CuSum test for normal margins, Pareto margins, and Cauchy margins. All calculations are based on 5,000 realizations of time series with sample sizes $n = 300$ and $n = 500$. Test decisions are based on an application of the sampling-window method for a significance level of 5%, meaning that the values of the test statistics are compared to the 95%-quantile of the empirical distribution function \tilde{F}_{m_n, l_n} defined by (14). Moreover, block lengths $l_n = \lfloor n^\gamma \rfloor$ with $\gamma \in \{0.4, 0.5, 0.6\}$ are considered. Under the alternative A of a change in the mean, the power of the testing procedures is analyzed by considering different

choices for the height of the level shift, denoted by h , and the location of the change-point, denoted by τ . In the tables, the columns that are superscribed by $h = 0$ correspond to the frequency of a type 1 error, i.e. the rejection rate under the hypothesis.

		$\tau = 0.25$									$\tau = 0.5$						
		$h = 0$			$h = 0.5$			$h = 1$			$h = 0.5$			$h = 1$			
n	l_n	W	V	C	W	V	C	W	V	C	W	V	C	W	V	C	
$H = 0.6$	300	9	0.043	0.062	0.061	0.266	0.316	0.304	0.701	0.762	0.752	0.500	0.565	0.547	0.955	0.973	0.967
	300	17	0.063	0.071	0.072	0.316	0.335	0.338	0.734	0.770	0.773	0.566	0.585	0.580	0.966	0.969	0.969
	300	30	0.070	0.075	0.077	0.319	0.330	0.334	0.699	0.725	0.728	0.556	0.571	0.571	0.952	0.952	0.950
	500	12	0.055	0.059	0.060	0.407	0.444	0.442	0.855	0.881	0.881	0.690	0.726	0.718	0.993	0.994	0.994
	500	22	0.064	0.066	0.065	0.429	0.448	0.445	0.853	0.876	0.878	0.708	0.721	0.720	0.992	0.992	0.993
500	41	0.069	0.070	0.068	0.422	0.428	0.430	0.824	0.840	0.843	0.700	0.709	0.708	0.985	0.986	0.985	
$H = 0.7$	300	9	0.055	0.074	0.063	0.159	0.202	0.177	0.417	0.478	0.443	0.293	0.345	0.307	0.757	0.800	0.759
	300	17	0.064	0.073	0.070	0.178	0.193	0.189	0.419	0.450	0.446	0.318	0.333	0.328	0.753	0.765	0.756
	300	30	0.073	0.077	0.077	0.182	0.191	0.191	0.402	0.420	0.422	0.319	0.322	0.321	0.730	0.732	0.727
	500	12	0.063	0.075	0.069	0.199	0.233	0.225	0.512	0.565	0.553	0.377	0.415	0.406	0.855	0.874	0.862
	500	22	0.068	0.072	0.070	0.207	0.222	0.220	0.508	0.541	0.542	0.386	0.403	0.400	0.855	0.860	0.853
500	41	0.074	0.077	0.077	0.207	0.214	0.216	0.474	0.495	0.498	0.380	0.386	0.385	0.821	0.820	0.821	
$H = 0.8$	300	9	0.073	0.096	0.072	0.122	0.158	0.119	0.263	0.318	0.259	0.224	0.259	0.206	0.531	0.577	0.500
	300	17	0.069	0.080	0.070	0.113	0.133	0.123	0.240	0.270	0.259	0.206	0.222	0.208	0.499	0.518	0.490
	300	30	0.071	0.077	0.075	0.113	0.120	0.120	0.225	0.245	0.242	0.207	0.213	0.207	0.471	0.473	0.458
	500	12	0.060	0.080	0.066	0.123	0.153	0.135	0.268	0.311	0.289	0.224	0.259	0.233	0.581	0.621	0.576
	500	22	0.061	0.069	0.065	0.120	0.132	0.124	0.251	0.279	0.282	0.222	0.237	0.226	0.563	0.574	0.556
500	41	0.064	0.070	0.068	0.119	0.127	0.124	0.236	0.253	0.259	0.215	0.222	0.216	0.540	0.543	0.530	
$H = 0.9$	300	9	0.095	0.113	0.069	0.130	0.153	0.100	0.211	0.253	0.189	0.209	0.237	0.164	0.476	0.501	0.382
	300	17	0.072	0.083	0.071	0.097	0.116	0.098	0.162	0.191	0.181	0.169	0.184	0.161	0.398	0.409	0.362
	300	30	0.073	0.076	0.072	0.097	0.108	0.103	0.148	0.165	0.170	0.160	0.165	0.153	0.366	0.363	0.342
	500	12	0.077	0.097	0.075	0.103	0.124	0.098	0.187	0.226	0.193	0.171	0.197	0.157	0.455	0.481	0.412
	500	22	0.066	0.079	0.069	0.089	0.098	0.092	0.160	0.187	0.183	0.151	0.163	0.147	0.413	0.422	0.392
500	41	0.066	0.073	0.068	0.089	0.096	0.096	0.145	0.161	0.163	0.147	0.152	0.148	0.383	0.386	0.365	

TABLE 1

Rejection rates of the self-normalized $CuSum$ (C), the self-normalized Wilcoxon (W) and the self-normalized Van der Waerden (V) change-point tests obtained by subsampling with block length $l_n = \lfloor n^\gamma \rfloor$, $\gamma \in \{0.4, 0.5, 0.6\}$, for transformed fractional Gaussian noise time series of length n with Hurst parameter H , marginal standard normal distribution and a change in location of height h after a proportion τ of the simulated data.

		$\tau = 0.25$									$\tau = 0.5$						
		$h = 0$			$h = 0.5$			$h = 1$			$h = 0.5$			$h = 1$			
n	l_n	W	V	C	W	V	C	W	V	C	W	V	C	W	V	C	
$H = 0.6$	300	9	0.045	0.063	0.017	0.843	0.919	0.214	0.974	0.989	0.688	0.985	0.986	0.516	1.000	1.000	0.923
	300	17	0.066	0.072	0.045	0.869	0.919	0.341	0.975	0.986	0.785	0.990	0.985	0.659	1.000	1.000	0.954
	300	30	0.082	0.084	0.061	0.828	0.883	0.350	0.936	0.959	0.733	0.980	0.970	0.666	0.999	0.998	0.931
	500	12	0.053	0.062	0.026	0.944	0.975	0.436	0.995	0.999	0.882	0.999	0.999	0.765	1.000	1.000	0.986
	500	22	0.058	0.062	0.042	0.948	0.972	0.497	0.993	0.997	0.902	0.998	0.996	0.815	1.000	1.000	0.987
	500	41	0.066	0.068	0.052	0.923	0.958	0.494	0.980	0.988	0.865	0.996	0.994	0.811	1.000	1.000	0.976
$H = 0.7$	300	9	0.061	0.077	0.026	0.574	0.691	0.130	0.809	0.879	0.471	0.875	0.878	0.344	0.992	0.987	0.787
	300	17	0.070	0.076	0.050	0.569	0.654	0.195	0.803	0.848	0.553	0.871	0.852	0.444	0.988	0.978	0.840
	300	30	0.075	0.080	0.064	0.528	0.612	0.201	0.726	0.777	0.508	0.839	0.807	0.454	0.972	0.955	0.800
	500	12	0.067	0.076	0.038	0.691	0.785	0.224	0.898	0.932	0.658	0.954	0.945	0.517	0.998	0.997	0.911
	500	22	0.071	0.075	0.051	0.691	0.769	0.263	0.888	0.921	0.683	0.949	0.936	0.562	0.997	0.994	0.924
	500	41	0.074	0.074	0.058	0.644	0.718	0.261	0.826	0.867	0.635	0.931	0.902	0.556	0.995	0.985	0.898
$H = 0.8$	300	9	0.078	0.098	0.039	0.350	0.434	0.084	0.574	0.664	0.352	0.690	0.697	0.277	0.930	0.916	0.655
	300	17	0.075	0.085	0.060	0.311	0.372	0.114	0.525	0.585	0.378	0.654	0.630	0.323	0.902	0.870	0.688
	300	30	0.077	0.084	0.073	0.285	0.340	0.121	0.447	0.507	0.324	0.614	0.585	0.326	0.873	0.831	0.632
	500	12	0.066	0.083	0.044	0.397	0.495	0.131	0.620	0.697	0.427	0.740	0.737	0.359	0.948	0.926	0.748
	500	22	0.069	0.073	0.054	0.379	0.448	0.145	0.587	0.645	0.430	0.721	0.696	0.380	0.937	0.906	0.747
	500	41	0.067	0.072	0.060	0.337	0.398	0.146	0.520	0.578	0.381	0.693	0.654	0.370	0.909	0.864	0.698
$H = 0.9$	300	9	0.097	0.121	0.052	0.268	0.331	0.138	0.395	0.470	0.380	0.608	0.601	0.348	0.829	0.809	0.646
	300	17	0.073	0.089	0.064	0.208	0.254	0.141	0.320	0.375	0.372	0.537	0.518	0.350	0.778	0.740	0.641
	300	30	0.072	0.081	0.070	0.174	0.213	0.123	0.264	0.316	0.285	0.504	0.475	0.325	0.732	0.676	0.558
	500	12	0.078	0.097	0.063	0.249	0.309	0.152	0.386	0.457	0.411	0.594	0.589	0.378	0.838	0.812	0.687
	500	22	0.067	0.077	0.065	0.212	0.255	0.147	0.334	0.383	0.388	0.550	0.521	0.372	0.801	0.756	0.667
	500	41	0.071	0.077	0.064	0.186	0.225	0.126	0.288	0.335	0.309	0.516	0.477	0.342	0.763	0.705	0.579

TABLE 2

Rejection rates of the self-normalized $CuSum$ (C), the self-normalized Wilcoxon (W) and the self-normalized Van der Waerden (V) change-point tests obtained by subsampling with block length $l_n = \lfloor n^\gamma \rfloor$, $\gamma \in \{0.4, 0.5, 0.6\}$, for transformed fractional Gaussian noise time series of length n with Hurst parameter H , marginal Pareto(3)-distribution and a change in location of height h after a proportion τ of the simulated data.

		$\tau = 0.25$									$\tau = 0.5$						
		$h = 0$			$h = 0.1$			$h = 0.2$			$h = 0.1$			$h = 0.2$			
n	l_n	W	V	C	W	V	C	W	V	C	W	V	C	W	V	C	
$H = 0.6$	300	9	0.034	0.044	0.013	0.202	0.262	0.010	0.464	0.488	0.010	0.414	0.586	0.014	0.784	0.867	0.012
	300	17	0.059	0.063	0.048	0.279	0.311	0.042	0.553	0.541	0.043	0.500	0.633	0.045	0.843	0.890	0.046
	300	30	0.070	0.072	0.063	0.286	0.319	0.053	0.544	0.530	0.057	0.503	0.626	0.060	0.830	0.877	0.067
	500	12	0.051	0.055	0.021	0.377	0.447	0.024	0.723	0.729	0.026	0.666	0.812	0.023	0.948	0.970	0.026
	500	22	0.062	0.062	0.043	0.414	0.471	0.048	0.759	0.743	0.047	0.697	0.826	0.043	0.956	0.970	0.048
	500	41	0.066	0.071	0.050	0.417	0.460	0.057	0.745	0.722	0.055	0.692	0.814	0.053	0.946	0.969	0.060
$H = 0.7$	300	9	0.034	0.044	0.014	0.155	0.205	0.013	0.373	0.402	0.010	0.336	0.491	0.012	0.685	0.784	0.010
	300	17	0.057	0.059	0.044	0.215	0.250	0.046	0.457	0.447	0.042	0.410	0.538	0.042	0.752	0.810	0.040
	300	30	0.069	0.070	0.051	0.232	0.260	0.057	0.454	0.437	0.056	0.422	0.533	0.057	0.749	0.795	0.053
	500	12	0.038	0.039	0.020	0.281	0.334	0.019	0.588	0.589	0.022	0.527	0.685	0.022	0.878	0.919	0.022
	500	22	0.048	0.049	0.037	0.315	0.357	0.040	0.618	0.605	0.046	0.562	0.706	0.044	0.885	0.920	0.042
	500	41	0.056	0.057	0.046	0.323	0.369	0.049	0.603	0.584	0.054	0.562	0.692	0.056	0.883	0.912	0.052
$H = 0.8$	300	9	0.042	0.057	0.018	0.121	0.164	0.015	0.243	0.276	0.015	0.220	0.324	0.014	0.471	0.557	0.013
	300	17	0.062	0.069	0.044	0.156	0.178	0.044	0.286	0.293	0.050	0.263	0.338	0.043	0.518	0.564	0.045
	300	30	0.067	0.072	0.056	0.171	0.190	0.055	0.295	0.293	0.062	0.275	0.345	0.056	0.515	0.549	0.055
	500	12	0.046	0.056	0.025	0.176	0.218	0.029	0.364	0.377	0.027	0.340	0.454	0.025	0.643	0.706	0.027
	500	22	0.053	0.059	0.039	0.200	0.225	0.049	0.388	0.385	0.045	0.362	0.460	0.044	0.658	0.699	0.042
	500	41	0.060	0.063	0.045	0.208	0.226	0.058	0.392	0.374	0.053	0.369	0.449	0.051	0.650	0.684	0.053
$H = 0.9$	300	9	0.066	0.090	0.032	0.112	0.147	0.033	0.183	0.212	0.032	0.175	0.234	0.031	0.316	0.374	0.029
	300	17	0.070	0.080	0.053	0.113	0.130	0.060	0.188	0.195	0.056	0.178	0.213	0.059	0.312	0.340	0.059
	300	30	0.073	0.079	0.062	0.123	0.127	0.069	0.191	0.190	0.065	0.182	0.205	0.064	0.316	0.330	0.069
	500	12	0.066	0.083	0.039	0.127	0.154	0.044	0.228	0.243	0.046	0.187	0.248	0.039	0.371	0.415	0.037
	500	22	0.067	0.077	0.049	0.128	0.144	0.058	0.233	0.233	0.059	0.191	0.230	0.050	0.368	0.393	0.048
	500	41	0.069	0.078	0.056	0.128	0.137	0.062	0.230	0.224	0.067	0.195	0.228	0.059	0.358	0.369	0.060

TABLE 3

Rejection rates of the self-normalized $CuSum$ (C), the self-normalized Wilcoxon (W) and the self-normalized Van der Waerden (V) change-point tests obtained by subsampling with block length $l_n = \lfloor n^\gamma \rfloor$, $\gamma \in \{0.4, 0.5, 0.6\}$, for transformed fractional Gaussian noise time series of length n with Hurst parameter H , marginal Cauchy-distribution and a change in location of height h after a proportion τ of the simulated data.

		$\tau = 0.25$									$\tau = 0.5$						
		$h = 0$			$h = 0.5$			$h = 1$			$h = 0.5$			$h = 1$			
n	l_n	W	V	C	W	V	C	W	V	C	W	V	C	W	V	C	
$H = 0.6$	300	9	0.036	0.042	0.013	0.976	0.992	0.577	0.999	1.000	0.966	1.000	1.000	0.889	1.000	1.000	0.999
	300	17	0.053	0.061	0.043	0.983	0.994	0.752	1.000	1.000	0.988	1.000	1.000	0.959	1.000	1.000	1.000
	300	30	0.071	0.072	0.061	0.971	0.987	0.755	0.994	0.997	0.966	1.000	0.999	0.954	1.000	1.000	0.999
	500	12	0.046	0.051	0.024	1.000	1.000	0.874	1.000	1.000	0.998	1.000	1.000	0.994	1.000	1.000	1.000
	500	22	0.069	0.060	0.042	0.999	1.000	0.911	1.000	1.000	0.998	1.000	1.000	0.996	1.000	1.000	1.000
500	41	0.069	0.067	0.053	0.997	0.999	0.909	0.999	0.999	0.992	1.000	1.000	0.993	1.000	1.000	1.000	
$H = 0.7$	300	9	0.038	0.044	0.016	0.933	0.970	0.416	0.995	0.997	0.900	0.995	0.995	0.773	1.000	1.000	0.992
	300	17	0.060	0.059	0.040	0.952	0.971	0.583	0.995	0.996	0.942	0.996	0.994	0.880	1.000	1.000	0.997
	300	30	0.072	0.072	0.056	0.929	0.958	0.601	0.980	0.987	0.910	0.992	0.990	0.879	0.999	0.999	0.989
	500	12	0.042	0.050	0.026	0.992	0.997	0.736	0.999	1.000	0.984	1.000	1.000	0.948	1.000	1.000	1.000
	500	22	0.051	0.061	0.042	0.993	0.997	0.790	0.999	1.000	0.987	1.000	1.000	0.966	1.000	1.000	1.000
500	41	0.063	0.066	0.055	0.982	0.991	0.786	0.996	0.998	0.973	1.000	0.999	0.961	1.000	1.000	0.999	
$H = 0.8$	300	9	0.035	0.054	0.020	0.756	0.830	0.236	0.919	0.942	0.675	0.925	0.928	0.520	0.988	0.986	0.899
	300	17	0.056	0.065	0.042	0.775	0.821	0.336	0.916	0.930	0.743	0.931	0.923	0.630	0.985	0.980	0.928
	300	30	0.072	0.069	0.055	0.739	0.785	0.354	0.870	0.888	0.691	0.910	0.899	0.634	0.972	0.964	0.894
	500	12	0.045	0.053	0.030	0.872	0.909	0.414	0.968	0.975	0.841	0.968	0.967	0.720	0.997	0.996	0.970
	500	22	0.049	0.064	0.045	0.870	0.902	0.467	0.965	0.972	0.848	0.966	0.962	0.758	0.997	0.995	0.972
500	41	0.066	0.068	0.055	0.835	0.871	0.466	0.939	0.947	0.808	0.954	0.947	0.748	0.993	0.989	0.957	
$H = 0.9$	300	9	0.065	0.092	0.042	0.469	0.540	0.139	0.666	0.710	0.440	0.718	0.725	0.358	0.880	0.879	0.698
	300	17	0.075	0.081	0.062	0.442	0.490	0.180	0.618	0.649	0.474	0.692	0.682	0.411	0.859	0.844	0.719
	300	30	0.075	0.082	0.069	0.409	0.441	0.184	0.553	0.578	0.419	0.658	0.647	0.409	0.816	0.797	0.664
	500	12	0.067	0.080	0.046	0.514	0.578	0.202	0.722	0.762	0.541	0.756	0.761	0.454	0.907	0.901	0.786
	500	22	0.069	0.074	0.057	0.499	0.540	0.220	0.685	0.711	0.531	0.736	0.725	0.466	0.892	0.878	0.783
500	41	0.075	0.083	0.066	0.461	0.488	0.216	0.623	0.643	0.478	0.708	0.690	0.459	0.863	0.844	0.732	

TABLE 4

Rejection rates of the self-normalized $CuSum$ (C), the self-normalized Wilcoxon (W) and the self-normalized Van der Waerden (V) change-point tests obtained by subsampling with block length $l_n = \lfloor n^\gamma \rfloor$, $\gamma \in \{0.4, 0.5, 0.6\}$, for transformed fractional Gaussian noise time series of length n with Hurst parameter H , marginal χ^2 -distribution and a change in location of height h after a proportion τ of the simulated data.

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