Comparing semi-actuated and fixed control for a tandem of intersections

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Abstract

In this paper, we present a new method to evaluate the performance of a network of intersections. We apply this method to a tandem of intersections. Using this simple example, we show difference between fixed and semi-actuated control. Our method takes the random nature of the traffic and the impact of offsets on the delays into account.

1. Introduction

In the world of growing population and congestion, it is very important to use the existing road infrastructure efficiently, especially, in urban areas such as big cities. To do so, one first needs to evaluate the performance of the road network with traffic lights and then find the optimal settings for these traffic lights. In practice, the performance evaluation of the network is mostly based on traffic microsimulation. In such simulations, complicated car-following models and randomness in the behaviour of drivers are incorporated to make simulations realistic. These features make such approach very accurate, but also time-consuming. Therefore, it is difficult to optimize the network.

Another type of simulations is more deterministic. For example, based on [4], it is common to compare a traffic flow with a water flow. In such models, the randomness in the behaviour of individual drivers is neglected. However, even few drivers that are departing slower than the others can influence the performance of the system especially for a high load. Therefore, it is important to take the randomness of the arrival and departure processes into account. This is considered in analytical approaches which are based on queueing theory. The first results are due to Webster [9] and Darroch [2]. They considered an isolated intersection with fixed control and showed that to find an exact solution for the expected queue length for even one intersection is already a challenging problem. This may be the reason why many authors focused on isolated intersections with different policies and were reluctant to consider a network of intersections. Those who consider a network of traffic lights use simplifying assumptions. For example, Osorio et al. [7] do not take the impact of the offsets into account.

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Our method is a generalization of the isolated fixed-cycle traffic-light model in [6]. In this sense, it is similar to [1]; however, there the authors consider only fixed control and assume that inter-departure times of the vehicles are identical (2 seconds). In this paper, we consider both fixed and semi-actuated control and provide longer departure times for the first vehicles departing from the lane after the traffic lights switch to green.

In this paper, we analysed a tandem of intersections, see Figure 1. There are several intersections that serve an arterial road in the west-east direction. We consider two ways of controlling these intersections. The first is a fixed control; the second is a semi-actuated control, which means that traffic lights switch to green for the main direction as soon as minor lanes are served.

![Figure 1: A tandem of intersections in SUMO. There is an arterial road (west-east) and the minor roads (from north and south). At minor lanes the induction loop detectors are installed.](image)

The paper is structured as follows. In Section 2, we present our analytical models. Then, in Section 3, we present numerical results and, in Section 4, we conclude the paper.

### 2. Models

In this section, we consider two discrete-time models. The first model represents the minor lanes equipped with detectors. In the second model, we consider the major lanes with platooned arrivals. Our goal is to be able to model an intersection with semi-actuated control. In what follows, we suppose that time is split in seconds, and, by convention, the numeration of seconds in a cycle starts at 0. We consider one phase schedule. It can be divided in several parts, each consisting of subsequent phases for lanes with detectors (minor lanes) ending with one phase for lanes without detectors (major lanes). All the saved time from the minor lanes is used for the major lanes. In Figure 2, we give an example of a green-time schedule consisting of one such part. Throughout the paper, the $n^{th}$ second has the same meaning as second $n$ for any $n$. 
Lanes 1, 2, 3 are conflicting. Lanes 1 and 2 have detectors and green-time limits $g_1, \text{max}$, $g_2, \text{max}$, respectively. Lane 1 emptied before the limit is reached. Therefore, the traffic lights switched to lane 2. The green-time limit for lane 2 is reached. Hence, the traffic lights switched to lane 3, even though there may be a non-empty queue at lane 2. Lane 3 emptied before the end of the green time and had free-flow departures during the rest of its green time. By $b_1$, $b_2$, $b_3$, we denote the (random) beginning times of green periods at lanes 1, 2, 3, respectively. The departure of the $k^{th}$ delayed vehicles from lane $l$ happens at the $(b_l + d_k)$th second.

The analysis of both models has the following common structure. We focus on one approaching lane and consider the queue length for this lane. Note that, in general, the queues at different lanes are correlated. However, to simplify analysis, we assume that they are independent. The numerical results show that our model gives a good approximation of reality. The lengths of the red and green times are random, because they depend on the minor lanes scheduled before this lane. However, the cycle length is fixed and equal to $c$ seconds. We assume that delayed vehicles can depart only during predetermined times \{b + d_k\}, $k = 0, \ldots$, during a green period, where $b$ is a random variable of the green-time beginning, $0 \leq b < c$, and the numbers $d_k$ are deterministic. More strictly, if at the $(b + d_k)$th second the traffic lights for this lane are still green and there is a queue at the lane, there will be one departure during this second. Note that using of $d_k$ allows us to model different departure times for the vehicles at the beginning of the queue. For a lane without detectors, if the queue is empty during a green second, all arriving vehicles can proceed without delay. These departures we call free-flow.

For the beginnings of the green times we use the following assumption:

**Assumption 1** (Independence of the green-time beginnings). The green-time beginning moments are independent of each other and the queue length at the lane. They are identically distributed and their distribution is known.
In the case of the fixed control, the green-time beginning \( b \) is a deterministic variable. Later, in Subsection 3.2, for the fixed control, we assume that \( b \) is a random variable with a small range due to the fact that the first vehicle in the queue may start moving later. In the case of semi-actuated control, \( b \) depends on the state of the other lanes, and, therefore, the state on the considered lane depends on the state of the other lanes and \( b \). However, as we show, the dependency does not have a big impact on the results. The distribution of \( b \) follows from the analysis of the preceding (in the green-time schedule) lanes. This assumption allows us to maintain tractability of the model. In Subsection 3.1, we consider this and other independence assumptions and show that the independence of the green-time beginnings gives a very good approximation of the results for the dependent beginnings. In fact, the results are almost indistinguishable.

2.1. Model for a lane with detectors

We assume that the minor lane has detectors, and the green time for this lane terminates as soon as all vehicles are served or a limit of the green-time length in seconds, denoted by \( g \), is reached, whatever occurs first. Using this time limit helps us to avoid too long waiting times for the other lanes.

For simplicity, we assume that the arrivals during different seconds to a minor lane are independent and identically distributed. A similar analysis can be done in the case of more complicated arrivals, as in Subsection 2.2. We denote the probability-generating function (pgf) of arrivals during one second by \( Y(z) \). In our paper, we only use Bernoulli arrivals, so \( Y(z) = \lambda z + 1 - \lambda \), where \( \lambda \) denotes the arrival rate for a current lane. We chose Bernoulli arrivals, since the inter-arrival times of vehicles to the lane of less than one second are unrealistic due to the safety reasons. We denote by \( Y \) the random variable of the number of arrivals.

Denote by \( X_{i}^{act}(z) \) the pgf of the queue length in a steady state at the beginning of the \( i^{th} \) second of a cycle, \( i = 0, 1, \ldots, c-1 \). Let \( \kappa_s \) be the probability \( P(b=s), s = 0, \ldots, c-1 \). Then, we can split the pgf in several parts, by conditioning on \( b \):

\[
X_{i}^{act}(z) = \sum_{s=0}^{c-g} \kappa_s X_{i,s}^{act}(z),
\]

(1)

where by \( X_{i,s}^{act}(z) \) we denote the pgf of the queue length in a steady state at the beginning of the \( i^{th} \) second of a cycle provided that the green time of this cycle begins at the \( s^{th} \) second. Note that \( X_{i,s}^{act}(z) = X_{i,s_2}^{act}(z) \) for \( i \leq s_1, s_2 \), since \( b \) is independent of the arrival process and the queue at the lane at the beginning of the cycle.

Let us focus on one realization of \( b \), e.g., \( b = s \), and consider \( X_{i,s}^{act}(z) \). During the red time, \( Y \) vehicles arrive each second. Therefore,

\[
X_{i,s}^{act}(z) = X_0^{act}(z)(Y(z))^i
\]

(2)

for each \( i \leq s \). During next \( g \) seconds we either have green time or it has already terminated due to the service of the last vehicle in the queue. Note that in the later case it is possible to have a non-empty queue, because some vehicles
could arrive after the termination of the green time. Since the system behaves differently in these two cases, we need to split the pgf in two parts:

\[ X_{i,s}^{\text{act}}(z) = G_{i,s}^{\text{act}}(z) + R_{i,s}^{\text{act}}(z), \]  

(3)

where \( G_{i,s}^{\text{act}}(z) \) and \( R_{i,s}^{\text{act}}(z) \) correspond to green and red light at second \( i \), respectively, \( i = s, \ldots, s + g - 1 \). More precisely,

\[ G_{i,s}^{\text{act}}(z) = \mathbb{E}(z^{X_{i,s}^{G}}1\{\text{Green at time } i\}), \quad R_{i,s}^{\text{act}}(z) = \mathbb{E}(z^{X_{i,s}^{R}}1\{\text{Red at time } i\}), \]

(4)

where \( X_{i,s} \) is the random variable of the queue length at time \( i \), \( 1 \) is the indicator function. At time \( i = s \), we have

\[ G_{s,s}^{\text{act}}(z) = X_{s,s}^{\text{act}}(z) - X_{s,s}^{\text{act}}(0), \quad R_{s,s}^{\text{act}}(z) = X_{s,s}^{\text{act}}(0). \]  

(5)

This means that we have no green time in a cycle with probability \( X_{s,s}^{\text{act}}(0) \). Note that by definition, \( G_{i,s}^{\text{act}}(0) = 0 \) since if the queue is empty, we switch to red time. For a non-departure second, i.e., second \( i \neq s + d_k \) for each \( k \), we need to multiply both parts by \( Y(z) \):

\[ C_{i+1,s}^{\text{act}}(z) = G_{i+1,s}^{\text{act}}(z)Y(z), \quad R_{i+1,s}^{\text{act}}(z) = R_{i+1,s}^{\text{act}}(z)Y(z), \]  

(6)

which corresponds to an arrival of \( Y \) vehicles. However, for a second with a departure, i.e., \( i = s + d_k \) for some \( k \), there is a probability that the next second will be red. Hence, for \( i = s + d_k \), we have

\[ G_{i+1,s}^{\text{act}}(z) = \frac{G_{i,s}^{\text{act}}(z)Y(z)}{z} - w_{k+1,s}, \quad R_{i+1,s}^{\text{act}}(z) = R_{i,s}^{\text{act}}(z)Y(z) + w_{k+1,s}, \]  

(7)

where \( w_{k+1,s} \) is the probability that queue empties after the \( k \)th departure.

For a minor lane, we assume that \( g = d_{n-1} + 1 \) for some \( n \), i.e., the \( (s + g - 1) \)th second is a second of the \( (n-1) \)th departure, since otherwise the extra time will not be used. Note that if we use (7) for \( i = s + d_{n-1} \), we will get \( G_{s+g,s}^{\text{act}}(z) \) and \( R_{s+g,s}^{\text{act}}(z) \), which do not correspond to green and red time since we force green-time termination. However, \( X_{s+g,s}^{\text{act}}(z) = G_{s+g,s}^{\text{act}}(z) + R_{s+g,s}^{\text{act}}(z) \). Note that in \( X_{s+g,s}^{\text{act}}(z) \), probabilities \( w_{z,s} \) are used only for \( k = 0, \ldots, n-1 \), where we define \( w_{0,s} \) as \( R_{0,s}^{\text{act}}(1) = X_{s,s}^{\text{act}}(0) \).

The rest of the cycle time is red, and the pgf changes in the same way as in (2):

\[ X_{i,s}^{\text{act}}(z) = X_{s+g,s}^{\text{act}}(z)(Y(z))^{s-g}, \]  

(8)

where \( i = s + g, \ldots, c \), and \( X_{c,s}^{\text{act}}(z) \) is the pgf of the queue length at the beginning of a cycle provided that the green time during the previous cycle began at the \( s \)th second.

Equations (5), (6) and (7) together give

\[ X_{s+g,s}^{\text{act}}(z) = \frac{X_{n,s}^{\text{act}}(z)(Y(z))^{s-g} + \sum_{k=0}^{n-1} w_{s,k} \left( 1 - \frac{1}{z^{s-k}} \right) (Y(z))^{s-d_k-1}}{z^n}, \]  

(9)

where for convenience \( d_0 = -1 \). Using (2), (8) and (9), we get that

\[ X_{c,s}^{\text{act}}(z) = \frac{X_{0,s}^{\text{act}}(z)(Y(z))^{c} + \sum_{k=0}^{n-1} w_{c,k} \left( 1 - \frac{1}{z^{s-k}} \right) (Y(z))^{c-s-d_k-1}}{z^n}. \]  

(10)
Finally, the combination of results for all $s$ yields

$$X^{act}_c(z) = \sum_{s=0}^{c-g} \kappa_s X^{act}_{c,s}(z) = \frac{X^{act}_0(z)(Y(z))^c}{z^n} +$$

$$+ \sum_{s=0}^{c-g} \kappa_s \sum_{k=0}^{n-1} w_{k,s} \left(1 - \frac{1}{z^{n-k}}\right)(Y(z))^{c-s-d_k-1}.$$  \hspace{1cm} (11)

Note that since $X^{act}_c(z)$ is the pgf of steady state distribution, $X^{act}_0(z) = X^{act}_0(z)$, which further gives us an equation for the function $X^{act}_0(z)$ with many unknown variables $w_{k,s}$. By careful consideration of possible arrivals, we can express these variables in terms of the probabilities $x_j, j = 0, \ldots, n-1$, of having $j$ vehicles in the queue at the beginning of a cycle. Note that it is possible to have an empty queue after the $k$th departure only if there were not more than $k$ vehicles in the queue at the beginning of a cycle. Let us denote by $\alpha_{jk,s}$ the conditional probability that queue empties after the $k$th departure provided that at the beginning of the cycle there were $j$ vehicles in the queue. Then, we have

$$w_{k,s} = \sum_{j=0}^{k} \alpha_{jk,s} x_j.$$  \hspace{1cm} (12)

We can now present the pgf of the queue length at the beginning of a cycle $X^{act}_0(z)$ as a function of $x_j, j = 0, \ldots, n-1$:

$$X^{act}_0(z) = \sum_{j=0}^{n-1} x_j f^{act}_j(z).$$  \hspace{1cm} (13)

where

$$f^{act}_j(z) = \sum_{s=0}^{c-g} \kappa_s \sum_{k=j}^{n-1} \alpha_{jk,s}(z^n - z^k)(Y(z))^{c-s-d_k-1}.$$  \hspace{1cm} (14)

**Remark 1.** To find the pgf, we need equations on the probabilities $x_j$. The common way to do it is to use the fact that a pgf must be analytical inside the unit disk $D_1 = \{z : |z| < 1\}$ and continuous up to unit circle. It is known (see, e.g., [8]) that for a stable system, i.e., $\lambda c < n$, there are $n$ roots $z_0, \ldots, z_{n-1}$ of equation

$$z^n = (Y(z))^c$$  \hspace{1cm} (15)

in the closed unit disk $\bar{D}_1 = \{z : |z| \leq 1\}$. One of the roots is equal to 1. We will denote this root by $z_0$. Note that the pgf can be continuous in the closed unit disk only if the numerator in (13) is equal to 0 at $z_r$, for any $r = 0, \ldots, n - 1$. This fact gives $n - 1$ equations on $x_j$, since the first equation, which corresponds to $z_0$, is trivial. The additional equation we get from the fact that $X^{act}_0(1) = 1$. Therefore, we can find the pgf of the queue length at the beginning of a cycle by finding the coefficients $\alpha_{jk,s}$, the roots $z_r$, and solving the resulting system of (linear) equations on $x_j$. To find coefficients $\alpha_{jk,s}$, one need to consider possible queue lengths depending on the second of the cycle provided that there were $j$ vehicles in the queue in the beginning of a cycle. Note that for Bernoulli arrivals, (15) contains polynomials at both sides and can be solved using standard methods to find the zeros of polynomials.
Figure 3: An example of a scenario and arrivals realization. There are two sources of platoons, and one platoon comes before the end of a cycle and continues to the next cycle. We split it in two independent platoons (0 and 2). As a result, we have 3 platoons per cycle, instead of 2. In this example, platoon 0 has both platooned and free-flow part; platoon 1 has only platooned arrival; platoon 2 has only free-flow arrivals. The arrivals of platoon 1 are coming from a busy lane, where not all vehicles waiting in the queue departed during the green time. Otherwise, there would be free-flow arrivals at the end of this platoon. By $b_{0,s}, b_{1,s}, b_{2,s}$, we denote the beginning of platoon 0, 1 and 2, respectively. The arrival of the $l$th (platooned) vehicle in platoon $u$ happens at second $b_{u,s} + a_l$.

2.2. Model for a lane without detectors

In this subsection, we consider a lane without detectors and derive the pgf of its queue length at the beginning of a cycle. As in the previous subsection, it is a rational function. However, compared to the case of the minor lane, the numerator has a more complicated form. Therefore, after giving preliminary notations, we first prove the fractional form of the pgf. Then, we derive in details the numerator.

Consider a lane without detectors. The green time can start randomly due to detectors on the other lanes, but we suppose that it terminates at a predetermined moment $e$ of the cycle. This lane can have either free-flow arrivals from outside the system or correlated arrivals from another intersection. Note that arrivals from an upstream lane form a platoon, which have platooned part consisting of vehicles that were delayed at the previous intersection, possibly followed by free-flow part of vehicles that went through the previous intersection without delay. The distribution of the lengths of these parts depends on the settings of the upstream intersection. Note that we can see free-flow arrivals from outside the system as a special case of correlated arrivals (with a probability 0 of having platooned arrivals). Therefore, in this subsection we consider only correlated arrivals.

As with the beginning of the green time, we consider an independence assumption:

Assumption 2 (Platoons’ independence). The arrivals are independent of the state of the queue at the beginning of a cycle. The arrivals during different cycles are identically distributed (with a known distribution) and independent of each other. The platoons from different upstream lanes are independent of each.
This assumption allows us to consider the queue length at the beginning of a cycle as a Markov chain.

The correlated arrivals can have a very difficult structure. Indeed, the arrival moments, durations of platooned and free-flow parts are random. For simplicity of analysis, we divide all the different possibilities in scenarios. For each scenario, we fix the arrival moments of platoons and the total duration of green time for each upstream lane, see Figure 3. We also fix the beginning of the green time $b$. Only the distribution of arrivals inside a platoon is random. We denote the set of scenarios by $\mathcal{S}$ and the probability of a scenario $s$ by $\kappa_s$. Note that different beginning times of a green time for a minor lane can be also viewed as scenarios. This is why, we use the same notation as in Subsection 2.1.

We enumerate platoons during a cycle starting from 0. Usually, for a standard four-directions intersection, the number of platoons is 3, i.e., $u = 0, 1, 2$. For platoon $u$, we denote by $b_{u,s}$ the arrival second of the platoon and by $e_{u,s}$ the last second of arrivals from this platoon. Due to distance and offset between intersections, it is possible that arrivals from one lane of upstream intersection arrive just before the end of a cycle and continue at the beginning of the next cycle. We want to maintain independence between cycles, and, for this reason, we split such a platoon in two independent platoons. One has a beginning second $b_{u_1,s} < 0$ and another has an end second $e_{u_2,s} \geq c$, see Figure 3. In Subsection 2.3, we discuss in details how to transform the output process of an upstream intersection to the input for a downstream intersection.

The platooned arrivals of platoon $u$ happen at moments $b_{u,s} + a_l$, $l = 0, \ldots$. Note that, for simplicity, we assume that arrival moments of delayed vehicles are independent of the source of arrivals and scenario. The analysis for dependent arrival moment is the same. Furthermore, we do not assume that $a_l = d_l$, where $d_l$ is a departure moment of a delayed vehicle. In this way, we can embed the possibility that the platoons are changing between the intersections, for more details see Subsection 3.2.

As for the arrivals inside the platoon, we denote by $Y_u(z)$ the pgf of free-flow arrivals of platoon $u$ and by $c_u$ the probability that a vehicle leaves platoon, for example, due to lane change. Here, the free-flow arrivals are those arrivals that came through the upstream intersection without stopping. Moreover, we consider here only possibility of two intersections in tandem. If at least one of the upstream lanes has correlated arrivals, the pgf of free-flow arrivals is not $Y_u(z)$, see Subsection 2.3. Note that since we consider only Bernoulli arrivals, $Y_u(z) = \lambda u z + (1 - \lambda u)$, where $\lambda u$ is the arrival rate for an upstream lane.

The introduction of probability $c_u$ allows us to consider turns on the main road. Therefore, if we have platooned arrivals, then at moments $b_{u,s} + a_l$ we have an arrival with probability $1 - c_u$. The corresponding pgf we denote by $C_u(z) = (1 - c_u)z + c_u$. If the arrivals are free-flow, then each second we have arrivals with pgf $\tilde{Y}_u(z) = Y_u(z)(1 - c_u) + c_u$. We assume that each driver makes decision to change the lane independently of each other. We denote by $\sigma_{m,u,s}$ the probability of having $m$ platooned arrivals, possibly followed by free-flow arrivals, in platoon $u$ in scenario $s$. Note that the actual number of platooned arrivals can be smaller due to leaving vehicles. Finally, we assume that if there are no vehicles in the queue during the green time, all arriving vehicles proceed without delay.
2.2.1. The pgf of the queue length at the beginning of a cycle

Denote by $X_0(z)$ the pgf of the queue length at the beginning of a cycle. We want to prove that, similar to Subsection 2.1, we can represent $X_0(z)$ as a fraction:

$$X_0(z) = \sum_{j=0}^{n-1} x_j f_j(z) z^n - A(z), \quad (16)$$

where $n$ is the maximum number of delayed departures per cycle, $x_j$, $j = 0, \ldots, n-1$, are probabilities of having $j$ vehicles in the queue at the beginning of a cycle, $f_j(z)$ and $A(z)$ are some functions. More precisely,

$$A(z) = \sum_{s \in S} \kappa_s z^{n-n_s} A_s(z), \quad (17)$$

where $n_s$ is the maximum number of delayed departures per cycle in case of scenario $s$, and $A_s(z)$ is the pgf of all arrivals during the cycle in case of scenario $s$.

The main difficulty in (16) is to find $f_j(z)$; we return to this question in Subsection 2.2.4. To prove the fraction form (16), we need to consider the changes in the queue length during a cycle. To do so, we first give some arrival-related notation in the following subsection.

2.2.2. Arrival process

Note that the arrivals are correlated. Therefore, the arrivals at second $i$ are not independent of the queue length. We can model such arrivals as generated by a Markov (non-stationary) chain. For each scenario $s$ and each second $i$, we denote the set of possible states of this Markov chain by $L_{i,s}$.

If there is a platoon, say number $u$, during second $i$, $L_{i,s} = \{-3, -2, \ldots, m_{u,s}\}$, where $m_{u,s}$ is the maximum possible number of platooned arrivals in platoon $u$. State $l \in L_{i,s}$ corresponds to max{$l, 0$} platooned arrivals during this platoon. The reason for several states in the case of only free-flow arrivals (0 platooned arrivals) is that these free-flow arrivals can be postponed due to the vehicles from the previous platoon. We do not allow to have two arrivals in the same second. As we will see in Subsection 3.2, the possible overlaps between platoons are 1, 2 and 4 seconds. This happens if the previous platoon has 3, 2 or 1 platooned arrivals, respectively, where we suppose that the yellow time (switch time between lanes) is 6 seconds. Note that these postponed arrivals do not cause delay for the next platoon.

If there are no arrivals during second $i$ in scenario $s$, the state space of the Markov chain is $L_{i,s} = \{-4\}$.

Let $\Omega_{i,s}^l$ be the event that the Markov chain is in state $l \in L_{i,s}$ and $\sigma_{i,s}^l = \mathbb{P}(\Omega_{i,s}^l)$. If there is a platoon during second $i$, then $\sigma_{i,s}^l = \sigma_{l,u,s}$ for $l > 0$, $\sigma_{i,s}^l = \sigma_{-i,u-1,s} \sigma_{0,u,s}$ for $l < 0$ and $\sigma_{0,s}^0 = 1 - \sum_{l \neq 0} \sigma_{i,s}^l$. In the beginning of the cycle, the state is chosen randomly. It changes only in the beginning of a platoon or if a platoon finishes and there is no more arrivals. In the first case, let $u'$ be the number of platoon during second $i + 1$. Then,

$$\mathbb{P}(\Omega_{i+1,s}^{l'}|\Omega_{i,s}^l) = \sigma_{l',u',s} \quad \text{if } l' > 0,$$

$$\mathbb{P}(\Omega_{i+1,s}^l|\Omega_{i,s}^l) = \sigma_{0,u',s} \quad \text{if } 1 \leq l \leq 3,$$

$$\mathbb{P}(\Omega_{i+1,s}^0|\Omega_{i,s}^l) = \sigma_{0,u',s} \quad \text{if } l \leq 0 \text{ or } l \geq 4.$$
In the second case, the transition probability \( P(\Omega_{l,s}^{-4}, \Omega_{l,s}^l) = 1 \).

Let us denote by \( A_{i,i',s}^l(z) \) the pgf of the arrivals between the beginnings of second \( i \) and second \( i' \) in case of event \( \Omega_{l,s}^l \). For simplicity, if \( i' = i + 1 \), we denote the pgf as \( A_{i,s}^l(z) \). Note that since the arrivals only depend on the state of the Markov chain which generates arrivals: 

\[
A_{i,i',s}^l(z) = A_{i,i',s}^l(z) \sum_{l' \in L_{i,i',s}} P(\Omega_{l',s}^l|\Omega_{l,s}^l)A_{l',i',s}^l(z). \tag{18}
\]

Hence, the functions \( A_{i,s}^l(z) \) are the building blocks of the arrivals.

If there is no platoon during second \( i \) in scenario \( s \), then \( A_{i,s}^{-4}(z) = 1 \). Now consider the case when during second \( i \) there is a platoon, say number \( u \). Suppose event \( \Omega_{l,s}^l \) happens. If \( l > 0 \), then during time interval \( [b_{u,s}, b_{u,s} + a_{l-1}] \) we have platooned arrivals and during time interval \( [b_{u,s} + a_{l-1} + 1, b_{u,s} + a_{l-1} + g_{u,s} - d_{l-1} - 1] \) there are free-flow arrivals, where \( g_{u,s} \) is the green time of an upstream lane from which platoon \( u \) arrived. Therefore,

\[
A_{i,s}^l(z) = \begin{cases} 
C_u(z) & \text{if } i = b_{u,s} + a_m, \quad 0 \leqslant m < l, \\
\bar{Y}_u(z) & \text{if } b_{u,s} + a_{l-1} < i < b_{u,s} + a_{l-1} + g_u - d_{l-1}, \\
1 & \text{otherwise.} 
\end{cases} \tag{19}
\]

\[
A_{i,s}^l(z) = \begin{cases} 
1 & \text{if } i = b_{u,s} + a_m, \quad 0 \leqslant m < l, \\
1 - \bar{Y}_u(z) & \text{if } b_{u,s} + a_{l-1} < i < b_{u,s} + a_{l-1} + g_u - d_{l-1}, \\
0 & \text{otherwise.} 
\end{cases} \tag{20}
\]

For given \( l \in L_{i,s} \), the pgf of arrivals \( A_{i,s}^l(z) \) is defined depending on which one of right-side conditions in \( (19) \) - \( (21) \) holds. If \( l \leqslant 0 \), then we have free-flow arrivals during time interval \( [b_{u,s} + d, b_{u,s} + d + g_{u,s} - 1] \), where \( d \) is 0, 1, 2 or 4 depending on \( l \). During seconds \( [b_{u,s}, b_{u,s} + d - 1] \) there are arrivals from the previous platoon. They are defined as in \( (19) \) - \( (21) \) with the only change that \( u - 1 \) is used instead of \( u \). Note that in case of more than 2 intersections in the tandem, the free-flow arrivals, to which the equation \( (20) \) corresponds, are more complicated, see Subsection 2.3.

2.2.3. The evolution of the queue length during a cycle

Denote by \( X_i(z) \) the pgf of the queue length at the beginning of the \( i^{th} \) second in a cycle. As before, it is easier to consider each scenario in isolation. Hence, we split \( X_i(z) \) in parts

\[
X_i(z) = \sum_{s \in S} \kappa_s X_{i,s}(z), \tag{22}
\]

where \( X_{i,s}(z) \) is the pgf of the queue length at the beginning of the \( i^{th} \) second in case of scenario \( s \). Further, we split \( X_{i,s}(z) \) in parts corresponding to different states of the Markov chain which generates arrivals:

\[
X_{i,s}(z) = \sum_{l \in L_{i,s}} \sigma_{l,s}^i X_{i,s}^l(z), \tag{23}
\]

where

\[
X_{i,s}^l(z) = E(z^{X_{i,s}^l} | \Omega_{l,s}^l). \tag{24}
\]

To prove form \( (16) \), we need to consider all possible types of seconds during a cycle (red second, green second with delayed departure, green second without delayed departure).
Consider a red second $i$. During such second there are only arrivals. Therefore, the pgf of the queue length changes as

$$X'_{i+1,s}(z) = \sum_{l \in L_{i,s}} P(\Omega^l_{i,s}|\Omega^l_{i+1,s}) X^l_{i,s}(z) A^l_{i,s}(z). \quad (25)$$

During the green time, there is a probability that the queue is empty and the arrivals proceed without delay. Let us denote by $w^l_{i,s}$ the probability of having an empty queue at the beginning of the $i$th second and event $\Omega^l_{i,s}$. Note that the total probability of having an empty queue, $\sum_{l \in L_{i,s}} w^l_{i,s}$, does not change in between departures due to our assumption on free-flow departures.

Consider a green second without departures, i.e., $b_0 + d_{i-1} < i < b_i + d_i$ for some $k$. Note that we have arrivals only if the queue is not empty since in the case of an empty queue the arrivals proceed without delay and the queue stays empty. Thus, we get

$$X'_{i+1,s}(z) = \sum_{l \in L_{i,s}} P(\Omega^l_{i,s}|\Omega^l_{i+1,s}) \left[ X^l_{i,s}(z) - \frac{w^l_{i,s}}{\sigma^l_{i,s}} \right] A^l_{i,s}(z). \quad (26)$$

Here, we used $w^l_{i,s}/\sigma^l_{i,s} = X^l_{i,s}(0)$.

Finally, consider a second of departure $i$, i.e., $i = b_i + d_i$. After an arrival the pgf changes as in (26). Afterwards, there is a departure if the queue is not empty. Therefore, the pgf changes as follows:

$$X'_{i+1,s}(z) = \sum_{l \in L_{i,s}} P(\Omega^l_{i,s}|\Omega^l_{i+1,s}) \left[ X^l_{i,s}(z) - \frac{w^l_{i,s}}{\sigma^l_{i,s}} \right] A^l_{i,s}(z) + \frac{w^l_{i,s}}{\sigma^l_{i,s}}. \quad (27)$$

Note that after multiplying both sides by $\sigma^l_{i+1,s}$, we can rewrite (26) and (27) as

$$\sigma^l_{i+1,s} X'_{i+1,s}(z) = \sum_{l \in L_{i,s}} P(\Omega^l_{i+1,s}|\Omega^l_{i+1,s}) \left( \sigma^l_{i,s} X^l_{i,s}(z) A^l_{i,s}(z) + w^l_{i,s} (1 - A^l_{i,s}(z)) \right) \quad (28)$$

and

$$\sigma^l_{i+1,s} X'_{i+1,s}(z) = \sum_{l \in L_{i,s}} P(\Omega^l_{i+1,s}|\Omega^l_{i+1,s}) \left( \sigma^l_{i,s} X^l_{i,s}(z) \frac{A^l_{i,s}(z)}{z} + w^l_{i,s} \left( 1 - \frac{A^l_{i,s}(z)}{z} \right) \right), \quad (29)$$

respectively. Note that we have changed the order of events in the conditional probability.

To see that (16) is true, observe that the pgf of the queue length at the previous second is always multiplied by arrivals during that second and then maybe divided by $z$ in case of a departure. Therefore, after summing up all the components, we get that $X_{0,s}(z)$ is multiplied by the pgf of all arrivals $A_0(z)$ and divided by $z^{n_s}$, where $n_s$ is defined after (17). Hence, $X_{0}(z)$ is multiplied by $A(z)/z^{n_s}$, and we only need to show that the numerator in (16) is right. Note that in the numerator there should be a sum of $w^l_{i,s}$ multiplied by some polynomials. Moreover, the probability $w^l_{i,s}$ can be written in terms of $x_j$, $j = 0, \ldots, n - 1$. Therefore, we showed that $X_{0}(z)$ has form (16).
2.2.4. The numerator of the pgf

In the following subsection, we show that

\[ f_j(z) = \sum_{s \in S} \kappa_s \sum_{k=0}^{n_s} \sum_{l \in L_{b_s}} \alpha_{j,k,s}^l (g_{k,s}(z)z^{1-\delta_{k,n_s}} - f_{k,s}^l(z))z^{k+n-n_s}, \quad (30) \]

where for simplicity we denoted \( L_{b_s+d_k-1+s} \) as \( \bar{L}_{b_s} \); \( \alpha_{j,k,s}^l \) are non-negative coefficients; \( \delta_{kn_s} \) is Kronecker delta, i.e., \( \delta_{kn_s} = 1 \) if \( k = n_s \) and 0 otherwise; \( g_{k,s}(z) \) and \( f_{k,s}^l(z) \) are pgf’s of arrivals during the rest of the cycle starting from second \( b_s + d_k + 1 \) (or \( b_s + g \) if \( k = n_s \)) and \( b_s + d_k - 1 + 1 \), respectively, provided \( \Omega_{b_s+d_k-1+s} \). Further, the coefficients \( \alpha_{j,k,s}^l \) satisfy the following equation:

\[ w_{b_s+d_k-1+1}^l = \bar{w}_{b_s}^l = \sum_{j=0}^k x_j \alpha_{j,k,s}^l. \quad (31) \]

This means that for each \( j, k \) and \( s \), the coefficient \( \alpha_{k,s}^l \) is the probability of having an empty queue at the beginning of the \( k \)th departure and have event \( \Omega_{b_s+d_k-1+s}^l \) provided that there were \( j \) vehicles in the queue at the beginning of the cycle. Note that for \( k = n_s \), the probability \( \bar{w}_{k,s}^l \) depends also on \( x_{n_s} \). However, by summing \( \bar{w}_{n_s,0}^l f_{n,s,0}(0) \) for all \( s \) and \( l \) we should obtain \( x_0 \), which gives an equation on \( x_{n_s} \) (\( n = \max_s n_s \)). Hence, all \( \bar{w}_{k,s}^l \) depend only on \( x_j \) for \( j = 0, \ldots, n - 1 \). As before, we note that coefficients \( \alpha_{j,k,s}^l \) can be found by careful consideration of possible arrivals.

Observe that it is sufficient to prove that the numerator of (16) has the following form:

\[ \sum_{j=0}^n x_j f_j(z) = \sum_{s \in S} \kappa_s \sum_{k=0}^{n_s} \sum_{l \in L_{b_s}} \bar{w}_{k,s}^l (g_{k,s}(z)z^{1-\delta_{k,n_s}} - f_{k,s}^l(z))z^{k+n-n_s}. \quad (32) \]

To do so, we first need to understand more clearly how the pgf of queue length changes between departures.

2.2.5. The queue-length evolution between departures

Let us fix \( s \) and \( k \) and consider a pair of seconds \( v, i \), such that \( b_s + d_k - 1 < v \leq i \leq b_s + d_k \). Using induction, we want to prove that

\[ X_{i,s}(z) = \sum_{l \in L_{b_s}} \left( a_{v,s}^l X_{v,s}^l(z)A_{v,i,s}^l(z) + w_{v,s}^l (1 - A_{v,i,s}^l(z)) \right). \quad (33) \]

First of all, if \( i = v \), then our statement is true, since \( A_{v,v,s}^l(z) = 1 \) for all \( l \). Then, for \( i = v+1 \), since \( A_{v,v+1,s}^l(z) = A_{v+1,s}^l(z) \), our statement follows from (23) and (28).

Now, suppose we proved our statement for all \( i \) and \( v \) that \( i - v = m \), and let us prove it for such \( i \) that \( i - v = m + 1 \). Instead of considering pair \( (v, i+1) \), we will consider pair \( (v - 1, i) \). Consider (33) for pair \( (v, i) \). In it, we can plug
for $v - 1$ instead of $i$. The result will be

$$X_{i,s}(z) = \sum_{l' \in L_{v,s}} \left( \sum_{l \in L_{v-1,s}} \left[ P_{v-1,s}(z) \sigma_{v-1,s}X_{v-1,s}(z)A_{v-1,s}(z) + w_{v-1,s}(1 - A_{v-1,s}(z)) \right] A'_{v,i,s}(z) + w'_{v,s}(1 - A'_{v,i,s}(z)) \right).$$

(34)

Note that from (18), it follows $\sum_{l' \in L_{v,s}} A'_{v-1,s}(z)A'_{v-1,i,s}(z) = A'_{v-1,i,s}(z)$. Hence, for a fixed $l$, $X_{v-1,s}(z)$ is multiplied by $\sigma_{v-1,s}A'_{v-1,i,s}(z)$ and $w_{v-1,s}A'_{v-1,i,s}(z)$ by $A'_{v-1,i,s}(z)$. Note also that from (26) for $z = 0$ it follows that $\sum_{l \in L_{v-1,s}} w_{v-1,s}P_{v-1,s}(z) = w'_{v,s}$. Thus, we can rewrite (34) as

$$X_{i,s}(z) = \sum_{l \in L_{v-1,s}} (\sigma_{v-1,s}X_{v-1,s}(z)A'_{v-1,i,s}(z) - w_{v-1,s}A'_{v-1,i,s}(z)) + \sum_{l' \in L_{v,s}} w'_{v,s}.$$

Now, (33) follows from the fact that the probability to have an empty queue does not change in between departures, i.e., $\sum_{l' \in L_{v,s}} w'_{v,s} = \sum_{l \in L_{v-1,s}} w_{v-1,s}$.

To prove (32), consider $v = b + d_{k-1} + 1$. For such $v$, probabilities $w'_{v,s}$ are equal to $w_{v,s}$. Note that if we plug $i = b + d_k + 1$ in the right hand side of (33), the result will be the pgf of the queue length just before the $k$th departure. Hence, the pgf after departure is equal to

$$X_{i,s}(z) = \sum_{l \in L_{v,s}} \left( \sigma_{v,s}X_{v,s}(z)A'_{v,i,s}(z) + w_{v,s} \left( 1 - A'_{v,i,s}(z) \right) \right),$$

(35)

where for simplicity we use $i = b + d_k + 1$ and $v = b + d_{k-1} + 1$. Note that after this departure $w_{v,s}$ from this equation will be multiplied by (possible) arrivals and divided by $z$ in power of number of departures afterwards. This is exactly what stays in (32) up to multiplying by $z^a$. In the case that there is no delayed departure, i.e., $k = n_1$, there may be some free-flow departures and the pgf changes as in (33). We can use $v = b + d_{k-1} + 1$ and any $i \leq e$, where $e$ is the (deterministic) end second of the green time. Afterwards, $w_{v,s}$ will be multiplied by the pgf of possible arrivals. Because there is no departure for $k = n_1$, we use $z^{1-\delta_{k,n_1}}$ in (32). This concludes the prove of (32).

### 2.3. Output process

In this subsection, we consider the output process of the described models. First, we consider only the case when the arrivals are free-flow.

In the model for a lane with detectors, the output is a batch of delayed vehicles. Consider scenario $s$. With probability $w_{k,s}$, there are $k$ vehicles in the bunch, $k = 0, \ldots, n-1$. With probability $1 - \sum_{k=0}^{n} w_{k,s}$, the green time is terminated due to the time limit and there are $n$ departures. Note that since for this lane the duration of the green time depends on departures, the arriving moment of the following platoon to a downstream lane also depends on the number of departures. Therefore, we need to split each scenario corresponding to different green-time beginnings of the downstream lane in $n$ scenarios. Probabilities $w_{k,s}$ for $k = 0, \ldots, n-1$ and $1 - \sum_{k=0}^{n} w_{k,s}$ provide such splitting.
In the model for a lane without detectors, the output consists of a mix of delayed and free-flow departures. Consider scenario $s$. Note that in the case of only free-flow arrivals, the scenarios correspond to a random beginning of the green time as it is for a lane with detectors. With probability $w_{k,s}^0$, there are $k$ delayed departure and $e - b_s - d_{k-1}$ free-flow departures, $k = 0, \ldots, n_s$, where $b_s$ is the beginning second of green time, and $e$ is the last second of green time. With probability $1 - \sum_{k=0}^{n_s} w_{k,s}^0$, there are $n_s$ delayed departures and no free-flow departures. Note that $\sigma_{i,s}^0 = 1$ for each $i, s$ since there are no platooned arrivals.

Consider a downstream lane. Note that the numeration of scenarios for a downstream lane can be different than for the current lane. Therefore, we consider any scenario $s'$ of the downstream lane such that the arriving moment of platoon from the current lane is fixed and corresponds to scenario $s$ of current lane. There may be multiple number of such scenarios since the beginning of green time for a downstream lane is random and independent of arriving moments of platoons. For input of the downstream lane, we set $\sigma_{k,u,s'}^0 = w_{k,s}^0$ for $k = 0, \ldots, n_s - 1$ and $\sigma_{n_s,u,s'}^0 = 1 - \sum_{k=0}^{n_s} w_{k,s}^0$, where $u$ is the number of platoon departing from this lane and arriving to the downstream lane. We set $g_{u,s'} = e - b_s$.

The output of a lane without detectors and with platooned arrivals also consists of delayed and free-flow departures. However, if the queue empties earlier than the end of green time, instead of one platoon with free-flow arrivals, we may get several platoons. Thus, the arrival Markov chain will consist of the states corresponding to the number of delayed arrivals and the states of the arrival processes for the upstream lanes. Hence, given the generality of analysis in Subsections 2.2.3 - 2.2.5, this case also can be calculated using our results. The only difference will be in the definitions of the arrival pgfs (more precisely in the equation (20)) and the Markov chain.

3. Numerical results

In this section, we present numerical results. In Subsection 3.1, we simulate our model and compare results with simulation of the whole system. In this way, we measure the impact of our independence assumptions, i.e., the assumptions that platoons and the green-time lengths are independent of each other and the state of the system. In Subsection 3.2, we find proper departure and arrivals moments, using the microsimulation suite SUMO (see [3] for more details).

3.1. Independence assumptions

In this subsection, we discuss our independence assumptions. In Assumption 1, we assume that the beginning moments of the green times are independent from each other. In reality, they depend on the queue lengths at the minor lanes.

The second assumption is independence between arrivals. The arrivals from outside the system are independent from each other, because the drivers decide on their own when to come. The arrivals from another intersection form platoons. We assume that platoons from different lanes are independent from each other. It is logical, since they arrive to upstream lanes independently of each other. Additionally, we assume that the arrivals during different cycles are
independent of each other, see Assumption 2. We base this assumption on the case of low load. Consider a lane without detectors. If the load is low, most of the cycles the queue is empty in the end of the green time. Therefore, the number of departures in the next cycle is determined by the arrivals during the red time and, hence, is completely independent of the current cycle departures.

3.1.1. Simulation of dependent and independent models

Figure 4: The structure of the simulated model. Queues 0 and 1 are upstream lanes and queue 2 is a downstream lane. Queues 1 and 2 are on the main road and queue 0 is a minor road. For a given load $\rho$, the arrival rates to queues 0 and 1 are $4\rho/60$ and $8\rho/60$ vehicles per second, respectively. The cycle length is 60 seconds. The capacities of queues 0, 1 and 2 are 4, 8 and 8 (delayed) vehicles per cycle, respectively. With probabilities 0.4 and 0.8 the departing vehicles (from queues 0 and 1, respectively) join queue 2.

To test our assumptions, we simulate our model with and without independence assumptions. For simplicity, we consider 3 lanes: two upstream lanes (minor 0 and major 1) and one downstream lane (2). We choose arrival rates and probabilities to change the lane such that the load for each lane is the same, see Figure 4. We used 12 seconds of green for a minor lane and 22 seconds of green for major lanes. The resulting capacities are 4 and 8 (delayed) vehicles per cycle. The offset between green times for queue 1 and 2 is 10 seconds and the “travel time” between queues is 6 seconds. First, we simulated the behaviour of all three queues together. Then, for queue 2 and for queue 1 in the case of semi-actuated control, the probabilities to terminate green time earlier and the distributions of platoons, acquired from the dependent simulation, are used to generate the beginning of the green time for queue 1 and the arrivals to queue 2 at the beginning of each cycle independently of the previous cycle. We consider semi-actuated and fixed controls for an upstream intersection and only fixed for a downstream lane. We considered several green times and offsets between intersections, and the results are the same. Independence between cycles gives a good results for loads up to 0.8 and gives an underestimation for load 0.9, see results for queue 2 in Figures 5 and 6. Note that even for a long simulation (2 days of simulated time), loads higher than 0.8 give a bigger confidence interval (we used 95% confidence interval), which means that the resulting waiting times are
Figure 5: The comparison between dependent and independent queues in case of the fixed control. The green times for queues 0, 1 and 2 are 12, 22 and 22 seconds, respectively.

Figure 6: The comparison between dependent and independent queues in case of the semi-actuated control. The maximum green time for queue 0 is 12 seconds; the minimum green time for queue 1 is 22 seconds; the green time for queue 2 is 22 seconds.
sensitive to random fluctuations in demand. Therefore, operating under such load is undesirable. As for the independence between green-time beginnings, the results are good for every load, see waiting times for queue 1 in Figure 6.

Note that for both Figures 5 and 6 the waiting time for queue 2 is slightly decreasing for medium load. This happens because the offset between green times of queues 1 and 2 is slightly longer than time needed for a free-flow departing vehicle to go to queue 2, but shorter than the time needed for a delayed vehicle. Therefore, for a lower load, there are more free-flow departures from queue 1 and they need to wait in queue 2, while for a bit higher load, the first departure is delayed and during this delay some vehicles from queue 0 can depart, which makes the average waiting time a bit less. This effect means that the optimal offset depend on the arrival rates to upstream intersection.

We see that the independence between green times gives a very good approximation for all loads, while the independence between arrivals only for loads up to 0.8. Hence, the correlation between service times has a less impact than the correlation between arrival times. The same results were obtained for a M/M/1 queue with autocorrelation (see [5]). In the following subsection we analyse these correlations.

3.1.2. Autocorrelation

For both independence assumptions, we consider the autocorrelation. Let $G$ be the length of a green time and $P$ be the number of platooned arrivals in a platoon. We plot for $X = G, P$ the autocorrelation

$$R_X(\tau) = \frac{(X_t - \mu_X)(X_{t+\tau} - \mu_X)}{\sigma_X^2}$$

depending on $\tau = 1, \ldots, 10$ and $R_X(1)$ as a function of load, see Figure 7. We see that the quantities are mostly positively autocorrelated. However, for a small load the number of delayed departures from queue 0 and, therefore, the beginning of the green time for queue 1 have a negative coefficient of autocorrelation.

Note that in the case of semi-actuated control, for any load, the shorter green time for the minor lane in the current cycle, the higher probability that there will be many vehicles in the queue at the beginning of the next green time and, therefore, the higher probability that there will be a longer green time. However, for a high load if the time is fully used in one cycle, it is most probably also fully used in another. Hence, the green time lengths are negatively-correlated for a low load, but positively-correlated for a high load.

The assumption of independent platoons gives good results up to load 0.8. For a higher load, the probability that if the green time was used fully by delayed vehicles, the probability that there will be many departures during the next cycle is higher than in average. This leads to long series of many arrivals to the downstream lane, which increases the queue length.

3.2. Choice of arrival and departure moments

In this subsection, we consider the problem of finding departure moments $d_k$ and arrival moments $a_l$. For this purpose, we use traffic microsimulation. We simulate a long queue and a long cycle with 38 seconds of green time. After the
Figure 7: The autocorrelation of the number of delayed departures, a), b), c), d), and the beginning of the green time, e) and f). The subfigures a), b), e) and f) correspond to semi-actuated control, and the subfigures c) and d) correspond to the fixed control. The subfigures a), c) and e) show the autocorrelation as a function of step ($\tau$). The subfigures b), d) and f) show the autocorrelation as a function of load for step $\tau = 1$. 
Figure 8: The trajectories of delayed vehicles departing from one lane of the intersection. The green time is 38 seconds.
beginning of a green time we record the trajectories of the departing vehicles depending on their departure number, see Figure 8. Note that the numbering of departures restart from 0 at the beginning of each green period. We see that the distribution of departure moments is very close to deterministic. The domains of the distributions consists of 2-4 values, see Figure 9. The only exception is the last departure (departure 15, see Figure 8). The bifurcating nature of the trajectories of this last departure is caused by vehicles that try but can not stop after switching to the yellow light. They depart from the lane with a lower speed. We are not able for now to incorporate this information in our model. However, since this phenomenon happens to a small fraction of arriving vehicles and only for long queues, we suppose that it does not have a big impact on our numerical results.

In our model, the departure and arrival moments are deterministic. Therefore, we will model the random nature of departure moments by assuming that the departures of the delayed vehicles can be postponed with some probability. We denote this random variable by $\xi$. Note that we only need to postpone the delayed departures, i.e., we consider that the delayed departure happen at moments $b + \xi + d_k$, while free-flow departures can happen starting second $b$. We achieve that by considering different sets of $d_k$ depending on value of $\xi$, i.e., for each value of $\xi$ we consider a separate scenario. Note that the results of Section 2 will not change.

Since the distributions, shown in Figure 9, have most of probability mass distributed between 3 values, we restrict the domain of $\xi$ to $\{0, 1, 2\}$ seconds. Let $p$ and $q$ be probabilities of postponing the beginning of the green period by 0 and 1 second, respectively. For each set of $d_k$, we consider the following objective. For each possible value of $\xi + d_k$ and the random variable of the $k^{th}$ departure, we take the absolute difference between probabilities of these variables to be equal to this value. We set the objective to be equal to the maximum of the differences. It can be seen as max metric between distributions. We also consider the sum of the differences as an objective, i.e., a sum metric. For each choice of metric, we find optimal values of $d_k$, $k = 0, \ldots$. The optimal value of the objective for $d_k$ then optimized for $p$ and $q$. Since we have about 200 observations of each departure number (except the 15th departure, where the number of observations is less), we decide to consider $p$ and $q$ only up to 2 digits after the decimal point and use an exhaustive search for the optimal values. The resulting optimal $p$ and $q$ are different for the metrics, however the shifts $d_k$ and $a_l$ are the same (except for $d_2$ and $a_{12}$, which is out of table), see Table 1. The arrival moments are shifted by 6 seconds because this is the time needed to go 100 meters with free-flow speed 60 kph.

Note that the parameters $p$, $q$ and $d_k$ completely determine the capacity of the lane. This capacity is different from the one given by SUMO. This happens due to the fact that each vehicle has a random departure time. As a result, a system with a fixed arriving rate may be stable for our model and unstable in SUMO or otherwise. Therefore, we use the found parameters $d_k$ and the capacity curve, determined by SUMO, see Figure 10, to find appropriate $p$ and $q$. For each green time $g$, we choose them in such a way that the capacity of green times $g$ and $g – 1$ are the same as in SUMO. This means that we incorporate all the randomness of the departures in the first departure delay, $\xi$. 

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Figure 9: The distributions of a) departure and b) arrival moments for departures with even number. The arriving intersection is 100 meters away from the departing intersection.
Table 1: Departure, arrival moments and optimal probabilities $p$ and $q$ of postponing the beginning of green time or arrivals for max and sum metrics.

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Figure 10: The average number of departures per cycle as a function of green time length.

4. Conclusions

In this paper, we considered two types of the traffic control for an arterial road: fixed and semi-actuated. We presented an analytical model for the queue length on each lane. We fine-tuned the parameters of the model using the microsimulation suite SUMO. Comparing to deterministic models, our model takes the randomness of the driver behaviour and the arrival process into account and, therefore, is more accurate. It is also more time-efficient compared to microsimulations.

We would like to extend the model for the case of inhomogeneous traffic. It is also interesting to consider other traffic control policies.

References


