THE RAMSEY NUMBERS OF LARGE CYCLES VERSUS SMALL WHEELS

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Abstract

For two given graphs G and H, the *Ramsey number* $R(G,H)$ is the smallest positive integer N such that for every graph F of order N the following holds: either F contains G as a subgraph or the complement of F contains H as a subgraph. In this paper, we determine the Ramsey number $R(C_n, W_m)$ for $m = 4$ and $m = 5$. We show that $R(C_n, W_4) = 2n-1$ and $R(C_n, W_5) = 3n-2$ for $n \geq 5$. For larger wheels it remains an open problem to determine $R(C_n, W_m)$.

1. Introduction

Throughout the paper, all graphs are finite and simple. Let G be such a graph. We write $V(G)$ or V for the vertex set of G and $E(G)$ or E for the edge set of G. The graph $\overline{G}$ is the *complement* of the graph G, i.e., the graph obtained from the complete graph $K_{|V(G)|}$ on $|V(G)|$ vertices by deleting the edges of G.

The graph $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. For any nonempty subset $S \subset V$, the *induced subgraph* by S is the maximal subgraph of G with vertex set S; it is denoted by $G[S]$.

[1]Part of the work was done while the first author was visiting the University of Twente.

If $e = \{u, v\} \in E$ (in short, $e = uv$), then u is called *adjacent to* v, and u and v are called *neighbors*. For $x \in V$ and a subgraph B of G, define $N_B(x) = \{y \in V(B) : xy \in E\}$ and $N_B[x] = N_B(x) \cup \{x\}$. The *degree* $d(x)$ of a vertex x is $|N_G(x)|$; $\delta(G)$ denotes the minimum degree in G.

A *cycle* C_n of length $n \geq 3$ is a connected graph on n vertices in which every vertex has degree two. A *wheel* W_n is a graph on $n+1$ vertices obtained from a C_n by adding one vertex x, called the *hub* of the wheel, and making x adjacent to all vertices of the C_n, called the *rim* of the wheel.

Given two graphs G and H, the *Ramsey number* $R(G, H)$ is defined as the smallest natural number N such that every graph F on N vertices satisfies the following condition: F contains G as a subgraph or $\overline{F}$ contains H as a subgraph.

We will also use the short notations $H \subseteq F$, $F \supseteq H$, $H \not\subseteq F$, and $F \not\supseteq H$ to denote that H is (not) a subgraph of F, with the obvious meanings.

Several results have been obtained for wheels. For instance, Burr and Erdős [1] showed that $R(C_3, W_m) = 2m + 1$ for each $m \geq 5$.

Ten years later Radziszowski and Xia [9] gave a simple and unified method to establish the Ramsey number $R(G, C_3)$, where G is either a path, a cycle or a wheel.

Hendry [5] showed $R(C_5, W_4) = 9$. Jayawardane and Rousseau [6] showed $R(C_5, W_5) = 11$. Surahmat et al. [13] showed $R(C_4, W_m) = 9, 10$ and 9 for $m = 4, 5$ and 6 respectively. Independently, Tse [14] showed $R(C_4, W_m) = 9, 10, 9, 11, 12, 13, 14, 15$ and 17 for $m = 4, 5, 6, 7, 8, 9, 10, 11$ and 12, respectively.

Recently, in [11], it was shown that the Ramsey number $R(S_n, W_4) = 2n - 1$ if $n \geq 3$ and n is odd, $R(S_n, W_4) = 2n + 1$ if $n \geq 4$ and n is even, and $R(S_n, W_5) = 3n - 2$ for each $n \geq 3$. Here S_n denotes a star on n vertices (i.e., $S_n = K_{1,n-1}$).

In [12] several Ramsey numbers of star-like trees versus large odd wheels were established, e.g., it was shown that $R(S_n, W_m) = 3n - 2$ for $n \geq 2m - 4$, $m \geq 5$ and m odd.

More information about the Ramsey numbers of other graph combinations can be found in [8].

2. Main Results

The aim of this paper is to determine the Ramsey number of a cycle C_n versus W_4 or W_5. We will show that $R(C_n, W_4) = 2n - 1$ and $R(C_n, W_5) = 3n - 2$ for $n \geq 5$.

For given graphs G and H, Chvátal and Harary [3] established the lower bound

$R(G,H) \geq (c(G)-1)(\chi(H)-1)+1$, where $c(G)$ is the number of vertices of the largest component of G and $\chi(H)$ is the chromatic number of H. In particular, if $n \geq 5$, $G = C_n$ and $H = W_4$ or W_5, then we have $R(C_n, W_4) \geq 2n-1$ and $R(C_n, W_5) \geq 3n-2$, respectively.

For the upper bounds we will present proofs by induction. In order to prove the main results of this paper, we need the following known results and lemmas.

Theorem 1 (Ore [7]).
If G is a graph of order $n \geq 3$ such that for all distinct nonadjacent vertices u and v, $d(u) + d(v) \geq n$, then G is hamiltonian.

Theorem 2 (Faudree and Schelp [4]; Rosta [10]).

$$R(C_n, C_m) = \begin{cases} 2n-1 \text{ for } 3 \leq m \leq n,\ m \textbf{ odd},\ (n,m) \neq (3,3). \\ n + \frac{m}{2} - 1 \text{ for } 4 \leq m \leq n,\ m \textbf{ even} \text{ and } n \textbf{ even},\ (n,m) \neq (4,4). \\ \max\{n + \frac{m}{2} - 1, 2m-1\} \text{ for } 4 \leq m < n,\ m \textbf{ even} \text{ and } n \textbf{ odd}. \end{cases}$$

Lemma 1 (Chvátal and Erdős [2]; Zhou [15]).
If $H = C_s \subseteq F$ for a graph F, while $F \not\supseteq C_{s+1}$ and $\overline{F} \not\supseteq K_r$, then $|N_H(x)| \leq r-2$ for each $x \in V(F)\backslash V(H)$.

Lemma 2 *Let F be a graph with $|V(F)| \geq R(C_n, C_m)+1$. If there is a vertex $x \in V(F)$ such that $|N_F[x]| \leq |V(F)| - R(C_n, C_m)$ and $F \not\supseteq C_n$, then $\overline{F} \supseteq W_m$.*

Proof. Let $A = V(F)\backslash N_F[x]$ and so $|A| \geq R(C_n, C_m)$. If the subgraph $F[A]$ of F induced by A contains no C_n, then by the definition of $R(C_n, C_m)$ we get that $\overline{F}[A]$ contains a C_m and hence $\overline{F}$ contains a W_m (with hub x). □

Lemma 3 *Let F and G be graphs with $2n-1$ and $3n-2$ vertices without a C_n, respectively. If $\overline{F}$ and $\overline{G}$ contain no W_m, then $\delta(F) \geq n - \frac{m}{2}$ for* **even** *$m \geq 4$ and $n \geq \frac{3m}{2}$, and $\delta(G) \geq n-1$ for* **odd** *$m \geq 5$ and $n \geq m$.*

Proof. By contraposition. Suppose $\delta(F) < n - \frac{m}{2}$ for $m \geq 4$ even and $n \geq \frac{3m}{2}$. Then, there exists a vertex $x \in V(F)$ such that $|N_F[x]| = d_F(x)+1 = \delta(F)+1 \leq n - \frac{m}{2} = (2n-1) - (n + \frac{m}{2} - 1)$. Using Theorem 2 we get that $N_F[x] \leq |V(F)| - R(C_n, C_m)$. By Lemma 2, we conclude that $\overline{F}$ contains a W_m with hub x.

Now, suppose $\delta(G) < n-1$ for m odd and $n \geq m$. Then, similarly, using Theorem 2 there exists a vertex $y \in V(G)$ such that $|N_G[y]| \leq n-1 = (3n-2)-(2n-1) = |V(G)| - R(C_n, C_m)$. By Lemma 2, we conclude that $\overline{F}$ contains a W_m with hub y. □

Before we deal with the general case of a cycle and W_4, we will first separately prove that $R(C_6, W_4) = 11$ and $R(C_7, W_4) = 13$.

Theorem 3 $R(C_6, W_4) = 11$.

Proof. Let F be a graph on 11 vertices containing no C_6. We will show that $\overline{F}$ contains W_4. To the contrary, assume $\overline{F}$ contains no W_4. It is known from [5] that $R(C_5, W_4) = 9$, implying that F contains C_5. Let $A = \{x_0, x_1, x_2, x_3, x_4\}$ be the set of vertices of $C_5 \subseteq F$ in a cyclic ordering, and let $B = V(F)\backslash A$. Then $|B| = 6$. By Theorem 1, there exists a vertex $b \in B$ such that $|N_B(b)| \leq 2$, since otherwise $F[B]$, and hence F, contains C_6. By Lemma 3, $\delta(F) \geq 6 - \frac{4}{2} = 4$, implying that $|N_A(b)| \geq 2$. If b is adjacent to x_i and x_{i+1} (indices modulo 5), then clearly $C_6 \subseteq F$. So we may assume without loss of generality that $N_A(b) = \{x_1, x_3\}$. Let $\{b_1, b_2, b_3\}$ denote the three vertices of $B \setminus N_B(b)$. Our next observation is that $x_2x_4 \notin E(F)$; otherwise we obtain a C_6 with edge set $(E(C_5) \setminus \{x_1x_2, x_3x_4\}) \cup \{x_1b, bx_3, x_2x_4\}$. Similarly, $x_0x_2 \notin E(F)$.

Since F contains no C_6, we have $|N_{\{b_1,b_2\}}(x_i) \cap N_{\{b_1,b_2\}}(x_j)| = 0$ for $i = 0, 2, 4$ and $i \neq j$. This implies that there exists an x_i ($i \in \{0, 2, 4\}$) with no neighbor in $\{b_1, b_2\}$, say x_4. Since $\overline{F}$ contains no W_4, x_0 must be adjacent to both b_1 and b_2. This implies that x_2 has no neighbor in $\{b_1, b_2\}$; otherwise F contains a C_6. Thus $\overline{F}$ contains a W_4 with hub b and rim $b_1x_4b_2x_2b_1$, our final contradiction. □

Theorem 4 $R(C_7, W_4) = 13$.

Proof. Let F be a graph on 13 vertices containing no C_7. We will show that $\overline{F}$ contains W_4. To the contrary, assume $\overline{F}$ contains no W_4. By the previous result, we know that F contains C_6. Let $A = \{x_0, x_1, x_2, x_3, x_4, x_5\}$ be the set of vertices of $C_6 \subseteq F$ in a cyclic ordering, and let $B = V(F)\backslash A$. Then $|B| = 7$. By Theorem 1, there exists a vertex $b \in B$ such that $|N_B(b)| \leq 3$, since otherwise $F[B]$ and hence F contains C_7. By Lemma 3, $\delta(F) \geq 7 - \frac{4}{2} = 5$, implying that $|N_A(b)| \geq 2$. If b is adjacent to x_i and x_{i+1} (indices modulo 5), then clearly $C_7 \subseteq F$. Now we distinguish three cases.

Case 1: *b has two neighbors in A at distance 3 along the C_6.*
We may assume without loss of generality that $N_A(b) = \{x_1, x_4\}$. Let b_1, b_2, b_3 denote three vertices of $B \setminus N_B(b)$. As in the proof of Theorem 3, we observe that $x_0x_3 \notin E(F)$; otherwise we obtain a C_7. Similarly, $x_2x_5 \notin E(F)$. Now one of x_0x_2, x_3x_5 is an edge of F; otherwise we obtain a W_4 in $\overline{F}$ with hub b and rim $x_0x_3x_5x_2x_0$. We next observe that precisely one of these edges exists in F; otherwise $x_0x_2x_3x_5x_4bx_1x_0$ is a C_7 in F. We may assume without loss of generality that $x_0x_2 \in E(F)$ and $x_3x_5 \notin E(F)$. Since $x_0x_3, x_3x_5 \notin E(F)$, at least one of x_0 and x_5 is a neighbor of b_i in F ($i = 1, 2, 3$). Suppose $x_0b_1, x_0b_2 \in E(F)$. Since there is no C_7 in F, we easily get that $x_5b_1, x_5b_2 \notin E(F)$. Now at least one of x_2b_1, x_2b_2 is an edge of F; otherwise we obtain a W_4 in $\overline{F}$ as in the proof of Theorem 3. But then $x_0b_ix_2x_3x_4bx_1x_0$ is a C_7 in F for $i = 1$ or $i = 2$, a contradiction. Since we do not use the edge x_0x_2 in the last arguments, the case that $x_5b_1, x_5b_2 \in E(F)$ is symmetric. This completes Case 1.

Case 2: *b has three neighbors in A.*
We may assume without loss of generality that $N_A(b) = \{x_1, x_3, x_5\}$. Let b_1, b_2, b_3 denote three vertices of $B \setminus N_B(b)$. As in the proof of Theorem 3, we observe that $x_0x_2 \notin E(F)$; otherwise we obtain a C_7. Similarly, $x_2x_4, x_4x_0 \notin E(F)$. Since $x_0x_2, x_2x_4 \notin E(F)$, at least one of x_0 and x_4 is a neighbor of b_i in F $(i = 1, 2, 3)$. Suppose by symmetry that $x_0b_1, x_0b_2 \in E(F)$. Similarly, at least one of $x_2b_1, x_4b_1 \in E(F)$. By symmetry and possibly reversing the orientation of the C_6, we may assume $x_2b_1 \in E(F)$. Clearly, $b_1x_1, b_1x_3, b_1x_5, b_2x_1, b_2x_5, x_1x_3, x_1x_5 \notin E(F)$. Also $x_3x_5 \notin E(F)$; otherwise $x_5x_3bx_1x_2b_1x_0x_5$ is a C_7 in F. Now $b_1b_2 \in E(F)$; otherwise we obtain a W_4 in $\overline{F}$ with hub b_1 and rim $b_2x_1x_3x_5b_2$. We conclude that $x_0b_2b_1x_2x_3x_4x_5x_0$ is a C_7 in F. This completes Case 2.

Case 3: *b has exactly two neighbors in A at distance 2 along the C_6.*
We may assume without loss of generality that $N_A(b) = \{x_1, x_3\}$. Let b_1, b_2, b_3 denote vertices of $B \setminus N_B(b)$. As in the proof of Theorem 3, we observe that $x_0x_2 \notin E(F)$; otherwise we obtain a C_7. Similarly, $x_2x_4 \notin E(F)$. Since $x_0x_2, x_2x_4 \notin E(F)$, at least one of x_0 and x_4 is a neighbor of b_1 in F. Suppose by symmetry that $x_0b_1 \in E(F)$.

Since $x_0x_2, x_2x_4 \notin E(F)$ and $\overline{F}$ contains no W_4, by the Pigeonhole Principle, there exists an $x \in \{x_0, x_4\}$ such that x is adjacent to at least two vertices in $\{b_1, b_2, b_3\}$. Let x_0 be adjacent to b_1 and b_2. If $x_1x_5 \in E(F)$, then x_2 and x_4 are not adjacent to b_1 and b_2, since otherwise F contains a C_7, so $\overline{F}$ contains a W_4 with hub b and rim $b_1x_4b_2x_2b_1$. In case $x_1x_5 \notin E(F)$, we get that $x_5b \in E(F)$, since otherwise we have a W_4 in $\overline{F}$ with hub x_5 and rim $b_1x_1b_2bb_1$. The case is now similar to Case 2. This completes Case 3 and the proof of Theorem 4. □

Lemma 4 *Let F be a graph on $2n-1$ vertices with $n \geq 8$, and suppose $\overline{F}$ contains no W_4. If $C_{n-1} \subseteq F$ and $F \not\supseteq C_n$, then $|N_{\mathcal{A}}(x)| \leq 2$ for each $x \in V(F)\backslash\mathcal{A}$, where $\mathcal{A} = V(C_{n-1})$.*

Proof. Let $\mathcal{A} = \{x_1, x_2, ..., x_{n-1}\}$ be the set of vertices of a cycle C_{n-1} in F in a cyclic ordering, and let $\mathcal{B} = V(F)\backslash\mathcal{A}$. Suppose there exists a vertex $b_1 \in \mathcal{B}$ with $|N_{\mathcal{A}}(b_1)| \geq 3$. Clearly, $b_1x_{i+1} \notin E(F)$ whenever $b_1x_i \in E(F)$ (indices modulo $n-1$). Since $n \geq 8$, $|\mathcal{A}| \geq 7$, and hence we can choose two neighbors x_i and x_j of b_1 in $\mathcal{A}$ such that $x_{i+1} \neq x_{j-1}$ and $x_{i-1} \neq x_{j+1}$ (indices modulo $n-1$). Let $A = \{x_{i-1}, x_{i+1}, x_{j-1}, x_{j+1}\}$. Then $|A| = 4$ and $xb_1 \notin E(F)$ for each $x \in A$. Moreover, since F contains no C_n, by standard long cycle arguments $x_{i-1}x_{j-1}, x_{i+1}x_{j+1} \notin E(F)$, If $|N_A(x)| \leq 1$ for all $x \in A$, then in $\overline{F}$ all vertices of A have at least $2 = \frac{1}{2}|A|$ neighbors, implying that $\overline{F}$ contains a W_4 with hub b_1. Hence $|N_A(x)| \geq 2$ for some $x \in A$. By symmetry, considering the two possible orientations of C_{n-1}, we may assume without loss of generality that $|N_A(x_{i+1})| \geq 2$, hence $x_{i-1}x_{i+1}, x_{i+1}x_{j-1} \in E(F)$. Then $x_ix_{j-1} \notin E(F)$; otherwise we can obtain a C_n from $E(C_{n-1}) \setminus \{x_{j-1}x_j, x_ix_{i+1}, x_{i-1}x_i\} \cup \{x_jb_1, b_1x_i, x_ix_{j-1}\}$. Similarly, $x_ix_{j+1} \notin E(F)$. Since $\delta(F) \geq n-2$ by Lemma 3 and $|N_{\mathcal{A}}(b)| \leq 5-2 = 3$ for each $b \in \mathcal{B}$ by Lemma 1, there exist distinct vertices $b_2, b_3 \in \mathcal{B}$ such that $b_1b_2, b_1b_3 \in E(F)$. This implies that x_{j-1} and x_{j+1} are not adjacent to any vertex in $\{b_2, b_3\}$ since otherwise F contains a C_n (extending the C_{n-1} by including b_1 and b_2 or b_3, while skipping x_i). Now, we will distinguish the following two cases.

Case 1: $x_{j-1}x_{j+1} \notin E(F)$.
Since $\overline{F}$ contains no W_4, $x_tb_2, x_tb_3 \in E(F)$ for each $t \in \{i-1, i+1\}$. Suppose to the contrary, e.g., that $x_{i-1}b_2 \notin E(F)$. Then $\overline{F}$ contains a W_4 with hub x_{j-1} and rim $\{x_{i-1}, b_2, x_{j+1}, b_1\}$. The other cases are symmetric. See Figure 1.

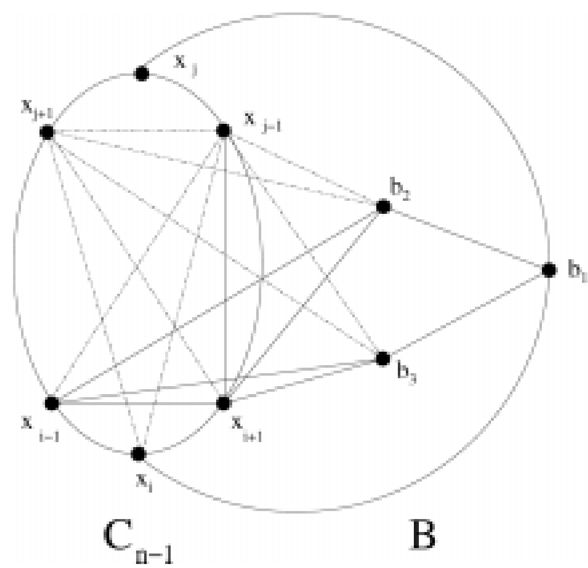

Figure 1: The proof of Lemma 4 for Case 1.

Clearly then $x_ib_2, x_ib_3 \notin E(F)$ since $F \not\supseteq C_n$. Thus, we have a W_4 in $\overline{F}$ with hub x_i and rim $\{x_{j-1}, b_2, x_{j+1}, b_3\}$, a contradiction.

Case 2: $x_{j-1}x_{j+1} \in E(F)$.
If $b_2x_{i-1} \in E(F)$, then we obtain a C_n in F with edge set
$E(C_{n-1}) \setminus \{x_{j-1}x_j, x_jx_{j+1}, x_{i-1}x_i\} \cup \{x_{i-1}b_2, b_2b_1, b_1x_i, x_{j-1}x_{j+1}\}$.
Hence $b_2x_{i-1} \notin E(F)$. Similarly, $b_2x_{i+1}, b_3x_{i-1}, b_3x_{i+1} \notin E(F)$. If $x_jx_{i-1} \in E(F)$, we obtain a C_n with edge set
$E(C_{n-1}) \setminus \{x_jx_{j+1}, x_{j-1}x_j, x_{i-1}x_i\} \cup \{x_jb_1, b_1x_i, x_{j-1}x_{j+1}\}$.
Hence, by symmetry, $x_jx_{i-1}, x_jx_{i+1} \notin E(F)$. Since $\overline{F}$ contains no W_4 (with hub x_i and rim $\{x_{j+1}, b_2, x_{j-1}, b_3\}$), x_i is adjacent to a vertex in $\{b_2, b_3\}$. Without loss of generality, let $x_ib_2 \in E(F)$. Since $\delta(F) \geq n-2$ by Lemma 3, x_{i+1} must be adjacent to two vertices in $\mathcal{B} \setminus \{b_1, b_2, b_3\}$. Let $x_{i+1}b_4, x_{i+1}b_5 \in E(F)$ for $b_4, b_5 \in \mathcal{B}$. By similar arguments as before, $C_n \not\subseteq F$ implies $b_1b, b_2b \notin E(F)$ for each $b \in \{b_4, b_5\}$. Suppose $b_4x_{i-1} \notin E(F)$. Then we have a W_4 in $\overline{F}$ with hub x_{i-1} and rim $\{b_4, b_1, x_{j-1}, b_2\}$. Similar case analyses

show that $b_4x, b_5x \in E(F)$ for each $x \in \{x_{i-1}, x_{j-1}\}$. Since F contains no C_n, we clearly have $b_4b_5 \notin E(F)$, and also $x_ix_j \notin E(F)$ (otherwise consider $E(C_{n-1}) \setminus \{x_{j-1}x_j, x_{i-1}x_i\} \cup \{x_ix_j, x_{i-1}b_4, b_4x_{j-1}\}$). Since $\delta(F) \geq n-2$ by Lemma 3, there exists a vertex $b_6 \in \mathcal{B} \setminus \{b_1, \ldots, b_5\}$ such that $b_4b_6 \in E(F)$. This clearly implies $b_6x_i, b_6x_j, b_6b_5 \notin E(F)$. See Figure 2.

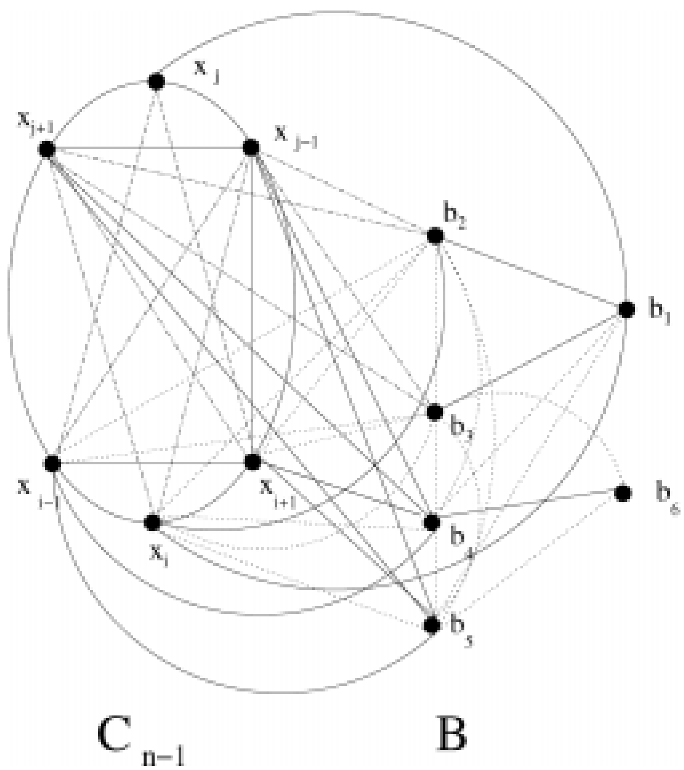

Figure 2: The proof of Lemma 4 for Case 2.

Thus, $\overline{F}$ contains a W_4 with hub b_5 and rim $\{x_i, b_6, x_j, b_4\}$, a contradiction. This completes the proof. □

Theorem 5 *$R(C_n, W_4) = 2n-1$ for $n \geq 5$.*

Proof. We use induction on $n \geq 5$. We already know that $R(C_n, W_4) \geq 2n-1$ for $n \geq 5$. For $n = 5$, 6, and 7, we respectively know from [5], Theorem 3, and Theorem 4 that $R(C_n, W_4) = 2n-1$. Now assume that $R(C_n, W_4) = 2n-1$ for $n < k$ with $k \geq 8$ and let F be a graph on $2k-1$ vertices containing no C_k. We shall show that $\overline{F}$ contains W_4. To the contrary, assume $\overline{F}$ contains no W_4. By the induction hypothesis, we have $F \supseteq C_{k-1}$. Let $A = V(C_{k-1})$, $B = V(F) \backslash V(C_{k-1})$ and so $|B| = k$. By Lemma 4, we have $|N_A(x)| \leq 2$ for each $x \in B$. Since by Lemma 3, $\delta(F) \geq k-2$, we obtain $|N_B(x)| \geq k-2-2 = k-4 \geq \frac{1}{2}k = \frac{1}{2}|B|$ for all $x \in B$. Now $F[B]$ and hence F contains a C_k by Theorem 1, a contradiction. This completes the proof. □

Theorem 6 *$R(C_n, W_5) = 3n - 2$ for $n \geq 5$.*

Proof. We use induction on n. We already know that $R(C_n, W_5) \geq 3n-2$ for $n \geq 5$. For $n = 5$, we know from [6] that $R(C_5, W_5) = 3.5-2$. Assume the theorem holds for $n < k$ with $k \geq 6$ and let F be a graph on $3k-2$ vertices containing no C_k. We shall show that $\overline{F}$ contains W_5. To the contrary, assume that $\overline{F}$ contains no W_5. Consequently, F must contain a C_{k-1}, and we let $A = \{a_1, a_2, ..., a_{k-1}\}$ denote the set of vertices of a cycle C_{k-1} in F, in a cyclic ordering. Let $B = V(F)\backslash A$, so $|B| = 2k-1$. Then, by Theorem 5, the complement of the subgraph $F[B]$ of F induced by B must contain a W_4. Let x_0 be the hub and $X = \{x_1, x_2, x_3, x_4\}$ be the rim of a W_4 in $\overline{F}[B]$. We distinguish the following cases.

Case 1: *k is* **even**.
Since F contains no C_k, within F: $|N_A(z)| \leq \lfloor \frac{k-1}{2} \rfloor$ for each $z \in B$. This implies that there exist vertices $a_j, a_{j+1} \in A$ for some $j \in \{1, 2, ..., k-1\}$ such that $a_j x_0, a_{j+1} x_0 \notin E(F)$. No C_k in F also implies $N_X(a_j) \cap N_X(a_{j+1}) = \emptyset$. No W_5 in $\overline{F}$ implies in F: $|N_X(a_j)| \geq 2$ and $|N_X(a_{j+1})| \geq 2$, and without loss of generality we may assume a_j is adjacent to x_1 and x_3, and a_{j+1} is adjacent to x_2 and x_4. This implies $x_1x_3, x_2x_4, x_0a_{j+2}, x_0a_{j-1} \in E(F)$ since otherwise $\overline{F} \supseteq W_5$ (Note that $F \not\supseteq C_k$ implies neither of a_{j-1} and a_{j+2} is adjacent to a vertex in X). Since F contains no C_k, it is not difficult to check $x_0a_{j-2}, a_{j-2}x_1, a_{j+1}a_{j-2} \notin E(F)$. This implies $\overline{F} \supseteq W_5$ with hub x_0 and rim $\{x_3, a_{j+1}, a_{j-2}, x_1, x_2\}$, a contradiction.

Case 2: *k is* **odd**.
We may assume $a_i x_0 \in E(F)$ for each odd $i \in \{1, 2, ..., k-1\}$, since otherwise we can use the same arguments as in the first case. Since F contains no C_k, $a_j a_h \notin E(F)$ for all even $j, h \in \{1, 2, ..., k-1\}$. If $k \geq 11$, we have K_6 in $\overline{F}$ which implies $\overline{F} \supseteq W_5$, a contradiction. Now assume $7 \leq k < 11$. In F we have $|N_X(a_j)| \geq 2$ for all even $j \in \{1, 2, ..., k-1\}$, since otherwise $\overline{F} \supseteq W_5$. By the same token, we may assume without loss of generality that a_j is adjacent to x_1 and x_3 for some even $j \in \{1, 2, ..., k-1\}$. We distinguish two subcases.

Subcase 2.1: *x_1 is adjacent to x_3.*
Then x_1 and x_3 are not adjacent to any vertex in $\{a_{j-1}, a_{j-2}, a_{j+1}, a_{j+2}\}$, since otherwise F clearly contains a C_k. Thus, we get $\overline{F} \supseteq W_5$ with hub x_0 and rim $\{x_3, a_{j+2}, a_{j-2}, x_1, x_2\}$, a contradiction.

Subcase 2.2: *x_1 is not adjacent to x_3.*
This implies x_2 and x_4 are adjacent to all vertices in $\{a_{j-1}, a_{j+1}\}$, since otherwise $\overline{F} \supseteq W_5$. Suppose, e.g., $x_2a_{j-1} \notin E(F)$. Then $\overline{F} \supseteq W_5$ with hub x_1 and rim $\{a_{j-1}, x_2, x_0, x_3, a_{j+1}\}$; the other cases are similar. Thus, we get $x_2a_j, x_4a_{j+2} \notin E(F)$; otherwise a C_k in F is immediate. Thus, we get $\overline{F} \supseteq W_5$ with hub x_0 and rim $\{x_4, a_{j+2}, a_j, x_2, x_3\}$, our final contradiction.

This completes the proof. □

3. Problem

We conclude the paper with the following open problem:

Find the Ramsey number $R(C_n, W_m)$ for $n \geq m \geq 6$.

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