THE RAMSEY NUMBERS OF LARGE_CYCLES VERSUS SMALL_WHEELS

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Abstract

For two given graphs G and H, the Ramsey number R(G, H) is the smallest positive integer N such that for every graph F of order N the following holds: either F contains G as a subgraph or the complement of F contains H as a subgraph. In this paper, we determine the Ramsey number R(Cₙ,Wₘ) for m = 4 and m = 5. We show that R(Cₙ,W₄) = 2n − 1 and R(Cₙ,W₅) = 3n − 2 for n ≥ 5. For larger wheels it remains an open problem to determine R(Cₙ,Wₘ).

1. Introduction

Throughout the paper, all graphs are finite and simple. Let G be such a graph. We write V(G) or V for the vertex set of G and E(G) or E for the edge set of G. The graph G is the complement of the graph G, i.e., the graph obtained from the complete graph K|V(G)| on |V(G)| vertices by deleting the edges of G.

The graph H = (V', E') is a subgraph of G = (V, E) if V' ⊆ V and E' ⊆ E. For any nonempty subset S ⊆ V, the induced subgraph by S is the maximal subgraph of G with vertex set S; it is denoted by G[S].

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If \( e = \{u, v\} \in E \) (in short, \( e = uv \)), then \( u \) is called \textit{adjacent to} \( v \), and \( u \) and \( v \) are called \textit{neighbors}. For \( x \in V \) and a subgraph \( B \) of \( G \), define \( N_B(x) = \{y \in V(B) : xy \in E\} \) and \( N_B[x] = N_B(x) \cup \{x\} \). The \textit{degree} \( d(x) \) of a vertex \( x \) is \(|N_G(x)|\); \( \delta(G) \) denotes the minimum degree in \( G \).

A \textit{cycle} \( C_n \) of length \( n \geq 3 \) is a connected graph on \( n \) vertices in which every vertex has degree two. A \textit{wheel} \( W_n \) is a graph on \( n + 1 \) vertices obtained from a \( C_n \) by adding one vertex \( x \), called the \textit{hub} of the wheel, and making \( x \) adjacent to all vertices of the \( C_n \), called the \textit{rim} of the wheel.

Given two graphs \( G \) and \( H \), the \textit{Ramsey number} \( R(G, H) \) is defined as the smallest natural number \( N \) such that every graph \( F \) on \( N \) vertices satisfies the following condition: \( F \) contains \( G \) as a subgraph or \( \overline{F} \) contains \( H \) as a subgraph.

We will also use the short notations \( H \subseteq F \), \( F \supseteq H \), \( H \nsubseteq F \), and \( F \nsubseteq H \) to denote that \( H \) is (not) a subgraph of \( F \), with the obvious meanings.

Several results have been obtained for wheels. For instance, Burr and Erdős [1] showed that \( R(C_3, W_m) = 2m + 1 \) for each \( m \geq 5 \).

Ten years later Radziszowski and Xia [9] gave a simple and unified method to establish the Ramsey number \( R(G, C_3) \), where \( G \) is either a path, a cycle or a wheel.

Hendry [5] showed \( R(C_5, W_4) = 9 \). Jayawardane and Rousseau [6] showed \( R(C_5, W_5) = 11 \). Surahmat et al. [13] showed \( R(C_4, W_m) = 9, 10 \) and \( 9 \) for \( m = 4, 5 \) and \( 6 \) respectively. Independently, Tse [14] showed \( R(C_4, W_m) = 9, 10, 9, 11, 12, 13, 14, 15 \) and \( 17 \) for \( m = 4, 5, 6, 7, 8, 9, 10, 11 \) and \( 12 \), respectively.

Recently, in [11], it was shown that the Ramsey number \( R(S_n, W_4) = 2n - 1 \) if \( n \geq 3 \) and \( n \) is odd, \( R(S_n, W_4) = 2n + 1 \) if \( n \geq 4 \) and \( n \) is even, and \( R(S_n, W_5) = 3n - 2 \) for each \( n \geq 3 \). Here \( S_n \) denotes a star on \( n \) vertices (i.e., \( S_n = K_{1,n-1} \)).

In [12] several Ramsey numbers of star-like trees versus large odd wheels were established, e.g., it was shown that \( R(S_n, W_m) = 3n - 2 \) for \( n \geq 2m - 4 \), \( m \geq 5 \) and \( m \) odd.

More information about the Ramsey numbers of other graph combinations can be found in [8].

2. Main Results

The aim of this paper is to determine the Ramsey number of a cycle \( C_n \) versus \( W_4 \) or \( W_5 \). We will show that \( R(C_n, W_4) = 2n - 1 \) and \( R(C_n, W_5) = 3n - 2 \) for \( n \geq 5 \).

For given graphs \( G \) and \( H \), Chvátal and Harary [3] established the lower bound...
Let $R(G,H) \geq (c(G)-1)(\chi(H)-1)+1$, where $c(G)$ is the number of vertices of the largest component of $G$ and $\chi(H)$ is the chromatic number of $H$. In particular, if $n \geq 5$, $G = C_n$ and $H = W_4$ or $W_5$, then we have $R(C_n,W_4) \geq 2n-1$ and $R(C_n,W_5) \geq 3n-2$, respectively.

For the upper bounds we will present proofs by induction. In order to prove the main results of this paper, we need the following known results and lemmas.

**Theorem 1 (Ore [7]).**
If $G$ is a graph of order $n \geq 3$ such that for all distinct nonadjacent vertices $u$ and $v$, $d(u)+d(v) \geq n$, then $G$ is hamiltonian.

**Theorem 2 (Faudree and Schelp [4]; Rosta [10]).**

$$R(C_n,C_m) = \begin{cases} 
2n-1 & \text{for } 3 \leq m \leq n, \text{ } m \text{ odd, } (n,m) \neq (3,3). \\
n + \frac{m}{2} - 1 & \text{for } 4 \leq m \leq n, \text{ } m \text{ even and } n \text{ even, } (n,m) \neq (4,4). \\
\max\{n + \frac{m}{2} - 1, 2m - 1\} & \text{for } 4 \leq m < n, \text{ } m \text{ even and } n \text{ odd.}
\end{cases}$$

**Lemma 1 (Chvátal and Erdős [2]; Zhou [15]).**
If $H = C_s \subseteq F$ for a graph $F$, while $F \not\supseteq C_{s+1}$ and $\overline{F} \not\supseteq K_r$, then $|N_H(x)| \leq r - 2$ for each $x \in V(F) \setminus V(H)$.

**Lemma 2** Let $F$ be a graph with $|V(F)| \geq R(C_n,C_m)+1$. If there is a vertex $x \in V(F)$ such that $|N_H(x)| \leq |V(F)| - R(C_n,C_m)$ and $F \not\supseteq C_n$, then $\overline{F} \supseteq W_m$.

**Proof.** Let $A = V(F) \setminus N_F[x]$ and so $|A| \geq R(C_n,C_m)$. If the subgraph $F[A]$ of $F$ induced by $A$ contains no $C_n$, then by the definition of $R(C_n,C_m)$ we get that $\overline{F[A]}$ contains a $C_m$ and hence $\overline{F}$ contains a $W_m$ (with hub $x$).

**Lemma 3** Let $F$ and $G$ be graphs with $2n-1$ and $3n-2$ vertices without a $C_n$, respectively. If $\overline{F}$ and $\overline{G}$ contain no $W_m$, then $\delta(F) \geq n - \frac{m}{2}$ for even $m \geq 4$ and $n \geq \frac{3m}{2}$, and $\delta(G) \geq n - 1$ for odd $m \geq 5$ and $n \geq m$.

**Proof.** By contraposition. Suppose $\delta(F) < n - \frac{m}{2}$ for $m \geq 4$ even and $n \geq \frac{3m}{2}$. Then, there exists a vertex $x \in V(F)$ such that $|N_F[x]| = d_F(x) + 1 = \delta(F) + 1 \leq n - \frac{m}{2} = (2n-1) - (n + \frac{m}{2} - 1)$. Using Theorem 2 we get that $|N_F[x]| \leq |V(F)| - R(C_n,C_m)$. By Lemma 2, we conclude that $\overline{F}$ contains a $W_m$ with hub $x$.

Now, suppose $\delta(G) < n - 1$ for $m$ odd and $n \geq m$. Then, similarly, using Theorem 2 there exists a vertex $y \in V(G)$ such that $|N_G[y]| \leq n - 1 = (3n - 2) - (2n - 1) = |V(G)| - R(C_n,C_m)$. By Lemma 2, we conclude that $\overline{F}$ contains a $W_m$ with hub $y$.

Before we deal with the general case of a cycle and $W_4$, we will first separately prove that $R(C_6,W_4) = 11$ and $R(C_7,W_4) = 13$. 
Theorem 3 \( R(C_6, W_4) = 11 \).

Proof. Let \( F \) be a graph on 11 vertices containing no \( C_6 \). We will show that \( F \) contains \( W_4 \). To the contrary, assume \( F \) contains no \( W_4 \). It is known from [5] that \( R(C_5, W_4) = 9 \), implying that \( F \) contains \( C_5 \). Let \( A = \{ x_0, x_1, x_2, x_3, x_4 \} \) be the set of vertices of \( C_5 \subseteq F \) in a cyclic ordering, and let \( B = V(F) \setminus A \). Then \( |B| = 6 \). By Theorem 1, there exists a vertex \( b \in B \) such that \( |N_B(b)| \leq 2 \), since otherwise \( F[B] \), and hence \( F \), contains \( C_6 \). By Lemma 3, \( \delta(F) \geq 6 - \frac{4}{3} = 4 \), implying that \( |N_A(b)| \geq 2 \). If \( b \) is adjacent to \( x_i \) and \( x_{i+1} \) (indices modulo 5), then clearly \( C_6 \subseteq F \). So we may assume without loss of generality that \( N_A(b) = \{ x_1, x_3 \} \). Let \( \{ b_1, b_2, b_3 \} \) denote the three vertices of \( B \setminus N_B(b) \). Our next observation is that \( x_2x_4 \notin E(F) \); otherwise we obtain a \( C_6 \) with edge set \( 6(E) \setminus \{ x_2x_3, x_3x_4 \} \subseteq \{ x_1b, b_2x_3, x_4x_2 \} \). Similarly, \( x_0x_2 \notin E(F) \).

Since \( F \) contains no \( C_6 \), we have \( |N_{\{b_1,b_2\}}(x_i) \setminus N_{\{b_1,b_2\}}(x_j)| = 0 \) for \( i = 0, 2, 4 \) and \( i \neq j \). This implies that there exists an \( x_i \) (\( i \in \{0, 2, 4\} \)) with no neighbor in \( \{b_1, b_2\} \), say \( x_0 \). Since \( F \) contains no \( W_4 \), \( x_0 \) must be adjacent to both \( b_1 \) and \( b_2 \). This implies that \( x_2 \) has no neighbor in \( \{b_1, b_2\} \); otherwise \( F \) contains a \( C_6 \). Thus \( F \) contains a \( W_4 \) with hub \( b \) and rim \( b_1x_4b_2x_2b_1 \), our final contradiction.

Theorem 4 \( R(C_7, W_4) = 13 \).

Proof. Let \( F \) be a graph on 13 vertices containing no \( C_7 \). We will show that \( F \) contains \( W_4 \). To the contrary, assume \( F \) contains no \( W_4 \). By the previous result, we know that \( F \) contains \( C_6 \). Let \( A = \{ x_0, x_1, x_2, x_3, x_4, x_5 \} \) be the set of vertices of \( C_6 \subseteq F \) in a cyclic ordering, and let \( B = V(F) \setminus A \). Then \( |B| = 7 \). By Theorem 1, there exists a vertex \( b \in B \) such that \( |N_B(b)| \leq 3 \), since otherwise \( F[B] \) and hence \( F \) contains \( C_7 \). By Lemma 3, \( \delta(F) \geq 7 - \frac{4}{3} = 5 \), implying that \( |N_A(b)| \geq 2 \). If \( b \) is adjacent to \( x_i \) and \( x_{i+1} \) (indices modulo 5), then clearly \( C_7 \subseteq F \). Now we distinguish three cases.

Case 1: \( b \) has two neighbors in \( A \) at distance 3 along the \( C_6 \).
We may assume without loss of generality that \( N_A(b) = \{ x_1, x_4 \} \). Let \( b_1, b_2, b_3 \) denote three vertices of \( B \setminus N_B(b) \). As in the proof of Theorem 3, we observe that \( x_0x_3 \notin E(F) \); otherwise we obtain a \( C_7 \). Similarly, \( x_2x_5 \notin E(F) \). Now one of \( x_0x_2, x_3x_5 \) is an edge of \( F \); otherwise we obtain a \( W_4 \) in \( F \) with hub \( b \) and rim \( x_0x_3x_5x_2x_0 \). We next observe that precisely one of these edges exists in \( F \); otherwise \( x_0x_2x_3x_5x_4bx_1x_0 \) is a \( C_7 \) in \( F \). We may assume without loss of generality that \( x_0x_2 \in E(F) \) and \( x_3x_5 \notin E(F) \). Since \( x_0x_3, x_2x_5 \notin E(F) \), at least one of \( x_0 \) and \( x_5 \) is a neighbor of \( b_i \) in \( F \) (\( i = 1, 2, 3 \)). Suppose \( x_0b_1, x_0b_2 \in E(F) \). Since there is no \( C_7 \) in \( F \), we easily get that \( x_3b_1, x_5b_2 \notin E(F) \). Now at least one of \( x_2b_1, x_2b_2 \) is an edge of \( F \); otherwise we obtain a \( W_4 \) in \( F \) as in the proof of Theorem 3. But then \( x_0b_1x_2x_3x_4bx_1x_0 \) is a \( C_7 \) in \( F \) for \( i = 1 \) or \( i = 2 \), a contradiction. Since we do not use the edge \( x_0x_2 \) in the last arguments, the case that \( x_3b_1, x_5b_2 \in E(F) \) is symmetric. This completes Case 1.
Case 2: b has three neighbors in A.
We may assume without loss of generality that \( N_A(b) = \{x_1, x_3, x_5\} \). Let \( b_1, b_2, b_3 \) denote three vertices of \( B \setminus N_B(b) \). As in the proof of Theorem 3, we observe that \( x_0x_2 \notin E(F) \); otherwise we obtain a \( C_7 \). Similarly, \( x_2x_4, x_4x_0 \notin E(F) \). Since \( x_0x_2, x_2x_4 \notin E(F) \), at least one of \( x_0 \) and \( x_4 \) is a neighbor of \( b_i \) in \( F \) (\( i = 1, 2, 3 \)). Suppose by symmetry that \( x_0b_1, x_0b_2 \in E(F) \). Similarly, at least one of \( x_2b_1, x_4b_1 \in E(F) \). By symmetry and possibly reversing the orientation of the \( C_6 \), we may assume \( x_2b_1 \in E(F) \). Clearly, \( b_1x_1, b_1x_3, b_1x_5, b_2x_1, b_2x_5, x_1x_3, x_1x_5 \notin E(F) \). Also \( x_3x_5 \notin E(F) \); otherwise \( x_3x_5bx_1x_2b_1x_0x_5 \) is a \( C_7 \) in \( F \). Now \( b_1b_2 \in E(F) \); otherwise we obtain a \( W_4 \) in \( F \) with hub \( b_1 \) and rim \( b_2x_1x_3x_5b_2 \). We conclude that \( x_0b_2b_1x_2x_3x_4x_5x_0 \) is a \( C_7 \) in \( F \). This completes Case 2.

Case 3: b has exactly two neighbors in A at distance 2 along the C6.
We may assume without loss of generality that \( N_A(b) = \{x_1, x_3\} \). Let \( b_1, b_2, b_3 \) denote vertices of \( B \setminus N_B(b) \). As in the proof of Theorem 3, we observe that \( x_0x_2 \notin E(F) \); otherwise we obtain a \( C_7 \). Similarly, \( x_2x_4 \notin E(F) \). Since \( x_0x_2, x_2x_4 \notin E(F) \), at least one of \( x_0 \) and \( x_4 \) is a neighbor of \( b_i \) in \( F \). Suppose by symmetry that \( x_0b_1 \in E(F) \).

Since \( x_0x_2, x_2x_4 \notin E(F) \) and \( \overline{F} \) contains no \( W_4 \), by the Pigeonhole Principle, there exists an \( x \in \{x_0, x_4\} \) such that \( x \) is adjacent to at least two vertices in \( \{b_1, b_2, b_3\} \). Let \( x_0 \) be adjacent to \( b_1 \) and \( b_2 \). If \( x_1x_5 \in E(F) \), then \( x_2 \) and \( x_4 \) are not adjacent to \( b_1 \) and \( b_2 \), since otherwise \( F \) contains a \( C_7 \), so \( \overline{F} \) contains a \( W_4 \) with hub \( b \) and rim \( bx_4bx_5 \). In case \( x_1x_5 \notin E(F) \), we get that \( x_3b \in E(F) \), since otherwise we have a \( W_4 \) in \( \overline{F} \) with hub \( x_5 \) and rim \( b_1x_1b_2b_1 \). The case is now similar to Case 2. This completes Case 3 and the proof of Theorem 4.

\[ \square \]

Lemma 4 Let \( F \) be a graph on \( 2n - 1 \) vertices with \( n \geq 8 \), and suppose \( \overline{F} \) contains no \( W_4 \). If \( C_{n-1} \subseteq F \) and \( F \not\supseteq C_n \), then \( |N_A(x)| \leq 2 \) for each \( x \in V(F) \setminus A \), where \( A = V(C_{n-1}) \).

\textbf{Proof.} Let \( A = \{x_1, x_2, ..., x_{n-1}\} \) be the set of vertices of a cycle \( C_{n-1} \) in \( F \) in a cyclic ordering, and let \( B = V(F) \setminus A \). Suppose there exists a vertex \( b_1 \in B \) with \( |N_A(b_1)| \geq 3 \). Clearly, \( b_1x_{i+1} \notin E(F) \) whenever \( b_1x_i \in E(F) \) (indices modulo \( n - 1 \)). Since \( n \geq 8 \), \( |A| \geq 7 \), and hence we can choose two neighbors \( x_i \) and \( x_j \) of \( b_1 \) in \( A \) such that \( x_{i+1} \neq x_{j-1} \) and \( x_{i-1} \neq x_{j+1} \) (indices modulo \( n - 1 \)). Let \( A = \{x_{i-1}, x_{i+1}, x_{j-1}, x_{j+1}\} \). Then \( |A| = 4 \) and \( xb_1 \notin E(F) \) for each \( x \in A \). Moreover, since \( F \) contains no \( C_n \), by standard long cycle arguments \( x_{i-1}x_{j-1}, x_{i+1}x_{j+1} \notin E(F) \). If \( |N_A(x)| \leq 1 \) for all \( x \in A \), then in \( \overline{F} \) all vertices of \( A \) have at least 2 = \( \frac{1}{2} |A| \) neighbors, implying that \( F \) contains a \( W_4 \) with hub \( b_1 \). Hence \( |N_A(x)| \geq 2 \) for some \( x \in A \). By symmetry, considering the two possible orientations of \( C_{n-1} \), we may assume without loss of generality that \( |N_A(x_{i+1})| \geq 2 \), hence \( x_{i-1}x_{i+1}, x_{i+1}x_{j-1} \in E(F) \). Then \( x_{i}x_{j-1} \notin E(F) \); otherwise we can obtain a \( C_n \) from \( E(C_{n-1}) \setminus \{x_{j-1}x_{i}, x_{i}x_{i+1}, x_{i-1}x_{i}\} \cup \{xb_1, b_1x_i, x_ix_{j-1}\} \). Similarly, \( x_i x_{j+1} \notin E(F) \). Since \( \delta(F) \geq n - 2 \) by Lemma 3 and \( |N_A(b)| \leq 5 - 2 = 3 \) for each \( b \in B \) by Lemma 1, there
exist distinct vertices $b_2, b_3 \in \mathcal{B}$ such that $b_1b_2, b_1b_3 \in E(F)$. This implies that $x_{j-1}$ and $x_{j+1}$ are not adjacent to any vertex in $\{b_2, b_3\}$ since otherwise $F$ contains a $C_n$ (extending the $C_{n-1}$ by including $b_1$ and $b_2$ or $b_3$, while skipping $x_i$). Now, we will distinguish the following two cases.

**Case 1:** $x_{j-1}x_{j+1} \notin E(F)$.
Since $\overline{F}$ contains no $W_4$, $x_ib_2, x_ib_3 \in E(F)$ for each $t \in \{i - 1, i + 1\}$. Suppose to the contrary, e.g., that $x_{i-1}b_2 \notin E(F)$. Then $\overline{F}$ contains a $W_4$ with hub $x_{j-1}$ and rim $\{x_{i-1}, b_2, x_{j+1}, b_1\}$. The other cases are symmetric. See Figure 1.

![Figure 1: The proof of Lemma 4 for Case 1.](image)

Clearly then $x_ib_2, x_ib_3 \notin E(F)$ since $F \not\supseteq C_n$. Thus, we have a $W_4$ in $\overline{F}$ with hub $x_i$ and rim $\{x_{j-1}, b_2, x_{j+1}, b_3\}$, a contradiction.

**Case 2:** $x_{j-1}x_{j+1} \in E(F)$.
If $b_2x_{i-1} \in E(F)$, then we obtain a $C_n$ in $F$ with edge set $E(C_{n-1}) \setminus \{x_{j-1}x_j, x_jx_{j+1}, x_{i-1}x_i\} \cup \{x_{i-1}b_2, b_2b_1, b_1x_i, x_{j-1}x_{j+1}\}$.
Hence $b_2x_{i-1} \notin E(F)$. Similarly, $b_2x_{i+1}, b_3x_{i-1}, b_3x_{i+1} \notin E(F)$. If $x_jx_{i-1} \in E(F)$, we obtain a $C_n$ with edge set $E(C_{n-1}) \setminus \{x_jx_{j+1}, x_{j-1}x_j, x_{i-1}x_i\} \cup \{x_jb_1, b_1x_i, x_{j-1}x_{j+1}\}$.
Hence, by symmetry, $x_jx_{i-1}, x_jx_{i+1} \notin E(F)$. Since $\overline{F}$ contains no $W_4$ (with hub $x_i$ and rim $\{x_{j+1}, b_2, x_{j-1}, b_3\}$), $x_i$ is adjacent to a vertex in $\{b_2, b_3\}$. Without loss of generality, let $x_ib_2 \in E(F)$. Since $\delta(F) \geq n - 2$ by Lemma 3, $x_{i+1}$ must be adjacent to two vertices in $\mathcal{B} \setminus \{b_1, b_2, b_3\}$. Let $x_{i+1}b_4, x_{i+1}b_5 \in E(F)$ for $b_4, b_5 \in \mathcal{B}$. By similar arguments as before, $C_n \not\subseteq F$ implies $b_4b_5 \notin E(F)$ for each $b \in \{b_4, b_5\}$. Suppose $b_4x_{i-1} \notin E(F)$.
Then we have a $W_4$ in $\overline{F}$ with hub $x_{i-1}$ and rim $\{b_4, b_1, x_{j-1}, b_2\}$. Similar case analyses
show that $b_4, b_5 \in E(F)$ for each $x \in \{x_{i-1}, x_{j-1}\}$. Since $F$ contains no $C_n$, we clearly have $b_4b_5 \notin E(F)$, and also $x_ix_j \notin E(F)$ (otherwise consider $E(C_{n-1}) \setminus \{x_{j-1}x_j, x_{i-1}x_i\} \cup \{x_ix_j, x_{i-1}b_4, b_4x_{j-1}\}$). Since $\delta(F) \geq n - 2$ by Lemma 3, there exists a vertex $b_6 \in B \setminus \{b_1, \ldots, b_5\}$ such that $b_4b_6 \in E(F)$. This clearly implies $b_6x_i, b_6x_j, b_6b_5 \notin E(F)$. See Figure 2.

![Figure 2](image_url)

Figure 2: The proof of Lemma 4 for Case 2.

Thus, $F$ contains a $W_4$ with hub $b_5$ and rim $\{x_i, b_6, x_j, b_4\}$, a contradiction. This completes the proof. □

**Theorem 5** $R(C_n, W_4) = 2n - 1$ for $n \geq 5$.

*Proof.* We use induction on $n \geq 5$. We already know that $R(C_n, W_4) \geq 2n - 1$ for $n \geq 5$. For $n = 5, 6,$ and 7, we respectively know from [5], Theorem 3, and Theorem 4 that $R(C_n, W_4) = 2n - 1$. Now assume that $R(C_n, W_4) = 2n - 1$ for $n < k$ with $k \geq 8$ and let $F$ be a graph on $2k - 1$ vertices containing no $C_k$. We shall show that $F$ contains $W_4$. To the contrary, assume $F$ contains no $W_4$. By the induction hypothesis, we have $F \supseteq C_{k-1}$. Let $A = V(C_{k-1})$, $B = V(F) \setminus V(C_{k-1})$ and so $|B| = k$. By Lemma 4, we have $|N_A(x)| \leq 2$ for each $x \in B$. Since by Lemma 3, $\delta(F) \geq k - 2$, we obtain $|N_B(x)| \geq k - 2 - 2 = k - 4 \geq \frac{1}{2}k = \frac{1}{2}|B|$ for all $x \in B$. Now $F[B]$ and hence $F$ contains a $C_k$ by Theorem 1, a contradiction. This completes the proof. □
**Theorem 6** \( R(C_n, W_5) = 3n - 2 \) for \( n \geq 5 \).

*Proof.* We use induction on \( n \). We already know that \( R(C_n, W_5) \geq 3n - 2 \) for \( n \geq 5 \). For \( n = 5 \), we know from [6] that \( R(C_5, W_5) = 3.5 - 2 \). Assume the theorem holds for \( n < k \) with \( k \geq 6 \) and let \( F \) be a graph on \( 3k - 2 \) vertices containing no \( C_k \). We shall show that \( \overline{F} \) contains \( W_5 \). To the contrary, assume that \( \overline{F} \) contains no \( W_5 \). Consequently, \( F \) must contain a \( C_{k-1} \), and we let \( A = \{a_1, a_2, \ldots, a_{k-1}\} \) denote the set of vertices of a cycle \( C_{k-1} \) in \( F \), in a cyclic ordering. Let \( B = V(F) \setminus A \), so \( |B| = 2k - 1 \). Then, by Theorem 5, the complement of the subgraph \( F[B] \) of \( F \) induced by \( B \) must contain a \( W_4 \). Let \( x_0 \) be the hub and \( X = \{x_1, x_2, x_3, x_4\} \) be the rim of a \( W_4 \) in \( \overline{F}[B] \). We distinguish the following cases.

**Case 1:** \( k \) is even.
Since \( F \) contains no \( C_k \), within \( F \): \( |N_A(z)| \leq \lceil \frac{k-1}{2} \rceil \) for each \( z \in B \). This implies that there exist vertices \( a_j, a_{j+1} \in A \) for some \( j \in \{1, 2, \ldots, k - 1\} \) such that \( a_jx_0, a_{j+1}x_0 \not\in E(F) \). No \( C_k \) in \( F \) also implies \( N_X(a_j) \cap N_X(a_{j+1}) = \emptyset \). No \( W_5 \) in \( \overline{F} \) implies in \( F \): \( |N_X(a_j)| \geq 2 \) and \( |N_X(a_{j+1})| \geq 2 \), and without loss of generality we may assume \( a_j \) is adjacent to \( x_1 \) and \( x_3 \), and \( a_{j+1} \) is adjacent to \( x_2 \) and \( x_4 \). This implies \( x_1x_3, x_2x_4, x_0a_{j+2}, x_0a_{j-1} \in E(F) \) since otherwise \( \overline{F} \supseteq W_5 \) (Note that \( F \nsubseteq C_k \) implies neither of \( a_{j-1} \) and \( a_{j+2} \) is adjacent to a vertex in \( X \)). Since \( F \) contains no \( C_k \), it is not difficult to check \( x_0a_{j-2}, a_{j-2}x_1, a_{j+1}a_{j-2} \not\in E(F) \). This implies \( \overline{F} \supseteq W_5 \) with hub \( x_0 \) and rim \( \{x_3, a_{j+1}, a_{j-2}, x_1, x_2\} \), a contradiction.

**Case 2:** \( k \) is odd.
We may assume \( x_i \in E(F) \) for each odd \( i \in \{1, 2, \ldots, k - 1\} \), since otherwise we can use the same arguments as in the first case. Since \( F \) contains no \( C_k \), \( a_ja_h \not\in E(F) \) for all even \( j, h \in \{1, 2, \ldots, k - 1\} \). If \( k \geq 11 \), we have \( K_6 \) in \( \overline{F} \) which implies \( \overline{F} \supseteq W_5 \), a contradiction. Now assume \( 7 \leq k < 11 \). If \( F \) we have \( |N_X(a_j)| \geq 2 \) for all even \( j \in \{1, 2, \ldots, k - 1\} \), since otherwise \( \overline{F} \supseteq W_5 \). By the same token, we may assume without loss of generality that \( a_j \) is adjacent to \( x_1 \) and \( x_3 \) for some even \( j \in \{1, 2, \ldots, k - 1\} \). We distinguish two subcases.

**Subcase 2.1:** \( x_1 \) is adjacent to \( x_3 \).
Then \( x_1 \) and \( x_3 \) are not adjacent to any vertex in \( \{a_{j-1}, a_{j-2}, a_{j+1}, a_{j+2}\} \), since otherwise \( F \) clearly contains a \( C_k \). Thus, we get \( \overline{F} \supseteq W_5 \) with hub \( x_0 \) and rim \( \{x_3, a_{j+2}, a_{j-2}, x_1, x_2\} \), a contradiction.

**Subcase 2.2:** \( x_1 \) is not adjacent to \( x_3 \).
This implies \( x_2 \) and \( x_4 \) are adjacent to all vertices in \( \{a_{j-1}, a_{j+1}\} \), since otherwise \( \overline{F} \supseteq W_5 \). Suppose, e.g., \( x_2a_{j-1} \not\in E(F) \). Then \( \overline{F} \supseteq W_5 \) with hub \( x_1 \) and rim \( \{a_{j-1}, x_2, x_0, x_3, a_{j+1}\} \); the other cases are similar. Thus, we get \( x_2a_j, x_4a_{j+2} \not\in E(F) \); otherwise a \( C_k \) in \( F \) is immediate. Thus, we get \( \overline{F} \supseteq W_5 \) with hub \( x_0 \) and rim \( \{x_4, a_{j+2}, a_j, x_2, x_3\} \), our final contradiction.

This completes the proof. \( \square \)
3. Problem

We conclude the paper with the following open problem:

Find the Ramsey number $R(C_n, W_m)$ for $n \geq m \geq 6$.

References


