The Ramsey numbers of Fans versus K_4

Surahmat[1,2⋆], **E.T. Baskoro**[1], **H.J. Broersma**[2]

[1] Department of Mathematics
Institut Teknologi Bandung,
Jalan Ganesa 10 Bandung, Indonesia,
{kana_s, ebaskoro}@dns.math.itb.ac.id
[2] Faculty of Mathematical Sciences,
University of Twente,
P.O. Box 217, 7500 AE Enschede, The Netherlands,
broersma@math.utwente.nl

Abstract. For two given graphs G and H, the *Ramsey number* $R(G,H)$ is the smallest positive integer N such that for every graph F of order N the following holds: either F contains G as a subgraph or the complement of F contains H as a subgraph. In this paper, we determine the Ramsey number $R(F_l, K_4)$, where F_l is the graph obtained from l disjoint triangles by identifying precisely one vertex of every triangle (F_l is the join of K_1 and lK_2). It is known that for fixed l, $R(F_l, K_n) \leq (1+o(1))\frac{n^2}{log n}(n \to \infty)$. We prove that $R(F_l, K_n) = 2l(n-1)+1$ for $n = 4$ and $l \geq 3$. We conjecture that $R(F_l, K_n) = 2l(n-1)+1$ for $l \geq n \geq 5$.

Keywords: *Ramsey number, fan, complete graph.*

AMS Subject Classifications: 05C55, 05D10.

1 Introduction

Throughout the paper, all graphs are finite and simple. Let G be such a graph. We write $V(G)$ or V for the vertex set of G and $E(G)$ or E for the edge set of G. The graph $\overline{G}$ is the *complement* of the graph G, i.e., the graph obtained from the complete graph $K_{|V(G)|}$ on $|V(G)|$ vertices by deleting the edges of G.

The graph $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. For any nonempty subset $S \subset V$, the *induced subgraph* by S is the maximal subgraph of G with vertex set S; it is denoted by $G[S]$.

⋆ Permanent address: Department of Mathematics Education UNISMA, Jalan MT Haryono 193 Malang 65144, Indonesia.

If $e = \{u, v\} \in E$ (in short, $e = uv$), then u is called *adjacent to* v, and u and v are called *neighbors*. For $x \in V$ and $B \subset V$, define $N_B(x) = \{y \in B : xy \in E\}$ and $N_B[x] = N_B(x) \cup \{x\}$.

We denote by K_n the *complete* graph on n vertices. A *fan* F_l is the graph on $2l + 1$ vertices obtained from l disjoint triangles (K_3's) by identifying precisely one vertex of every triangle (F_l is the join of K_1 and lK_2). By S_n we denote a *star* on n vertices (i.e., $S_n = K_{1,n-1}$, the join of K_1 and $(n-1)K_1$).

Given two graphs G and H, the *Ramsey number* $R(G, H)$ is defined as the smallest natural number N such that every graph F on N vertices satisfies the following condition: F contains G as a subgraph or $\overline{F}$ contains H as a subgraph.

We will also use the short notations $H \subseteq F$, $F \supseteq H$, $H \not\subseteq F$, and $F \not\supseteq H$ to denote that H is (not) a subgraph of F, with the obvious meanings.

Chvátal and Harary [2] studied Ramsey numbers for graphs and established the lower bound: $R(G, H) \geq (\chi(G) - 1)(c(H) - 1) + 1$, where $\chi(G)$ is the chromatic number of G and $c(H)$ is the number of vertices of the largest component of H. More specifically, Chvátal [1] showed that $R(K_m, T_n) = (m-1)(n-1) + 1$ where T_n is a tree on n vertices. Radziszowski and Xia [6] gave a simple and unified method to establish the Ramsey number $R(G, K_3)$, where G is either a path, a cycle or a wheel. Li and Rousseau [5] used probabilistic arguments to show that $R(F_l, K_n) \leq (1 + o(1))\frac{n^2}{logn}(n \to \infty)$. Gupta et al. [3] showed $R(F_l, K_3) = 4l + 1$ for any integer $l \geq 2$. For other interesting results see a survey paper of [7]. In this paper, we study the first open case for fans versus larger complete graphs, namely $R(F_l, K_4)$.

2 Main Result

The aim of this paper is to determine the Ramsey number of a fan F_l with $2l + 1$ vertices versus K_n for $n = 4$. We will show that $R(F_l, K_4) = 6l + 1$ for any integer $l \geq 3$.

For the lower bound, consider the graph $G = (n-1)K_{2l}$. Clearly, G has $2l(n-1)$ vertices and it contains no fan F_l, whereas its complement contains no K_n. Thus $R(F_l, K_n) \geq 2l(n-1) + 1$.

It is known that $R(F_1, K_4) = R(K_3, K_4) = 9$. Hendry [4] found the Ramsey number $R(F_2, K_4)$. Applying the above lower bound we get $R(F_3, K_4) = 19$.

To prove the upper bound for $n = 4$ we will use the result on trees due to Chvátal [1] as well as the result on F_l versus K_3 from [3] as follows.

Theorem 1. *For any integer* $l \geq 4$, $R(F_l, K_4) = 6l + 1$.

Proof. Let G be a graph on $6l+1$ vertices containing no fan F_l. We will show that $\overline{G}$ contains a K_4. Suppose to the contrary that $\overline{G}$ contains no K_4. Since $R(S_{2l+1}, K_4) = 6l+1$ by [1], G must contain an S_{2l+1}. Let x_0 be the vertex of highest degree in an S_{2l+1} and denote by $X = \{x_1, x_2, ..., x_{2l-1}, x_{2l}\}$ the set of neighbors of x_0 in S_{2l+1}. Since $\overline{G}$ contains no K_4, there exists at least one edge in any subgraph $G[X_1]$ of G induced by $X_1 \subseteq X$ with $|X_1| = 4$. Thus, $G[X \cup \{x_0\}]$ contains a fan F_{l-1}. Without loss of generality, let $x_i x_{i+1} \in E(G)$ for each $i = 1, 3, 5, ..., 2l-3$. Then, since $F_l \nsubseteq G$, $x_{2l-1}x_{2l} \notin E(G)$. Let $B = V(G) \backslash (X \cup \{x_0\})$. We have $|N_B(x_0)| \leq 1$, since otherwise considering x_{2l-1}, x_{2l} and two vertices from $N_B(x_0)$ we obtain $G \supseteq F_l$. Let $D = B \backslash N_B(x_0)$. We also obtain $\overline{G}[D] \not\supseteq K_3$, otherwise combined with x_0 we find a K_4 in $\overline{G}$. Note that $|D| \geq 4l-1$. Since $R(F_l, K_3) = 4l+1$ for $l \geq 2$ by the result in [3], we have $G[D] \supseteq F_{l-1}$. Let y_0 denote the vertex of highest degree in an F_{l-1} and $Y = \{y_1, y_2, ..., y_{2l-2}\}$ the set of neighbors of y_0 in F_{l-1}. Next, let $P = D \backslash (Y \cup \{y_0\})$. We obtain that $|N_P(y_0)| \leq 2$ since otherwise $\overline{G}[D] \supseteq K_3$. Let $Q = P \backslash N_P(y_0)$. Now, $G[Q]$ is a complete graph; otherwise vertices $q_1, q_2 \in Q$ and y_0 for $q_1 q_2 \notin E(G)$ form a K_3 in $\overline{G}[D]$. Since $|Q| = |B| - |N_B(x_0)| - (2l-1) - |N_P(y_0)| = 2l+1 - |N_B(x_0)| - |N_P(y_0)|$, depending on the neighborhoods of x_0 and y_0 we find a complete graph $G[Q]$ on at least $2l-2$ and at most $2l+1$ vertices. We distinguish the following two cases and subcases.

Case 1. $|N_B(x_0)| = 0$.

For this case we distinguish the following three subcases.

Subcase 1.1. $|N_P(y_0)| = 0$.
We obtain $|Q| = 2l+1$. This implies $G[Q] = K_{2l+1} \supseteq F_l$, a contradiction.

Subcase 1.2. $|N_P(y_0)| = 1$.
Let $p \in N_P(y_0)$. We obtain $G[Q] = 2l$. To avoid an $F_l \subseteq G$, every vertex in $V(G) \backslash Q$ has at most one neighbor in Q. We claim that $G[Y \cup \{y_o, p\}] = K_{2l}$. Suppose to the contrary that some distinct $y_i, y_j \in Y \cup \{p\}$ are nonadjacent. Then these two vertices together with a common nonneighbor in Q (existing by the previous statement) induce a K_3 in $\overline{G}[D]$. Both of $\{x_{2l-1}, x_{2l}\}$ have at most one neighbor in Q and in $Y \cup \{y_0, p\}$; otherwise an F_l in G is immediate. Now it is easy to obtain a vertex $q \in Q$ and a vertex $y \in Y \cup \{y_0, p\}$ such that $\{x_{2l-1}, x_{2l}, q, y\}$ is an independent set, contradicting that $\overline{G} \not\supseteq K_4$.

Subcase 1.3. $|N_P(y_0)| = 2$.
Let $p_1, p_2 \in N_P(y_0)$. It is clear that p_1 is nonadjacent to p_2 and $G[Q] = K_{2l-1}$. Since $\overline{G}[D] \not\supseteq K_3$, for each $y \in Y$ is adjacent to one of p_1, p_2. If

$N_Y(p_1) \cap N_Y(p_2) \neq \emptyset$, then we easily obtain an F_l in G. Hence $N_Y(p_1) \cap N_Y(p_2) = \emptyset$. Since $\overline{G}[D] \not\supseteq K_3$, $|N_Q(p_i)| \geq 4$ for some $i \in \{1, 2\}$. Assume $Q' = \{q_1, q_2, q_3, q_4\} \subseteq N_Q(p_2)$. We obtain that p_1 is adjacent to at most one vertex in Q' since otherwise $G[P] \supseteq F_l$ with q is the center for some $q \in Q'$.

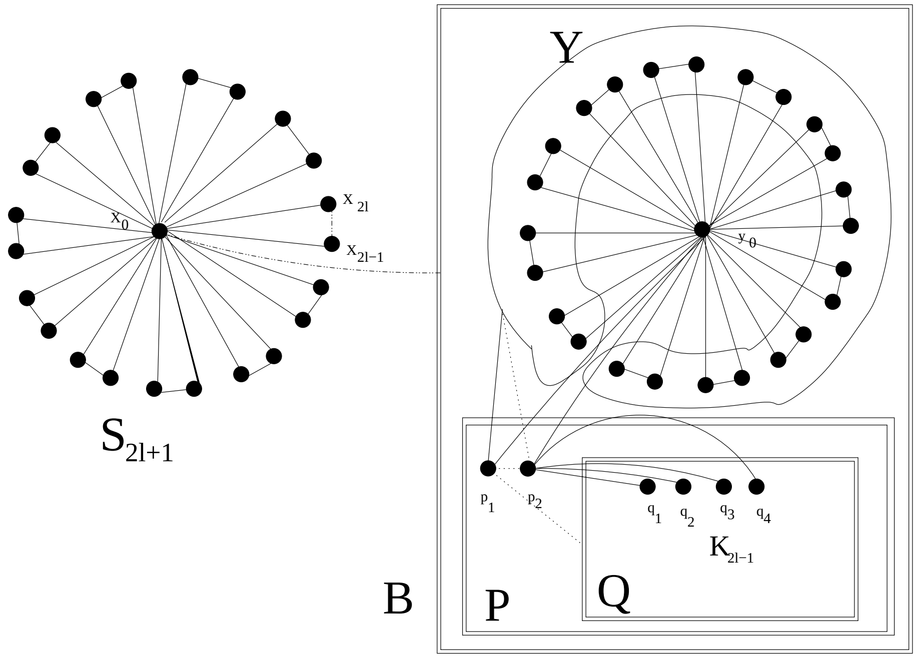

Fig. 1. The proof of Theorem 1 Subcase 1.3.

We claim that $N_Y(p_2) = \emptyset$. Suppose to the contrary $y' \in N_Y(p_2)$. Since G contains no F_l, y' is nonadjacent to any vertex in Q'. Since p_1 is adjacent to at most one vertex in Q' and $N_Y(p_1) \cap N_Y(p_2) = \emptyset$, we obtain a K_3 in $\overline{G}[D]$ formed by vertices p_1, y' and q_i for some $i \in \{1, 2, 3, 4\}$.

The previous claim implies that p_1 is adjacent to all vertices in Y. If $yz \notin E(G)$ for $y, z \in Y$, then similarly we have that $\{y, z, p_2\}$ is an independent set in $G[D]$. This implies $G[Y \cup \{p_1, y_0\}] = K_{2l}$. Clearly all $a \in V(G) \backslash (Y \cup \{p_1, y_0\})$ have at most one neighbor in this K_{2l}, since otherwise $G \supseteq F_l$. See Figure 1. Because $Q' \subseteq N_Q(p_2)$ and $G[Q] = K_{2l-1}$, for each vertex in $V(G) \setminus (Q' \cup \{p_2\})$ are adjacent to at most one vertex in Q', since otherwise $G \supseteq F_l$ with some $q \in Q'$ as its center. Now we can find four vertices which are independent in G, namely $\{x_{2l-1}, x_{2l}, q^*, y^*\}$

for some suitable $q^* \in Q'$ and $y^* \in Y \cup \{p_1, y_0\}$, a contradiction.

Case 2. $|N_B(x_0)| = 1$.

Let $b \in N_B(x_0)$. Then by obvious arguments, the set $\{b, x_{2l-1}, x_{2l}\}$ is an independent set of vertices in G, and every other vertex is adjacent to at least one of them. We again distinguish three subcases.

Subcase 2.1. $|N_P(y_0)| = 0$.
We obtain $G[Q] = K_{2l}$. At least one of $\{b, x_{2l-1}, x_{2l}\}$ has at least two neighbors in Q, since otherwise $\overline{G} \supseteq K_4$. This yields an F_l in G, a contradiction.

Subcase 2.2. $|N_P(y_0)| = 1$.
Let $p \in N_P(y_0)$. We obtain $G[Q] = K_{2l-1}$. As in the previous case, $|N_Q(x)| \geq 3$ for some $x \in \{b, x_{2l-1}, x_{2l}\}$. Suppose $Q_1 = N_Q(x_{2l-1})$ with $|Q_1| \geq 3$. Now, let $Y^* = Y \cup \{p, y_0\}$ and so $|Y^*| \geq 8$. We claim that $G[Y^*] = K_{2l}$. Suppose to the contrary that $yz \notin E(G)$ for some $y, z \in Y^*$. We know that $|N_{Q_1}(y)| \leq 1$, since otherwise $G[Q \cup \{x_{2l-1}, y\}] \supseteq F_l$ with some $q \in N_{Q_1}(y)$ as its center; similarly, $|N_{Q_1}(z)| \leq 1$. But then $\{y, z, t\}$ induces a K_3 in $\overline{G}[D]$ for some $t \in Q_1$, a contradiction. Since $\overline{G}$ contains no K_4, at least one of $\{x_{2l-1}, x_{2l}, b\}$ has at least two neighbors in $G[Y^*]$, and we obtain an F_l in G, a contradiction.

Subcase 2.3. $|N_P(y_0)| = 2$.
As in Subcase 1.3, let $p_1, p_2 \in N_P(y_0)$. We obtain that p_1 is nonadjacent to p_2 and $G[Q] = K_{2l-2}$. Since $\overline{G}[D] \not\supseteq K_3$, every vertex of Y is adjacent to p_1 or p_2. We first observe that this implies that $N_Y(p_1) \cap N_Y(p_2) = \emptyset$; otherwise, using the previous statement we easily obtain an F_l in G. Since $\overline{G}[D] \not\supseteq K_3$, we also obtain that $|N_Q(p_i)| \geq 3$ for some $i \in \{1, 2\}$. Let $N_Q(p_2) \supseteq Q_2$ with $Q_2 = \{q_1, q_2, q_3\}$. We claim that $G[Y \cup \{y_0\}] = K_{2l-1}$. Suppose to the contrary that $t_1z_1 \notin E(G)$ for some $t_1, z_1 \in Y$. Consider $t_2, z_2 \in Y$ such that $t_1t_2, z_1z_2 \in E(G)$. As we argued before, each of t_1, t_2, z_1, z_2 is adjacent to exactly one of p_1, p_2. If p_i is adjacent to only one of t_1, t_2, then p_{3-i} is adjacent to the other, and an F_l is immediate. We get that one of p_1, p_2 is adjacent to t_1, t_2 and the other to z_1, z_2. By similar arguments, no $t \in \{t_1, t_2\}$ is adjacent to any vertex in $\{z_1, z_2\}$. Suppose without loss of generality that $p_1z_1, p_1z_2, p_2t_1, p_2t_2 \in E(G)$. Each vertex of Q_2 is adjacent to all vertices in $\{t_1, t_2\}$ or in $\{z_1, z_2\}$; otherwise $qt, qz \notin E(G)$ for some $q \in Q_2$, $t \in \{t_1, t_2\}$ and $z \in \{z_1, z_2\}$, and so $\overline{G}[D] \supseteq K_3$. We obtain an F_l in G from Q, p_2 and t_1, t_2 (or z_1, z_2), a contradiction. This proves our claim that $G[Y \cup \{y_0\}] = K_{2l-1}$. Our next claim is that $N_Y(p_2) = \emptyset$. Suppose to the contrary that $N_Y(p_2) \neq \emptyset$. If $|N_Y(p_1)| \geq 1$, then we obtain from Y, y_0, p_1, p_2 that G contains an F_l. So $N_Y(p_1) = \emptyset$

and $N_Y(p_2) = Y$ since $N_Y(P_1)$ and $N_Y(P_2)$ partition Y. This implies that $G[Y \cup Q_2 \cup \{p_2\}]$ contains an F_l, a contradiction. Thus $N_Y(p_2) = \emptyset$, and hence $N_Y(p_1) = Y$ since $N_Y(P_1)$ and $N_Y(P_2)$ partition Y. We get that $G[Y \cup \{y_0, p_1\}] = K_{2l}$. See Figure 2. At least one of $\{b, x_{2l-1}, x_{2l}\}$ has at least two neighbors in $Y \cup \{y_0, p_1\}$; otherwise we have $K_4 \subseteq \overline{G}$. Now clearly we obtain an F_l in G, our final contradiction. ∎

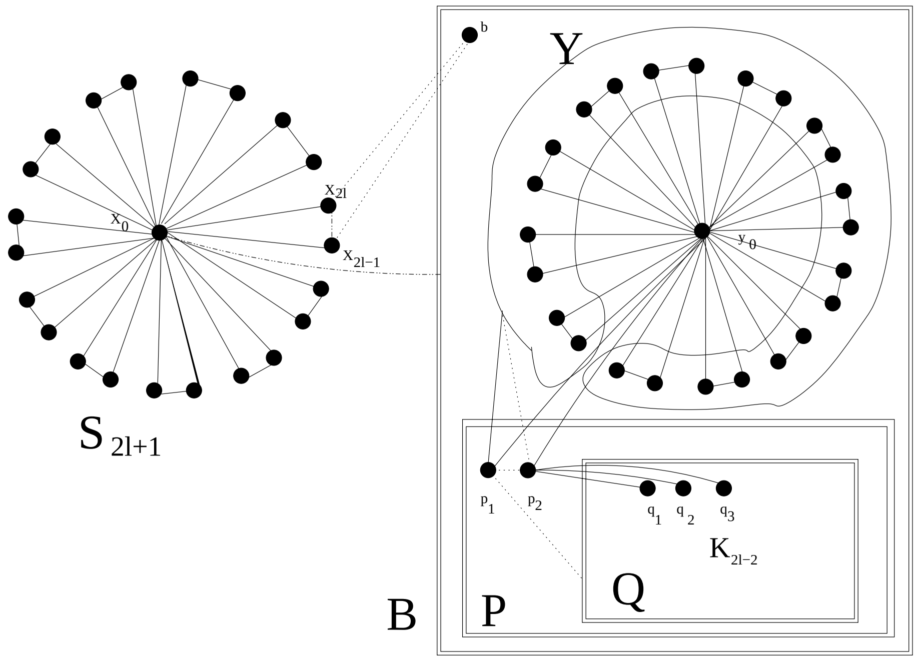

Fig. 2. The proof of Theorem 1 Subcase 2.3.

3 Conjecture

To conclude the paper, we conjecture that $R(F_l, K_n) = 2l(n-1) + 1$, if $l \geq n \geq 5$.

References

1. V. Chvátal, Tree-Complete Graph Ramsey Numbers, *Journal of Graph Theory* **1** (1977) 93.

2. V. Chvátal and F. Harary, Generalized Ramsey Theory for Graphs, III: Small off-Diagonal Numbers, *Pacific Journal of Mathematics*, **41**(1972) 335-345.
3. S.K. Gupta, L. Gupta and A. Sudan, On Ramsey Numbers for Fan-Fan Graphs, *Journal of Combinatorics, Information & System Sciences* **22** (1997) 85-93.
4. G.R.T. Hendry, Ramsey Numbers for Graphs with Five Vertices, *Journal of Graph Theory*, **13** (1989) 245-248.
5. Li Yusheng and C.C. Rousseau, Fan-Complete Graph Ramsey Numbers, *Journal of Graph Theory* **23** (1996) 413-420.
6. S.P. Radziszowski and J. Xia, Paths, Cycles and Wheels without Antitriangles, *Australasian Journal of Combinatorics* **9** (1994) 221-232.
7. S. P. Radziszowski, Small Ramsey Numbers, *The Electronic Journal of Combinatorics*, July (2002) #DS1.9, http://www.combinatorics.org/