An existence theorem for Volterra integrodifferential equations with infinite delay *

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Abstract

Using Schauder’s fixed point theorem, we prove an existence theorem for Volterra integrodifferential equations with infinite delay. As an application, we consider an \( n \) species Lotka-Volterra competitive system.

1 Introduction

Vrabie [10, page 265] studied the partial integrodifferential equation

\[
\dot{u}(t) = -Au(t) + \int_{a}^{t} k(t-s)g(s, u(s))ds
\]

\( u(a) = u_0, \tag{1.1} \)

where \( u : [a, b] \to X \), \( X \) is a Banach space, \( A : D(A) \subset X \to X \) is an \( M \)-accretive operator; \( t \in [a, b] \), \( g : [a, b] \times X \to X \), \( k : [0, a] \to \mathcal{L}(X) \) are continuous functions. The result, existence of solutions on some interval \( [a, c) \) was obtained by using the Schauder’s fixed point theorem.

Schauder’s fixed point theorem is a usual tool for proving existence theorems in infinite delay case. In [8], Teng applied it to prove existence theorems for Kolmogorov systems. Another frequently used method (especially for integrodifferential equations) is the Leray-Schauder alternative, see [5] and its references.

Modifying (1.1) we investigate the case when the initial function is given on \( (-\infty, 0] \), which means infinite delay, moreover in the right-hand side we take a function of the integral. This form allows us proving existence theorems for systems. In this case \( g, k \) in the right hand side have to be also modified. The spirit of the proof is similar to [10, pages 265–268] but we need some assumptions on \( k \) and \( g \) and additional spaces and operators have to be introduced to carry out the proof.

In section 3 we apply the result to a system (a competition model arising from population dynamics); existence of global solution will be proved. In the compactness arguments we need the following definition.

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Definition  A family of functions $H \subset L^1([a,b];X)$ is 1-equiintegrable if the following two conditions are satisfied:

- For all $\epsilon > 0$, there exists $\delta$ such that for all $f \in H$, $\lambda(E) < \delta \rightarrow \int_E \|f(t)\|dt < \epsilon$
- For all $\epsilon > 0$, there exists $h > 0$ such that for all $f \in H$ and all $h_0 < h$,
  $$\int_a^{b-h_0} \|f(t+h_0) - f(t)\|dt < \epsilon.$$

In this paper, let $X$ be a Banach space, $A : \mathcal{D}(A) \subset X \to X$ an $M$-accretive operator [10, page 21]. Further, the spaces equipped with the supremum-norm are denoted by denoted by $C$. We study of the abstract Cauchy problem ([7, page 90], [2, pages 390–398])

$$\dot{u}^f(t) + Au^f(t) = f(t) \quad \text{if} \quad t \geq a$$

$$u^f(a) = u(a).$$  \hspace{1cm} (1.2)

Here $u^f$ denotes the $f$ dependence of the solution. We also use the following theorem [10, page 65] which is the basis of the compactness method employing in the following section.

**Theorem 1.1** Let $A : X \to X$ be an $M$-accretive operator, and $(I-\lambda A)^{-1}$ compact for each $\lambda > 0$. Let $u_0 \in \mathcal{D}(A)$ and $K \subset L^1([a,b];X)$ be 1-equiintegrable. Then the set $M(K) = \{u^f : u^f$ is the mild solution of (1.2), $f \in K\}$ is relatively compact in $C([a,b];X)$.

2 An existence result for a class of Volterra-type integrodifferential equations

A class of Volterra-type integrodifferential equations

Let $U$ be an open subset of $X$, and $U_A = U \cap \mathcal{D}(A)$, with $(I-\lambda A)^{-1}$ compact. Let $b > a$ and $g = (g_1, g_2, \ldots, g_n)$ be Lipschitz-continuous functions in the second variable, where $g_i : (-\infty, b] \times U_A \to X$ are bounded and continuous. Let $k = (k_1, k_2, \ldots, k_n)$ be a function such that $k_i \in L^1([0, \infty), \mathcal{L}(X))$ and

$$k(t)g(s,u(s)) = (k_1(t)g_1(s,u(s)), k_2(t)g_2(s,u(s)), \ldots, k_n(t)g_n(s,u(s))). \hspace{1cm} (2.1)$$

Let the space $X^n$ be equipped with the maximum norm, $\|x\| = \max_{1 \leq i \leq n} \|x_i\|$, where $x = (x_1, x_2, \ldots, x_n)$. Let $F : X^n \to X$ be a function such that for some constant $M_F \in \mathbb{R}$,

$$\|F(x)\| \leq M_F \|x\| \quad \text{and} \quad M_F \int_{-\infty}^{0} \|k(-\tau)\|d\tau \leq 1. \hspace{1cm} (2.2)$$
Consider the problem

\[\dot{u}(t) = -Au(t) + F\left(\int_{-\infty}^{t} k(t-s)g(s,u(s))ds\right) \quad \text{for} \quad t \geq a \]  
(2.3)

\[u(t) = u_0(t-a) \quad \text{for} \quad t \leq a, \]  
(2.4)

where \(u_0 \in \mathcal{C}((\infty,0],X)\) is a given bounded, equiintegrable function which fulfills the matching condition

\[u_0(0) = F\left(\int_{-\infty}^{0} k(-s)g(a+s,u_0(s))ds\right). \]  
(2.5)

**Theorem 2.1** Under assumptions (2.1) and (2.2), there is a value \(c\) in \((a,b]\) such that (2.3)-(2.4) has a weak solution on \((-\infty,c]\).

**Proof:** Note that \(k_i \in L_1([0,\infty),\mathcal{L}(X))\) implies \(k \in L_1([0,\infty),\mathcal{L}(X^n,\mathbb{R}^n))\) and (2.2) makes sense. This is only a technical supposition because (2.3) could be rewrite with \(k/M\) and \(Mg\) (instead of \(k, g\), resp.; \(M \in \mathbb{R}\) is sufficiently big) fulfilled (2.3). Let

\[P : \mathcal{C}(((-\infty,b],U) \mapsto \mathcal{C}((\infty,b],U)\]

defined by

\[Pf(t) = \begin{cases} 
F\left(\int_{-\infty}^{t} k(t-s)g(s,u^f(s))ds\right) & \text{if } t \geq a \\
\int_{-\infty}^{t} k(t-s)g(s,u^f(s))ds & \text{if } t \leq a,
\end{cases} \]  
(2.6)

where \(u^f\) is the weak solution of (1.2).

Observe that \(Pf = f\) holds if and only if \(u^f\) is the weak solution of the equation (2.3)-(2.4). Let us choose \(\rho > 0\) such that

\[B(u(a),\rho) := \{v \in X : \|v - u(a)\| \leq \rho\} \subset U. \]  
(2.7)

Since \(g\) is bounded there is \(M \in \mathbb{R}\) such that

\[\|g(s,v)\| \leq M \quad \text{for} \quad (s,v) \in ((\infty, b] \times [U_A \cap B(u_0,\rho)]). \]  
(2.8)

Denote by \(S(t)\) the semigroup generated by \(-A\) on \(\mathcal{D}(A)\). Let us choose further \(b \geq c_0 \geq a\) such that for all \(t \in [a,c_0]\)

\[\|S(t-a)u_0 - u_0\| + (c_0-a)M \leq \rho, \]  
(2.9)

and \(c \in [a,c_0]\) such that

\[(c-a)MF\|k\|_{L_1} \leq 1. \]  
(2.10)

Let us define

\[\mathcal{C}_{u_0}((\infty, b],U) = \{u \in \mathcal{C}((\infty, b],U) : u(t) = u_0(t-a) \quad \text{for} \quad t \leq a\}. \]  
(2.11)
Let

\[ H : C_u((−∞, b], U) \mapsto C([a, b], U) \]

be a natural homeomorphism with \((Hf)(t) = f(t)\) for \(t \in [a, b]\) and let

\[ K_{u_0}^r := \{ f \in C([−∞, c], X) : \|Hf(t)\|_∞ ≤ r \& f(b) = u_0(d - a) \text{ for } d ≤ a \}. \quad (2.12) \]

Obviously \(K_{u_0}^r\) is nonempty, bounded, closed and convex subset of the space \(C_u([−∞, c], X)\).

Observe that \(P = P_1 \circ P_2\), where (using the matching condition (2.5)) we define \(P_1 : C_u((−∞, b], U) \mapsto C_u((−∞, b], U)\) as

\[ P_1v(t) = \begin{cases} F \left( \int_{−∞}^t k(t - s)g(s, v(s))\,ds \right) & \text{if } t ≥ a \\ v(t) & \text{if } t < a \end{cases} \quad (2.13) \]

and \(P_2 : C_u((−∞, b], U) \mapsto C_u((−∞, b], U)\) is defined as \(P_2 = H^{-1} P_2^* H\), where

\[ P_2^* : C([a, b], U) \mapsto C([a, b], U) \]

and \(P_2^* g(t)\) is the weak solution of the abstract Cauchy problem

\[ \dot{u}(t) + Au(t) = g(t) \text{ for } t ≥ a \]
\[ u(a) = g(a) = u_0(0). \quad (2.14) \]

For details on this problem, we refer the reader to Barbu [1, page 124] and for some applications of this result to [10, page 35].

Let \( f, h \in L_1([a, b], X) \) and let \( u, v \) be solutions, in the weak sense, of

\[ \begin{align*}
\dot{u}(t) + Au(t) &= f(t) \\
\dot{v}(t) + Av(t) &= h(t)
\end{align*} \quad (2.15) \]

with some initial conditions \(u(a), v(a)\). Then for \( s, t \in [a, b] \) we have

\[ \|u(t) - v(t)\| ≤ \|u(s) - v(s)\| + \int_s^t \|f(\tau) - h(\tau)\|\,d\tau. \quad (2.16) \]

From this inequality, it follows that

\[ \|P_2^* h_1(t) - P_2^* h_2(t)\| ≤ \|h_1(a) - h_2(a)\| + \int_a^t \|h_1(\tau) - h_2(\tau)\|\,d\tau \]
\[ ≤ \|h_1 - h_2\|_∞ (t - a + 1), \]

which implies the continuity of \(P_2^*\) on \(C([a, b], U)\) and so \(P_2\) on \(K_{u_0}^r\). Using (2.8), (2.9) and (2.16) for \(u \in K_{u_0}^r, t \in [a, c_0]\) we get

\[ \|P_2^* u(t) - u(a)\| ≤ \|P_2 u(t) - S(t - a)u(a)\| + S(t - a)u(a) - u(a)\|
\]
\[ ≤ \|S(t - a)u(a) - u(a)\| + \int_a^t \|g(t)\|\,dt \]
\[ ≤ \|S(t - a)u(a) - u(a)\| + (c_0 - a)M ≤ ρ. \quad (2.17) \]
Then we conclude that $P^*_2 u(t) \in B(u(a), \rho) \cap D(A)$. Consequently, $P_2 u(t) \in D(g)$ for $t \geq a$. By (2.2), (2.8), (2.10) and (2.13), for $t \geq a$ we have

$$
\|P u(t)\| = \|P_1 P_2 u(t)\| = \|F( \int_{-\infty}^{t} k(t-s)g(s, P_2 u(s))ds)\| \\
\leq M_F \sup_{s \in (-\infty, t]} \|g(s, P_2 u(s))\| \int_{-\infty}^{0} \|k(-\tau)\|d\tau \\
\leq M_F M \int_{-\infty}^{0} \|k(-\tau)\|d\tau \leq M.
$$

and (2.5) implies that $Pu(t) = u(t)$ for $t \leq a$; i.e., $P$ maps $K_{u_0}$ into itself. Since

$$
\|(P_1 v - P_1 w)(t)\| = \|F( \int_{-\infty}^{t} k(t-s)[g(s, v(s)) - g(s, w(s))]ds)\| \\
= M_F \int_{-\infty}^{a} k(t-s)[g(s, v(s)) - g(s, w(s))]ds \\
+ \int_{a}^{t} k(t-s)[g(s, v(s)) - g(s, w(s))]ds \\
\leq M_F [v(a) - w(a)] \\
+ \max_{s \in [a, t]} [g(s, v(s)) - g(s, w(s))](t-a)\|k(t-s)\|_{L_1},
$$

the function $P_1$ is continuous from $C_{u_0}((\infty, b]; U)$ into itself. Using the continuity of $P_2$ we have that $P : K_{u_0}^M \rightarrow K_{u_0}^M$ is continuous. Since

$$
\int_{E} Pf(t)dt \leq \lambda(E) \max_{t} Pf(t) \leq \lambda(E)\|k\|_{L_1} M
$$

and

$$
\int_{a}^{b-h_0} \|Pf(t + h_0) - Pf(t)\|dt \\
\leq \|F\| \|(a-b)( \int_{-\infty}^{t+h_0} k(t-s)g(s, u^f(s))ds - \int_{-\infty}^{t} k(t-s)g(s, u^f(s))ds)\| \\
\leq h_0 \|F\| \|(a-b)\|k\|_{L_1} M
$$

we get that $HP(K_{u_0}^M)$ is 1-equiintegrable. Let us define

$$
K_{u_0} := cl(\text{conv } P(K_{u_0}^M)).
$$

Easy calculations shows that $H(K_{u_0}) = cl(\text{conv } HP(K_{u_0}^r))$ is equiintegrable and Theorem 1.1 implies the relative compactness of $P^*_2 H(K_{u_0}) = HP_2(K_{u_0}).$
Since $H$ is homeomorphism, $P_2(K_{u_0})$ and $P(K_{u_0}) = P_1P_2(K_{u_0})$ are relative compact. Since $P(K_{u_0})$ is a subset of the closed, bounded and convex set $K_{u_0}$, the Schauder fixed point theorem ensures the existence of a fixed point of $P$.

### 3 Application to an $n$ species Lotka-Volterra competitive system

We prove local existence of solutions for a system, which is a model of an $n$ species competition arising in the population dynamics. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Feng [3] studied the system $(i = 1, \ldots, N)$

$$
\begin{align*}
(u_i)_t &= D_i \left[ \Delta u_i + u_i (a_i - u_i - \sum_{j \neq i}^N \kappa_{ij} u_j^{\tau_{ij}}) \right] \quad \text{on } (0, \infty) \times \Omega \\
\quad u_i &= 0 \quad \text{in } (0, \infty) \times \partial \Omega \\
\quad u_i(s, x) &= \eta_i(s, x) \quad \text{on } [-\tau, 0] \times \Omega,
\end{align*}
$$

where $u_i(t, x)$ denotes the density of the $i$-th species at time $t$ and position $x$ (inside a bounded domain $\Omega$ of $\mathbb{R}^3$), $u_j^{\tau_{ij}}(t, x) = u_j(t - \tau_{ij}, x)$, $\tau_{ij} > 0$, $\tau = \max \{\tau_{ij}\}$, $D_i, a_i$ are positive, and $\kappa_{ij}$ are nonnegative real numbers. Supposing the existence of a solution (a sufficient condition for this - using upper and lower semisolutions - is formulated in [6]) the authors describe the attractors of (3.1).

In [8], Teng studies

$$
\begin{align*}
\frac{dx_i(t)}{dt} &= x_i(t) [a_i(t) - g_i(t, x_i(t)) - \sum_{j=1}^m c_{ij} P_j(x(t - \tau_{i,j}(t))) \\
&\quad - \sum_{j=1}^m \int_{-\sigma_{ij}}^0 \kappa_{ij}(t, s) Q_j(x_j(t + s)) ds], \quad (i = 1, \ldots, n)
\end{align*}
$$

an $n$-species Lotka-Volterra competitive system with delays as an application of existence result for periodic Kolmogorov systems with delay. Detailed study of the non-autonomous Lotka-Volterra models with delay (focused on existence of positive periodic solutions) can be found in [9].

We rewrite (3.1) taking into account that a bounded attractor $A$ has a bounded neighborhood $U$ and $B \in \mathbb{R}$ such that $u(t, x) \in U$ for $t \leq t_0$ implies $|u(t, x)| < B$ for all $t > t_0$. $B$ can be considered as a bound determined by the carrying capacity of the territory. Let $b: \mathbb{R} \to \mathbb{R}$ be a bounded, continuous such that $b(x) = x$ for $|x| < B$. The new form of (3.1) is

$$
\begin{align*}
(u_i)_t &= D_i \left[ \Delta u_i + b(u_i)(a_i - b(u_i) - \sum_{j=1}^N \kappa_{ij} b(u_j^{\tau_{ij}})) \right] \quad \text{on } (0, \infty) \times \Omega \\
\quad u_i &= 0 \quad \text{on } (0, \infty) \times \partial \Omega \\
\quad u_i(s, x) &= \eta_i(s, x) \quad \text{on } [-\tau, 0] \times \Omega.
\end{align*}
$$
We reformulate (3.3) again in accordance to the notations and assumptions of Theorem 2.1. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded open subset, \( X = [L^2(\Omega)]^n, u = (u_1, \ldots, u_n) : \mathbb{R} \to X \) \( u(s)(x) = (u_1(s,x), \ldots, u_n(s,x)) \) and

\[
D(A) = [C^2(\Omega)]^n, \quad A(u_1, u_2, \ldots, u_n) = (D_1 u_1, \ldots, D_n u_n).
\]

Let \( g = (g_1, g_2, \ldots, g_{n+1}) \) be such that \( g_i : (-\infty, \infty] \times X \to X \) are bounded and continuous, Lipschitz-continuous in the second variable and \( g_i(s,u(s))|_{\mathbb{R} \times B} = u(s) \), where \( B \) is an a priori bound of the solutions of (3.3), \( k = (k_1, k_2, \ldots, k_{n+1}) \), where \( k_i \in L_1([0, \infty), \mathcal{L}(X)) \).

We rewrite (2.3)-(2.4) in the form

\[
\begin{align*}
(u_i(t,x))_t &= D_1 u_i(t,x) + F_i\left( \int_{-\infty}^{t} g_i(t-s) g(s,u(s))ds \right) \quad (i = 1, \ldots, n) \\
u_i(s,x) &= \eta_i(s,x) \quad \text{on } [-\tau, 0] \times \Omega,
\end{align*}
\]

where we take \( A \) as defined above and \( n + 1 \) instead of \( n \). In a special case we get a perturbed version of (3.3), supposed that the right-hand side of (3.4) is approximated such that

\[
\left[ \int_{-\infty}^{t} k_i(t-s) g_i(s, u(s))ds \right]_j \approx \kappa_{ij} b(u_j^\tau_i(t)) \quad (i, j = 1, \ldots, n)
\]

and

\[
\left[ \int_{-\infty}^{t} k_{n+1}(t-s) g_{n+1}(s, u(s))ds \right]_j \approx b(u_j(t)) \quad (j = 1, \ldots, n).
\]

According to the choice of \( g \) requirements (3.5) and (3.6) can be rewritten as

\[
\int_{-\infty}^{t} k_i(t-s)(u_1(s), u_2(s), \ldots, u_n(s))ds \approx (\kappa_{11} b(u_1^\tau_1(t)), \kappa_{12} b(u_2^\tau_1(t)), \ldots, \kappa_{1n} b(u_n^\tau_1(t))) \quad (i = 1, \ldots, n)
\]

and

\[
\int_{-\infty}^{t} k_{n+1}(t-s)(u_1(s), \ldots, u_n(s))ds \approx (b(u_1(t)), \ldots, b(u_n(t))).
\]

Obviously \( k_1, k_2, \ldots, k_{n+1} \) can be chosen such that \( k_i \in L_1([0, \infty), \mathcal{L}(X)) \) and approximations (3.7) and (3.8) are sharp; namely, for all \( \epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1} > 0 \) there are \( k_i \in L_1([0, \infty), \mathcal{L}(X)) \) such that for any bounded \( (u_1, u_2, \ldots, u_n) \) and for all \( t > t_0, \)

\[
\int_{-\infty}^{t} k_i(t-s)(u_1(s), u_2(s), \ldots, u_n(s))ds = (\kappa_{11} b(u_1^\tau_1(t)), \kappa_{12} b(u_2^\tau_1(t)), \ldots, \kappa_{1n} b(u_n^\tau_1(t))) < \epsilon_i \quad (i = 1, \ldots, n)
\]
and
\[ \int_{-\infty}^{t} k_{n+1}(t-s)(u_1(s),\ldots,u_n(s))ds - (b(u_1(t)),\ldots,b(u_n(t))) < \epsilon_{n+1}. \]

Moreover, the terms on the left-hand side of (3.7) and (3.8) lead to a more precise model than the original equation did (3.1) or (3.3) since the new terms keep track the past of the population. Finally let \( F = (F_1, \ldots, F_n) \) where
\[ \int_{\infty}^{t} k(t-s)g(s,u(s))ds \in [L^2(\Omega)]^{n\times(n+1)} \]
and
\[ F_i : [L^2(\Omega)]^{n\times(n+1)} \to L^2(\Omega), \]
\[ F_i(x_1,x_2,\ldots,x_n,x_{n+1}) = a_i(x_{n+1})_i - (x_{n+1})_i^2 - \sum_{j=1}^{n} (x_{n+1})_i(x_i)_j. \] (3.9)

Since \( k = (k_1,k_2,\ldots,k_{n+1}) \) and \( g = (g_1,g_2,\ldots,g_{n+1}) \) fulfill every requirements listed in Theorem 2.1 we get the following

**Theorem 3.1** Let \( u_i(s,x) = \eta_i(s,x) \) on \([-\tau,0] \times \Omega \) be an initial condition with a priori bound \( B \) of the possible solutions of (3.4). Let further \( k, g \) and \( F \) be as defined by (3.5), (3.6) and (3.9) satisfying the conditions of Theorem 2.1. Then (3.4) - a modified version of (3.1) - has a global solution.

We have to prove only the existence of a global solution. Observe that the condition \( b > c_0 \) (required in (2.9) and in (2.17)) plays no role here because we have not restricted the domain of \( g \). By repeating the method for seeking local solution one can choose a constant \( c - a \) in each steps, i.e. we have a local solution on \([a,c]\) and then \([a,2c-a]\), \([a,3c-2a]\) and so on, where every local solution fulfills the conditions of Theorem 2.1 which ensures the existence of a global solution.

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