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ARTICLE

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Edge-colored complete graphs without properly colored even cycles: A full characterization

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Abstract

The structure of edge-colored complete graphs containing no properly colored triangles has been characterized by Gallai back in the 1960s. More recently, Čada et al. and Fujita et al. independently determined the structure of edge-colored complete bipartite graphs containing no properly colored C_4. We characterize the structure of edge-colored complete graphs containing no properly colored even cycles, or equivalently, without a properly colored C_4 or C_6. In particular, we first deal with the simple case of 2-edge-colored complete graphs, using a result of Yeo. Next, for $k \geq 3$, we define four classes of k-edge-colored complete graphs without properly colored even cycles and prove that any k-edge-colored complete graph without a properly colored even cycle belongs to one of these four classes.

KEYWORDS

complete graph, edge-colored graph, forbidden subgraph, properly colored cycle

1 | INTRODUCTION

We use the textbook [2] for terminology and notation not defined here and we consider finite undirected graphs without loops or multiple edges only.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. If a mapping *col*: $E(G) \to \mathbb{N}$ is specified for the graph G, then G (together with *col*) is called an *edge-colored graph* (or *colored graph* for short). We say G is a *properly colored graph* (or *PC graph*) if each pair of incident edges, i.e., edges sharing precisely one end vertex, are assigned distinct colors.

Let G be a colored graph. Denote by $col(G)$ the set of colors assigned to $E(G)$. We say G is a *k-colored* graph if the cardinality of $col(G)$ is k. The *color degree* of a vertex v in G, denoted by $d_G^c(v)$, is the number of distinct colors assigned to the edges incident with v. If $d^c(v) = 1$ for a vertex $v \in V(G)$, then we say v is a *monochromatic vertex*. We use $\delta^c(G) = \min\{d_G^c(v) | v \in V(G)\}$ to denote the *minimum color degree* of G. In this article, we are interested in characterizing the structure of colored complete graphs containing no even PC cycles. Note that a monochromatic vertex is not contained in any PC cycle. So in the following we only consider colored graphs of minimum color degree at least 2. We use C_ℓ to denote a cycle of length ℓ. We sometimes use triangle instead of C_3.

In graph-theoretical approaches, forbidding certain subgraphs or induced subgraphs is a commonly used method, because graphs without certain specified subgraphs may have very nice structural properties. Well-known examples are forests, bipartite graphs, planar graphs, perfect graphs, and claw-free graphs, to name just a few. Forbidden subgraphs are also well studied in graph coloring [9] and in the research on hamiltonian properties [4].

To stay closer to the subject of this article, we introduce the following results for colored graphs. Yeo [11] proved in 1997 that each colored graph G containing no PC cycle at all must contain a vertex z such that each component of $G - z$ is joined to z with edges of one color or no edge. Already back in the 1960s, Gallai [6] showed that each colored complete graph containing no PC triangle can be partitioned into $k(\geq 2)$ parts such that between all these parts there are (edges of) at most two colors and between each pair of parts there is exactly one color. More recently, Čada et al. [3] and Fujita et al. [5] independently characterized the structure of colored complete bipartite graphs containing no PC C_4. As a corollary, they showed that a colored complete bipartite graph G contains a PC C_4 if $\delta^c(G) \geq 3$. Interestingly, Axenovich et al. [1] proved that the minimum color degree guaranteeing that a colored complete graph contains a PC C_4 is also 3. A natural question is: can we characterize the structure of colored complete graphs which contain no PC C_4? Recently, Magnant et al. [8] studied the existence of monochromatic cliques, cycles, and stars in colored complete graphs that contain no PC C_4. Xu et al. [10] determined the structure of an n-colored K_n containing no PC C_4 and gave sufficient conditions for the existence of PC C_4's in edge-colored graphs. From a computational complexity angle, Gutin et al. [7] studied the complexity of determining the existence of odd PC cycles in edge-colored graphs.

In this article, our two main results deal with characterizing the structure of k-colored complete graphs without PC even cycles, for $k \geq 3$. This turns out to be equivalent to characterizing the structure of k-colored complete graphs without a PC C_4 or C_6. To be able to determine this structure, we first focus on structural properties of colored complete graphs containing no PC C_4. We start with the following observation from [5].

Observation 1 (Fujita et al. [5]). Let G be a colored complete bipartite graph. If $\delta^c(G) \geq 2$, then G contains a PC C_4 or a PC C_6.

Note that each even PC cycle $C_{2k} = a_1b_1a_2b_2 \cdots a_kb_ka_1$ in a colored K_n corresponds to a colored $K_{k,k}$ with partition (A, B), where $A = \{a_i | i \in [1, k]\}$ and $B = \{b_j | j \in [1, k]\}$. Hence, by Observation 1, a colored K_n that contains a PC even cycle also contains a PC C_4 or a PC C_6. So, in a colored K_n, the existence of an even PC cycle is equivalent to the existence of a PC C_4 or a PC C_6. Using this equivalence, for a 2-colored complete graph it is easy to verify that the existence of an even PC cycle is equivalent to the existence of a PC C_4, by considering a PC C_6 if it exists and a chord that splits the C_6 into two C_4s.

The next observation is an easy consequence of the result of Yeo [11] mentioned above, that each colored graph G containing no PC cycle must contain a vertex z such that each component of $G - z$ is joined to z with edges of one color or no edge.

Observation 2. Let G be a 2-colored complete graph with $\delta^c(G) = 2$. Then G contains a PC C_4.

So Yeo's result settles our problem when the minimum color degree is 2. The result clearly implies that a 2-colored complete graph containing no even PC cycle must contain a monochromatic vertex. After deleting this vertex, the remaining graph again contains a monochromatic vertex, etcetera. The structure of the graph is obvious.

For the above reasons, in the remainder of the article we focus on k-colored complete graphs with $k \geq 3$. By the following constructions, we introduce four classes of k-colored complete graphs containing no even PC cycles. These are depicted in Figure 1. Our two main results show that every k-colored complete graph G (with $\delta^c(G) \geq 2$) containing no even PC cycles must belong to one of these classes.

In the sequel, for a nonempty vertex set S and a color c, we sometimes write $col(G[S]) \subseteq \{c\}$ to indicate that $col(G[S]) = \{c\}$ when $|S| \geq 2$ (without specifying the cardinality of S, thus allowing that $|S| = 1$ and $col(G[S]) = \varnothing$). For two disjoint nonempty vertex sets X and Y in a colored graph G, we use (X, Y) to denote the set of edges with one end vertex in X and one end vertex in Y. We use (x, Y) as shorthand for $(\{x\}, Y)$ and we use $col_G(X, Y)$ to denote the set of different colors assigned to the edges of (X, Y).

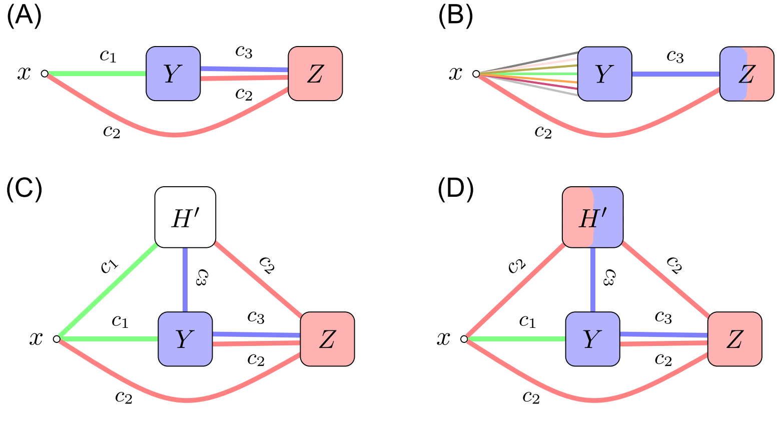

FIGURE 1 The four graph classes of Constructions 1–4. (A) The class $\mathcal{G}_1$; (B) the class $\mathcal{G}_2$; (C) the class $\mathcal{G}_3$; and (D) the class $\mathcal{G}_4$ [Color figure can be viewed at wileyonlinelibrary.com]

Construction 1. *Let $\{x\}$, Y, and Z be three disjoint nonempty vertex sets and let c_1, c_2, c_3 be three distinct colors. Construct a colored complete graph G with $V(G) = \{x\} \cup Y \cup Z$ such that the following conditions hold (See Figure 1A):*

(a) $col(x, Y) = \{c_1\}$, $col(x, Z) = \{c_2\}$, $col(G[Y]) \subseteq \{c_3\}$ *and* $col(G[Z]) \subseteq \{c_2\}$;
(b) $col(Y, Z) \subseteq \{c_2, c_3\}$ *and* (Y, Z) *contains no PC cycle;*
(c) $c_3 \in col(z, Y)$ *for every vertex* $z \in Z$.

Let $\mathcal{G}_1$ denote the set of all colored complete graphs G that are constructed this way.

Construction 2. *Let $\{x\}$, Y and Z be three disjoint nonempty vertex sets and let $c_1, c_2, \ldots, c_k$ be $k \geq 3$ distinct colors. Construct a colored complete graph G with $V(G) = \{x\} \cup Y \cup Z$ such that the following conditions hold (See Figure 1B):*

(a) $col(x, Y) = \{c_1\} \cup \{c_i | 4 \leq i \leq k\}$, $col(x, Z) = \{c_2\}$ *and* $col(Y, Z) = \{c_3\}$;
(b) $col(G[Y]) \subseteq \{c_3\}$, $col(G[Z]) \subseteq \{c_2, c_3\}$ *and* $G[Z]$ *contains no PC cycle.*
Let $\mathcal{G}_2$ denote the set of all colored complete graphs G that are constructed this way.

Note that, in particular, if $k = 3$, then $col(x, Y) = \{c_1\}$ (indicated by green in Figure 1) in the above construction.

Construction 3. *Let $H \in \mathcal{G}_1$ with $V(H) = \{x\} \cup Y \cup Z$ and $col(H) = \{c_1, c_2, c_3\}$ as in Construction 1. Let H' be an arbitrarily colored complete graph containing no even PC cycles such that $|V(H')| \geq 1$ and $V(H) \cap V(H') = \varnothing$. Construct a colored complete graph G by joining H and H' as follows (See Figure 1(C)): $col(x, V(H')) = \{c_1\}$, $col(Z, V(H')) = \{c_2\}$, and $col(Y, V(H')) = \{c_3\}$. Let $\mathcal{G}_3$ denote the set of all colored complete graphs G that are constructed this way.*

Construction 4. *Let $H \in \mathcal{G}_1$ with $V(H) = \{x\} \cup Y \cup Z$ and $col(H) = \{c_1, c_2, c_3\}$ as in Construction 1. Let H' be an arbitrarily colored complete graph with $col(H') \subseteq \{c_2, c_3\}$ such that H' contains no even PC cycles and $V(H) \cap V(H') = \varnothing$. Construct a colored complete graph G by joining H and H' as follows (See Figure 1(D)): $col(x, V(H')) = \{c_2\}$, $col(Z, V(H')) = \{c_2\}$ and $col(Y, V(H')) = \{c_3\}$. Let $\mathcal{G}_4$ denote the set of all colored complete graphs G that are constructed this way.*

We will prove the following two main results.

Theorem 1. *Let G be a 3-colored complete graph with $\delta^c(G) \geq 2$. Then G contains no even PC cycle if and only if $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$.*

Theorem 2. *Let G be a k-colored complete graph with $k \geq 4$ and $\delta^c(G) \geq 2$. Then G contains no even PC cycle if and only if $G \in \mathcal{G}_2 \cup \mathcal{G}_3$.*

Before delivering the proofs of the above two theorems in Section 3, we start off in the next section with presenting some auxiliary definitions and lemmas. The lemmas reveal some useful structural properties of graphs containing no PC C_4 that we use in our proofs of Theorems 1 and 2. In particular, we introduce the concept of vertices that are "friendly" to two of the three colors of a 3-colored complete graph. Based on this concept, the below Lemmas 1 and 2 imply a partition of the vertex set into four classes that will form the basis for our proof of Theorem 1. In that proof, Lemma 3 can be applied to deal with 3-colored complete graphs satisfying an additional assumption on the color properties of one specific vertex. For the remaining cases, we were not able to avoid a rather tedious proof by case distinctions. Based on the result of Theorem 1, we prove Theorem 2 by induction on k.

2 | TERMINOLOGY AND LEMMAS

Let G be a colored complete graph. Recall that for an edge e of G, by $col(e)$ or $col_G(e)$ we denote the color of e and $col(G) = \{col(e)|e \in E(G)\}$. For a color $\alpha \in col(G)$, let $E_G^\alpha = \{e \in E(G)|col_G(e) = \alpha\}$ be the set of edges of color α in G. Also recall that for two disjoint sets $S, T \subseteq V(G)$, we use $col_G(S, T)$ to denote the set of colors appearing in G on the edges between S and T. If $S = \{v\}$, we write $col_G(v, T)$ instead of $col_G(\{v\}, T)$. We use the same notation for two vertex-disjoint subgraphs F and H of G, so we use $col_G(F, H)$ to denote $col_G(V(F), V(H))$. When there is no ambiguity, we often write col instead of col_G. Let v be a vertex in G. We say a color α *appears* at v if $\alpha \in col(v, G - v)$. Let $\{\alpha, \beta\}$ be a pair of distinct colors in $col(G)$. We say v is *friendly* to $\{\alpha, \beta\}$ if there exists a PC triangle $vuwv$ in G such that $col(vu) = \alpha$ and $col(vw) = \beta$.

Our first lemma reveals that in 3-colored complete graphs without a PC C_4, vertices cannot be friendly to more than one pair of colors.

Lemma 1. *Let G be a 3-colored complete graph. If G contains no PC C_4, then each vertex of G is friendly to at most one pair of colors in $col(G)$.*

Proof. Suppose to the contrary that G contains no PC C_4 and there are two PC triangles $vuwv$ and $vu'w'v$ satisfying $\{col(vw), col(vu)\} \neq \{col(vw'), col(vu')\}$. Without loss of generality, assume that $col(G) = \{1, 2, 3\}$, $col(vu) = col(vu') = 1$, $col(vw) = 2$, and $col(vw') = 3$ (and possibly $u = u'$). Then, clearly $col(uw) = 3$ and $col(u'w') = 2$. If $u = u'$, then $wuw'vw$ is a PC C_4, a contradiction. So $u \neq u'$. However, in this case, by considering the color of ww', it is easy to check that either $wvu'w'w$ or $wuvw'w$ is a PC C_4, a contradiction. □

In our second lemma, we use the result of Lemma 1 to partition the vertex set of G and obtain some additional structural properties concerning the colors that appear between the sets in the partition.

Lemma 2. *Let G be a 3-colored complete graph with $col(G) = \{c_1, c_2, c_3\}$, $\delta^c(G) \geq 2$ and containing no PC C_4. Let*

$$\begin{aligned} X &= \{u \in V(G)|u \quad \text{is friendly to} \quad \{c_1, c_2\}\}, \\ Y &= \{u \in V(G)|u \quad \text{is friendly to} \quad \{c_1, c_3\}\}, \\ Z &= \{u \in V(G)|u \quad \text{is friendly to} \quad \{c_2, c_3\}\}, \textit{and} \\ U &= V(G) \backslash (X \cup Y \cup Z). \end{aligned}$$

Then there exist vertices $x \in X$, $y \in Y$ and $z \in Z$ with $col(x, G - x) = \{c_1, c_2\}$, $col(y, G - y) = \{c_1, c_3\}$ and $col(z, G - z) = \{c_2, c_3\}$, such that one of the following statements holds:

(a) $col(x, Y \cup U) = \{c_1\}$, $col(y, Z \cup U) = \{c_3\}$ *and* $col(z, X \cup U) = \{c_2\}$;
(b) $col(x, Z \cup U) = \{c_2\}$, $col(y, X \cup U) = \{c_1\}$ *and* $col(z, Y \cup U) = \{c_3\}$.

Proof. First, we show that there exist three distinct vertices x, y and z such that $col(x, G - x) = \{c_1, c_2\}$, $col(y, G - y) = \{c_1, c_3\}$ and $col(z, G - z) = \{c_2, c_3\}$. Since $\delta^c(G) \geq 2$ and $col(G) = \{c_1, c_2, c_3\}$, it is sufficient to show that for each $c_i (i = 1, 2, 3)$, there exists a vertex v_i in G such that c_i does not appear at v_i in G. Suppose, to the contrary, without loss of generality, that c_1 appears at every vertex of G. Recoloring all the edges in $\{e \in E(G) | col(e) \neq c_1\}$ with a new color α, we obtain a 2-colored complete graph G' with $\delta^c(G') = 2$. By Observation 2, G' contains a PC C_4, implying that G also contains a PC C_4, a contradiction. Now let x, y and z be three distinct vertices with $col(x, G - x) = \{c_1, c_2\}$, $col(y, G - y) = \{c_1, c_3\}$ and $col(z, G - z) = \{c_2, c_3\}$. Then $xyzx$ is a PC triangle satisfying $col(xy) = c_1$, $col(xz) = c_2$ and $col(yz) = c_3$. So, x, y, and z are friendly to $\{c_1, c_2\}$, $\{c_1, c_3\}$ and $\{c_2, c_3\}$, respectively (consequently, $x \in X$, $y \in Y$ and $z \in Z$). To complete the proof, it is sufficient to verify that if neither (a) nor (b) holds, then G contains a PC C_4. We first prove the following claim.

Claim 1. If (b) does not hold, then there exists a vertex $w \in V(G) \backslash \{x, y, z\}$ such that $col(wx) = c_1$, $col(wy) = c_3$ and $col(wz) = c_2$.

Proof. Suppose that (b) does not hold. Then either $col(x, Z \cup U) \neq \{c_2\}$ or $col(y, X \cup U) \neq \{c_1\}$ or $col(z, Y \cup U) \neq \{c_3\}$.

Suppose that $col(x, Z \cup U) \neq \{c_2\}$. Since $col(x, G - x) = \{c_1, c_2\}$, this implies there exists a vertex $w \in Z \cup U$ such that $col(xw) = c_1$. Consider the triangle $wzxw$. Since w is not friendly to $\{c_1, c_3\}$ (otherwise $w \in Y$, a contradiction), it is clear that $col(zw) \neq c_3$. Recalling that $col(z, G - z) = \{c_2, c_3\}$, we obtain that $col(zw) = c_2$. Consider the triangle $wzyw$. Since w is not friendly to $\{c_1, c_2\}$ (otherwise $w \in X$, a contradiction), it is clear that $col(yw) \neq c_1$. This implies that $col(yw) = c_3$ (since $col(y, G - y) = \{c_1, c_3\}$). In summary, w is a vertex in $(Z \backslash \{z\}) \cup U$ such that $col(wx) = c_1$, $col(wy) = c_3$ and $col(wz) = c_2$.

Lemma 1 implies that X, Y, and Z are three disjoint sets. It is easy to check that the statement of (b) is symmetric, in the sense that if we suppose $col(y, X \cup U) \neq \{c_1\}$ (or $col(z, Y \cup U) \neq \{c_3\}$), we can also find a vertex w satisfying the statement in Claim 1. □

Observing the symmetry between (a) and (b) and using Claim 1, we conclude that if (a) does not hold, then there exists a vertex $v \in V(G) \backslash \{x, y, z\}$ such that $col(vx) = c_2$, $col(vy) = c_1$ and $col(vz) = c_3$. If both (a) and (b) do not hold, then $wxvyw$ is a PC C_4, a contradiction. This completes the proof of Lemma 2. □

Our next lemma gives a partial solution to our aim of characterizing the structure of 3-colored complete graphs without PC even cycles. It shows that under slightly stronger assumptions, these graphs are contained in one of the four classes that we constructed earlier.

Lemma 3. *Let G be a 3-colored complete graph with $col(G) = \{c_1, c_2, c_3\}$, $\delta^c(G) \geq 2$ and containing no PC C_4. If there exists a vertex $x \in V(G)$ such that $col(x, G - x) = \{c_1, c_2\}$ and $col(G - x) \subseteq \{c_2, c_3\}$, then $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$.*

Proof. Let $x \in V(G)$ be the vertex satisfying $col(x, G - x) = \{c_1, c_2\}$ and $col(G - x) \subseteq \{c_2, c_3\}$. Let $U = \{u \in V(G) | col(xu) = c_1\}$ and $W = \{w \in V(G) | col(xw) = c_2\}$. Suppose that there exist edges $uu' \in E(G[U])$ and $ww' \in E(G[W])$ such that $col(uu') = c_2$ and $col(ww') = c_3$. Then, noting that $col(G - x) \subseteq \{c_2, c_3\}$ and considering the color of edge uw, it is easy to verify that either $xu'uwx$ or $xw'wux$ is a PC C_4, a contradiction. Hence, we have $col(G[U]) \subseteq \{c_3\}$ or $col(G[W]) \subseteq \{c_2\}$. We complete the proof by distinguishing these two cases.

Case 1. $col(G[U]) \subseteq \{c_3\}$.

In this case, let $W_1 = \{w \in W | \exists\, w' \in W$ such that $col(ww') = c_3\}$. Then either $|W_1| \geq 2$ or $|W_1| = 0$. Let $W_2 = W \backslash W_1$. Then $col(W_1, W_2) \subseteq \{c_2\}$, $col(G[W_2]) \subseteq \{c_2\}$ and $col(W_1, U) \subseteq \{c_3\}$ (otherwise, let $u \in U$, $w_1 \in W_1$ and $w' \in W_1 \backslash \{w_1\}$ be vertices satisfying $col(uw_1) = c_2$ and $col(w_1 w') = c_3$. Then $xuw_1w'x$ is a PC C_4). If $W_2 = \emptyset$, then $W_1 \neq \emptyset$ and we see that $G \in \mathcal{G}_2$ with $Y = U$ and $Z = G[W_1]$. Now assume that $W_2 \neq \emptyset$. Recall that $\delta^c(G) \geq 2$. For each vertex $w_2 \in W_2$, there exists a vertex $u \in U$ such that $col(uw_2) \neq c_2$ (in fact, $col(uw_2) = c_3$). If $W_1 = \emptyset$, then $G \in \mathcal{G}_1$ with $Y = U$ and $Z = W_2$. If $W_1 \neq \emptyset$, then $G \in \mathcal{G}_4$ with $Y = U$, $Z = W_2$ and $H' = G[W_1]$.

Case 2. $col(G[W]) \subseteq \{c_2\}$.

In this case, let $U_1 = \{u \in U | \exists\, u' \in U$ such that $col(uu') = c_2\}$. *Then either* $|U_1| \geq 2$ *or* $|U_1| = 0$. *Let* $U_2 = U \backslash U_1$. *Then* $col(U_1, U_2) \subseteq \{c_3\}$, $col(G[U_2]) = \{c_3\}$ *and* $col(U_1, W) \subseteq \{c_2\}$ *(otherwise, let* $w \in W$, $u_1 \in U_1$ *and* $u' \in U_1 \backslash \{u_1\}$ *be vertices satisfying* $col(wu_1) = c_3$ *and* $col(u_1 u') = c_2$. *Then* $xwu_1u'x$ *is a PC* C_4*). Since* $W \neq \emptyset$ *and* $\delta^c(G) \geq 2$, *for each vertex* $w \in W$, *there must exist a vertex* $u \in U_2$ *such that* $col(wu) \neq c_2$. *Thus* $U_2 \neq \emptyset$. *If* $U_1 = \emptyset$, *then* $G \in \mathcal{G}_1$. *If* $U_1 \neq \emptyset$, *then* $G \in \mathcal{G}_3$ *with* $Y = U_2$, $Z = W$ *and* $H' = G[U_1]$. □

Our final lemma of this section is a simple but useful observation that we use in our proof of Theorem 2.

Lemma 4. *Let G be a k-colored complete graph with* $k \geq 4$. *Then there exist two distinct colors* $a, b \in col(G)$ *such that* $col(v, G - v) \neq \{a, b\}$ *for all* $v \in V(G)$.

Proof. Let $col(G) = \{1, 2, \ldots, k\}$. Suppose to the contrary that for each pair of distinct colors $a, b \in col(G)$, there exists a vertex $v_{a,b}$ such that $col(v_{a,b}, G - v_{a,b}) = \{a, b\}$. Then consider the vertices $v_{1,2}$ and $v_{3,4}$. The color of the edge $v_{1,2}v_{3,4}$ should be contained in $\{1, 2\}$ and also in $\{3, 4\}$, a contradiction. □

We now have all the necessary ingredients to present our proofs of Theorem 1 and Theorem 2.

3 | PROOFS OF THEOREMS 1 AND 2

For convenience of the reader, we first give a rough outline of the proofs of Theorems 1 and 2. The "if" parts of the stated equivalences in both theorems are easy to check. Both "only if" parts require rather involved technical proofs in which we were not able to avoid a number of tedious case distinctions. Apart from these case distinctions, the general structure of the proof of Theorem 1 is as follows. We refer to Figure 2 for an illustration of the different sets of vertices that play a key role in our proof.

Using Lemma 2, we obtain a partition of the vertex set $V(G)$ into three mutually disjoint nonempty sets X, Y, and Z, and a set $U = V(G)\backslash(X \cup Y \cup Z)$, as indicated in Figure 2. For three specifically chosen vertices $x \in X$, $y \in Y$, and $z \in Z$, we then define a set S consisting of vertices u with $col(ux) = 1$, $col(uy) = 3$ and $col(uz) = 2$, and we let $R = V(G)\backslash(S \cup \{x, y, z\})$. Dealing with the case $R = \emptyset$ first, we next assume that $R \neq \emptyset$ and we let $R_x = R \cap X$, $R_y = R \cap Y$ and $R_z = R \cap Z$. We distinguish cases based on this partition, by first showing that we may assume that $R_x = \emptyset$ and $R_z \neq \emptyset$. For R_y, we have to deal with the two options separately, but the main case distinction is between $S = \emptyset$ (Case 1 in the proof) and $S \neq \emptyset$ (Case 2 in the proof). In Case 1, both options for R_y lead to the conclusion that G is in one of the four sets. In Case 2, we first establish two facts on the colors at the edges incident with vertices of R_z and of R_y (if $R_y \neq \emptyset$). Based on these facts, we finally define a vertex set P and deal with the two cases that $P = \emptyset$ and $P \neq \emptyset$. We frequently use Lemma 3 to deal with specific cases.

For our proof of the "only if" part of Theorem 2, we apply induction on k. The proof is based on Lemma 4 of the previous section. This lemma implies that for each k-colored complete graph G with $k \geq 4$ and $\delta^c(G) \geq 2$, there exist two colors a and b such that a $(k-1)$-colored complete graph G' with $\delta^c(G') \geq 2$ can be obtained by recoloring all the edges of color b with color a. The case $k = 4$ leads to a 3-colored complete graph G', to which we can apply Theorem 1. The case $k \geq 5$ leads to a $(k-1)$-colored complete graph G', to which we can apply the induction hypothesis. We analyze G' for (four) different choices of b (because the colors are not symmetric in the definitions of the graphs in $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$), and in each case conclude that $G \in \mathcal{G}_2 \cup \mathcal{G}_3$. Next we present our proof of Theorem 1. For convenience, we recall the statement of Theorem 1.

Theorem 1. *Let G be a* 3*-colored* complete graph with $\delta^c(G) \geq 2$. Then G contains no even PC cycle if and only if $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$.

Proof of Theorem 1. By Constructions 1–4, we know that each colored graph in $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$ contains no even PC cycle. Now let G be a 3-colored complete graph without even PC cycles with $\delta^c(G) \geq 2$ and assume that $col(G) = \{1, 2, 3\}$. We will prove that $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$. Referring to Lemma 2, let X, Y, and Z be the nonempty sets of vertices that are friendly to $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$, respectively. Let $U = V(G)\backslash(X \cup Y \cup Z)$. Lemma 1 implies that X, Y, and Z are three mutually

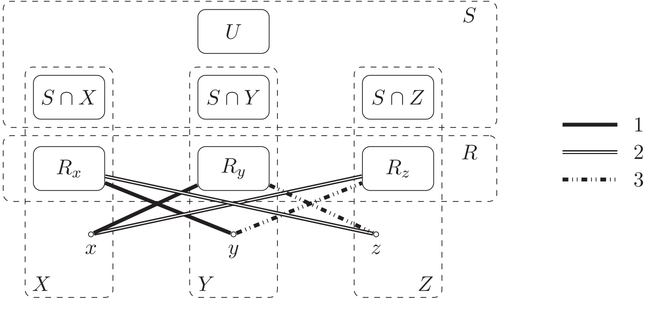

FIGURE 2 The vertex partition of G

disjoint sets. By the symmetry of statements (a) and (b) in Lemma 2, without loss of generality, we assume that there exist vertices $x \in X$, $y \in Y$, and $z \in Z$ such that:

(i) $col(x, G - x) = \{1, 2\}$, $col(y, G - y) = \{1, 3\}$ and $col(z, G - z) = \{2, 3\}$;
(ii) $col(x, Y \cup U) = \{1\}$, $col(y, Z \cup U) = \{3\}$ and $col(z, X \cup U) = \{2\}$.

Based on this basic structure, we can see that $xyzx$ is a PC triangle with $col(xy) = 1$, $col(yz) = 3$ and $col(xz) = 2$. Now we are going to derive additional structural properties and we will consider several cases to conclude that $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$. To start, we define $S = \{u \in V(G) | col(ux) = 1, col(uy) = 3, \text{ and } col(uz) = 2\}$. Then by (ii), it is easy to verify that $U \subseteq S$. Let $R = V(G) \backslash (S \cup \{x, y, z\})$. Then $R \subseteq (X \cup Y \cup Z) \backslash \{x, y, z\}$. See Figure 2 for the relation between these vertex sets.

If $R = \varnothing$, then $V(G) = S \cup \{x, y, z\}$ and it is easy to check that $G \in \mathcal{G}_3$. Next, we assume that $R \neq \varnothing$ and we let $R_x = R \cap X$, $R_y = R \cap Y$, and $R_z = R \cap Z$. Note that not all of these sets need to be nonempty. In fact, some of these sets will turn out to be empty, as we will see after proving the following claim.

Claim 1. The following statements hold (see Figure 2):

(a) $col(R_x, z) \subseteq \{2\}$ and $col(R_x, y) \subseteq \{1\}$;
(b) $col(R_y, x) \subseteq \{1\}$ and $col(R_y, z) \subseteq \{3\}$;
(c) $col(R_z, y) \subseteq \{3\}$ and $col(R_z, x) \subseteq \{2\}$;
(d) if there exist vertices $x' \in R_x$, $y' \in R_y$, and $z' \in R_z$, then $xy'zx'yz'x$ is a PC C_6.

Proof. Suppose there exist vertices $x' \in R_x$, $y' \in R_y$, and $z' \in R_z$. We first prove Claim 2(a). Note that by (ii), $col(z, X \cup U) = \{2\}$, so we have that $col(x'z) = 2$. Suppose that $col(x'y) \neq 1$. Then, since $col(y, G - y) = \{1, 3\}$ by (i), we have that $col(x'y) = 3$. Now consider the color of xx'. Since $col(x'z) = 2$, $col(x'y) = 3$, and $x' \notin S$, we have $col(xx') \neq 1$. Recall that $col(x, G - x) = \{1, 2\}$ by (i). We have $col(xx') = 2$. Then $xx'yx$ is a PC triangle, implying that x' is friendly to $\{2, 3\}$. This contradicts that $x' \in X$. Hence, $col(x'y) = 1$ and Claim 1(a) holds. Similarly, it is easy to validate Claims 1(b) and 1(c). Claim 1(d) follows immediately from (a), (b) and (c) of Claim 1. □

Since G contains no PC C_6, Claim 1 implies that R has a nonempty intersection with at most two of the sets X, Y, and Z. Thus, by the symmetry of X, Y, and Z, and the assumption that $R \neq \varnothing$, without loss of generality, we assume that $R_x = \varnothing$ and $R_z \neq \varnothing$. We prove one more claim before we start a case distinction.

Claim 2. $col(G[R_y \cup \{y\}]) \subseteq \{1, 3\}$ and $col(G[R_z \cup \{z\}]) \subseteq \{2, 3\}$.

Proof. For each pair of distinct vertices $v, v' \in R_y \cup \{y\}$, by Claim 1(b), $col(vx) = 1$ and $col(v'z) = 3$. Recall that $col(xz) = 2$. Since $vxzv'v$ is not a PC C_4, we conclude that $col(vv') \in \{1, 3\}$, hence $col(G[R_y \cup \{y\}]) \subseteq \{1, 3\}$. Similarly, $col(G[R_z \cup \{z\}]) \subseteq \{2, 3\}$. □

We continue the proof by distinguishing the cases that S is empty and nonempty.

Case 1. $S = \varnothing$.

We first consider the subcase that $R_y = \varnothing$. Then $R = R_z$ and $V(G)\backslash\{x\} = R_z \cup \{y, z\}$. Combining (*ii*), Claim 1(c) and Claim 2, we obtain that $col(G[R_z \cup \{y, z\}]) \subseteq \{2, 3\}$, so we have $col(G - x) \subseteq \{2, 3\}$. Recalling that $col(x, G - x) = \{1, 2\}$, using Lemma 3, we conclude that $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$.

Next, we consider the subcase that $R_y \neq \varnothing$. We first prove that either $col(G[R_y \cup \{y\}]) = \{3\}$ or $col(G[R_z \cup \{z\}]) = \{3\}$. Suppose the contrary. Then by Claim 2, there are edges $e \in E(G[R_y \cup \{y\}])$ and $f \in E(G[R_z \cup \{z\}])$ such that $col(e) = 1$ and $col(f) = 2$. We analyze the four possible cases and derive a contradiction in each of the cases, as follows.

- If $e = yu$ and $f = zv$ for some vertices $u \in R_y$ and $v \in R_z$, then $yuzvy$ is a PC C_4, a contradiction.
- If $col(y, R_y) = \{3\}$, $e = uu'$ for $u, u' \in R_y$ and $f = zv$ for $v \in R_z$, then consider the even cycles $uu'zvu$ and $uyxvu$. We obtain that $col(uv) \notin \{1, 3\}$. This implies that $col(uv) = 2$, but then $uu'zxyvu$ is a PC C_6, a contradiction.
- If $col(z, R_z) = \{3\}$, $e = yu$ for $u \in R_y$ and $f = vv'$ for $v, v' \in R_z$, then consider the even cycles $vv'yuv$ and $vzxuv$. We obtain that $col(uv) \notin \{2, 3\}$. This implies that $col(uv) = 1$, but then $vv'yxzuv$ is a PC C_6, a contradiction.
- The final case is that $col(y, R_y) = \{3\}$, $col(z, R_z) = \{3\}$, $e = uu'$ and $f = vv'$ for $u, u' \in R_y$ and $v, v' \in R_z$. Consider the even cycles $uu'zxyvu$ and $vv'yxzuv$. We obtain that $col(uv) \notin \{1, 2\}$, hence $col(uv) = 3$. By the symmetry of $\{u, v\}$ and $\{u', v'\}$, we also have $col(u'v') = 3$. This implies that $uu'v'vu$ is a PC C_4, a contradiction.

So we know that either $col(G[R_y \cup \{y\}]) = \{3\}$ or $col(G[R_z \cup \{z\}]) = \{3\}$. Arbitrarily choose vertices $y' \in R_y$ and $z' \in R_z$. If $col(G[R_y \cup \{y\}]) = \{3\}$, then consider the cycle $xyy'z'x$. We get $col(y'z') \in \{2, 3\}$. Thus $col(R_y, R_z) \subseteq \{2, 3\}$. Combining this with Claims 1 and 2, we get $col(G - x) \subseteq \{2, 3\}$. Note that $col(x, G - x) = \{1, 2\}$. By Lemma 3, we know that $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$. If $col(G[R_z \cup \{z\}]) = \{3\}$, then similarly, we have $col(G - x) \subseteq \{1, 3\}$. By Lemma 3, we know that $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$. This completes the proof for Case 1.

Case 2. $S \neq \varnothing$.

We first establish the following three facts. Recall that we assume $R_x = \varnothing$ and $R_z \neq \varnothing$.

Fact 1. $col(R_z, z) = \{2\}$, $col(R_z, S) \subseteq \{2, 3\}$ and $col(G[R_z]) \subseteq \{2, 3\}$.

Proof. Let s and v be arbitrarily chosen vertices in S and R_z, respectively. By considering the even cycles $vxszv$ and $syxvs$, we observe that $col(zv) = 2$ and $col(sv) \in \{2, 3\}$. Thus $col(R_z, z) = \{2\}$ and $col(R_z, S) \subseteq \{2, 3\}$. Suppose that v' is a vertex in $R_z\backslash\{v\}$. Then consider the even cycle $xv'vsx$. We get $col(vv') \in \{2, 3\}$. So $col(G[R_z]) \subseteq \{2, 3\}$. □

Fact 2. $col(R_y, S \cup \{y\}) \subseteq \{3\}$, $col(G[R_y]) \subseteq \{3\}$ and $col(R_y, R_z) \subseteq \{2, 3\}$.

Proof. It is sufficient to deal with the case that $R_y \neq \varnothing$. Let s, u and v be arbitrarily chosen vertices in S, R_y and R_z, respectively. By considering the even cycle $uzsyu$, we observe that $col(yu) = 3$. Thus $col(R_y, y) = \{3\}$. Consider the even cycle $sxzus$. We get $col(su) \in \{1, 3\}$. If $col(us) = 1$, then consider the even cycle $suzxyvs$. We obtain that $col(vs) \neq 2$. Thus $col(vs) = 3$ (by Fact 1 that $col(R_z, S) \subseteq \{2, 3\}$). Considering the cycle

$uzsvu$, we get $col(uv) = 3$. However, this implies that $suvxyzs$ is a PC C_6, a contradiction. So $col(su) = 3$. In summary, we have $col(R_y, S \cup \{y\}) \subseteq \{3\}$.

Assume that u' is a vertex in $R_y \setminus \{u\}$. Considering the cycle $uszu'u$, we get $col(uu') = 3$. Hence, $col(G[R_y]) \subseteq \{3\}$.

Suppose that there exist vertices $u \in R_y$ and $v \in R_z$ such that $col(uv) \notin \{2, 3\}$. Then $col(vu) = 1$. By Claim 1 and Fact 1, we know that $col(zu) = 3$ and $col(zv) = 2$. Thus $zuvz$ is a PC triangle, which implies that v is friendly to $\{1, 2\}$. This contradicts that $v \in Z$. We conclude that $col(R_y, R_z) \subseteq \{2, 3\}$. □

Let $W = \{w \in S \mid \exists\ u \in R_z \quad \text{such that} \quad col(wu) = 3\}$ and $P = S \setminus W$.

Fact 3. $col(P, R_z) \subseteq \{2\}$, $col(G[W]) \subseteq \{3\}$ and $col(W, P) \subseteq \{3\}$.

Proof. From the definition of W and P, and Fact 1, we conclude that $col(P, R_z) \subseteq \{2\}$. For a vertex $w \in W$, let u be a vertex in R_z such that $col(uw) = 3$. If there is a vertex $w' \in S \setminus \{w\}$, then, since neither $wuxw'w$ nor $wuxyzw'w$ is a PC cycle, we have that $col(ww') = 3$. Hence, $col(w, S \setminus \{w\}) \subseteq \{3\}$ for every vertex $w \in W$, so $col(G[W]) \subseteq \{3\}$ and $col(W, P) \subseteq \{3\}$.

We are going to complete the proof by distinguishing the subcases that $P = \emptyset$ and $P \neq \emptyset$. First suppose that $P = \emptyset$. Then Facts 1, 2, and 3 imply that $col(G - x) \subseteq \{2, 3\}$. Recall that $col(x, G - x) = \{1, 2\}$. Now, G satisfies the conditions of Lemma 3, so $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$. Next, assume that $P \neq \emptyset$. Let p be a vertex in P. For each pair of distinct vertices $u, u' \in R_z \cup \{z\}$, consider the cycle $upxu'u$. We obtain that $col(uu') = 2$. Thus $col(R_z \cup \{z\}) = \{2\}$. Now, we observe that $G \in \mathcal{G}_3$, with $R_y \cup W \cup \{y\}$ in the role of Y in Construction 3, $R_z \cup \{z\}$ in the role of Z and $G[P]$ in the role of H'.

This completes the proof of Theorem 1. □

We finish this section and article by presenting our proof of Theorem 2. For convenience, we recall the statement of Theorem 2.

Theorem 2. *Let G be a k-colored complete graph with $k \geq 4$ and $\delta^c(G) \geq 2$. Then G contains no even PC cycle if and only if $G \in \mathcal{G}_2 \cup \mathcal{G}_3$.*

Proof of Theorem 2. By Constructions 2 and 3, we know that each colored graph in $\mathcal{G}_2 \cup \mathcal{G}_3$ contains no even PC cycle. Now let G be a k-colored complete graph without even PC cycles with $k \geq 4$ and $\delta^c(G) \geq 2$. We will prove that $G \in \mathcal{G}_2 \cup \mathcal{G}_3$ by using Theorem 1 and applying induction on k. For $k \geq 5$, assume that Theorem 2 holds for each colored graph with at most $k - 1$ colors. Let $col(G) = \{1, 2, ..., k\}$. Since $k \geq 4$, by Lemma 4, there exist two distinct colors $a, b \in col(G)$ such that $col(v, G - v) \neq \{a, b\}$ for all $v \in V(G)$. Without loss of generality, assume that $a = k$ and $b \in \{1, 2, ..., k - 1\}$. Recolor all the edges with color k in G using color b. This way we obtain a colored graph G' without even PC cycles with $col(G') = \{1, 2, ..., k - 1\}$ and $\delta^c(G') \geq 2$. By Theorem 1 and the induction hypothesis, $G' \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$ when $k = 4$ and $G' \in \mathcal{G}_2 \cup \mathcal{G}_3$ when $k \geq 5$. For each $i \in [1, 4]$, if $G' \in \mathcal{G}_i$, then assume $c_j = j$ for $j = 1, 2, ..., k - 1$. The notations x, Y, Z and H' referring to Constructions 1–4 will be frequently used in our proof.

Note that $E_G^k \subset E_{G'}^b$. We say an edge $e \in E(G)$ is *recolored* if $col_G(e) = k$ and $col_{G'}(e) = b$. For two edges $e, f \in G'$, if e being recolored implies that f is recolored too when $G' \in \mathcal{G}_i$ for some $i \in [1,4]$, then we write $e \rightsquigarrow^i f$. If $e \rightsquigarrow^i f$ for all $i \in [1,4]$, then we write $e \rightsquigarrow f$.

Let y and z be arbitrary vertices in Y and Z, respectively, and let s be an arbitrary vertex in $V(H')$ when $G' \in \mathcal{G}_3 \cup \mathcal{G}_4$. We distinguish four cases, depending on the value of b.

Case 1. $b = 1$.

First suppose that $G' \in \mathcal{G}_1 \cup \mathcal{G}_4$. Then $|col(G')| = 3$ and $k = 4$. Since $E_G^k \subset E_{G'}^1$, we get $col_G(x, Y) = \{1, k\}$. If $col_G(z, Y) = \{2, 3\}$, then the bipartite graph $B = (\{x, z\}, Y)$ satisfies that $\delta^c(B) \geq 2$. Thus, by Observation 2, B contains an even PC cycle (actually, a PC C_4), a contradiction. Recall Constructions 1 and 4. We can see that $3 \in col_G(z, Y)$. So $col_G(z, Y) = \{3\}$. Since z is an arbitrarily chosen vertex in Z, we have $col_G(Y, Z) = \{3\}$. This implies that $G \in \mathcal{G}_2$ with $c_i = i$ for $i = 1, 2, 3, 4$.

Next, suppose that $G' \in \mathcal{G}_3$ and let $\{x\} \cup Y \cup Z \cup V(H')$ be a partition of $V(G)$ with $|V(H')|$ as large as possible. Then we deduce that $3 \in col(y, Z)$ (otherwise, we should move y to H'). Let $z' \in Z$ satisfying $col(yz') = 3$. Then, by considering the cycle $xyz'sx$, we get that $xy \rightsquigarrow^3 xs$ and $xs \rightsquigarrow^3 xy$. So, either all the edges between x and $V(H') \cup Y$ are recolored or none of them are. In both cases, we conclude that $G \in \mathcal{G}_3$.

The remaining case is that $G' \in \mathcal{G}_2$. In this case, it is easy to see that $G \in \mathcal{G}_2$.

Case 2. $b = 2$.

In this case, let z' be an arbitrary vertex in $Z \backslash \{z\}$ when $|Z| \geq 2$. Let $y' \in Y$ such that $col_G(zy') = 3$ (by Constructions 1–4, y' always exists). We first prove the following claim.

Claim 1. If z is incident with a recolored edge, then all the edges in $E_{G'}^2 \cap (z, G - z)$ are recolored.

Proof. It is sufficient to prove the following three statements:

(i) if $col_G(zy) = 2$, then $zx \rightsquigarrow zy$ and $zy \rightsquigarrow zx$;
(ii) if $|Z| \geq 2$ and $col_{G'}(zz') = 2$, then $zx \rightsquigarrow zz'$ and $zz' \rightsquigarrow zx$;
(iii) $zx \rightsquigarrow^i zs$ and $zs \rightsquigarrow^i zx$ for $i = 3, 4$.

If $col_G(zy) = 2$, then $y \neq y'$. By considering the cycle $xy'yzx$, we get (i). If $|Z| \geq 2$ and $col_{G'}(zz') = 2$, then let $y'' \in Y$ satisfy that $col_G(y''z') = 3$. By considering the cycle $xy''z'zx$, we get (ii). When $G' \in \mathcal{G}_3 \cup \mathcal{G}_4$, the vertex s exists and $col(sz) = 2$. By considering the cycle $syxzs$, we get (iii). So all the edges in $E_{G'}^2 \cap (z, G - z)$ are recolored.

Now we proceed with the proof of Case 2 by distinguishing the following two subcases. □

Subcase 2.1.
None of the vertices in Z is incident with a recolored edge.

In this case, it is easy to see that either $G' \in \mathcal{G}_3$ with all the recolored edges contained in $E(H')$, or $G' \in \mathcal{G}_4$ with the recolored edges contained in $E(H') \cup (x, H')$. In the former case, clearly, $G \in \mathcal{G}_3$ with $c_i = i$ for $i = 1, 2, 3$. For the latter case, by considering the cycle $sxy'zs$, we know that xs is not recolored. This implies that ss' is not recolored for all edges $ss' \in E(H')$ (considering the cycle $xss'yx$). Thus $E^2_{G'} = E^2_G$ and $E^k_G = \varnothing$, a contradiction.

Subcase 2.2.
The vertex z is incident with a recolored edge.

First suppose $G' \in \mathcal{G}_1 \cup \mathcal{G}_3 \cup \mathcal{G}_4$. Then $col(G'[Z]) \subseteq \{2\}$. By Claim 1, every vertex in $Z \backslash \{z\}$ is joined to z by a recolored edge. Applying Claim 1 to every vertex in $Z \backslash \{z\}$, we obtain that all the edges in $E(G[Z]) \cup \left(E^2_{G'} \cap (Z, G - Z)\right)$ are recolored edges. Now, it is impossible that $G' \in \mathcal{G}_1$; otherwise $E^k_G = E^2_{G'}$, a contradiction. It is also impossible that $G' \in \mathcal{G}_4$; otherwise, we assert that all the other edges of color 2 which are not incident to Z in G' are also recolored edges. Since $sxy'zs$ is not a PC C_4, the edge xs must be recolored. Let s' be an arbitrary vertex in $V(H') \backslash \{s\}$ when $|V(H')| \geq 2$. If $col(ss') = 2$, then we can see that ss' is also recolored by considering the cycle $xss'yx$. In summary, all the edges in $E^2_{G'}$ are recolored edges, so $E^2_{G'} = E^k_G$, a contradiction. The remaining subcase is that $G' \in \mathcal{G}_3$. This implies that $G \in \mathcal{G}_3$ with $c_1 = 1$, $c_2 = k$, and $c_3 = 3$.

Next, suppose $G' \in \mathcal{G}_2$. Then let G_0 be the subgraph of $G[Z]$ induced by $E^2_{G'}$. Clearly, G_0 is connected (otherwise, let uv and $u'v'$ be edges from two distinct components of G_0. Then $uvu'v'u$ is a PC C_4 in G, a contradiction). If $|V(G_0)| = 0$, then $col(G[Z]) = \{3\}$ and G is obtained by joining x to a monochromatic clique. So $G \in \mathcal{G}_2$. The remaining case is that $|V(G_0)| \geq 2$. If $z \in V(G_0)$, then for each vertex $u \in V(G_0) \backslash \{z\}$, there exists a path $zu_1u_2\ldots u_t$ in G_0 with $u = u_t$. Recall that all the edges in $E^2_{G'} \cap (z, G - z)$ are recolored. So zu_1 is recolored. Apply Claim 1 to u_1. We get that all the edges in $E^2_{G'} \cap (u_1, G - u_1)$ are recolored. Thus u_1u_2 is recolored. By repeating this process, we can finally prove that all the edges in $E^2_{G'} \cap (u, G - u)$ are recolored. In summary, all the edges in G_0 are recolored and $col_G(x, V(G_0)) = \{k\}$. Since $E^2_{G'} \neq E^k_G$, the set $U = \{u \in Z \,|\, col_G(xu) = 2\}$ is nonempty. Let $Y^* = Y \cup U$ and $Z^* = Z \backslash U$. Then $\{x\} \cup Y^* \cup Z^*$ is a partition of G showing that $G \in \mathcal{G}_2$ with $c_2 = k$, $c_3 = 3$ and $\{3, k\} \cap col_G(x, Y^*) = \varnothing$.

Case 3. $b = 3$.

Since the bipartite graph induced by (Y, Z) contains no PC cycles, there must exist a vertex $y_0 \in Y$ such that $col_{G'}(y_0, Z) = \{3\}$. Let y' be an arbitrary vertex in $Y \backslash \{y\}$ when $|Y| \geq 2$. If $G' \in \mathcal{G}_3$, then assume that $V(H') \cup \{x\} \cup Y \cup Z$ is a partition of $V(G')$ as in Construction 3 such that $V(H')$ is as large as possible. Then for each vertex $y \in Y$, $3 \in col_{G'}(y, Z)$ (otherwise, we can move y into $V(H')$ and G' would still satisfy the rules of Construction 3). We distinguish three subcases.

Subcase 3.1.
None of the vertices of Y is incident with a recolored edge.

In this case, since $E_G^k \neq \emptyset$, it is impossible that $G' \in \mathcal{G}_1$. It is also impossible that $G' \in \mathcal{G}_2 \cup \mathcal{G}_4$; otherwise, we can find a recolored edge uu' in $G[Z]$ when $G \in \mathcal{G}_2$ (in H' when $G \in \mathcal{G}_4$). Thus $xyuu'x$ is a PC C_4, a contradiction. The only remaining case is that $G' \in \mathcal{G}_3$. In this case, all the recolored edges are contained in H'. Then $G \in \mathcal{G}_3$ with $c_i = i$ for $i = 1, 2, 3$.

Subcase 3.2.
$|Y| \geq 2$ and the vertex y is incident with some recolored edges.

We will prove that all the edges in $E(G[Y]) \cup \left(E_{G'}^3 \cap (Y, G - Y)\right)$ are recolored. Recall that s is an arbitrarily chosen vertex in H' when $G' \in \mathcal{G}_3 \cup \mathcal{G}_4$. Consider the even cycles $xy'yzx$, $xsyzx$, and $xy'ysx$. We easily obtain the following three assertions:

(i) if $col_G(zy) = 3$, then $yy' \rightsquigarrow yz$ and $yz \rightsquigarrow yy'$;
(ii) if $col_G(zy) = 3$, then $yz \rightsquigarrow^3 ys$ and $ys \rightsquigarrow^3 yz$;
(iii) $yy' \rightsquigarrow^4 ys$ and $ys \rightsquigarrow^4 yy'$.

If $G' \in \mathcal{G}_1 \cup \mathcal{G}_2$, then (i) implies that all the edges in $E_{G'}^3 \cap (y, G - y)$ are recolored. If $G' \in \mathcal{G}_3 \cup \mathcal{G}_4$, then recall the assumption that $3 \in col_{G'}(y, Z)$ and apply (i), (ii), and (iii) to y. We conclude that all the edges in $E_{G'}^3 \cap (y, G - y)$ are recolored. In all cases, every vertex in $Y \backslash \{y\}$ is incident with a recolored edge. Now apply (i), (ii) and (iii) to every vertex in $Y \backslash \{y\}$. We conclude that all the edges in $E(G[Y]) \cup (E_{G'}^3 \cap (Y, G - Y))$ are recolored edges.

If $G' \in \mathcal{G}_1$, then $E_{G'}^3 = E_G^k$, a contradiction. If $G' \in \mathcal{G}_2 \cup \mathcal{G}_4$, then let ww' be an edge of color 3 in $G'[Z]$ when $G' \in \mathcal{G}_2$ (in H' when $G' \in \mathcal{G}_4$). Consider the cycle $xww'yx$. The edge ww' must be recolored. Thus $E_{G'}^3 = E_G^k$, again a contradiction. So, the only possible case is that $G' \in \mathcal{G}_3$ with all the recolored edges contained in H'. Then $G \in \mathcal{G}_3$ with $c_1 = 1$, $c_2 = 2$, and $c_3 = k$.

Subcase 3.3.
$Y = \{y_0\}$ and y_0 is incident with some recolored edges.

Since $|Y| = 1$, we have $|col_G(x, Y)| = 1$. Recall that in Constructions 1, 2, 3, and 4, for each vertex $u \in Z$, there exists a vertex $v \in Y$ such that $col(uv) = 3$. So we have $col_{G'}(y_0, Z) = \{3\}$. If $G' \in \mathcal{G}_4$, then by merging H' into Z, we can see that G' is also in $\mathcal{G}_2$ with $c_i = i$ for $i = 1, 2, 3$. Hence it is sufficient to distinguish the three subcases that $G' \in \mathcal{G}_i$ for $i = 1, 2, 3$.

If $G' \in \mathcal{G}_1$, then $col(G[Z]) \subseteq \{2\}$ and all the recolored edges in G form a proper subset of (y_0, Z). Let $x^* = y_0$, $Y^* = Z$ and $Z^* = \{x\}$. Then the partition $\{x^*\} \cup Y^* \cup Z^*$ shows that $G \in \mathcal{G}_2$ with $k = 4$, $c_1 = 3$, $c_2 = 1$, $c_3 = 2$, and $c_4 = 4$.

If $G' \in \mathcal{G}_3$, then consider the cycle xzy_0sx. We get $y_0z \rightsquigarrow^3 y_0s$ and $y_0s \rightsquigarrow^3 y_0z$. This implies that all the edges in $E_{G'}^3 \cap (y_0, G - y_0)$ are recolored. Thus $G \in \mathcal{G}_3$ with $c_1 = 1$, $c_2 = 2$, and $c_3 = k$.

If $G' \in \mathcal{G}_2$, then define $x^* = \{y_0\}$, $Y^* = \{x\}$, and $Z^* = Z$. Then the partition $\{x^*\} \cup Y^* \cup Z^*$ shows that $G' \in \mathcal{G}_2$ with $c_1 = 1$, $c_2 = 3$, and $c_3 = 2$. Let $c_4 = 4$. We observe that G' is obtained by recoloring edges of color c_4 in G with color c_2. So this case can be verified as that in Case 2.

Case 4. $b \in [4, k-1]$.

In this case, $|col(G')| \geq 4$. So $G' \in \mathcal{G}_2 \cup \mathcal{G}_3$. If $G' \in \mathcal{G}_3$, then all the recolored edges are contained in $E(H')$ and $G \in \mathcal{G}_3$. If $G' \in \mathcal{G}_2$, then it is easy to see that $G \in \mathcal{G}_2$ with $c_i = i$ for $i = 1, 2, \ldots, k$. This completes the proof of Theorem 2. □

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