Modeling of Geometric Compliant Contacts for Grasping Applications

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1 Introduction

The modeling of multidimensional contacts is of fundamental importance in order to be able to simulate grasping strategies. Such models should also be reasonably simple in order to be used for control purposes. This implies that FEM models are not suitable for this goal, but at the same time we would like to reproduce the basic viscoelastic features of contacts in 3D.

In the literature different contact models can be found. Most of them assume that the contacting objects have very specific shapes like for example the contact models described in [7] and [12]: [7] which is used inside the ERA Simulation Facility (ESF), can only model the contact between points and planes, while [12] assumes that the surfaces of the objects are linear or quadratic. Montana\textsuperscript{14, 15} on the other hand only assumes that the objects have finite curvature, which means that the objects may not have infinitely sharp edges. This is the most general model because in reality edges are never infinitely sharp even if this has numerical consequences. Montana's model however has the following two limitations. First it assumes that the objects are in contact. It cannot detect contact while [7] and [12] can. For the points and planes in [7] the contact detection is trivial and [12] uses a numerical optimization algorithm to detect contact. Secondly Montana's model is only a kinematic model while [7] and [12] are fully dynamic models. [7] treats the objects as infinitely stiff but [12] takes the finite object stiffness into account. Both contact detection and dynamics are essential in creating controllers for dextrous robots performing contact tasks. This chapter, which is based on previous work\textsuperscript{21, 6}, generalizes Montana's contact kinematics so that it can also detect contact, and then uses the generalized contact kinematics in a dynamic viscoelastic setting. Furthermore it extends [21, 6] for anisotropic materials. It presents some novel work whose contributions reside in geometric, power consistent, port description of contact dynamics. It is also pointed out in [2], that a tractable model of contact compliance, particularly in the tangential direction is not trivial. It is important to stress that the goal of this
work is NOT to give an exact and detailed model of contact, but rather to give a simplified spatial model which can be used efficiently to describe and simulate contact situations in grasping tasks.

For issues related to the restitution coefficient the reader can consult [10] which presents the Hunt-Crossley model which extends the linear Kelvin-Voight model. A good reference for a detailed tangential friction model can be found in [1]. An excellent reference on soft finger contacts is [4]. A nice analysis on controllability of rolling contacts can be found in [13] and for a general review on grasping and contacts the reader is addressed to [2].

The chapter is organized as follows: Sect. 2 reviews background material and can be skipped for readers familiar with the topics or just quickly browsed for the notation introduced. Sect. 3 introduces the basics of the presented model. Sect. 4 which is based on [6] presents an analytical method to detect the contact of the bodies. Sect. 5 which is the main section of the chapter presents how to handle the elastic coupling during contact and Sect. 6 the dissipative part. To conclude Sect. 7 briefly introduces the basic ideas which will be further developed about slipping, Sect. 8 briefly illustrates a grasping simulation and Sect. 9 concludes the chapter pointing out issues for future research.

2 Background

This section reviews background material which can be useful for the understanding of the material concerning the modeling of contacts. Some parts will be also more extended than strictly necessary for the sake of completeness.

2.1 Matrix Lie Groups

A manifold is intuitively a smooth space which is locally homeomorphic to \( \mathbb{R}^n \) and brings with itself nice differentiability properties. Proper definitions of manifolds can be found on [5, 3]. A group is an algebraical structure defined on a set. Definitions of groups can be found on any basic book of algebra.

A Lie group is a group, whose set on which the operation are defined is a manifold \( \mathcal{G} \). This manifold \( \mathcal{G} \) has therefore a special point ‘e’ which is the identity of the group.

Using the structure of the group, and by denoting the group operation as:

\[
\circ : \mathcal{G} \times \mathcal{G} \to \mathcal{G} ; \ (h, g) \mapsto h \circ g,
\]

we can define two mappings within the group which are called respectively left and right mapping:

\[
L_g : \mathcal{G} \to \mathcal{G} ; \ h \mapsto g \circ h
\]

\[
R_g : \mathcal{G} \to \mathcal{G} ; \ h \mapsto g \circ h
\]

(1)

and
As we will see later the differential of these mappings at the identity, plays an important role in the study of mechanics.

The tangent space $T_eG$ to $G$ at $e$, which is indicated with $\mathfrak{g}$, has furthermore the structure of a Lie algebra which is nothing else than a vector space together with an internal, skew-symmetric operation called the commutator:

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} ; (g_1, g_2) \mapsto [g_1, g_2]$$

For $\mathfrak{g}$ to be a Lie algebra, the commutator should furthermore satisfy what is called the Jacobi identity:

$$[[g_1, g_2], g_3] + [[g_2, g_3], g_1] + [[g_3, g_1], g_2] = 0 \quad \forall g_1, g_2, g_3 \in \mathfrak{g}$$

Lie groups are important because we can use them as acting on a manifold $\mathcal{M}$, which in our case will be the Euclidean space. An action of $G$ on $\mathcal{M}$ is a smooth application of the following form:

$$a : G \times \mathcal{M} \to \mathcal{M}$$

such that

$$a(e, x) = x \quad \forall x \in \mathcal{M},$$

and

$$a(g_1, a(g_2, x)) = a(g_1g_2, x) \quad \forall x \in \mathcal{M}, g_1, g_2 \in G.$$
\[ L_G(H) = GH \quad \text{and} \quad R_G(H) = HG. \]

We can now consider how velocities are mapped using the previous maps. Suppose that we want to map a velocity vector \( \dot{H} \in T_H \mathcal{G} \) to a velocity vector in \( T_{GH} \mathcal{G} \) using the left translation and to a vector in \( T_{HG} \mathcal{G} \) using right translation. We obtain:

\[
(L_G)_*(H, \dot{H}) = (GH, G\dot{H}) \quad \text{and} \quad (R_G)_*(H, \dot{H}) = (HG, \dot{HG})
\]

In particular, if we take a reference velocity at the identity, we obtain:

\[
(L_G)_*(I, T) = (G, GT) \quad \text{and} \quad (R_G)_*(I, T) = (G, TG)
\]

where \( T \in \mathfrak{g} \), the Lie algebra. With an abuse of notation, we will often indicate:

\[
(L_G)_* T = GT \quad \text{and} \quad (R_G)_* T = TG
\]

when it is clear that we consider mappings from the identity of the group. On a Lie group, we can define left invariant or right invariant vector fields. These vector fields are such that the differential of the left invariant and right invariant map leaves them invariant. If we indicate with

\[
V : \mathcal{G} \to T \mathcal{G} ; \quad x \mapsto (x, v)
\]

a smooth vector field on the Lie group \( \mathcal{G} \), we say that this vector field is left invariant if:

\[
V(L_g(h)) = (L_g)_* V(h) \quad \forall g, h \in \mathcal{G},
\]

and similarly it is right invariant if:

\[
V(R_g(h)) = (R_g)_* V(h) \quad \forall g \in \mathcal{G}.
\]

For a matrix group, if we take in the previous definitions \( h = I \) we obtain respectively:

\[
V(G) = GT_L \quad \text{and} \quad V(G) = T_R G
\]

where we indicated the representative of the left and right invariant vector fields at the identity with \( T_L \) and \( T_R \). We can conclude from this that any left or right invariant vector field is characterized completely by its value at the identity of the group. We could now ask ourselves: what are the integrals of a left or invariant vector field? From what just said, the integral of a left invariant vector field, can be calculated as the integral of the following matrix differential equation:

\[
\dot{G} = GT_L \Rightarrow G(t) = G(0)e^{T_L t}
\]

(5)

where \( T_L \) is the value of the vector field at the identity. In a similar way, the integral of a right invariant vector field is:
\[ G(t) = e^{T_R t} G(0). \]

From this it is possible to conclude that if we take an element \( T \in \mathfrak{g} \), its left and right integral curves passing through the identity coincide and they represent the exponential map from the Lie algebra to the Lie group:

\[ e : \mathfrak{g} \to G; T \mapsto e^T. \]

It is easy to show, and important to notice, that integral curves passing through points \( H = e^{T_1} \) of right and left invariant vector fields which have as representative in the identity \( T_2 \), are coincident iff \( e^{T_1} e^{T_2} = e^{T_2} e^{T_1} \) which is true iff \([T_1, T_2] = 0\), where the last operation is the commutator of the Lie algebra. But how does the commutator look like for a matrix Lie algebra? Being a Lie group a manifold, we can compute the Lie brackets of vector fields on the manifold. Furthermore, we know that elements of the Lie algebra \( \mathfrak{g} \) have a left and right vector field associated to them. We can then calculate the Lie bracket of two left or right invariant vector fields, and if the solution is still left or right invariant, consider the value of the resulting vector field at the identity as the solution of the commutator. We will start with the left invariant case first. Consider we are in a point \( G(t) \in \mathfrak{g} \) at time \( t \). If we have two left invariant vector fields characterized by \( T_1, T_2 \in \mathfrak{g} \), the Lie bracket of these two vector fields, can be calculated by moving from \( G(t) \) along the vector field correspondent to \( T_1 \) for \( \sqrt{s} \) time, then along the one correspondent to \( T_2 \), then along \(-T_1\) and eventually along \(-T_2\). In mathematical terms we have:

\[ G(t + \sqrt{s}) = G(t)e^{T_1 \sqrt{s}} \Rightarrow G(t + 2\sqrt{s}) = G(t + \sqrt{s})e^{T_2 \sqrt{s}} \Rightarrow \]
\[ G(t + 3\sqrt{s}) = G(t + 2\sqrt{s})e^{-T_1 \sqrt{s}} \Rightarrow G(t + 4\sqrt{s}) = G(t + 3\sqrt{s})e^{-T_2 \sqrt{s}} \Rightarrow G(t + 4\sqrt{s}) = G(t)e^{T_1 \sqrt{s}} e^{T_2 \sqrt{s}} e^{-T_1 \sqrt{s}} e^{-T_2 \sqrt{s}} \]

If we look at \( \frac{d}{ds} G(t + 4\sqrt{s}) \bigg|_{s=0} \), we can approximate the exponentials with the first low order terms and we obtain:

\[ G(t + 4\sqrt{s}) \simeq G(t) \left( \left( I + T_1 \sqrt{s} + \frac{T_1^2}{2} s \right) \left( I + T_2 \sqrt{s} + \frac{T_2^2}{2} s \right) \right. \]
\[ \left. \left( I - T_1 \sqrt{s} + \frac{T_1^2}{2} s \right) \left( I - T_2 \sqrt{s} + \frac{T_2^2}{2} s \right) \right) \]
\[ \simeq G(t)(I + (T_1 T_2 - T_2 T_1)s + o(s)) \]

which implies

\[ \frac{d}{ds} G(t + 4\sqrt{s}) \bigg|_{s=0} = G(t)(T_1 T_2 - T_2 T_1). \]

From the previous equation, we can conclude that the resulting vector field is still left invariant and it is characterized by the Lie algebra element \( T_1 T_2 - \).
$T_2T_1$. We can therefore define the commutator based on left invariant vector fields as:

$$[T_1, T_2]_L = T_1T_2 - T_2T_1.$$ 

With similar reasoning, it is possible to show for right invariant vector fields that:

$$\left. \frac{d}{ds} G(t + 4\sqrt{s}) \right|_{s=0} = (T_2T_1 - T_1T_2)G(t).$$

and therefore, in this case:

$$[T_1, T_2]_R = T_2T_1 - T_1T_2.$$ 

We have therefore that:

$$[T_1, T_2]_L = -[T_1, T_2]_R.$$ 

In the literature, $[,]_L$ is used as the standard commutator and we will adopt this convention.

**Matrix Group Actions**

A group action we can consider for an $n$ dimensional matrix Lie group is the linear operation on $\mathbb{R}^n$. We can therefore define as an action:

$$a(G, P) = GP \quad G \in \mathcal{G}, P \in \mathbb{R}^n$$

It is easy to see that this group action trivially satisfies all the properties required.

**Adjoint representation**

Using the left and right maps, we can define what is called the conjugation map as $K_g := R_g^{-1}L_g$ which for matrix groups results:

$$K_G : \mathcal{G} \rightarrow \mathcal{G} ; H \mapsto GHG^{-1}.$$ 

But what is the importance of this conjugation map? To answer this question, we need the matrix group action. Suppose we have a certain element $H \in \mathcal{G}$ such that $Q = HP$ where $Q, P \in \mathbb{R}^n$. What happens if we move all the points of $\mathbb{R}^n$ and therefore also $Q$ and $P$ using an element of $\mathcal{G}$? What will the corresponding mapping of $H$ look like? If we have $Q' = GQ$ and $P' = GP$, it is straightforward to see that:

$$Q' = K_G(H)P'.$$

The conjugation map is therefore related to global motions or equivalently changes of coordinates. We clearly have that $K_G(I) = I$ and therefore the
differential of $K_G()$ at the identity is a Lie algebra endomorphism. This linear map is called the Adjoint group representation:

$$Ad_G : \mathfrak{g} \to \mathfrak{g} ; \ T \mapsto GTG^{-1}.$$  

The Adjoint representation of the group shows how an infinitesimal motion changes moving the references of a finite amount $G$. Eventually, it is possible to consider the derivative of the previous map at the identity

$$ad_T := \left. \frac{d}{ds} Ad_{e^s T} \right|_{s=0} .$$

This map is called the adjoint representation of the Lie algebra and it is a map of the form:

$$ad_T : \mathfrak{g} \to \mathfrak{g} \quad T \in \mathfrak{g}$$

If we use the definitions we can see that:

$$\frac{d}{ds} Ad_{e^s T_1} T_2 \right|_{s=0} = \frac{d}{ds} e^{T_1 s} T_2 e^{-T_1 s} \right|_{s=0} = T_1 T_2 - T_2 T_1 = [T_1, T_2]_L$$

which shows that:

$$ad_{T_1; T_2} = [T_1, T_2]_L \quad (9)$$

### 2.2 Motions of Rigid Bodies

In this chapter, we deal with rigid bodies moving in the Euclidean space $E$, which means we can describe the position and orientation of every body by an element of the matrix Lie Group called Special Euclidean group and denoted with $SE(3)$, once a reference frame has been chosen. As shown for example in [16, 19], elements of this group can be represented by a homogeneous matrix of the form

$$H_i^j = \begin{bmatrix} R_i^j & p_i^j \\ 0 & 1 \end{bmatrix}$$

where $R_i^j$ is a rotation matrix which is an element of the matrix Lie group called Special Orthonormal and denoted with

$$SO(3) := \{ R \in \mathbb{R}^{3 \times 3} \text{ s.t. } R^{-1} = R^T, \det(R) = 1 \} \quad (10)$$

and $p_i^j$ is a vector in $\mathbb{R}^3$.

$H_i^j$ denotes the change of coordinates from a right-handed coordinate frame $\Psi_j$ to another right-handed coordinate frame $\Psi_i$ and can thus be used for example to describe the position and orientation of a body (with attached coordinate frame $\Psi_j$) relative to a reference (inertial) coordinate frame ($\Psi_i$).

The instantaneous velocity of a body $B_i$ relative to a body $B_j$ can be numerically represented by a twist $T_i^{k,j}$ in frame $\Psi_k$, with
\[ T_{i,j}^{k,i} = \begin{bmatrix} \omega_{k,j}^{i} \\ v_{i,j}^{k} \end{bmatrix} \]

where \( \omega_{k,j}^{i} \) denotes the angular velocity of body \( B_i \) relative to body \( B_j \) expressed in coordinate frame \( \Psi_k \), and \( v_{i,j}^{k} \) denotes the instantaneous velocity (relative to body \( B_j \)) of the point fixed with respect to body \( B_i \) that passes through the origin of frame \( \Psi_k \). A twist can be regarded as the derivative of a homogeneous matrix in the following way, using right translations of a Lie group as explained in Sect. 2.1:

\[ \tilde{T}_{i,j}^{k,j} := \begin{bmatrix} \tilde{\omega}_{j,j}^{i} \\ v_{j,j}^{i} \end{bmatrix} = \dot{H}_{j}^{i} H_{j}^{i} \tag{11} \]

where \( \tilde{\omega} = -\tilde{\omega}^T \) is the matrix equivalent to \((\omega \times \cdot)\).

We can also define a wrench \( W_{i}^{k} \) (the dual of a twist), which describes the generalized forces acting on body \( B_i \) and expressed in frame \( \Psi_k \), as

\[ W_{i}^{k} = \begin{bmatrix} \tau_{i}^{k} \\ F_{i}^{k} \end{bmatrix} \]

where \( F_{i}^{k} \) denotes the linear force and \( \tau_{i}^{k} \) the momentum, acting on the point in the origin of frame \( \Psi_k \). The dual product of a twist and a wrench (when expressed in the same coordinate frame) is equal to a power flow and the pair \((T_{i,j}^{k,j}, W_{i}^{k})\) is called a power port and it is the basic concept used in bondgraphs [18] and port Hamiltonian systems [20]. More information on twists and wrenches can be found in [16, 19].

### 2.3 Surfaces description

Consider a rigid body \( B \) with a smooth, oriented surface \( S \), embedded in the Euclidean space \( E \). To this body, we rigidly attach a coordinate frame \( \Psi \). In the frame \( \Psi \), we can describe the surface (locally) as a bijective mapping \( f : D \rightarrow S \), which assigns to each set of local coordinates \( u \in D \subset \mathbb{R}^2 \) a point of the surface. The mapping \( f \) is a (local) parameterization of the surface, and we assume this parameterization to be well-defined, i.e. the derivative mapping \( f_* = \frac{\partial f}{\partial u} \) is continuous and has kernel zero, i.e. the partial derivatives \( \frac{\partial f}{\partial u} \) are independent at all points.

At each point of the surface, we can find the unit vector \( n(p) \) normal to the surface (we can compute this for example by taking the right order of the cross product between the partial derivatives of \( f \)). We can identify these unit vectors with points on the unit sphere \( S^2 \), if we think of the point on the sphere as the tip of the normal vector with its base point in the center of the sphere.

The Gauss map \( g : S \rightarrow S^2 \) is defined as the mapping which takes a point \( p \) on the surface and returns a point \( g(p) \) on the sphere, corresponding
Fig. 1. Relation diagram showing the mappings between the coordinate patch $D$, the surface $S$, the unit sphere $S^2$ and their tangent spaces. The canonical projection $\pi$ is added for completeness; it takes an element $(p, \zeta)$ of the tangent bundle and returns its base point $p$.

to the unit normal at $p$. The smoothness and orientability of the surface ensure that the normal vector varies smoothly over the surface, and hence the mapping $g$ is smooth. This means that we can also define the derivative mapping $g_* : TS \rightarrow T S^2$. This derivative can be interpreted as follows: if we move tangent to the surface at velocity $\zeta \in TS$, then the normal vector changes with velocity $g_*\zeta \in T S^2$. Since the vector $g(p)$ is perpendicular to the surface at $p$ as well as to the sphere at $g(p)$, we can directly regard an element $g_*\zeta \in T_{g(p)}S^2$ as an element $Pg_*\zeta \in T_pS$, where $P$ denotes the mapping from $T_{g(p)}S^2$ to $T_pS$.

The intuitive meaning of the differential $g_*$ of the Gauss map is curvature: the vector $g_\zeta$ for some $\zeta \in TS$ describes the curvature of the surface when moving at velocity $\zeta$. If $g_\zeta(p) = 0$ for all $\zeta \in T_pS$, then the surface is locally flat at $p$. If $\langle \zeta, Pg_\zeta(p)\zeta \rangle > 0$ for all $\zeta \in T_pS$, then we say that the surface is locally absolutely convex\footnote{We explicitly use the term ‘absolutely convex’ (which just means ‘convex’ in the usual sense) to distinguish it from the term ‘relatively convex’, which we define in Section 4.1.} at $p$. The following set which will be called radius-curvature quadric at $p$, which is strictly related to the Dupin indicatrix) is important to characterize the shape of the contact patch later on:
Table 1. Numerical implementation of the geometrical equations. A diamond (⋄) denotes an arbitrary number.

<table>
<thead>
<tr>
<th>type</th>
<th>example</th>
<th>numerical format</th>
</tr>
</thead>
<tbody>
<tr>
<td>local coordinates</td>
<td>u</td>
<td>⋄⋄</td>
</tr>
<tr>
<td>point in $\mathcal{E}$</td>
<td>$p$</td>
<td>⋄⋄⋄1 7</td>
</tr>
<tr>
<td>free vector in $\mathcal{E}$</td>
<td>$\dot{p}$</td>
<td>⋄⋄0 7</td>
</tr>
<tr>
<td>surface parameterization</td>
<td>$f(u)$</td>
<td>⋄⋄1 7</td>
</tr>
<tr>
<td>Gauss map</td>
<td>$g(p)$</td>
<td>⋄⋄0 7</td>
</tr>
<tr>
<td>tangent mapping</td>
<td>$f_\ast$</td>
<td>⋄⋄0 7</td>
</tr>
<tr>
<td>inverse tangent mapping (M.P. pseudo-inverse)</td>
<td>$f_{\ast}^{-1}$</td>
<td>$(f_{\ast}^T f_{\ast})^{-1} f_{\ast}^T$</td>
</tr>
</tbody>
</table>

$$R(p) := \{ \zeta \in T_p\mathcal{S} \text{ s.t. } \langle \zeta, Pg_\ast(p)\zeta \rangle = 1 \}.$$ (12)

This quadric can be an ellipse in case the curvature is positive in both directions or could be an infinite circle in case the surface would be flat in the point $p$.

Fig. 1 shows the relations between the various mappings and spaces. It is important to note that $f$ is a bijective mapping, hence it uniquely identifies coordinate-pairs to points on the surface and its derivative mapping $f_\ast$ is invertible. This means that although the equations in the following sections do not contain local coordinates $u$ or the surface parameterization $f$, we can always find coordinate expressions for these equations using $f$, $f_\ast$ and their inverses.

Numerical Notation

Since we want to use the formal results in numerical simulation, we need ways to represent the geometrical ideas in vectors and matrices. If we use the formats denoted in Table 1, then the geometric equations in the following sections can be implemented directly into numerical equations.

Once a coordinate $\Psi$ fixed to the body $B$ has been chosen, we can represent $f$ using homogeneous coordinates and obtain a map of the following form:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^4; (u, v) \mapsto \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \\ 1 \end{pmatrix}$$ (13)

It is now possible to find a numerical representation of $f_\ast$ which becomes:

$$f_\ast = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ 0 & 0 \end{pmatrix}$$ (14)
If the parameterization has been correctly chosen, this last matrix will have rank equal to 2 since the 2 column vectors should be linear independent. This implies that if we calculate the Moore-Penrose pseudo-inverse $f^+ \in \mathbb{R}^{2\times4}$ and multiply it for an existing vector $v \in T_p\mathcal{S}$, we get the unique vector $(\dot{u}_1, \dot{u}_2)^T$ such that

$$v = f^+ (\dot{u}_1, \dot{u}_2)^T.$$  

As long as a numerical representation of the Gauss map is concerned, it will turn out to be useful to represent it as a map of the following form:

$$g : \mathbb{R}^4 \rightarrow \mathbb{R}^4; (x, y, z, 1) \mapsto (\hat{n}^T, 1)^T. \quad (15)$$

Clearly this map is ‘extremely redundant’ since the surface is clearly 2-dimensional and the normal vector has only 2 independent coordinates, but the advantages of this representation will become clear in the sequel. It is clear what Eq. (15) should return when the element of the domain belongs to the surface of the body, but it is not clear a priori what $g(p)$ should be if $p$ does NOT belong to the surface. This is necessary in order to be able to consider its differential $g_*$. Around the surface, $g$ should be such to have constant values along directions normal to the surface. Furthermore, the vector $\hat{n}$ is numerically expressed in the same reference system which effectively consider a numerical representation of the map $P$ as the identity matrix.

After $g$ has been properly defined, it is possible to calculate its differential $g_* \in \mathbb{R}^{4\times4}$ which is such that $\text{rank}(g_*) \leq 2$. Restricting motions along the surface of the body and projections of the variations of $\hat{n}$ on the Gauss sphere, we could define $\bar{g}_* \in \mathbb{R}^{2\times2}$ with $\text{rank}(\bar{g}_*) \leq 2$. The definiteness of $\bar{g}$ characterizes again the curvature at the desired point and is a numerical representation of the quadric $\mathcal{R}(p)$.

### 3 Contact Description

The contact model which will be introduced is based on a differential geometric description of the surfaces of the bodies. A complete model which can be used for simulating grasps must be able to describe the detection and release together with the forces taking place during contact.

In this work we present a visco-elastic model which is complete enough to describe transitions from no contact to contact, rolling and slipping and at the same time simple enough so that can be simulated in real time.

The model is visco-elastic and therefore will have a viscous part which will describe energy dissipation and an elastic part which will model energy storage due to material deformations. Furthermore, the model is port-based and geometric. These last features are useful for using the model as part of more complex systems and makes it coordinate independent.

In this work the elastic part will be given the most attention being by far the most complex.
4 Contact Detection

In computer graphics, methods based on bounding boxes which are also implemented via hardware are used to detect collisions between bodies. This is also used in the collision detection system of the European Space Arm. These methods could certainly be combined with what will be presented in this work. For big distances the use of bounding boxes is very efficient and can handle a lot of different shapes. On the other hand, when the distances become small, in order to model the collision in a physical way, a more detailed description is necessary.

Here we suppose that the contacting surfaces are relatively convex in a neighborhood where the contact is likely to take place. Under this hypothesis, it is possible to find a relation between the relative twist of the two bodies and the velocity of the points on the two surface bodies which do have minimal distance (see Fig. 2).

Such a relation is useful since it allows an efficient tracking of the points of the surfaces which will get in contact. This is furthermore also used during the contact dynamics in the undeformable part as explained later in Sect. 5.1. We will hereafter explain this method following the ideas presented in [21, 6] and the analysis reported in [6].

4.1 Regular Contact Kinematics

We first consider the case of two rigid bodies in point contact, moving with a relative velocity represented by a twist $T_2^{12} = -T_2^{11}$ (this is exactly the case described by Montana [14]). We attach to each body $i$ a coordinate frame $\Psi_i$ and we assume to have a description $f_i$ of the surface of this body, expressed in frame $\Psi_i$.

If we then express the location of the point of contact as two points $p_1$ (expressed in frame $\Psi_1$) and $p_2$ (expressed in frame $\Psi_2$), then if the two bodies are in point contact, we have

\[
\begin{align*}
  p_1 &= H_1^2 p_2 \\
  g_1 &= -H_2^3 g_2
\end{align*}
\]

where we abbreviated $g_i := g_i(p_i)$. These equations just say that for point contact, the two contact points must be the same (when expressed in the same coordinate frame, in this case $\Psi_1$) and the normal vectors to the surfaces must be opposite.\(^5\)

\(^5\)Note that we use a homogeneous matrix $H_2^1$ to change coordinates for points ($p_2$) as well as for free vectors ($g_2$). Normally, free vectors only need to be rotated (using the rotation part of the coordinate transformation), but since we express these vectors numerically as a four-by-one matrix with its last element zero, we can just as well use multiplication by the full homogeneous matrix.
To obtain the kinematic equation relating the velocities of the contact points to the velocity of the bodies, we only need to take the time-derivative of (16) to obtain:

\[
\begin{align*}
\dot{p}_1 &= \dot{H}_1^2 p_2 + H_1^2 \dot{p}_2 \\
\dot{g}_1 &= -H_2^1 g_2 - H_2^2 \dot{g}_2
\end{align*}
\]  

(17)

If we now consider the definition of the differential map we have that \( \dot{g}_i = g_i \dot{p}_i \) and furthermore, using the definition of the twists we have \(-H_1^1 g_2 = H_2^1 H_2^2 g_1 = T_2^{1,1} g_1 \) and therefore we obtain:

\[
\begin{align*}
\dot{p}_2 &= H_2^2 \dot{p}_1 - H_2^1 \dot{H}_1^2 p_2 \\
(1,1)\dot{p}_1 &= T_2^{1,1} g_1 - H_2^1 g_2, \dot{p}_2
\end{align*}
\]  

(18)

where we pre-multiplied (17) by \( H_2^2 \) to obtain an expression for \( \dot{p}_2 \). If we now substitute this expression into the second line of (18) and repeat the whole derivation with objects 1 and 2 switched, we obtain the desired kinematic equations:

\[
\begin{align*}
(g_1 + H_2^1 g_2, H_2^2)\dot{p}_1 &= T_2^{1,1} g_1 - H_2^1 g_2, T_1^{2,2} p_2 \\
(g_2 + H_2^1 g_1, H_2^1)\dot{p}_2 &= T_1^{2,2} g_2 - H_2^1 g_1, T_2^{1,1} p_1
\end{align*}
\]  

(19)

Let us now briefly discuss the conditions under which these equations have unique solutions \( \dot{p}_1, \dot{p}_2 \). Because of the symmetry, we only consider the first equation, i.e. the equation for \( \dot{p}_1 \). Because the four-by-four matrix \((g_1 + H_2^1 g_2, H_2^2)\) has a non-zero kernel, we cannot simply invert this matrix and always get a unique result. Instead, we must look at the equation from a geometrical point of view.

First of all, since we look at the motion of the contact point over the surface, we must have \( \dot{p}_1 \in T_{p_1} S_1 \).

Secondly, since the domain of the mapping \( g_2 + H_2^1 g_2, H_2^2 \) is \( T_{p_2} S_2 \) and not all vectors in \( E \), we must have \( T_1^{2,2} p_2 \in T_{p_2} S_2 \). This constraint means that the velocity of the instantaneous contact point \( (T_1^{2,2} p_2) \) can not have a component perpendicular to the surface, thus constraining the allowed relative motion to five degrees of freedom, which is clear from a physical point of view since we consider the two bodies to maintain contact.

Finally, we need to ensure that a unique solution \( \dot{p}_1 \) exists for any twist satisfying the constraint above. Since both \( T_2^{1,1} g_1 \) and \( H_2^1 g_2, T_1^{2,2} p_2 \) are tangent to the surface, the co-domain of the matrix \((g_1 + H_2^1 g_2, H_2^2)\) must be the whole tangent plane to the surface, i.e. the matrix must have rank two. For physical reasons (no intersection of the surfaces) this means that the two surfaces must be relatively convex: the two non-zero eigenvalues of \((g_1 + H_2^1 g_2, H_2^2)\) must be larger than zero. Physically, relative convexity means that if one surface is concave, then the other body must be extra convex. Absolute convexity (as defined in Sect. 2.3) can be considered as a special case: an absolutely convex surface is relatively convex to a plane.
4.2 Generalized Contact Kinematics

In this section, we extend the results of Section 4.1 to the more general case as depicted in Fig. 2: we do not consider just the kinematics of the point of contact between the two bodies, but we look at the kinematics of the points on the surfaces which have the shortest (in the Euclidean sense) distance between them. We call this problem the generalized contact kinematics problem.

We use $p_i, i = 1, 2$ to denote the point on body $i$ expressed in frame $i$ such that the distance between $p_1$ and $p_2$ is the minimum distance between the bodies. This implies that the line connecting $p_1$ and $p_2$ must be perpendicular to both surfaces, which can be translated into the following equations:

$$
\begin{align*}
    p_1 + \Delta g_1 &= H^1_2 p_2 \\
    g_1 &= -H^1_2 g_2
\end{align*}
$$

where $\Delta \in \mathbb{R}$ denotes the ‘signed distance’ between the generalized contact points:

$$
\Delta = \langle g_1, H^1_2 p_2 - p_1 \rangle
$$

i.e. $\Delta > 0$ means there is a distance $|\Delta|$ between the bodies, and $\Delta < 0$ means the bodies have a maximum penetration distance of $|\Delta|$. The use of this definition for distance (instead of the usual $\|H^1_2 p_2 - p_1\|$) turns out to be very useful in the modeling of contact dynamics later on.

**Theorem 1.** Given two rigid bodies and the generalized contact points as defined in (20). If the bodies are absolutely convex, then the velocity of the generalized contact points is uniquely determined by the following equations:
We first compute the time derivative of $\dot{g}_1$ in (21). preserves the inner product. This shows that

$$\dot{T}_{2,1}^1 g_1 + H_2^1 g_2, (\dot{\Delta} g_2 - \dot{T}_{1,2}^2 p_2)$$

and

$$\dot{T}_{2,1}^2 g_2 + H_1^2 g_1, (\dot{\Delta} g_1 - \dot{T}_{2,1}^1 p_1)$$

where $\dot{T}_{2,1}^1 = -H_2^1 \dot{T}_{2,2}^{2,2} H_1^1$ can be any relative twist of the two bodies and $\Delta > \Delta_{min}$ for some $\Delta_{min} < 0$ depending on the surfaces and $\Delta$ is defined as in (21).

Proof. We first compute the time derivative of $\Delta$, e.g. the change of distance between the bodies.

$$\dot{\Delta} = \langle g_1, H_2^1 p_2 - p_1 \rangle + \langle g_1, \dot{H}_2^1 p_2 + H_2^1 \dot{p}_2 - \dot{p}_1 \rangle$$

$$= \langle g_1, \dot{H}_2^1 p_2 \rangle$$

(23)

$$= \langle g_1, \dot{T}_{2,1}^2 H_2^1 p_2 \rangle$$

(24)

where (23) results since the normal vector $g_1$ is always perpendicular to the velocities ($\dot{p}_1$ and $H_2^1 \dot{p}_2$) of the contact points over the surface and since $\dot{g}_1$ is perpendicular to $H_2^1 \dot{p}_2 - \dot{p}_1$, and (24) results by applying (11).

Using this result for $\dot{\Delta}$, we can compute the time derivative of (20) to obtain the kinematics equation:

$$\begin{align*}
\dot{p}_1 + \dot{\Delta} g_1 + \Delta g_1 \dot{p}_1 &= \dot{H}_2^1 p_2 + H_2^1 \dot{p}_2 \\
\dot{p}_1 &= -\dot{H}_2^1 g_2 - H_2^1 g_2, \dot{p}_2 \\
\dot{p}_2 &= H_2^1 (\dot{p}_1 + \dot{\Delta} g_1 + \Delta g_1 \dot{p}_1 - H_2^1 \dot{p}_2) \\
\dot{g}_1, \dot{p}_1 &= \dot{T}_{2,1}^1 g_1 - H_2^1 \dot{g}_2, \dot{p}_2
\end{align*}$$

(25)

If we now substitute the first equation of (25) into the second, and repeat the whole derivation with objects 1 and 2 switched, we immediately obtain (22). Note that for $\Delta = 0$, we recover the regular contact kinematics (19).

Now consider again the requirements for a unique solution $\dot{p}_1$. First we look at the term $\dot{\Delta} g_2 - \dot{T}_{2,2}^{1,2} p_2$ in (22) and take the inner product with $g_2$:

$$\langle g_2, \dot{\Delta} g_2 - \dot{T}_{2,2}^{1,2} p_2 \rangle = \dot{\Delta} - \langle H_2^1 g_2, H_2^1 \dot{T}_{2,2}^{1,2} p_2 \rangle$$

$$= \dot{\Delta} - \langle g_1, \dot{T}_{2,1}^2 H_2^1 p_2 \rangle$$

$$= 0$$

where we used (20), (24), and the fact that a homogeneous transformation preserves the inner product. This shows that $\dot{\Delta} g_2 - \dot{T}_{2,2}^{1,2} p_2$ is always tangent to the surface, so the right-hand side of (22) is well-defined for all twists $T_{2,2}^{1,2}$.

Whether $(g_1, H_2^1 g_2, H_2^1 (I + \Delta g_1))$ has rank two cannot be easily related to properties of the objects, since it also depends on the distance $\Delta$. Even
Fig. 3. The Geometrical contact models used: (Left) Undeformable contact model which is used for the decomposition in motions which do have elastic resistance and rotation which do not. (Right) Deformable contact model which takes place during contact and gives rise to an elastic force.

though an object may be relative convex (i.e. the contact points vary smoothly as the objects roll over each other), the contact points can jump when the objects are not in contact and move at a certain distance from each other. However, if the objects are absolutely convex, then invertibility is ensured for any $\Delta > \Delta_{\text{min}}$ for some $\Delta_{\text{min}} < 0$, i.e. for any positive distance, and for small enough penetrations, where $\Delta_{\text{min}}$ is the largest distance for which the matrix has rank less than two.

Although the kinematic equation (22) is similar to the results obtained in [21], the approach used here does not depend on extra coordinates and orthogonal parameterization, and is therefore more transparent and easier to interpret and understand geometrically.

5 Elastic Part

The elastic part of the contact model describes the forces in the contact region which are the consequence of elastic deflection of the material. As described in Sect. 4, the instant in which contact occurs is when the minimal distance $\Delta$ between the bodies, also called separation function, becomes zero. For the purpose of modeling and analysis, we use two different descriptions of the contact region:

- Undeformable
- Deformable

The Undeformable part is used in order to find a geometrical decomposition of relative motions of the bodies. In this description, using the rigidity of the surfaces, we allow the minimum distance $\Delta$ to become negative. Such a situation, is clearly plausible only from a modeling point of view. In Fig. 3 (Left), a drawing of this model is reported. For relative convex surfaces around the contact, it is again possible to define uniquely two points $p_1$ and $p_2$ which
have a maximum absolute distance within the patch region or in other words
whose distance is minimal (and negative).

The Deformable description is instead used to find a characterization of
the compliance between the bodies which gives rise to the repulsive elastic
wrench $W$. A drawing of this model is reported in Fig. 3 (Right).

For what follows, consistently with what explained in Sect. 4 let us indicate
the volume of the two bodies with $B_1 \subset E$ and $B_2 \subset E$, where $E$ is the
Euclidean space where the bodies belong. $B_1$ can be the object to be grasped
and $B_2$ the tip of a robotic’s hand. We indicate also with $S_1 := \partial B_1$ and $S_2 :=
\partial B_2$ respectively the surfaces of the two objects. We can consider $p_1 \in S_1$ and
$p_2 \in S_2$ the two initial contact points which do coincide when the contact
occurs.

We will now analyze both descriptions separately and combine them to-
gether at the end.

5.1 The Undeformable Description

If we consider a geometric description of the bodies as Undeformable for the
purpose of modeling, we allow $\Delta$ to become negative. This means that we
virtually allow $B_1 \cap B_2 \neq \emptyset$. Clearly this should result in a repulsive elastic
wrench $W$. This can be achieved in a geometric way as described hereafter.

Under the assumptions previously explained, there are two unique points
$p_1 \in S_1$ and $p_2 \in S_2$ in the region $\partial (B_1 \cup B_2)$ (see Fig. 3 (Left)) whose
connecting line $l_n$ is normal to the surfaces in $p_1$ and $p_2$.

Furthermore, given a point $c \in l_n$, there is a unique plane $O$ orthogonal
to $l_n$ and passing through $c$.

The point $c$ will correspond in the Deformable contact description to the
centroid of contact. We will analyze later how to choose the point $c$.

Motion Decomposition

In Lie group terms, the relative configuration of the two contacting bodies can
be studied using $SE(3)$ as introduced in Sect. 2.2. The relative instantaneous
motion instead, can be studied using the Lie algebra $se(3)$ associated to $SE(3)$.
This algebra is six dimensional and corresponds to the 6 possible motions of a
rigid body. We can therefore choose 6 basis vectors (screws) belonging to $se(3)$.
In order to decompose the motion between relative motions involving elastic
storage of energy and not, we will choose two screws representing pure distinct
rotations around two axis lying on $O$ and passing through $c$ ($r_x, r_y$) which are
two screws with zero pitch, and the other basis screws as the rotation around
$l_n$ ($r_z$) again a screw with zero pitch, and the three translations ($t_x, t_y, t_z$)
which are screws with infinite pitch.
We can now decompose $se(3)$ in the direct sum of two subspaces $R := \text{span}\{r_x, r_y\}$ and $\text{span}\{t_x, t_y, r_z, t_z\}$ which turns out to be equal to the Lie algebra $se(2) \times T$ of motions on $O$ ($se(2)$) together with the normal motion along $l_1 (T)$:

$$se(3) = R \oplus (se(2) \times T).$$

Due to this decomposition, it is possible to take uniquely the projection of any relative twist $T_2^1$ of $B_1$ with respect to $B_2$ on $se(2) \times T$ along $R$. Since this projection is uniquely defined by the plane $O$ together with the point $c \in O$, we can indicate the projection operator with the linear operator $P_{O,c}$:

$$P_{O,c} : se(3) \rightarrow se(2) \times T; T_2^1 \rightarrow PT_2^1.$$  

For any linear operator, there is an adjoint operator which maps dual elements corresponding to ‘wrenches’

$$P_{O,c}^* : se^*(2) \times T^* \rightarrow se^*(3); W \rightarrow P^*W$$  

in such a way that power is conserved:

$$\langle W | PT_2^1 \rangle = \langle P^*W | t_1^2 \rangle.$$  

The pairing of $P$ and $P^*$ is a well known concept in the language of bond graphs called a transformer.

**Elastic storage**

Due to the Lie group structure of $SE(2) \times T$, it is possible to meaningfully and geometrically integrate this projected velocity and find an elastic state which can be used to study the elastic reaction forces of the contact.

In order to do so, we consider a potential energy function of the following form:

$$V : SE(2) \times T \rightarrow \mathbb{R}$$

where $SE(2) \times T$ represents the elastic state of the contact.

Using Lie group left or right translations depending on the relative contact motion which has been projected, it is possible to map an element of the Lie algebra $se(2) \times T$ to the tangent space in the current state of the spring $T(SE(2) \times T)$ which can then be integrated. After this it is possible to find a well defined relation between the state and the corresponding elastic wrench $W$ based on the choice of $V$. It is important to notice that the only requirement of $V$ is to have a minimum at the identity element of $SE(2) \times T$ and to be lower bounded. This implies that the elastic relation does not have to be in general linear and it is therefore very general. For details on spatial geometric springs, the reader is addressed to [19].

---

6 It is important to note that this decomposition is only dependent on the choice of the position of $c$ and NOT on the choices of $r_x$ and $r_y$ as long as they are linear independent and lying on the plane $O$.

7 Notice that this is not a semi-direct group product, but a normal group product.
5.2 The Deformable Description

Using Herzian theory \[8\], we can study each body elastic properties: we can consider a compression of each of the bodies separately against a flat, infinitely rigid plane. Assuming no tangential load for the moment as it is done in the Herzian theory, we can consider an elliptical contact patch\(^8\). This patch will have a shape corresponding to the radius curvature quadric \(R(p)\) as introduced in Eq. (12).

We assume that the characterization of the elliptic contact patch and its forces are related by three factors:

- The Normal Compression (\(-\Delta\))
- The Curvature of the body in the contact point \(p_i\) (\(g_\ast(p_i)\)).
- The possible anisotropical properties of the material at the contact point.

If we now consider direct contact and loading of \(B_1\) and \(B_2\) in the points \(p_1\) and \(p_2\), we assume that an elliptic contact patch will increase around the initial contact points and that this patch lies in the plane \(O\). The shape of the patch is related to the patches obtained in the punch and differential geometrically speaking is directly dependent on the relative curvature of the surfaces in the following way:

\[
R(c) = (R_1(p_1) + R_2(p_2)).
\] (28)

Where \(R_i\) indicates the radius-curvature of body \(B_i\).

Clearly, in order to be able to compute Eq. (28) as the sum of two quadrics, we have to consider \(p_1 = p_2\) which can be done considering the initial contacting situation.

In a similar way, due to possible anisotropy of the materials there can be direction dependent stiffnesses in the contact.

In order to take these effect into account, we can associate a stiffness information to each point of the contacting surfaces and then calculate a corresponding geometrical anisotropical stiffness during contact based on them. Once this stiffness is defined, it can be used in the projection plane in order to integrate the projected twists and calculate the corresponding wrenches as explained.

In mathematical terms we can proceed as follows. We can associate to each point of the surfaces a two covariant tensor based on \(se(2) \times T\) corresponding to a stiffness:

\[
K_i : S_i \rightarrow (se(2) \times T)_2, \quad i = 1, 2
\] (29)

The previous mappings are in differential geometric terms called tensor bundles. By clearly making an approximation, we can then consider both tensors defined in the same point \(c\) considering the initial contact situation as also done to calculate Eq. (28). Under this assumption, it is meaningful to consider

\(^8\) This can be considered correct in a first approximation, but more general consideration can be made.
as a representative stiffness of the contact. To understand this, it is sufficient to realize that in case one of the two contacting materials is much softer than the other, the resulting combined stiffness \( K(c) \) is almost equal to the the one with the smallest stiffness.

**The choice of the point \( c \in l_n \)**

It is now possible to find a physical way to uniquely identify the position of \( c \in l_n \) (see Fig. 3) in order to decompose motions based on the elastic properties of the material. In order to give a mathematical expression we first need to define a projection operator which gives the normal component of the stiffness tensor:

\[
P_n := (\text{se}(2) \times T)_2 \rightarrow \mathbb{R} ; K \mapsto K(\hat{T}, \hat{T})
\]

where \( \hat{T} \) indicates a unit vector in the direction of \( l_n \). Using this operator, we can then uniquely define the position of \( c \) as:

\[
c := (1 - \alpha)p_1 + \alpha p_2 \quad \text{where} \quad \alpha := \frac{P_n(K_1(p_1))}{P_n(K(c))}
\]

The intuition of Eq. (32) is easily explained: suppose that \( B_2 \) is much harder than \( B_1 \). This implies that \( P_n(K_2) \) will be much bigger than \( P_n(K_1) \). This implies that \( P_n(K) \approx P_n(K_1) \) and therefore \( \alpha \approx 1 \). This means that \( c \) will be very close to \( p_2 \) which makes a lot of sense since \( B_2 \), being much harder, will deform the least.

**Handling anisotropy**

A crucial point at this stage is that the elastic anisotropy of the material can be handled as a metrical property and coordinate deformations can be applied in such a way that the contact would be described in new coordinates for which the materials would have relative unity uniform stiffness. Clearly, this change of coordinates does have effect on the contact patch shape which would change accordingly. In geometrical terms, the new principal directions of the relative contact patch can be calculated by looking at the eigen vectors of the original undeformed patch quadric \( R(c) \) with respect to the relative stiffness metric \( K(c) \). In this new situation we obtain an equivalent contact with a different rotated contact patch, between two homogeneous materials. Analytically we can proceed as follows. Under the condition that there is no coupling among the normal stiffness, the rotational one and the tangential one, it is possible to find two lines \( l_x, l_y \in O \) by means of which we can define four screws \( r_n, t_n, t_x, t_y \) which are an orthonormal base of \( \text{se}(2) \times T \) with respect to the metric \( K(c) \). \( r_n \) is a zero pitch screw along \( l_n \) corresponding to
a pure rotation around \( l_n \) and \( t_n, t_x, t_y \) are infinite pitch screws corresponding to translations respectively in the directions \( l_n, l_x \) and \( l_y \). Using the base \( \mathcal{B}_{K(c)} := \{t_x, t_y, t_n, r_n\} \) for \( se(2) \times T \), a numerical representation for \( K(c) \) becomes per construction the identity matrix \( I_{4 \times 4} \).

On the other hand, a numerical representation of \( R(c) \) using the base elements \( t_x, t_y \) would in general not result in a diagonal matrix which would correspond with the principal curvature directions along the basis vector. For this reason, we can implement a second partial change of coordinates which implements a pure rotation in the plane spanned by \( t_x \) and \( t_y \) in order to have coordinates in which the radius-curvature is oriented with the coordinate axis. Such a map can be implemented by:

\[
\bar{R} : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \text{ s.t. } \begin{pmatrix} x \\ y \\ z \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ \theta \end{pmatrix}
\]

where \( R \) is an orthonormal matrix which has as rows the normalized eigenvectors of \( R(c) \) calculated with respect to the metric \( K(c) \). In this way, we can define for the compliant contact an energy function

\[
\bar{V} : \mathbb{R}^4 \rightarrow \mathbb{R}
\]

which abstracts from the compliant properties of the materials of the two bodies and which has the relative radius-curvature aligned with the first two coordinates. The total normalizing change of coordinates is therefore

\[
N(c) : se(2) \times T \rightarrow \mathbb{R}^4 \text{ s.t. } v \mapsto \bar{R} \cdot (t_x t_y t_n r_n) v
\]

which is a linear map and has as such an adjoint \( N^* \) which maps the corresponding wrenches in the opposite direction:

\[
N^*(c) : (\mathbb{R}^4)^* \rightarrow se^*(2) \times T^* \text{ s.t. } f \mapsto (t_x t_y t_n r_n)^* \bar{R}^T f.
\]

The pair \((N, N^*)\) is nicely represented by a transformer in the bondgraphs formalism.

The only step left is the definition of an energy function \( \bar{V} \) which can be either quadratic (giving rise to a linear spring) or not.

**Remark 1.** The change of coordinates has used the tensor \( K(c) \) which is representing the stiffness of the material. In general the stiffness is not a tensor, but it can be defined as such when a geometric connection is considered [9, 22]. In our case, the natural connection which could be used is the one associated to the exponential coordinates of the Lie group \( SE(2) \times T \) which being a commutative group gives rise to basis coordinates and therefore a symmetric stiffness. For a non quadratic energy function this would be position dependent and not equal to \( K(c) \), but for the geometrical considerations we made we consider \( K(c) \) as representative.
5.3 The complete picture

The previous considerations can be applied to any relative contacting situation of the bodies. This implies that at each instant the point $p_1, p_2$ can be computed integrating equations Eq. (22) and therefore the line $l_n$ is consequently defined. Based on $K_1(p_1)$ and $K_2(p_2)$ the point $c$ can be calculated using Eq. (32). Once $c$ is available, the plane $O$ is uniquely determined and therefore it is possible to uniquely project a relative motion belonging to $se(3)$ on $se(2) \times T$ along $R$ using the projection operator of Eq. (26). This projection can then be transformed through $N$ as defined in Eq. (34). The resulting vector can be directly integrated due to the commutativity in the exponential coordinates of $SE(2) \times T$. This results in the elastic state which generates a force which is calculated using $d\bar{V}(x, y, z, \theta)$. The corresponding elastic repulsive force is then equal to

$$W = P_{O,c}^\ast N^\ast d\bar{V}(x, y, z, \theta),$$

and this complete the elastic model of the contact. It is important to realize that the elastic function $\bar{V}$ is left general. This implies that different elastic models can be implemented based on the linear Kelvin-Voight model or the more general non linear Hunt-Crossley model [10].

Clearly, when the elastic load reaches a certain threshold, slipping occurs. This will be briefly handled in Sect. 7.

6 Dissipative Part

The dissipative part can be handled in a similar way to the elastic one. As we did for the elastic part, using the tensor fields reported in Eq. (29) we can define damping fields for the surfaces.

$$B_i : S_i \rightarrow (se(2) \times T)^2 \quad i = 1, 2 \quad (37)$$

The resulting field which will characterize the damping will be resultant of the series of the two which similarly to Eq. (30) can be calculated as

$$B(c) := (B_1^{-1}(p_1) + B_2^{-1}(p_2))^{-1} \quad (38)$$

This can be directly used as a linear dissipation following the line of the Kelvin-Voigt model by considering the linear map corresponding to the previous quadratic form which is a map like:

$$B^\sharp(c) : se(2) \times T \rightarrow se^\ast(2) \times T^\ast \quad (39)$$

or this information can be used to create a geometrical extension of the Hunt-Crossley model [10] by considering for example:
Compliant Contact Modeling for Grasping Applications

![Diagram](image)

**Fig. 4.** The threshold function for slip detection.

\[
\begin{align*}
B^H_{\eta}(c) : se(2) \times T &\to se^*(2) \times T \text{ s.t.} \\
v &\mapsto B^R(c)N^{-1}(c) \begin{pmatrix} x^n & 0 & 0 \\ 0 & y^n & 0 \\ 0 & 0 & z^n \\ 0 & 0 & 0 & \theta^n \end{pmatrix} N(c)v
\end{align*}
\] (40)

where \((x, y, z, \theta)\) is the state of the elastic energy \(\bar{V}\) as introduced in Eq. (33).

### 7 Slipping

A lot of research is going on in the geometrical modeling of slipping by the authors and a detailed description will be reported in a forthcoming paper. In this section we briefly give the basic ideas on how slipping can be handled within the presented framework.

From a microscopical point of view, slip occurs when the elastic coupling between the two bodies reaches a threshold of extension. In such a situation the elastic bindings break and motion occurs. When motion occurs, the elastic extension up to the moment of slip is retained and will play a role during the stick phase. A simplified efficient model of the slip effect can be obtained from a microscopical point of view defining the following two functions:

\[
V_{\text{slip}} : se(2) \to \mathbb{R}
\] (41)

and

\[
S : T \to \mathbb{R}
\] (42)

The first function \(V_{\text{slip}}\) associates to a tangential elastic load an energy value. This function could be also strictly related to the elastic energy function \(\bar{V}\), but not necessarily.

The threshold function \(S\) associates instead to the current compression \(\Delta \in T\) gives a maximum energetical value after which slip occurs. This function will clearly be strictly decreasing and have a shape similar to the one
Fig. 5. Simulation of a Grasping strategy using the presented contact model

reported in Fig. 4. An analytical expression of $S$ and $V_{\text{slip}}$ based on physical principles will be presented in future work. Slip is then detected when the following condition is satisfied:

$$V_{\text{slip}}(h) > S(\Delta)$$

where $(h, \Delta) \in se(2) \times T$ indicates the geometrical state of the elastic spring.

8 Simulations of Grasping

As an application we show a simulation of a robotic hand manipulating an object, in this case a sphere. In Fig. 5 we see three frames: the first one is the reference frame, the third the object frame and the second the virtual object frame which exists only in the controller. The used controller is a special form of impedance control. For more information on the controller see [19]. The hand has to make the object follow the reference by rolling it between its fingers and without dropping it. The simulations have been performed with 20-sim (http://www.20sim.com), which is a powerful modeling and simulation package which supports bondgraphs.

9 Conclusions and Future Work

This chapter has presented a geometrical, energetically consistent model of the contact dynamics between two convex bodies whose surface viscoelastic properties are described by two, possibly anisotropic, tensor fields defined on the surfaces. The model is able to handle a lot of linear and non-linear models and can be the basis for a more physical description of the contact dynamics.

Slipping has been only introduced and a future paper will report a detailed analysis on how to handle slip and stick in this framework.
A future and important extension to this work would clearly be an identification and validation stage which would prove the validity of the model in real experiments. This would be of great value since the model is geometrically complete and at the same time computationally not very heavy and this has great advantages for real time applications like the space application RokViss [11].

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References