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Regulated State Synchronization for Discrete-time Homogeneous Networks of Non-introspective Agents in Presence of Unknown Non-uniform Input Delays: A Scale-free Protocol Design

Zhenwei Liu[1], Donya Nojavanzadeh[2], Dmitri Saberi[3], Ali Saberi[2], Anton A. Stoorvogel[4]

1. College of Information Science and Engineering, Northeastern University, Shenyang 110819, China
E-mail: liuzhenwei@ise.neu.edu.cn

2. School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164, USA
E-mail: donya.nojavanzadeh@wsu.edu; saberi@eecs.wsu.edu

3. Stanford University, Stanford, CA 94305, USA
E-mail: dsaberi@stanford.edu

4. Department of Electrical Engineering, Mathematics and Computer Science, University of Twente, Enschede, The Netherlands
E-mail: A.A.Stoorvogel@utwente.nl

Abstract: This paper studies regulated state synchronization of discrete-time homogeneous networks of non-introspective agents in presence of unknown non-uniform input delays. A scale free protocol is designed based on localized information exchange, which does not need any knowledge of the directed network topology and the spectrum of the associated Laplacian matrix. The proposed protocol is scalable and achieves state synchronization for any arbitrary number of agents. Meanwhile, an upper bound for the input delay tolerance is obtained, which explicitly depends on the agent dynamics.

Key Words: Multi-agent systems, Regulated state synchronization, Unknown non-uniform input delays, Discrete-time

1 Introduction

Cooperative control of multi-agent systems (MAS) is an active research topic because of its widespread application in different areas such as sensor networks, automotive vehicle control, satellite or robot formation, power distribution systems and so on. See for instance the books [29] and [45] or the survey paper [24].

We identify two classes of MAS: homogeneous and heterogeneous. State synchronization inherently requires homogeneous networks (i.e. networks with identical agent models). Therefore, in this paper, our focus is on homogeneous networks of MASs. State synchronization based on diffusivefull-state coupling has been considered in the literature where the agent dynamics progress from single- and double-integrator (e.g. [25], [27], [28]) to more general dynamics (e.g. [31], [39], [43]). State synchronization based on diffusive partial-state coupling has also been considered, including static design ([19] and [20]), dynamic design ([9], [32], [33], [37], [40]), and the design with localized communication ([5] and [31]). The solvability conditions are studied for general dynamic in [35] and [34]. Recently, scale-free collaborative protocol designs are developed for continuous-time heterogeneous MAS [23] and for homogeneous continues-time MAS with actuator saturation [17].

A common assumption, especially for heterogeneous MAS, is that agents are introspective; that is, agents possess some knowledge about their own states. So far there exist many results about this type of agents, see for instance [3, 10, 13, 26, 48]. On the other hand, for non-introspective agents, designs can also be found, such as an internal model principle based design [44], distributed high-gain observer based design [8], low-and-high gain based, purely distributed, linear time invariant protocol design [7].

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In practical applications, the network dynamics are not perfect and may be subject to delays. Time delays may afflict systems performance or even lead to instability. As discussed in [2], two kinds of delay have been considered in the literature: input delay and communication delay. Input delay is the processing time to execute an input for each agent whereas communication delay can be considered as the time for transmitting information from origin agent to its destination. Some researches have been done in the case of communication delay [4, 11, 15, 21, 22, 36, 46]. In the case of input delays, many efforts have been done (see [1, 14, 25, 38, 47]) where they are mostly restricted to simple agent models such as first and second-order dynamics for both linear and nonlinear agent dynamics. [41, 42] studied state synchronization problems in the presnce of unknown uniform constant input delay for continuous- and discrete-time networks with higher-order linear agents. Recently, [49] has studied synchronization in homogeneous networks of both continuous- or discrete-time agents with unknown non-uniform constant input delays.

A common characteristic in all of the aforementioned works either with input delay or communication delay is that the proposed protocols require some knowledge of communication networks that is the spectrum of associated Laplacian matrix and obviously the number of agents. In contrast, by virtue of a localized information exchange for MAS with both full- and partial-state coupling, we design and present protocols with the following distinctive characteristics:

- The design is independent of information about communication networks. That is to say, the dynamical protocol can work for any communication network such that all of its nodes have path to the exosystem.
- The dynamic protocols are designed for networks with unknown non-uniform input delays where the admissible upper bound on delays only depends on agent model

and does not depend on communication network and the number of agents.
- The proposed protocols are scale-free: they achieve regulated state synchronization for any MAS with any number of agents, any admissible non-uniform input delays, and any communication network.

Due to space limitation, we have omitted numerical simulation which is available in the extended version, see [16].

Notations and definitions

Given a matrix $A \in \mathbb{R}^{m\times n}$, A^{T} denotes its conjugate transpose and $\|A\|$ is the induced 2-norm while $\sigma_{\min}(A)$ denotes the smallest singular value of A. Let j indicate $\sqrt{-1}$. A square matrix A is said to be Schur stable if all its eigenvalues are in the closed unit disc. $A \otimes B$ depicts the Kronecker product between A and B. I_n denotes the n-dimensional identity matrix and 0_n denotes $n\times n$ zero matrix; sometimes we drop the subscript if the dimension is clear from the context. Moreover, $\ell_\infty^n(K)$ denote the Banach space of finite sequences $\{y_1,\dots,y_K\} \subset \mathbb{C}^n$ with norm $\|.\|_\infty = \max_i\{\|y_i\|\}$.

To describe the information flow among the agents we associate a *weighted graph* $\mathcal{G}$ to the communication network. The weighted graph $\mathcal{G}$ is defined by a triple $(\mathcal{V},\mathcal{E},\mathcal{A})$ where $\mathcal{V} = \{1,\dots,N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix with non negative elements a_{ij}. Each pair in $\mathcal{E}$ is called an *edge*, where $a_{ij} > 0$ denotes an edge $(j,i) \in \mathcal{E}$ from node j to node i with weight a_{ij}. Moreover, $a_{ij} = 0$ if there is no edge from node j to node i. We assume there are no self-loops, i.e. we have $a_{ii} = 0$. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1,\dots,i_k\}$ such that $(i_j,i_{j+1}) \in \mathcal{E}$ for $j = 1,\dots,k-1$. A *directed tree* is a subgraph (subset of nodes and edges) in which every node has exactly one parent node except for one node, called the *root*, which has no parent node. The *root set* is the set of root nodes. A *directed spanning tree* is a subgraph which is a directed tree containing all the nodes of the original graph. If a directed spanning tree exists, the root has a directed path to every other node in the tree.

For a weighted graph $\mathcal{G}$, the matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j,\\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. The Laplacian matrix L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$ [6]. Moreover, if the graph contains a directed spanning tree, the Laplacian matrix L has a single eigenvalue at the origin and all other eigenvalues are located in the open right-half complex plane [29].

2 Problem Formulation

Consider a MAS consisting of N identical discrete-time linear dynamic agents with input delay:

$$\begin{cases} x_i(k+1) = Ax_i(k) + Bu_i(k-\kappa_i),\\ y_i(k) = Cx_i(k),\\ x_i(\psi) = \phi_i(\psi+\bar{\kappa}), \quad \psi \in \overline{[-\bar{\kappa},0]} \end{cases} \tag{1}$$

where $x_i(k) \in \mathbb{R}^n$, $y_i(k) \in \mathbb{R}^q$ and $u_i(k) \in \mathbb{R}^m$ are the state, output, and the input of agent $i = 1,\dots,N$, respectively.

Moreover, κ_i represent the input delays with $\kappa_i \in \overline{[0,\bar{\kappa}]}$, where $\bar{\kappa} = \max_i\{\kappa_i\}$, $\phi_i \in \ell_\infty^n(\bar{\kappa})$ and the notation $\overline{[k_1,k_2]}$ means

$$\overline{[k_1,k_2]} = \{k \in \mathbb{Z} : k_1 \leqslant k \leqslant k_2\}.$$

Assumption 1 *We assume that:*

(i) (A,B) *are stabilizable and* (C,A) *are detectable.*

(ii) All eigenvalues of A *are in the closed unit disc.*

In this paper, we consider regulated state synchronization. The reference trajectory is generated by the an exosystem:

$$\begin{aligned} x_r(k+1) &= Ax_r(k)\\ y_r(k) &= Cx_r(k). \end{aligned} \tag{2}$$

with $x_r(k) \in \mathbb{R}^n$. Our objective is that the agents achieve regulated state synchronization, that is

$$\lim_{k\to\infty}(x_i(k) - x_r(k)) = 0, \tag{3}$$

for all $i \in \{1,\dots,N\}$. Clearly, we need some level of communication between the exosystem and the agents. We assume that a nonempty subset $\mathscr{C}$ of the agents have access to their own output relative to the output of the exosystem. Specially, each agent i has access to the quantity

$$\psi_i = \iota_i(y_i(k) - y_r(k)), \qquad \iota_i = \begin{cases} 1, & i \in \mathscr{C},\\ 0, & i \notin \mathscr{C}. \end{cases} \tag{4}$$

The network provides agent i with the following information,

$$\bar{\zeta}_i(k) = \sum_{j=1}^{N} a_{ij}(y_i - y_j) + \iota_i(y_i(k) - y_r(k)). \tag{5}$$

where $a_{ij} \geqslant 0$ and $a_{ii} = 0$. This communication topology of the network can be described by a weighted graph $\mathcal{G}$ with the a_{ij} being the coefficients of the weighting matrix $\mathcal{A}$ (not of the dynamics matrix A introduced in (1)).

We refer to (5) as *partial-state coupling* since only part of the states are communicated over the network. When $C = I$, all states are communicated over the network, we call it *full-state coupling* and the original agents are expressed as

$$x_i(k+1) = Ax_i(k) + Bu_i(k-\kappa_i) \tag{6}$$

meanwhile, (5) will change as

$$\bar{\zeta}_i(k) = \sum_{j=1}^{N} a_{ij}(x_i(k) - x_j(k)) + \iota_i(x_i(k) - x_r(k)). \tag{7}$$

To guarantee that each agent can achieve the required regulation, we need to make sure that there exists a path to each node starting with node from the set $\mathscr{C}$. Motivated by this requirement, we define the following set of graphs.

Definition 1 *Given a node set* $\mathscr{C}$*, we denote by* $\mathbb{G}_{\mathscr{C}}^{\mathbb{N}}$ *the set of all graphs with N nodes containing the node set* $\mathscr{C}$*, such that every node of the network graph* $\mathscr{G} \in \mathbb{G}_{\mathscr{C}}^{\mathbb{N}}$ *is a member of a directed tree which has its root contained in the node set* $\mathscr{C}$*.*

Remark 1 *Note that Definition 1 does not require necessarily the existence of directed spanning tree.*

From now on, we will refer to the node set $\mathscr{C}$ as the *root* *set* in view of Definition 1. For any graph $\mathcal{G} \in \mathbb{G}^N_{\mathscr{C}}$, with the associated Laplacian matrix L, we define the expanded Laplacian matrix as

$$\bar{L} = L + \text{diag}\{\iota_i\} = [\bar{\ell}_{ij}]_{N\times N}.$$

and we define

$$\bar{D} = I - (2I + D_{\text{in}})^{-1}\bar{L}. \tag{8}$$

where $D_{\text{in}} = \text{diag}\{d_{in}(i)\}$ with $d_{in}(i) = \sum_{j=1}^N a_{ij}$. It is easily verified that the matrix $\bar{D}$ is a matrix with all elements nonnegative and the sum of each row is less than or equal to 1. Note that based on [18, Lemma 1], matrix $\bar{D}$ has all eigenvalues in the open unit disc if and only if $\mathscr{G} \in \mathbb{G}^{\mathbb{N}}_{\mathscr{C}}$.

We can obtain the new information exchange

$$\bar{\zeta}^d_i(k) = \frac{1}{2 + d_{in}(i)} \sum_{j=1}^N a_{ij}(y_i(k) - y_j(k)) + \iota_i(y_i(k) - y_r(k)). \tag{9}$$

and

$$\bar{\zeta}^d_i(k) = \frac{1}{2 + d_{in}(i)} \sum_{j=1}^N a_{ij}(x_i(k) - x_j(k)) + \iota_i(x_i(k) - x_r(k)) \tag{10}$$

In this paper, we also introduce a localized information exchange among protocols. In particular, each agent $i = 1, \dots, N$ has access to localized information, denoted by $\hat{\zeta}_i(k)$, of the form

$$\hat{\zeta}_i(k) = \frac{1}{2 + d_{in}(i)} \sum_{j=1}^N \bar{\ell}_{ij}\xi_j(k) \tag{11}$$

where $\xi_j(k) \in \mathbb{R}^n$ is a variable produced internally by agent j and to be defined in next sections.

Next, we formulate the problem for regulated state synchronization of a MAS with full- and partial-state coupling:

Problem 1 *Consider a MAS described by* (1) *satisfying Assumption 1, with a given $\bar{\kappa}$ and the associated exosystem* (2). *Let a set of nodes $\mathscr{C}$ be given which defines the set $\mathbb{G}^N_{\mathscr{C}}$ and let the asssociated network communication graph $\mathcal{G} \in \mathbb{G}^N_{\mathscr{C}}$ be given by* (5).

The ***scalable regulated state synchronization problem with localized information exchange*** *of a discrete-time MAS is to find, if possible, a linear dynamic protocol for each agent $i \in \{1, \dots, N\}$, using only knowledge of agent model, i.e., (A, B, C), and upper bound of delays $\bar{\kappa}$, of the form:*

$$\begin{cases} x_{c,i}(k+1) = A_{c,i}x_{c,i}(k) + B_{c,i}u_i(k - \kappa_i) \\ \qquad\qquad\qquad +C_{c,i}\bar{\zeta}^d_i(k) + D_{c,i}\hat{\zeta}_i(k), \\ u_i(k) \quad = F_{c,i}x_{c,i}(k), \end{cases} \tag{12}$$

where $\hat{\zeta}_i(k)$ is defined in (11) *with $\xi_i(k) = H_c x_{i,c}(k)$, and $x_{c,i}(k) \in \mathbb{R}^{n_i}$, such that regulated state synchronization* (3) *is achieved for any N and any graph $\mathcal{G} \in \mathbb{G}^N_{\mathscr{C}}$.*

3 Protocol Design

In this section, we will consider the regulated state synchronization problem for a MAS with input delays. In particular, we cover separately systems with full-state coupling and those with partial-state coupling.

3.1 Full-state coupling

Firstly, we define

$$\omega_{\max} = \begin{cases} 0, & \text{A is Schur stable,} \\ \max\{\omega \in [0, \pi] | \det(e^{j\omega}I - A) = 0\}, & \text{otherwise.} \end{cases}$$

Protocol 1 for MAS with full-state coupling

Then, we design a dynamic protocol with localized information exchanges for agent $i \in \{1, \dots, N\}$ as follows.

$$\begin{cases} \chi_i(k+1) = A\chi_i(k) + Bu_i(k - \kappa_i) + A\bar{\zeta}^d_i(k) - A\hat{\zeta}_i(k) \\ u_i(k) \quad = -\rho K_\varepsilon \chi_i(k), \end{cases} \tag{13}$$

where

$$K_\varepsilon = (I + B^{\mathrm{T}} P_\varepsilon B)^{-1} B^{\mathrm{T}} P_\varepsilon A$$

and P_ε is the unique solution of the following H_2 discrete algebraic Riccati equation (H_2-DARE)

$$A^{\mathrm{T}} P_\varepsilon A - P_\varepsilon - A^{\mathrm{T}} P_\varepsilon B(I + B^{\mathrm{T}} P_\varepsilon B)^{-1} B^{\mathrm{T}} P_\varepsilon A + \varepsilon I = 0 \tag{14}$$

and ρ and ε are positive parameters which their values depends on $\bar{\kappa}$ and are given explicitly in the proof of Theorem 1.
The agents communicate $\xi_i(k)$, which are chosen as $\xi_i(k) = \chi_i(k)$, therefore each agent has access to the following information:

$$\hat{\zeta}_i(k) = \frac{1}{2 + d_{in}(i)} \sum_{j=1}^N \bar{\ell}_{ij}\chi_j(k). \tag{15}$$

while $\bar{\zeta}^d_i(k)$ is defined by (10).

Remark 2 (14) *is a special case of the general low-gain H_2-DARE, which is written as follows:*

$$A^T P_\varepsilon A - P_\varepsilon - A^T P_\varepsilon B(R_\varepsilon + B^T P_\varepsilon B)^{-1} B^T P_\varepsilon A + Q_\varepsilon = 0 \tag{16}$$

where $R_\varepsilon > 0$, and $Q_\varepsilon > 0$ is such that $Q_\varepsilon \to 0$ as $\varepsilon \to 0$. In our case, we restrict our attention to $Q_\varepsilon = \varepsilon I$ and $R_\varepsilon = I$. However, as shown in [30], when A is neutrally stable, there exists a suitable (nontrivial) choice of Q_ε and R_ε which yields an explicit solution of (16), *of form*

$$P_\varepsilon = \varepsilon P \tag{17}$$

where P is a positive definite matrix that satisfies $A^T P A \leqslant P$.

Our formal result is stated in the following theorem.

Theorem 1 *Consider a MAS described by* (6) *satisfying Assumption 1, with a given $\bar{\kappa}$ and the associated exosystem* (2). *Let a set of nodes $\mathscr{C}$ be given which defines the set $\mathbb{G}^N_{\mathscr{C}}$ and let the asssociated network communication graph $\mathcal{G} \in \mathbb{G}^N_{\mathscr{C}}$ be given by* (10).

Then the scalable regulated state synchronization problem as stated in Problem 1 is solvable if

$$\bar{\kappa}\omega_{\max} < \frac{\pi}{2}. \tag{18}$$

In particular, there exist a $\rho^*(\omega_{\max}, \bar{\kappa}) > 0.5$ and for any fixed $\rho > \rho^*(\omega_{\max}, \bar{\kappa})$, there exists a $\varepsilon^*(\rho)$ such that for any $\varepsilon \in (0, \varepsilon^*(\rho)]$, dynamic protocol given by (13) *and* (14) *solves the scalable regulated state synchronization problem for any N and any graph $\mathcal{G} \in \mathbb{G}^N_{\mathscr{C}}$.*

To obtain this result, we need the following lemma.

Lemma 1 ([49]) Consider a linear time-delay system

$$x(k+1) = Ax(k) + \sum_{i=1}^{m} A_i x(k-\kappa_i), \tag{19}$$

where $x(k) \in \mathbb{R}^n$ and $\kappa_i \in \mathbb{N}^+$. Suppose $A + \sum_{i=1}^{m} A_i$ is Schur stable. Then, (19) *is asymptotically stable if*

$$\det[e^{j\omega} I - A - \sum_{i=1}^{m} e^{-j\omega\kappa_i^r} A_i] \neq 0,$$

for all $\omega \in [-\pi, \pi]$ and for all $\kappa_i \in \overline{[0, \bar{\kappa}]}$ for $(i = 1, \ldots, N)$.

Lemma 2 ([12, 42]) Consider a linear uncertain system,

$$x(k+1) = Ax(k) + \lambda Bu(k), \qquad x(0) = x_0, \tag{20}$$

where $\lambda \in \mathbb{C}$ is unknown. Assume that (A, B) is stabilizable and A has all its eigenvalues in the closed unit disc. A low-gain state feedback $u = F_\delta x$ is constructed, where

$$F_\delta = -(B^T P_\delta B + I)^{-1} B^T P_\delta A, \tag{21}$$

with P_δ being the unique positive definite solution of the H_2 algebraic Riccati equation,

$$P_\delta = A^T P_\delta A + \delta I - A^T P_\delta B (B^T P_\delta B + I)^{-1} B^T P_\delta A. \tag{22}$$

Then, $A + \lambda B F_\delta$ is Schur stable for any $\lambda \in \mathbb{C}$ satisfying,

$$\lambda \in \Omega_\delta := \left\{ z \in \mathbb{C} : \left| z - \left(1 + \tfrac{1}{\gamma_\delta}\right) \right| < \tfrac{\sqrt{1+\gamma_\delta}}{\gamma_\delta} \right\}, \tag{23}$$

where $\gamma_\delta = \lambda_{\max}(B^T P_\delta B)$. As $\delta \to 0$, Ω_δ approaches the set

$$H_1 := \{ z \in \mathbb{C} : \operatorname{Re} z > \tfrac{1}{2} \}$$

in the sense that any compact subset of H_1 is contained in Ω_δ for a δ small enough.

Proof of Theorem 1: Firstly, let $\tilde{x}_i = x_i - x_r$, we have

$$\tilde{x}_i(k+1) = A\tilde{x}_i(k) + Bu_i(k - \kappa_i)$$

We define

$$\tilde{x}(k) = \begin{pmatrix} \tilde{x}_1(k) \\ \vdots \\ \tilde{x}_N(k) \end{pmatrix}, \chi(k) = \begin{pmatrix} \chi_1(k) \\ \vdots \\ \chi_N(k) \end{pmatrix},$$

$$\tilde{x}^\kappa(k) = \begin{pmatrix} \tilde{x}_1(k-\kappa_1) \\ \vdots \\ \tilde{x}_N(k-\kappa_N) \end{pmatrix}, \text{ and } \chi^\kappa(k) = \begin{pmatrix} \chi_1(k-\kappa_1) \\ \vdots \\ \chi_N(k-\kappa_N) \end{pmatrix}$$

then we have the following closed-loop system

$$\begin{aligned} \tilde{x}(k+1) &= (I \otimes A)\tilde{x}(k) - \rho(I \otimes BK_\varepsilon)\chi^\kappa(k) \\ \chi(k+1) &= (I \otimes A)\chi(k) - \rho(I \otimes BK_\varepsilon)\chi^\kappa(k) \\ &\quad + [(I - \bar{D}) \otimes A](\tilde{x}(k) - \chi(k)). \end{aligned} \tag{24}$$

Let $\delta(k) = \tilde{x}(k) - \chi(k)$, we can obtain

$$\begin{aligned} \tilde{x}(k+1) &= (I \otimes A)\tilde{x}(k) - \rho(I \otimes BK_\varepsilon)\tilde{x}^\kappa(k) \\ &\quad + \rho(I \otimes BK_\varepsilon)\delta^\kappa(k) \\ \delta(k+1) &= (\bar{D} \otimes A)\delta(k) \end{aligned} \tag{25}$$

where $\delta^\kappa(k) = \tilde{x}^\kappa(k) - \chi^\kappa(k)$. The proof has two steps.

Step 1: First, we prove the stability of system (25) without delays, i.e.

$$\begin{aligned} \tilde{x}(k+1) &= (I \otimes A)\tilde{x}(k) - \rho(I \otimes BK_\varepsilon)\tilde{x}(k) + \rho(I \otimes BK_\varepsilon)\delta(k) \\ \delta(k+1) &= (\bar{D} \otimes A)\delta(k) \end{aligned} \tag{26}$$

where $\bar{D} = [\bar{d}_{ij}] \in \mathbb{R}^{N \times N}$ and we have that the eigenvalues of $\bar{D}$ are in open unit disk. The eigenvalues of $\bar{D} \otimes A$ are of the form $\lambda_i \mu_j$, with λ_i and μ_j eigenvalues of $\bar{D}$ and A, respectively. Since $|\lambda_i| < 1$ and $|\mu_j| \leqslant 1$, we find $\bar{D} \otimes A$ is Schur stable. Then we have $\delta_i(k) \to 0$ as $t \to \infty$. Therefore, we have that the dynamics for $\delta_i(k)$ is asymptotically stable.

According to the above result, for (26) we just need to prove the stability of

$$\tilde{x}(k+1) = [I \otimes (A - \rho BK_\varepsilon)]\tilde{x}(k)$$

or the stability of $A - \rho BK_\varepsilon$. Based on Lemma 2, there exist $\rho > 0.5$ and $\varepsilon^* > 0$ such that $A - \rho BK_\varepsilon$ is Schur stable for $\varepsilon \in (0, \varepsilon^*]$.

Step 2: In this step, since we have that dynamics of $\delta_i(k)$ is asymptotically stable, we just need to prove the stability of

$$\tilde{x}_i(k+1) = A\tilde{x}_i(k) - \rho BK_\varepsilon \tilde{x}_i(k - \kappa_i)$$

for $i = 1, \ldots, N$. Following Lemma 1 we need to prove

$$\det[e^{j\omega} I - A + \rho e^{-j\omega\kappa_i} BK_\varepsilon] \neq 0 \tag{27}$$

for $\omega \in [-\pi, \pi]$ and $\kappa_i \in \overline{[0, \bar{\kappa}]}$. We define

$$\rho^*(\bar{\kappa}, \omega_{\max}) = \frac{1}{2\cos(\bar{\kappa}\omega_{\max})} \tag{28}$$

Next choose a fixed ρ such that $\rho > \rho^*(\bar{\kappa}, \omega_{\max})$. Meanwhile, we note that there exists a θ such that

$$\rho > \frac{1}{2\cos(\bar{\kappa}\omega)}, \forall |\omega| < \omega_{\max} + \theta$$

Then, we split the proof of (27) into two cases where $\pi \geqslant |\omega| \geqslant \omega_{\max} + \theta$ and $|\omega| < \omega_{\max} + \theta$ respectively.

If $\pi \geqslant |\omega| \geqslant \omega_{\max} + \theta$, we have $\det(e^{-j\omega} I - A) \neq 0$, which yields $\sigma_{\min}(e^{j\omega} I - A) > 0$. Because $\sigma_{\min}(e^{j\omega} I - A)$ depends continuously on ω and the set $\{\pi \geqslant |\omega| \geqslant \omega_{\max} + \theta\}$ is compact. Hence, there exists a $\mu > 0$ such that

$$\sigma_{\min}(e^{j\omega} I - A) > \mu, \quad \forall \omega \text{ such that } |\omega| \geqslant \omega_{\max} + \theta.$$

Given ρ, for a small enough ε we have that $\|\rho e^{-j\omega\kappa_i} BK_\varepsilon\| \leqslant \mu/2$. Then, we obtain

$$\sigma_{\min}(e^{j\omega} I - A - \rho e^{-j\omega\kappa_i} BK_\varepsilon) \geqslant \mu - \tfrac{\mu}{2} \geqslant \tfrac{\mu}{2}.$$

Therefore, condition (27) holds for $\pi \geqslant |\omega| \geqslant \omega_{\max} + \theta$.

Now, it remains to show that condition (27) holds for $|\omega| < \omega_{\max} + \theta$. We find that

$$-\omega\kappa_i < |\omega|\bar{\kappa} \leqslant \frac{\pi}{2},$$

and hence $\rho\cos(-\omega\kappa_i) > \rho\cos(|\omega|\bar{\kappa}) > \frac{1}{2}$.

It implies that for a fixed ρ and small enough ε, we have $A - \rho e^{-j\omega\kappa_i} BK_\varepsilon$ is Schur stable based on Lemma 2. Therefore, (27) holds for $|\omega| < \omega_{\max} + \theta$ for a small enough ε and a fixed ρ satisfying $\rho > \frac{1}{2\cos(\bar{\kappa}\omega_{\max})}$.

Thus, for any $\varepsilon \in (0, \varepsilon^*(\rho)]$, we can obtain the regulated state synchronization result based on Lemma 1. ■

3.2 Partial-state coupling

In this subsection, we will consider the case via partial-state coupling.

Protocol 2 for MAS with partial-state coupling

We design the following dynamic protocol with localized information exchanges as follows.

$$\begin{cases} \hat{x}_i(k+1) = A\hat{x}_i(k) + B\hat{\zeta}_{i2}(k) + F(\bar{\zeta}_i^d(k) - C\hat{x}_i(k)) \\ \chi_i(k+1) = A\chi_i(k) + Bu_i(k-\kappa_i) + A\hat{x}_i(k) - A\hat{\zeta}_{i1}(k) \\ u_i(k) = -\rho K_\varepsilon \chi_i(k), \end{cases} \tag{29}$$

for $i = 1,\dots,N$ where F is a matrix such that $A - FC$ is Schur stable, and

$$K_\varepsilon = (I + B^{\mathrm{T}} P_\varepsilon B)^{-1} B^{\mathrm{T}} P_\varepsilon A,$$

and P_ε is the unique solution of H_2-DARE (14), and ρ and ε are positive parameters which their values depends on $\bar{\kappa}$ and are given explicitly in the proof of Theorem 2.

In this protocol, the agents communicate $\xi_i = (\xi_{i1}^{\mathrm{T}}, \xi_{i2}^{\mathrm{T}})^{\mathrm{T}}$ where $\xi_{i1}(k) = \chi_i(k)$ and $\xi_{i2}(k) = u_i(k-\kappa_i)$, therefore each agent has access to the localized information $\hat{\zeta}_i = (\hat{\zeta}_{i1}^{\mathrm{T}}, \hat{\zeta}_{i2}^{\mathrm{T}})^{\mathrm{T}}$:

$$\hat{\zeta}_{i1}(k) = \frac{1}{2 + d_{in}(i)} \sum_{j=1}^{N} \bar{\ell}_{ij}\chi_j(k), \tag{30}$$

and

$$\hat{\zeta}_{i2}(k) = \frac{1}{2 + d_{in}(i)} \sum_{j=1}^{N} \bar{\ell}_{ij}u_j(k-\kappa_j). \tag{31}$$

$\bar{\zeta}_i^d(k)$ is also defined as (9).

Then we have the following theorem for MAS via partial-state coupling.

Theorem 2 *Consider a MAS described by* (1) *satisfying Assumption 1, with a given $\bar{\kappa}$ and the associated exosystem* (2). *Let a set of nodes $\mathscr{C}$ be given which defines the set $\mathbb{G}_{\mathscr{C}}^N$ and let the asssociated network communication graph $\mathcal{G} \in \mathbb{G}_{\mathscr{C}}^N$ be given by* (9).

Then the scalable regulated state synchronization problem as stated in Problem 1 is solvable if (18) *holds. In particular, there exist a $\rho^*(\omega_{\max}, \bar{\kappa}) > 0.5$ and for any fixed $\rho > \rho^*(\omega_{\max}, \bar{\kappa})$, there exists a $\varepsilon^*(\rho)$ such that for any $\varepsilon \in (0, \varepsilon^*(\rho)]$, dynamic protocol given by* (13) *and* (14) *solves the scalable regulated state synchronization problem for any N and any graph $\mathcal{G} \in \mathbb{G}_{\mathscr{C}}^N$.*

Proof of Theorem 2: Similar to Theorem 1, let $\tilde{x}_i(k) = x_i(k) - x_r(k)$, we have

$$\begin{cases} \tilde{x}_i(k+1) = A\tilde{x}_i(k) + Bu_i(k-\kappa_i) \\ \hat{x}_i(k+1) = A\hat{x}_i(k) + B\hat{\zeta}_{i2}(k) + F(\bar{\zeta}_i^d(k) - C\hat{x}_i(k)) \\ \chi_i(k+1) = A\chi_i(k) + Bu_i(k-\kappa_i) + \hat{x}_i(k) - \hat{\zeta}_{i1}(k) \end{cases}$$

Then we have the following closed-loop system

$$\begin{aligned} \tilde{x}(k+1) &= (I \otimes A)\tilde{x}(k) - \rho(I \otimes BK_\varepsilon)\chi^\kappa(k) \\ \hat{x}(k+1) &= I \otimes (A - FC)\hat{x}(k) - \rho[(I - \bar{D}) \otimes BK_\varepsilon]\chi^\kappa(k) \\ &\quad + [(I - \bar{D}) \otimes FC]\tilde{x}(k) \\ \chi(k+1) &= [(I - \bar{D}) \otimes A]\chi(k) - \rho(I \otimes BK_\varepsilon)\chi^\kappa(k) + \hat{x}(k) \end{aligned} \tag{32}$$

by defining $\delta = \tilde{x} - \chi$ and $\bar{\delta} = [(I - \bar{D}) \otimes I]\tilde{x} - \hat{x}$, we obtain

$$\begin{aligned} \tilde{x}(k+1) &= (I \otimes A)\tilde{x}(k) \\ &\quad - \rho(I \otimes BK_\varepsilon)\tilde{x}^\kappa(k) + \rho(I \otimes BK_\varepsilon)\delta^\kappa(k) \\ \bar{\delta}(k+1) &= I \otimes (A - FC)\bar{\delta}(k) \\ \delta(k+1) &= (\bar{D} \otimes A)\delta(k) + \bar{\delta}(k) \end{aligned} \tag{33}$$

As before, we first prove stability of (33) without delays,

$$\begin{aligned} \tilde{x}(k+1) &= (I \otimes A)\tilde{x}(k) - \rho(I \otimes BK_\varepsilon)\tilde{x} + \rho(I \otimes BK_\varepsilon)\delta(k) \\ \bar{\delta}(k+1) &= I \otimes (A - FC)\bar{\delta}(k) \\ \delta(k+1) &= (\bar{D} \otimes I)\delta(k) + \bar{\delta}(k) \end{aligned} \tag{34}$$

Since we have $A - FC$ and $\bar{D} \otimes A$ are Schur stable, one can obtain $\bar{\delta}(k) \to 0$ and $\delta(k) \to 0$ as $t \to \infty$, i.e. we just need to prove the stability of

$$\tilde{x}_i(k+1) = (A - \rho BK_\varepsilon)\tilde{x}_i(k).$$

Then, similar to Theorem 1, we can obtain the result. ■

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