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ScienceDirect
IFAC PapersOnLine 54-19 (2021) 173–179

Exterior and vector calculus views of incompressible Navier-Stokes port-Hamiltonian models

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Abstract: In this paper we address the modeling of incompressible Navier-Stokes equations in the port-Hamiltonian framework. Such model not only allows describing the energy dissipation due to viscous effects but also incorporates the non-zero energy exchange through the boundary of the spatial domain for generic boundary conditions. We present in this work the coordinate-free representations of this port-Hamiltonian model using both exterior calculus and vector calculus as well as their corresponding coordinate-based descriptions.

Keywords: port-Hamiltonian, Navier-Stokes, exterior calculus, vector calculus

1. INTRODUCTION

In the past two decades, there has been an increasing interest in addressing the control and simulation of distributed parameter systems using the port-Hamiltonian formalism (Rashad et al., 2020). Unlike the standard Hamiltonian formalism, port-Hamiltonian theory is able to incorporate non-conservative dynamical systems and allows for non-zero energy exchange through the boundary of a spatial domain. The theory is therefore an ideal framework for the development of simulation and model-based control tools for distributed parameter systems (Duindam et al., 2009). A unique feature of the port-Hamiltonian modeling approach is that it represents a complex physical system as a network of interconnected subsystems which explicates the topology of energy exchanges within the system.

Advanced applications of the port-Hamiltonian theory for distributed systems, however, additionally require to cast the port-Hamiltonian theory of distributed systems in a differential geometric language that combines conceptual rigour with technical flexibility. The theoretical study and simulation of robotic birds flying in air (Califano et al., 2021a) provides one striking example. Both the conceptual and technical advantages of a differential geometric formulation are related to the coordinate-independence of this language. Indeed, its use was an essential ingredient for the development of distributed port-Hamiltonian theory in the foundational work of van der Schaft and Maschke (2002).

* This work was supported by the PortWings project funded by the European Research Council [Grant Agreement No. 787675].

Precisely the same conceptual and technical advantages propelled differential geometry in general, and exterior differential calculus in particular, to a method of choice at the research frontier of general relativity and particle physics about three decades ago. While exterior differential calculus is also at the heart of celebrated foundational work on geometric fluid and solid mechanics (Marsden et al., 1984; Marsden and Hughes, 1994), computational methods and engineering applications still largely cling to a vector calculus formulation and its severe limitations (Rashad et al., 2020). In our previous works (Rashad et al., 2021a,b; Califano et al., 2021b), we demonstrated how exterior calculus can be employed to derive port-Hamiltonian models for compressible and incompressible Euler and Navier Stokes equations from first principles.

In this paper, we show how the vector calculus counterparts of our previous exterior calculus-based models can be derived. As a representative for the wide range of port-Hamiltonian models in (Rashad et al., 2021a,b; Califano et al., 2021b) we focus in this paper on the port-Hamiltonian modeling of incompressible viscous flow only. More explicitly, the contributions of this paper are:

- Provide the different coordinate-free views of modeling fluid dynamical systems in the port-Hamiltonian framework using exterior and vector calculus and show how to change between both representations.
- Highlight the mathematical elegance of exterior calculus compared to vector calculus and coordinate-based expressions.
- Present the exterior calculus-based model for incompressible Navier Stokes in a pedagogical minimalis-

10.1016/j.ifacol.2021.11.074

tic manner, without burdening the reader with the technical details in (Rashad et al., 2021a,b; Califano et al., 2021b). Thus, we aim to make the subject more accessible to experts focused on simulation or control.

The rest of the paper is organized as follows: In Sec.2, we present the coordinate-free and coordinate-dependent descriptions of fluid flow on a general (possibly curved, as the surface of Earth) domain. Then we present the port-Hamiltonian model for incompressible Navier-Stokes equations using exterior calculus in Sec. 3 and using vector calculus in Sec. 4, and we conclude in Sec. 5.

2. COORDINATE-FREE AND COORDINATE-DEPENDENT DESCRIPTIONS

The spatial domain in which a non-relativistic fluid flows is represented mathematically by an n-dimensional compact manifold M, possibly including a boundary ∂M. For most engineering applications, one considers, of course, either dimension $n = 2$ or $n = 3$. We denote the space of smooth functions by $C^\infty(M)$, the vector space of smooth vector fields by $\mathfrak{X}(M)$, and the module of smooth differential k-forms by $\Omega^k(M)$. The manifold is equipped with a Riemannian metric $g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ which induces a volume form $\mu_{\text{vol}} \in \Omega^n(M)$, a Hodge star operator $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ and the Levi-Civita covariant derivative ∇. Furthermore, the metric-induced isomorphism $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ and its inverse $\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ allow identifying vector-fields with one-forms. For any $u \in \mathfrak{X}(M), \alpha \in \Omega^1(M)$, we have that $\flat(u) := g(u, \cdot)$ and $g(\sharp(\alpha), u) = \alpha(u)$.

The coordinate-based description of the fluid flow can be achieved by introducing coordinate maps $\varphi_q : U \to \mathbb{R}^n$, which assign an n-tuple of real numbers $(q^1, \cdots, q^n)$ to each physical point p of sufficiently many regions $U \subseteq M$ such that together they cover the entire fluid domain. In every thus constituted chart (U, φ_q), we can use the chart-induced basis vectors $\partial/\partial x^i$ and basis covectors dx^i (in each case $i = 1, \ldots, n$) to express any vector field $u \in \mathfrak{X}(M)$ and any k-form $\alpha \in \Omega^k(M)$ on the respective chart domains U as

$$u = u^i \frac{\partial}{\partial q^i}, \qquad \alpha = \alpha_{i_1 \cdots i_k} \mathrm{d}q^{i_1} \wedge \cdots \wedge \mathrm{d}q^{i_k},$$

with the Einstein convention to sum over repeated indices being used here and in the remainder of this article. The duality relation $\mathrm{d}q^j(\frac{\partial}{\partial q^i}) = \delta_i^j$, where δ_i^j the Kronecker delta symbol.

The Riemannian metric g and the induced volume form μ_{vol} can be expressed on a chart domain U as

$$g = g_{ij} \mathrm{d}q^i \otimes \mathrm{d}q^j, \qquad \mu_{\text{vol}} = \sqrt{g} \mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^n,$$

where $\sqrt{g}$ is a convenient short hand for the scalar density $\sqrt{\det(g_{ij})}$ constructed from the components g_{ij} of the metric tensor.

The fluid state is described fully by the Eulerian velocity vector field $v \in \mathfrak{X}(M)$ and the mass density function $\rho \in C^\infty(M)$. Using the manifold's Riemannian metric structure, we can alternatively describe the fluid's velocity using the one-form $\tilde{v} := \flat(v) \in \Omega^1(M)$ and the fluid's mass density by the mass top-form $\mu := \rho\mu_{\text{vol}} = \star\rho \in \Omega^n(M)$. The state variables (v, ρ) will be referred to as the contravariant states while $(\tilde{v}, \mu)$ will be referred to as the covariant states. In local coordinates, we have that

$$v = v^i \frac{\partial}{\partial q^i}, \qquad \tilde{v} = g_{ij} v^j \mathrm{d}q^i.$$

	Cartesian	Cylindrical	Spherical
q^i	(x, y, z)	(r, θ, z)	(r, θ, ϕ)
h_i	$(1, 1, 1)$	$(1, r, 1)$	$(1, r, r\sin(\theta))$
v^i	(v^x, v^y, v^z)	(v^r, v^θ, v^z)	(v^r, v^θ, v^ϕ)
$\tilde{v}_i$	(v^x, v^y, v^z)	$(v^r, r^2 v^\theta, v^z)$	$(v^r, r^2 v^\theta, r^2 \sin^2(\theta) v^\phi)$
μ_{vol}	$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$	$r\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}z$	$r^2 \sin(\theta) \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$

Table 1. Coordinate expressions for several coordinate maps on $M = \mathbb{R}^3$

The kinetic co-energy/energy [1] of the fluid in the spatial domain M can be represented using either (v, ρ) or $(\tilde{v}, \mu)$, which in local coordinates amounts to

$$E_{\mathrm{k}}^* = \int_M \tfrac{1}{2}\rho g(v, v)\mu_{\text{vol}} = \int_M \tfrac{1}{2}(\star\mu)\tilde{v} \wedge \star\tilde{v} = \int_M \tfrac{1}{2}\rho g_{ij} v^i v^j \underbrace{\sqrt{g}\mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^n}_{\mu_{\text{vol}}}. \quad (1)$$

Finally, we conclude with some remarks on the use of coordinate-based expressions in practical applications. First, the abstract coordinate-free treatment presented above allows for a global description and analysis of fluid flow on curved surfaces. Second, even in Euclidean flat space $\mathbb{R}^n$ it is still superior to coordinate-based treatments because it allows performing calculus in an arbitrary coordinate system adapted to the problem at hand.

In such general coordinate-system, the chart induced basis vectors $(\frac{\partial}{\partial q^i})_p \in T_pM$ or covectors $(\mathrm{d}q^i)_p \in T_p^*M$ are in general not the same at different points, not of unit length and not mutually orthogonal. In the most general case, the (symmetric) metric tensor has $\frac{1}{2}n(n+1)$ independent components in n dimensions. Consequently, the coordinate expressions of the different mathematical objects and operators might differ significantly across different coordinate-systems, while the coordinate-independent expressions keep their simple and invariant form.

In the more special (and popular) case of orthogonal non-unitary coordinate-systems (e.g. cylindrical or spherical coordinates in $\mathbb{R}^3$), the metric tensor components and volume form's density simplify to:

$$g_{ij} = h_i^2 \delta_{ij}, \qquad \sqrt{g} = \left(\prod_{i=1}^n h_i\right),$$

where $\delta_{ij} \equiv \delta_i^j$ while $h_i \in C^\infty(M)$ are often referred to as the Lamé coefficients defined by $h_i(p) := \|(\frac{\partial}{\partial q^i})_p\|$. Note that in the above expression the indices of h_i, δ_{ij}, and δ_i^j do not correspond to tensor components but are merely definitions of symbols. Thus, there is no implicit summation over the indices. An example of the difference in expressions between Cartesian, cylindrical and spherical coordinates is shown in Table 1.

[1] We will not further discriminate between energy, which is a function of thermodynamically extensive variables (momenta) and the corresponding co-energy, which is a function of intensive variables (velocities).

3. EXTERIOR CALCULUS PORT-HAMILTONIAN MODEL

In this section, we present the port-Hamiltonian model for incompressible viscous fluid flow using exterior calculus where all variables will be differential forms that are either scalar-valued, vector-valued or covector-valued. With reference to Fig. 1, the port-Hamiltonian model consists of a network of energetic subsystems, interconnected to each other by a pair of dual variables, that are called effort and flow variables and together are called a port. For each port, the intrinsic duality pairing between an effort and a flow corresponds to the power exchanged between the two subsystems connected via this port. Furthermore, the port-Hamiltonian model has open ports that characterize the power exchange between the port-Hamiltonian model and the external world.

In what follows, we present an overview of each individual subsystem leaving out many technical details in Rashad et al. (2021a,b); Califano et al. (2021b). For the reader's convenience, Table 2 summarizes all effort and flow variables of the port-Hamiltonian model.

3.1 Energy storage subsystem

The energy storage subsystem, represented graphically by $\mathbb{C}$ in Fig. 1, is defined by the manifold of the covariant state variables $x := (\tilde{v}, \mu) \in \Omega^1(M) \times \Omega^n(M)$ and the Hamiltonian functional

$$H_{\mathrm{k}}(\tilde{v},\mu) = \int_M \tfrac{1}{2}(\star\mu)\tilde{v} \wedge \star\tilde{v},$$

which represents the kinetic energy of the fluid. The flow variables of the storage subsystem are given by the rate of change of the state variables $\dot{x} := (\dot{\tilde{v}}, \dot{\mu})$ while their corresponding effort variables are given by the variational derivative of H_{k} with respect to x:

$$\delta_{\tilde{v}} H_{\mathrm{k}} = \rho \star \tilde{v} \in \Omega^{n-1}(M), \qquad \delta_\mu H_{\mathrm{k}} = \tfrac{1}{2}\iota_v \tilde{v} \in \Omega^0(M),$$

where $\iota_v : \Omega^k(M) \to \Omega^{k-1}(M)$ denotes the interior product. Note that $\iota_v\tilde{v} = \tilde{v}(v) = g(v,v)$.

In local coordinates, we have that $\iota_v\tilde{v} = \tilde{v}(v) = \tilde{v}_j v^j = g_{ij}v^iv^j$, whereas the $n-1$ form $\star\tilde{v}$ has the form

$$\star\tilde{v} = \begin{cases} \sqrt{g}\epsilon_{ij}v^i \mathrm{d}q^j, & (n=2) \\ \frac{1}{2}\sqrt{g}\epsilon_{ijk}v^i \mathrm{d}q^j \wedge \mathrm{d}q^k, & (n=3) \end{cases}$$

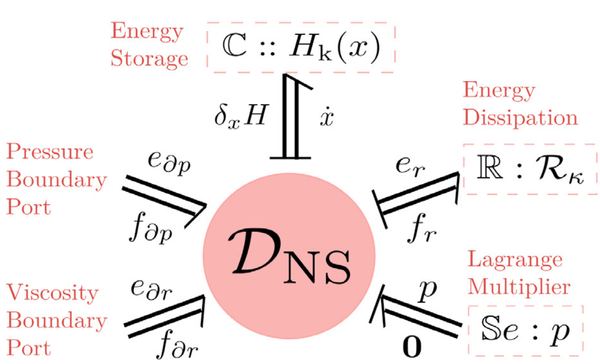

Fig. 1. Bond Graph model for the port-Hamiltonian model for Navier Stokes' equations with open boundary ports

Variable	Description
$\dot{x} \in \Omega^1(M) \times \Omega^n(M)$	rate of change of state variables
$\delta_x H_{\mathrm{k}} \in \Omega^{n-1}(M) \times \Omega^0(M)$	co-energy variables
$p \in \Omega^0(M)$	static pressure function
$e_{\partial p} \in \Omega^0(\partial M)$	boundary effort due to pressure
$f_{\partial p} \in \Omega^{n-1}(\partial M)$	boundary flow due to pressure
$e_r \in \Omega^1(M) \otimes \Omega^{n-1}(M)$	distributed effort due to viscosity
$f_r \in \mathfrak{X}(M) \otimes \Omega^1(M)$	distributed flow due to viscosity
$e_{\partial r} \in \Omega^1(M) \otimes \Omega^{n-1}(\partial M)$	boundary effort due to viscosity
$f_{\partial r} \in \mathfrak{X}(M) \otimes \Omega^0(\partial M)$	boundary flow due to viscosity

Table 2. Port variables of the port-Hamiltonian model in exterior calculus form

where $\epsilon_{ijk} \in C^\infty(M)$ denote the Levi-Civita symbols. The duality pairing between the efforts $(\delta_{\tilde{v}}H_{\mathrm{k}}, \delta_\mu H_{\mathrm{k}})$ and flows $(\dot{\tilde{v}}, \dot{\mu})$ is given mathematically by the integral of the wedge product and corresponds physically to the rate of change of kinetic energy stored:

$$\begin{aligned} \dot{H}_{\mathrm{k}} &= \int_M \dot{\tilde{v}} \wedge \delta_{\tilde{v}} H_{\mathrm{k}} + \dot{\mu} \wedge \delta_\mu H_{\mathrm{k}}, \\ &= \int_M \left(\rho g_{ij} v^i \dot{v}^j + \tfrac{1}{2} g_{ij} v^i v^j \dot{\rho}\right) \mu_{\mathrm{vol}}. \end{aligned} \tag{2}$$

3.2 Energy dissipation subsystem

The energy dissipation subsystem, represented graphically by $\mathbb{R}$ in Fig. 1, corresponds to the internal energy dissipation or resistance that occurs within the spatial domain due to shear viscosity only, considering the hypothesis of incompressible fluid.

In the exterior calculus representation of the port-Hamiltonian model, the flow variable f_r is chosen to be the velocity gradient $\nabla v \in \mathfrak{X}(M) \otimes \Omega^1(M)$ which is the vector-valued one-form, corresponding to the covariant derivative operator $\nabla_{(\cdot)} v : \mathfrak{X}(M) \to \mathfrak{X}(M)$, defined for any $u \in \mathfrak{X}(M)$ by $\nabla v(u) := \nabla_u v$. In local coordinates, we have that

$$\nabla v = (\partial_i v^j + \Gamma^j_{ki} v^k)\frac{\partial}{\partial q^j} \otimes \mathrm{d}q^i,$$

where $\Gamma^i_{kj} \in C^\infty(M)$ denote the Christoffel symbols defined by

$$\Gamma^i_{kj} = \tfrac{1}{2} g^{jm}(\partial_k g_{mi} + \partial_i g_{mk} - \partial_m g_{ki}),$$

where $g^{jm} \in C^\infty(M)$ denote the local components of the inverse metric tensor. In the case of Cartesian coordinates, we have that every $\Gamma^i_{kj} = 0$ and thus the components of the velocity gradient ∇v take the more common form $(\nabla v)^j_i = (\partial_i v^j)$. For cylindrical coordinates, one has that $\Gamma^r_{\theta\theta} = -r, \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$, while all other Christoffel symbols are zero.

The effort variable e_r is chosen to be the covector-valued $n-1$ form $\mathcal{T}_\kappa \in \Omega^1(M) \otimes \Omega^{n-1}(M)$ corresponding to the Cauchy shear stress. This stress describes the force on surface elements in a fluid flow due to shear viscosity. Note that for the class of incompressible fluid flows that we consider, the bulk viscosity does not play a role at all. The covector-valued form $e_r = \mathcal{T}_\kappa$ is related to the *true* Cauchy stress $(0,2)$ tensor field $\sigma \in T^0_2(M)$ by the relation(Califano et al., 2021b)

$$\mathcal{T}_\kappa = \star_2 \sigma, \tag{3}$$

where $\star_2$ denotes the Hodge star operator applied to the second leg of σ considered as a covector-valued 1-form. We denote by $T^p_q(M)$ the space of (p,q) tensor fields on M.

In local coordinates we have that $\sigma = \sigma_{ij}\, \mathrm{d}q^i \otimes \mathrm{d}q^j$ and

$$\mathcal{T}_\kappa = \begin{cases} \left(\sqrt{g}\epsilon_{ab} g^{aj}\sigma_{ij}\right) \mathrm{d}q^i \otimes \mathrm{d}q^b, & (n=2) \\ \left(\frac{1}{2}\sqrt{g}\epsilon_{abc} g^{aj}\sigma_{ij}\right) \mathrm{d}q^i \otimes \mathrm{d}q^b \wedge \mathrm{d}q^c. & (n=3) \end{cases}$$

For the motivation behind representing the Cauchy-stress geometrically as a covector-valued $n-1$ form, the reader is referred to (Frankel, 2004, Appendix A).

For incompressible flow, the Cauchy stress tensor σ is assumed to depend on the velocity gradient ∇v and not directly on the fluid's velocity v, considering that v is a relative quantity depending on the observer. Furthermore, σ is assumed to be symmetric due to the balance of angular momentum. Consequently, σ depends only on the symmetric part of the velocity gradient (Marsden and Hughes, 1994). In the Riemannian geometric setting, this symmetric part is encoded by the Lie derivative of the metric $\mathcal{L}_v g \in T^0_2(M)$. In local coordinates, we have that

$$\mathcal{L}_v g = (\nabla_i \tilde{v}_j + \nabla_j \tilde{v}_i) \mathrm{d}q^i \otimes \mathrm{d}q^j.$$

where $\tilde{v}_i = g_{ij} v^j$. In Cartesian coordinates, the components of $\mathcal{L}_v g$ reduce to the more common expression

$$(\mathcal{L}_v g)_{ij} = \partial_i(\delta_{jk} v^k) + \partial_j(\delta_{ik} v^k).$$

Therefore, the Lie derivative of the metric extends the concept of the rate-of-strain tensor to general (curvilinear) coordinates and curved spatial manifolds (Gilbert and Vanneste, 2019).

The relation between $\mathcal{L}_v g$ and ∇v can be derived using the property $\nabla \tilde{v} = (\nabla v)^{\flat_1}$ and the identity (Gilbert and Vanneste, 2019):

$$2\nabla \tilde{v} = \underbrace{\mathcal{L}_v g}_{\text{sym.}} - \underbrace{\mathrm{d}\tilde{v}}_{\text{skew-sym.}},$$

where all terms above are considered as covector-valued one-forms. Note that the term $\mathrm{d}\tilde{v} \in \Omega^2(M)$ can be associated to a covector-valued one-forms by means of the identification $\Omega^2(M) = \Omega^1(M) \otimes \Omega^1(M)$. Consequently, we have that

$$\mathrm{sym}(\nabla v) := \tfrac{1}{2}(\mathcal{L}_v g)^{\sharp_1}. \tag{4}$$

It is also important to note that the identity above holds for the induced Levi-Civita connection by the metric.

With the above construction, the geometric version of Stokes' stress constitutive relation for incompressible viscous fluid on a general Riemannian manifold is given by

$$\sigma = \kappa \mathcal{L}_v g, \tag{5}$$

where $\kappa \in C^\infty(M)$ is the dynamic viscosity function which in general depends on the mass density and pressure of the fluid. In terms of the chosen effort and flow variables, $e_r = \mathcal{T}_\kappa$ and $f_r = \nabla v$, the aforementioned constitutive relation takes the form

$$e_r = \mathcal{R}_\kappa(f_r), \qquad \mathcal{R}_\kappa := 2\kappa \star_2 \circ \flat_1 \circ \mathrm{sym},$$

as depicted in Fig. 1.

The duality pairing between e_r and f_r is given by

$$\int_M e_r \dot{\wedge} f_r \geq 0, \tag{6}$$

where $\dot{\wedge} : \Omega^1(M) \otimes \Omega^p(M) \times \mathfrak{X}(M) \otimes \Omega^q(M) \to \Omega^{p+q}(M)$ denotes the *dot wedge* product (Califano et al., 2021b). The non-decreasing property of (6) implies that energy always goes towards the $\mathbb{R}$ element in Fig.1 which represents the irreversible transfer of energy to the thermal domain. This property will become clear later when we present the alternative vector calculus expression of (6).

3.3 Open ports

The port-Hamiltonian model in Fig. 1 has three open ports, two of which are boundary ports while one is a distributed port. The distributed port $(p, \mathbf{0}) \in \Omega^0(M) \times \Omega^n(M)$ has the zero top form ($\mathbf{0} := 0 \cdot \mu_{\mathrm{vol}}$) as its flow variable, while its effort variable is the static pressure function p of the fluid within the spatial domain. It is straightforward to see that the power flowing through the port $(p, \mathbf{0})$ is zero. This reflects the fact that, for incompressible flow, the pressure acts as a Lagrange multiplier that enforces the incompressibility constraint of the fluid and does not have the same thermodynamic nature as in compressible flow (Rashad et al., 2021b).

The two boundary ports $(e_{\partial p}, f_{\partial p})$ and $(e_{\partial r}, f_{\partial r})$ represent the energy supplied to the fluid dynamical system through the boundary ∂M of the spatial domain. The first port corresponds to the power flow due to (total) pressure while the second one is due to viscosity. The duality pairings between the effort and flow variables of the two ports are given by

$$\int_{\partial M} e_{\partial p} \wedge f_{\partial p}, \qquad \int_{\partial M} e_{\partial r} \dot{\wedge} f_{\partial r}. \tag{7}$$

The aforementioned boundary port variables correspond to the boundary conditions of the partial differential equations (PDEs) represented by the port-Hamiltonian model.

3.4 Stokes Dirac structure $\mathcal{D}_{NS}$

The mathematical object used to interconnect the previously mentioned subsystems of the port-Hamiltonian model of Navier-Stokes equations is the Stokes-Dirac structure $\mathcal{D}_{\mathrm{NS}}$. This key structure is the infinite-dimensional subspace of the port-space corresponding to the five ports $(\delta_x H_{\mathrm{k}}, \dot{x})$, (e_r, f_r), $(p, \mathbf{0})$, $(e_{\partial p}, f_{\partial p})$, and $(e_{\partial r}, f_{\partial r})$ that encodes the power balance:

$$-\int_M \dot{x} \wedge \delta_x H_{\mathrm{k}} + p \cdot 0 - e_r \dot{\wedge} f_r + \int_{\partial M} e_{\partial p} \wedge f_{\partial p} + e_{\partial r} \dot{\wedge} f_{\partial r} = 0,$$

which equivalently can be expressed as

$$\dot{H}_{\mathrm{k}} = -\int_M e_r \dot{\wedge} f_r + \int_{\partial M} e_{\partial p} \wedge f_{\partial p} + e_{\partial r} \dot{\wedge} f_{\partial r}, \tag{8}$$

which states that the rate of kinetic energy stored is equal to the sum of the internal dissipated power due to viscosity and the external supplied power via the boundary ∂M due to pressure and shear stress forces.

The equations of the Stokes-Dirac structure $\mathcal{D}_{\mathrm{NS}}$ are given by(Califano et al., 2021b)

$$\begin{pmatrix}\partial_t \tilde{v}\\ \partial_t \mu\end{pmatrix} = -J\begin{pmatrix}\delta_{\tilde{v}} H_{\mathrm{k}}\\ \delta_\mu H_{\mathrm{k}}\end{pmatrix} + \begin{pmatrix}-\frac{1}{\star\mu}\mathrm{d}p + \frac{1}{\star\mu}\star_2 \mathrm{d}_\nabla e_r\\ 0\end{pmatrix}, \quad (9)$$

$$0 = \mathrm{d}\circ \frac{1}{\star\mu}(\delta_{\tilde{v}} H_{\mathrm{k}}), \quad (10)$$

$$f_r = \nabla \circ \sharp \circ \star \circ \frac{1}{\star\mu}(\delta_{\tilde{v}} H_{\mathrm{k}}), \quad (11)$$

$$e_{\partial p} = i^*(\tfrac{1}{2}(\star\mu)\iota_v \tilde{v} + p), \quad f_{\partial p} = -i^*(\frac{1}{\star\mu}\delta_{\tilde{v}} H_{\mathrm{k}}), \quad (12)$$

$$e_{\partial r} = i_2^*(e_r), \quad f_{\partial r} = i_2^*(\sharp \circ \star(\delta_{\tilde{v}} H_{\mathrm{k}})), \quad (13)$$

where $J : \Omega^{n-1}(M) \times \Omega^0(M) \to \Omega^1(M) \times \Omega^n(M)$ denotes the skew-symmetric Lie-Poisson operator given by

$$J\begin{pmatrix}\delta_{\tilde{v}} H_{\mathrm{k}}\\ \delta_\mu H_{\mathrm{k}}\end{pmatrix} = \begin{pmatrix}\frac{1}{*\mu}\iota_{\sharp\circ\star(\cdot)} d\tilde{v} & \mathrm{d}\\ \mathrm{d} & 0\end{pmatrix}\begin{pmatrix}\delta_{\tilde{v}} H_{\mathrm{k}}\\ \delta_\mu H_{\mathrm{k}}\end{pmatrix},$$

while $\mathrm{d}_\nabla : \Omega^1(M) \otimes \Omega^{n-1}(M) \to \Omega^1(M) \otimes \Omega^n(M)$ denotes the exterior covariant derivative operator, i^* denotes the pullback of the inclusion map $i : \partial M \to M$, and i_2^* denotes the pullback action of a two-point tensor. It is important to note that the exterior covariant derivative is the formal adjoint of the covariant derivative which implies that the overall interconnection operator is skew-symmetric (Califano et al., 2021b).

3.5 Corresponding partial-differential equations

In order to explicitly write the PDEs and their corresponding boundary conditions represented by the port-Hamiltonian model (9-13) we proceed as follows. First we replace the expressions for $\delta_{\tilde{v}} H_{\mathrm{k}}, \delta_\mu H_{\mathrm{k}}$ in Sec. 3.1 and e_r, f_r in Sec. 3.2. Second, we use the following identity (valid for incompressible flow)

$$\star_2 \mathrm{d}_\nabla \mathcal{T}_\kappa = \kappa \star_2 \mathrm{d}_\nabla \star_2 \mathcal{L}_v g = \kappa \Delta_R \tilde{v},$$

where $\Delta_R : \Omega^1(M) \to \Omega^1(M)$ is the Ricci Laplacian operator related to the Hodge Laplacian operator $\Delta := \star\mathrm{d}\star\mathrm{d} + \mathrm{d}\star\mathrm{d}\star$ by the Weitzenböck identity:

$$\Delta_R \tilde{v} = \Delta\tilde{v} + \mathrm{Ric}(v) = \star\mathrm{d}\star\mathrm{d}\tilde{v} + \mathrm{Ric}(v),$$

where $\mathrm{Ric} : \mathfrak{X}(M) \to \Omega^1(M)$ is the Ricci tensor field and $\mathrm{d}\star\tilde{v} = 0$ due to incompressibility.

Therefore, we can rewrite (9-11) as the PDEs:

$$\partial_t \tilde{v} = -\tfrac{1}{2}\mathrm{d}\iota_v \tilde{v} - \iota_v \mathrm{d}\tilde{v} - \frac{\mathrm{d}p}{\rho} + \frac{\kappa \Delta_R \tilde{v}}{\rho}, \quad (14)$$

$$\partial_t(\star\rho) = -\mathrm{d}(\rho\star\tilde{v}), \quad (15)$$

$$0 = \mathrm{d}\star\tilde{v}. \quad (16)$$

In local coordinates, the PDEs (14-16) are expressed as

$$\partial_t \tilde{v}_i = -\tfrac{1}{2}\partial_i(\tilde{v}_k v^k) - v^j \partial_j(\tilde{v}_i) - \frac{1}{\rho}\partial_i p + \frac{\kappa}{\rho}(\Delta\tilde{v}_i + R_i),$$

$$\partial_t(\rho\mu_{\mathrm{vol}}) = -\frac{1}{\sqrt{g}}\partial_j(\sqrt{g}\rho v^j)\mu_{\mathrm{vol}},$$

$$0 = \partial_i(\sqrt{g}v^i),$$

where $\Delta\tilde{v}_i, R_i \in C^\infty(M)$ denote the local components of the one-forms $\Delta\tilde{v}$ and $\mathrm{Ric}(v)$ given by

$$\Delta\tilde{v}_i := \sqrt{g}g^{am}\partial_m\left(\frac{1}{\sqrt{g}}\partial_a(g_{ij}v^j)\right),$$

$$R_i := \nabla_j \nabla_i v^j - \nabla_i \nabla_j v^j.$$

Note the fact that we are considering incompressible non-homogeneous fluid flow (i.e. $\partial_t\rho \neq 0$). In case of homogeneous flow, (15) and (16) would degenerate to the same equation. The boundary conditions associated to the PDEs (14-16) are given by

$$e_{\partial p} = i^*(p_{\mathrm{tot}}), \quad f_{\partial p} = -i^*(\star\tilde{v}) \quad (17)$$

$$e_{\partial r} = i_2^*(\mathcal{T}_\kappa), \quad f_{\partial r} = i_2^*(v), \quad (18)$$

which have the following physical interpretations.

First, the boundary effort $e_{\partial p}$ is the total pressure function at the boundary, i.e. $e_{\partial p} = i^*(p_{\mathrm{tot}}) := p_{\mathrm{tot}} \circ i \in C^\infty(\partial M)$, where

$$p_{\mathrm{tot}} := \tfrac{1}{2}\rho\iota_v\tilde{v} + p \in C^\infty(M),$$

is defined as the sum of the dynamic and static pressures. Second, the boundary flow $f_{\partial p}$ corresponds to the normal component of the vector field v at ∂M which can be seen from the identity:

$$i^*(\star\tilde{v}) = i^*(\iota_v\mu_{\mathrm{vol}}) = i^*(g(v,n))\mu_{\mathrm{vol}}^{\partial M}, \quad (19)$$

where n denotes the (outward) normal vector field to ∂M and $\mu_{\mathrm{vol}}^{\partial M} \in \Omega^{n-1}(\partial M)$ is the induced volume form on ∂M. Thus, the power supplied to the fluid through the boundary port $(e_{\partial p}, f_{\partial p})$ is given by

$$\int_{\partial M} e_{\partial p} \wedge f_{\partial p} = -\int_{\partial M} p_{\mathrm{tot}} \cdot g_{ij}v^i n^j \mu_{\mathrm{vol}}^{\partial M}.$$

The minus sign above is due to the choice of n as the outward normal vector field such that fluid flowing out of M results in a decrease in kinetic energy as seen from (8).

Third, the boundary effort $e_{\partial r} = i_2^*(\mathcal{T}_\kappa)$ is the pullback of the second leg of $\mathcal{T}_\kappa$ which corresponds to the pullback of the $n-1$ form part of it. Following the same reasoning of (19), we have that

$$i_2^*(\mathcal{T}_\kappa) = i_2^*(\star_2\sigma) = i^*(\sigma(n,\cdot))\mu_{\mathrm{vol}}^{\partial M}.$$

Whereas, the boundary flow $f_{\partial r} = i_2^*(v)$ represents the vector field v evaluated at the boundary ∂M.

Finally, the power supplied to the fluid through the boundary port $(e_{\partial r}, f_{\partial r})$ is given by

$$\int_{\partial M} e_{\partial r} \dot{\wedge} f_{\partial r} = \int_{\partial M} \sigma_{ij} n^i v^j \mu_{\mathrm{vol}}^{\partial M}.$$

4. VECTOR CALCULUS PORT-HAMILTONIAN MODEL

Now we turn attention to the vector calculus formulation of the incompressible viscous flow port-Hamiltonian model presented in the previous section. In contrast to the exterior calculus formulation, the state- and port-variables will be now either functions, vector-fields or second-rank tensor fields as summarized in Table 3.

Following the same line of thought of the previous section, we present the vector-calculus formulation of the individual subsystems next.

Distributed ports	$\dot{x}, \delta_x H_{\mathrm{k}}$	p	e_r, f_r
	$\mathfrak{X}(M) \times C^\infty(M)$	$C^\infty(M)$	$T_2^0(M)$
Boundary ports	$e_{\partial p}, f_{\partial p}$	$e_{\partial r}$	$f_{\partial r}$
	$C^\infty(\partial M)$	$T_{01}^{00}(\partial M)$	$T_{00}^{01}(\partial M)$

Table 3. Port variables of the port-Hamiltonian model in vector calculus form

4.1 Energy storage subsystem

The state of the energy storage subsystem will consist now of the contra-variant variables $x := (v, \rho) \in \mathfrak{X}(M) \times C^\infty(M)$ which are related to their differential-form counterparts by

$$v = \sharp(\tilde{v}), \qquad \rho = \star\mu. \tag{20}$$

In terms of the states (v, ρ), the Hamiltonian kinetic energy functional takes the form

$$H_\mathrm{k}(v, \rho) = \int_M \tfrac{1}{2}\rho g(v, v)\mu_\mathrm{vol}.$$

The flow variables are given by $(\dot{v}, \dot{\rho})$ while their corresponding effort variables are $(\delta_v H_\mathrm{k}, \delta_\rho H_\mathrm{k}) \in \mathfrak{X}(M) \times C^\infty(M)$ which are related to their differential-form counterparts by

$$\delta_v H_\mathrm{k} = \sharp \circ \star(\delta_{\tilde{v}} H_\mathrm{k}) = \rho v, \qquad \delta_\rho H_\mathrm{k} = \delta_\mu H_\mathrm{k} = \tfrac{1}{2} g(v, v) \tag{21}$$

The rate of change of kinetic energy in terms of the new port-variables is then given by

$$\dot{H}_\mathrm{k} = \int_M (g(\delta_v H_\mathrm{k}, \dot{v}) + \delta_\rho H_\mathrm{k} \cdot \dot{\rho})\mu_\mathrm{vol}$$

4.2 Energy dissipation subsystem

The flow and effort variables of the energy dissipation subsystem will be now given by the symmetric (0,2) tensor fields:

$$f_r = \tfrac{1}{2}\mathcal{L}_v g \in T^0_2(M), \qquad e_r = 2\sigma \in T^0_2(M), \tag{22}$$

with the constitutive relation between them given by (5) i.e.

$$e_r = \mathcal{R}_\kappa(f_r), \qquad \mathcal{R}_\kappa = \kappa.$$

The effort and flow tensor fields e_r, f_r are related to their differential-form counterparts by (3) and (4), respectively. Furthermore, their duality pairing is given by

$$\int_M \langle\langle e_r, f_r \rangle\rangle \mu_\mathrm{vol} = \int_M \tfrac{1}{2}\kappa \langle\langle \mathcal{L}_v g, \mathcal{L}_v g \rangle\rangle \mu_\mathrm{vol} \geq 0, \tag{23}$$

where $\langle\langle \cdot, \cdot \rangle\rangle : T^0_2(M) \times T^0_2(M) \to C^\infty(M)$ denotes the contraction of tensors defined using the metric as

$$\langle\langle \sigma, \tilde{\sigma} \rangle\rangle := g^{ai} g^{bj} \sigma_{ab} \tilde{\sigma}_{ij}, \qquad \forall \sigma, \tilde{\sigma} \in T^0_2(M).$$

Note that the duality pairing (23) is equivalent to (6) where the non-decreasing property is straightforward to assert in (23).

4.3 Open ports

With respect to the two boundary ports of the port-Hamiltonian model, both port-variables of $(e_{\partial p}, f_{\partial p})$ are now both smooth functions on ∂M with their duality pairing given by

$$\int_{\partial M} e_{\partial p} \cdot f_{\partial p} \mu^{\partial M}_\mathrm{vol},$$

with $\mu^{\partial M}_\mathrm{vol}$ being the induced volume form on ∂M.

Whereas the other boundary port-variables $(e_{\partial r}, f_{\partial r})$ need to be represented by *two-point tensor fields* over the inclusion map $i : \partial M \to M$ (Marsden and Hughes, 1994, Chapter 1). More precisely, the boundary flow $f_{\partial r}$ assigns to each point $m \in \partial M$ a two point tensor of type $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ over i. Thus, $f_{\partial r}(m) \in T_{i(m)}M$ is a tangent vector to M assigned to $m \in \partial M$. Similarly, the boundary effort $e_{\partial r}$ assigns a two point tensor of type $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ over i to each point $m \in \partial M$ given by the 1-form $e_{\partial r}(m) \in T^*_{i(m)}M$. The duality pairing between $e_{\partial r}$ and $f_{\partial r}$ is given by

$$\int_{\partial M} e_{\partial r}(f_{\partial r}) \mu^{\partial M}_\mathrm{vol},$$

where $e_{\partial r}(f_{\partial r}) \in C^\infty(\partial M)$ denotes the contraction of the two dual tensor fields. We denote by $T^{pr}_{qs}(\partial M)$ the space of two-point $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ tensor fields over the inclusion map $i : \partial M \to M$.

4.4 Stokes Dirac structure $\mathcal{D}_{NS}$

In terms of vector calculus notation, the new Stokes Dirac structure, connecting the aforementioned subsystems, can be derived from its exterior calculus counterpart in (9-13) as follows. First, we apply $\sharp$ to both sides of the first equation in (9) and apply $\star$ to both sides of the second equation in (9) as well as (10). Then using (3,4,20,21), we can write the equations of the Stokes-Dirac structure as

$$\begin{pmatrix} \partial_t v \\ \partial_t \rho \end{pmatrix} = -J \begin{pmatrix} \delta_v H_\mathrm{k} \\ \delta_\rho H_\mathrm{k} \end{pmatrix} + \begin{pmatrix} -\frac{1}{\rho}\mathrm{grad}(p) + \frac{1}{\rho}\mathbf{div}(e_r) \\ 0 \end{pmatrix}, \tag{24}$$

$$0 = -\,\mathrm{div}(\frac{1}{\rho}\delta_v H_\mathrm{k}) \tag{25}$$

$$f_r = \mathrm{sym} \circ \mathbf{grad}(\frac{1}{\rho}\delta_v H_\mathrm{k}) \tag{26}$$

$$e_{\partial p} = p_\mathrm{tot} \circ i, \qquad f_{\partial p} = -g(v, n) \circ i \tag{27}$$

$$e_{\partial r} = i^*(\sigma(n, \cdot)), \qquad f_{\partial r} = v \circ i \tag{28}$$

with the skew-symmetric operator $J : \mathfrak{X}(M) \times C^\infty(M) \to \mathfrak{X}(M) \times C^\infty(M)$ given by

$$J \begin{pmatrix} \delta_v H_\mathrm{k} \\ \delta_\rho H_\mathrm{k} \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho}\mathcal{S}_v & \mathrm{grad} \\ \mathrm{div} & 0 \end{pmatrix} \begin{pmatrix} \delta_v H_\mathrm{k} \\ \delta_\rho H_\mathrm{k} \end{pmatrix},$$

where we used the following operators:

$$\begin{aligned} \mathrm{grad}(f) &:= (\mathrm{d}f)^\sharp, & \mathrm{grad} &: C^\infty(M) \to \mathfrak{X}(M) \\ \mathrm{div}(v) &:= \star\,\mathrm{d} \star \tilde{v}, & \mathrm{div} &: \mathfrak{X}(M) \to C^\infty(M) \\ \mathcal{S}_v(u) &:= (\iota_u \mathrm{d}\tilde{v})^\sharp, & \mathcal{S}_v &: \mathfrak{X}(M) \to \mathfrak{X}(M) \\ \mathbf{div}(\sigma) &:= (\star_2 \mathrm{d}_\nabla \star_2 \sigma)^\sharp & \mathbf{div} &: T^0_2(M) \to \mathfrak{X}(M) \\ \mathbf{grad}(v) &:= \flat_1 \circ \nabla & \mathbf{grad} &: \mathfrak{X}(M) \to T^0_2(M) \end{aligned}$$

with grad denoting the gradient operator of scalar fields, div and **grad** denoting the divergence and gradient operators of vector fields, while **div** denotes the divergence operator of second rank tensor fields. Note that sym$\circ$**grad**$(\cdot)$ and **div**$(\cdot)$ are formal adjoint operators which implies the skew-symmetry of the overall interconnection operator. In local coordinates, we have that

$$\mathrm{grad}(f) = g^{ij}\partial_i f \frac{\partial}{\partial q^j}, \qquad \mathrm{div}(v) = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}v^i),$$

$$\mathbf{div}(\sigma) = g^{mk}\left(\partial_i \sigma_{im} - \Gamma^k_{ii}\sigma_{km} - \Gamma^k_{im}\sigma_{ik}\right)\frac{\partial}{\partial q^k},$$

$$\mathbf{grad}(v) = (\partial_i \tilde{v}_j - \Gamma^k_{ji}\tilde{v}_k)\mathrm{d}q^j \otimes \mathrm{d}q^i.$$

Whereas, the skew-symmetric operator $\mathcal{S}_v$ is defined as the (1,1) tensor field, or equivalently the vector-valued one-form, with local expression

$$\mathcal{S}_v = g^{mj}\partial_i(g_{jk}v^k)\frac{\partial}{\partial q^m} \otimes \mathrm{d}q^i.$$

For the special case $M = \mathbb{R}^3$, we have that

$$\mathcal{S}_v(\delta_v H_\mathrm{k}) = \mathrm{curl}(v) \times \delta_v H_\mathrm{k}, \qquad \mathrm{curl} : \mathfrak{X}(M) \to \mathfrak{X}(M),$$

where $\times$ denotes the cross-product of vector fields and $\mathrm{curl}(v) := (\star \mathrm{d}\tilde{v})^\sharp$ denotes the curl operator. Note that the identification of the action of $\mathcal{S}_v$ with the cross-products of vector fields is a generalization of the action of skew-symmetric matrices on $\mathbb{R}^3$ with the cross-products of three-dimensional vectors. Furthermore, the operator $\mathcal{S}_v$ is valid for any n-dimensional manifold unlike the use of the curl operator which is valid for $n = 3$ only.

Finally, the power balance encoded by the Stokes-Dirac structure (24-28) is given by

$$\dot{H}_\mathrm{k} = -\int_M \langle\langle e_r, f_r\rangle\rangle \mu_\mathrm{vol} + \int_{\partial M} \left[e_{\partial p} \cdot f_{\partial p} + e_{\partial r}(f_{\partial r})\right] \mu_\mathrm{vol}^{\partial M},$$

which is equivalent to (8) and the PDEs corresponding to (24-28) can be derived similar to Sec. 3.5.

5. CONCLUSION

In this paper, we presented the port-Hamiltonian model of incompressible viscous flow on a general Riemannian manifold using both exterior calculus, vector calculus and their respective coordinate-based expressions. While both representations of the port-Hamiltonian model in (9-13) and (24-28) are coordinate-free, equivalent, and valid for general arbitrary coordinate systems, the former representation using exterior calculus can be argued to be more elegant due to the minimal number of operators used. Furthermore, in the exterior calculus version one can see a clear separation between topological and metric operators.

An advantage of the presented work is that it acts as a guide for computational engineers and researchers focused more on simulation and control. However, we hope with this paper to increase the interest of such researchers, more proficient in vector calculus than exterior calculus, in developing structure-preserving discretization techniques that mirror the continuous exterior calculus operators to the discrete case. The interested reader can refer to Šešlija et al. (2012, 2014); Nitschke et al. (2017); Jagad et al. (2021) for this active area of research. Future work includes simulation of the presented model using techniques that preserve the pH structure of the system and comparing the results to other techniques e.g. Mohamed et al. (2016). Another possible direction is extending the fluid model to magneto-hydrodynamic systems e.g. Siuka et al. (2010).

REFERENCES

Califano, F., Rashad, R., Dijkshoorn, A., Koerkamp, L.G., Sneep, R., Brugnoli, A., and Stramigioli, S. (2021a). Decoding and realising flapping flight with port-hamiltonian system theory. *Annual Reviews in Control*, 51, 37–46.

Califano, F., Rashad, R., Schuller, F.P., and Stramigioli, S. (2021b). Geometric and energy-aware decomposition of the navier–stokes equations: A port-hamiltonian approach. *Physics of Fluids*, 33(4), 047114.

Duindam, V., Macchelli, A., Stramigioli, S., and Bruyninckx, H. (2009). *Modeling and control of complex physical systems: the port-Hamiltonian approach*. Springer Science & Business Media.

Frankel, T. (2004). *The geometry of physics: an introduction*. Cambridge university press.

Gilbert, A.D. and Vanneste, J. (2019). A geometric look at momentum flux and stress in fluid mechanics. *arXiv preprint arXiv:1911.06613*.

Jagad, P., Abukhwejah, A., Mohamed, M., and Samtaney, R. (2021). A primitive variable discrete exterior calculus discretization of incompressible navier–stokes equations over surface simplicial meshes. *Physics of Fluids*, 33(1), 017114.

Marsden, J.E. and Hughes, T.J. (1994). *Mathematical foundations of elasticity*. Courier Corporation.

Marsden, J.E., Raţiu, T., and Weinstein, A. (1984). Semidirect products and reduction in mechanics. *Transactions of the american mathematical society*, 281(1), 147–177.

Mohamed, M.S., Hirani, A.N., and Samtaney, R. (2016). Discrete exterior calculus discretization of incompressible navier–stokes equations over surface simplicial meshes. *Journal of Computational Physics*, 312, 175–191.

Nitschke, I., Reuther, S., and Voigt, A. (2017). Discrete exterior calculus (dec) for the surface navier-stokes equation. In *Transport processes at fluidic interfaces*, 177–197. Springer.

Rashad, R., Califano, F., Schuller, F.P., and Stramigioli, S. (2021a). Port-hamiltonian modeling of ideal fluid flow: Part i. foundations and kinetic energy. *Journal of Geometry and Physics*, 104201.

Rashad, R., Califano, F., Schuller, F.P., and Stramigioli, S. (2021b). Port-hamiltonian modeling of ideal fluid flow: Part ii. compressible and incompressible flow. *Journal of Geometry and Physics*, 104199.

Rashad, R., Califano, F., van der Schaft, A., and Stramigioli, S. (2020). Twenty Years of Distributed Port-Hamiltonian Systems: A Literature Review. *IMA Journal of Mathematical Control and Information*, 37(4), 1400–1422.

Šešlija, M., Scherpen, J.M., and van der Schaft, A. (2014). Explicit simplicial discretization of distributed-parameter port-Hamiltonian systems. *Automatica*, 50(2), 369–377.

Šešlija, M., van der Schaft, A., and Scherpen, J.M. (2012). Discrete exterior geometry approach to structure-preserving discretization of distributed-parameter port-Hamiltonian systems. *Journal of Geometry and Physics*, 62(6), 1509–1531.

Siuka, A., Schöberl, M., and Schlacher, K. (2010). Hamiltonian evolution equations of inductionless magnetohydrodynamics. In *Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems–MTNS*, volume 5.

van der Schaft, A. and Maschke, B.M. (2002). Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *Journal of Geometry and physics*, 42(1-2), 166–194.