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ROOTS, SYMMETRY, AND CONTOUR INTEGRALS IN QUEUING-TYPE SYSTEMS[*]

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Abstract. Many (discrete) stochastic systems are analyzed using the probability generating function (pgf) technique, which often leads to expressions in terms of the (complex) roots of a certain equation. In this paper, for a class of pgfs with a rational form, we show that it is not necessary to compute the roots in order to evaluate these expressions. Instead, one can use contour integrals, which is computationally a more reliable method than the classical root-finding approach. We also give the necessary and sufficient condition for the mean of the corresponding random variable, e.g., queue length, to be an additive function of the roots. In this case, the mean is found using one contour integral. Finally, we give the necessary and sufficient condition for the mean to be independent of the roots.

Key words. queuing systems, roots, contour integrals, probability generating functions

AMS subject classifications. 60K25, 30C15, 30E20

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1. Introduction.

In this paper, we consider a particular class of stochastic systems, where finding the probability generating function (pgf) $X(z)$ of the queue length (or another random variable of interest) leads to a rational form with several unknown coefficients x_j, $j = 0, \dots, n$, in the numerator:

$$X(z) = \frac{\sum_{j=0}^{n} x_j f_j(z)}{D(z)}(z-1), \tag{1.1}$$

where $f_j(z)$ for $j = 0, \dots, n$ and $D(z)$ are known analytic functions, $f_j(1) \neq 0$ for some j, and $D(z)$ has $n+1$ zeros in the closed unit disk, denoted by $\hat{z}_0 = 1, \hat{z}_1, \dots, \hat{z}_n$. Equation (1.1) is well known in queuing theory, which motivated this research. An example of such a system is the bulk-service queue; see [3]. For this example, $X(z)$ is the pgf of the queue length, $n+1$ is the size of service bulk, x_j is the probability of having j customers in the queue, $f_j(z) = A(z)\sum_{k=j}^{n} z^k$, and $D(z) = z^{n+1} - A(z)$, where $A(z)$ is the pgf of the number of the arrivals during a time slot. This type of pgf occurs in many other queuing systems, e.g., in traffic models (see [19], [18]), but also in more general stochastic systems; see section 5 below.

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The classical approach to finding the unknowns in the numerator is to consider the analyticity of the pgf in the unit disk and compute the zeros $\hat{z}_1, \ldots, \hat{z}_n$ of $D(z)$. Due to the analyticity of the pgf, the zeros of the denominator are also zeros of the numerator. This yields n linear equations for the unknowns:

$$\sum_{j=0}^{n} x_j f_j(\hat{z}_k) = 0, \qquad k = 1, \ldots, n. \tag{1.2}$$

One more equation follows from the normalization equation $X(1) = 1$ and L'Hôpital's rule:

$$1 = X(1) = \lim_{z \to 1} \frac{\sum_{j=0}^{n} x_j f_j(z)(z-1)}{D(z)} = \frac{\sum_{j=0}^{n} x_j f_j(1)}{D'(1)},$$

which yields

$$\sum_{j=0}^{n} x_j f_j(1) = D'(1). \tag{1.3}$$

In a similar way, the roots of a characteristic equation are used in the analysis of many queuing systems of which the pgf not necessarily has the form (1.1). This often occurs in discrete-time models such as the $G_D/G_D/1$ queue [21] and the multiserver queue $M/D/s$ [12], but also in continuous models; see, for example, [8], where the authors consider general interarrival times and a Markovian service process, or [20], where two interrelated queues are analyzed. This root-finding problem is not unique to queuing systems, but also occurs in the analysis of some Markov chains; for an example in risk theory, see [16]. In some special cases, there are formulas to compute the roots; see, e.g., [12] for the case of Poisson and binomial arrivals at the bulk-service queue. However, there is no explicit formula in general. Moreover, the derived solution can be very sensitive to the precision of the roots, which, in turn, can be poor even due to a small error in the coefficients; see, e.g., the study of the so-called Wilkinson polynomial [23]. For an example in queuing theory, see [18], where the authors consider the bulk-service queue and show that there are instances where the classical root-finding approach leads to a negative expected queue length.

The system of (1.2) and (1.3) can be rewritten in matrix form:

$$M(1, \hat{z}_1, \ldots, \hat{z}_n)(x_0, \ldots, x_n)^T = (D'(1), 0, \ldots, 0)^T, \tag{1.4}$$

where

$$M(z, z_1, \ldots, z_n) = \begin{pmatrix} f_0(z) & f_1(z) & \ldots & f_n(z) \\ f_0(z_1) & f_1(z_1) & \ldots & f_n(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(z_n) & f_1(z_n) & \ldots & f_n(z_n) \end{pmatrix}. \tag{1.5}$$

Here and later, we use $\hat{z}_1, \ldots, \hat{z}_n$ for the zeros of the denominator and $z_1, \ldots, z_n$ for complex variables. In this paper, we use the properties of the matrix $M(z, z_1, \ldots, z_n)$ and of symmetric polynomials to find the pgf without computing the roots. We represent the pgf using a determinant of a certain matrix, where each entry is a symmetric function of the roots, which can be computed using contour integrals. The same matrix can be used to find the coefficients $x_0, \ldots, x_n$. The advantages of using contour integrals are that the results are generally more reliable compared to the classical root-finding approach (see [18]) and can be used as an intermediate step for further results (see, e.g., [13]). The entries of the matrix can also be computed using the roots, when they are known or easy to find. This modified root-finding approach avoids the numerical inaccuracies that arise when solving (1.4). In the numerical results, we compare the classical root-finding method with our contour-integral and modified root-finding approaches for the case of the classical bulk-service queue. We show that our approaches give accurate results for the whole range of considered service-batch sizes, while the classical system (1.4) becomes ill-conditioned for large n.

This paper extends the results of [18], which focuses on the special case that the functions $f_j(z)$ form a geometric sequence with a certain common ratio. In that case, the pgf has two properties. First, the pgf can be represented in a special product form, where each term of the product depends on not more than one root. Second, the mean of the corresponding random variable, e.g., the queue length, is an additive function of the roots and, under additional conditions, can be found using one contour integral. In this paper, we give a necessary and sufficient condition in terms of the functions $f_j(z)$ for these properties to hold. The special product form of the pgf exists if and only if the functions $f_j(z)$ form, after a linear transformation, a geometric sequence, where the common ratio of the sequence may be more general than in [18]. We provide an algorithm that, for a set of functions $f_j(z)$, checks the existence of such a linear transformation. For the additive-mean property, we distinguish two cases. In the first case, which we call the *degenerate* case, the mean is independent of the roots and, therefore, is always an additive function of the roots. We give a simple necessary and sufficient condition for the degenerate case in terms of values $f_j(1)$ and $f_j'(1)$ for $j = 0, \ldots, n$. For the nondegenerate case, the additive-mean property is equivalent to the existence of the special product form of the pgf. The systems with these properties include the bulk-service queue (see [3]); the multiserver $M/D/s$ and $Geo/D/s$ queues, which are in some sense equivalent to the bulk-service queue with Poisson and binomial arrivals (see [12] and [14]); and the fixed-cycle traffic-light queue (see [18]). The traffic-light queue for a lane with detectors (see [19]) is an example of a queuing system with a pgf of the form (1.1) without these properties. For this queue, we can use the results for the systems with a general matrix $M(z, z_1, \ldots, z_n)$; see section 3.

The paper is structured as follows. In section 2, we give the definitions and required properties of symmetric polynomials and symmetric functions. In section 3, we obtain the pgf and the unknowns in terms of symmetric functions for a general numerator and give numerical results. Then we analyze a special subclass of pgfs in section 4. In section 5, we apply the results of sections 3 and 4 to several stochastic systems. Finally, we conclude the paper in section 6. The proofs of the intermediate results are given in Appendix A, with the exception of some short proofs that are included in the text.

2. Preliminaries. In this section, we give the required definitions and the preliminary results. First, in subsection 2.1, we define symmetric, skew-symmetric, and additive functions and alternant matrices. Then we relate linearly dependent functions and singular matrices; see subsection 2.2. In subsection 2.3, we describe two types of symmetric polynomials and their properties. In subsection 2.4, we analyze the determinant of an alternant matrix, which will be used later in section 3 to obtain the pgf and the unknowns as symmetric functions of the roots. Finally, in subsection 2.5, we obtain the values of symmetric functions at certain points in terms of contour integrals.

2.1. Definitions. Consider a function $f(z_1, \dots, z_n)$ of n complex variables. We focus on two types of functions: symmetric and skew-symmetric.

DEFINITION 2.1. *A function* $f(z_1, \dots, z_n)$ *is called* symmetric *if*

$$f(z_1, \dots, z_n) = f(z_{s(1)}, \dots, z_{s(n)})$$

for any permutation $s \in \boldsymbol{S}_n$*, where* $\boldsymbol{S}_n$ *is the set of all permutations of the set* $\{1, \dots, n\}$.

DEFINITION 2.2. *A function* $f(z_1, \dots, z_n)$ *is called* skew-symmetric *if*

$$f(z_1, \dots, z_n) = \operatorname{sgn}(s) f(z_{s(1)}, \dots, z_{s(n)})$$

for any permutation $s \in \boldsymbol{S}_n$*. Here,* $\operatorname{sgn}(s)$ *is the sign of the permutation* s *and is equal to* $(-1)^{m_s}$*, where* m_s *denotes the number of transpositions, i.e., permutations that interchange two elements, needed to construct* s*. The sign is independent of the representation of* s *as a product of transpositions.*

In the analysis in the following sections, we also use a subtype of symmetric functions, namely, additive functions.

DEFINITION 2.3. *A function* $f(z_1, \dots, z_n)$ *is called* additive *if*

$$f(z_1, \dots, z_n) = \sum_{k=1}^{n} g(z_k)$$

for some function $g(z)$.

In the analysis in sections 3 and 4, we mainly work with alternant matrices.

DEFINITION 2.4. *Consider the functions* $f_1(z), \dots, f_n(z)$ *and points* $z_1, \dots, z_n$*. The matrix*

$$\Lambda(z_1, \dots, z_n) = \begin{pmatrix} f_1(z_1) & \dots & f_n(z_1) \\ \vdots & \ddots & \vdots \\ f_1(z_n) & \dots & f_n(z_n) \end{pmatrix}$$

is called an alternant *matrix.*

An example of an alternant matrix is a Vandermonde matrix, where $f_j(z) = z^{j-1}$. The determinant of the Vandermonde matrix is denoted by $V(z_1, \dots, z_n) = \prod_{1 \leqslant j < k \leqslant n} (z_k - z_j)$. Note that the determinant of an alternant matrix is a skew-symmetric function of $z_1, \dots, z_n$. This follows immediately from the fact that if one interchanges two rows (or columns) in a square matrix, such an operation changes the sign of the determinant; see [22].

Remark 2.5 (equality of rational functions). In what follows, we will work with rational functions of several variables, i.e., $f(z_1, \dots, z_n)/g(z_1, \dots, z_n)$. Suppose that both the numerator and the denominator are analytic functions and $g(z_1, \dots, z_n)$ is not identically equal to 0. For the case of one variable, i.e., $n = 1$, there are not more than a finite number of points where this rational function is not defined, namely, where $g(z_1) = 0$. However, for $n > 1$ this is not true. For example, the function $1/(z_1 + z_2)$ is not defined on the plane $z_1 = -z_2$. Suppose that the functions $f(z_1, \dots, z_n)$ and $g(z_1, \dots, z_n)$ are defined on the set $\Delta_1^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \colon |z_k| < 1, k = 1, \dots, n\}$. Then one can prove that the function $f(z_1, \dots, z_n)/g(z_1, \dots, z_n)$ is defined on a dense open subset of Δ_1^n, because the set of the zeros of the function

$g(z_1,\ldots,z_n)$ is closed and has an empty interior; see [10]. In what follows, when we say that two rational functions are equal, we mean that they are equal on an open dense subset of Δ_1^n, where both of them are defined.

2.2. Singular matrices and linear independence. In this subsection, we relate linear dependency between functions and singular matrices; see the following properties. For the proofs of Properties 2.6 and 2.7, we refer the reader to [6] and Appendix A.1, respectively.

PROPERTY 2.6. *The analytic functions $f_1(z),\ldots,f_n(z)$ are linearly dependent if and only if the Wronskian*

$$\det W(z) = \det \begin{pmatrix} f_1(z) & \dots & f_n(z) \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)}(z) & \dots & f_n^{(n-1)}(z) \end{pmatrix} \tag{2.1}$$

is identically equal to 0.

PROPERTY 2.7. *Consider numbers $a_j \in \mathbb{C}$, $j = 0,\ldots,n$. Suppose the matrix*

$$\Lambda_a = \begin{pmatrix} a_0 & \dots & a_n \\ f_0(z_1) & \dots & f_n(z_1) \\ \vdots & \ddots & \vdots \\ f_0(z_n) & \dots & f_n(z_n) \end{pmatrix}$$

is singular for all $z_1,\ldots,z_n$. If $a_j \neq 0$ for some j, then the functions $f_0(z),\ldots, f_n(z)$ are linearly dependent.

From Property 2.7, it follows that if the matrix Λ_a is singular for all $z_1,\ldots,z_n$ and the functions $f_0(z),\ldots,f_n(z)$ are linearly independent, then $a_j = 0$ for $j = 0,\ldots,n$.

2.3. Symmetric polynomials and their properties. In this subsection, we introduce two types of symmetric polynomials and their properties. The *elementary symmetric polynomials* are given by

$$\sigma_m = \sigma_m(z_1,\ldots,z_n) = \sum_{1\leqslant k_1<\cdots<k_m\leqslant n} z_{k_1}\cdots z_{k_m}. \tag{2.2}$$

The above formula is used for $m = 1,\ldots,n$. For convenience, $\sigma_0 = 1$, and $\sigma_m = 0$ if either $m > n$ or $m < 0$. The elementary symmetric polynomials naturally arise in Vieta's formulas that relate the coefficients of a polynomial with its roots. Namely, consider a polynomial $\sum_{j=0}^n a_j z^j$ with roots $\tilde z_1,\ldots,\tilde z_n$, then it can be written as

$$a_n \prod_{k=1}^{n} (z-\tilde z_k) = a_n \sum_{j=0}^{n} (-1)^j \sigma_j(\tilde z_1,\ldots,\tilde z_n)\, z^{n-j}. \tag{2.3}$$

The proof of (2.3) requires the expansion of the left-hand side of (2.3); see [22].

In the analysis below, we mainly use the *complete homogeneous symmetric polynomials* defined as

$$\zeta_m = \zeta_m(z_1,\ldots,z_n) = \sum_{1\leqslant k_1\leqslant\cdots\leqslant k_m\leqslant n} z_{k_1}\cdots z_{k_m}.$$

Note the difference in the definitions of the elementary and complete homogeneous symmetric polynomials: the indexes k_j and k_{j+1} for the latter case may coincide for some or all j. This allows us to use the above formula for $m > n$. For $m = 0$, we define $\zeta_0 = 1$, and, for $m < 0$, $\zeta_m = 0$. The following property can be used to recursively find all complete homogeneous polynomials from the elementary symmetric polynomials. The proof is given in Appendix A.2.

PROPERTY 2.8. *For $m > 0$, the following equality holds:*

$$\zeta_m = \sum_{j=1}^{n} (-1)^{j-1} \sigma_j \zeta_{m-j}. \tag{2.4}$$

The upper limit of summation in (2.4) can be changed to any number that is at least $\min\{n, m\}$, since $\zeta_{m-j} = 0$ for $j > m$ and $\sigma_j = 0$ for $j > n$.

When we are interested in the values of the complete homogeneous and elementary symmetric polynomials at point $(\hat{z}_1, \ldots, \hat{z}_n)$, where $\hat{z}_1, \ldots, \hat{z}_n$ are the roots of a certain equation, these values can be found without actually knowing the roots; see subsection 2.5. In subsection 2.4 below, for an analytic function of one variable, we construct a symmetric function of n variables using the complete homogeneous symmetric polynomials. Such functions on Δ_1^n are later used to rewrite the determinant of an alternant matrix as a product of a skew-symmetric Vandermonde determinant and a symmetric function.

2.4. Symmetric functions and alternant matrices. In this subsection, we represent the determinant of the alternant matrix in terms of the Vandermonde determinant and symmetric functions. We use such determinants in section 3 to give an alternative representation of the considered type of pgf. First, we define a transformation rule of an analytic function in $\Delta_1 = \{z \in \mathbb{C} : |z| < 1\}$ to a symmetric function defined in $\Delta_1^m \subset \mathbb{C}^m$. Then we give several properties of this transformation. The main result of this subsection is presented in Lemma 2.11.

Consider an analytic function $f(z)$, with the Taylor expansion at 0 given by $f(z) = \sum_{l=0}^{\infty} \alpha_l z^l$. Let

$$F_k^m = F_k^m(z_1, \ldots, z_m) = \sum_{l=k}^{\infty} \alpha_l \zeta_{l-k}(z_1, \ldots, z_m), \tag{2.5}$$

where m corresponds to the number of variables. We call F_k^m the (k, m)-transformation of the function $f(z)$. The (k, m)-transformation of the function $f_j(z)$ is denoted by $F_{j,k}^m$. Note that the function $F_k^m(z_1, \ldots, z_m)$ is a symmetric function of $z_1, \ldots, z_m$.

In the analysis in section 4, we use the following properties.

PROPERTY 2.9. *If $m \leqslant n$, then*

$$F_k^n(z_1, \ldots, z_m, 0, \ldots, 0) = F_k^m(z_1, \ldots, z_m).$$

PROPERTY 2.10. *For $k \geqslant 0$, consider the function $f(z) = \sum_{l=0}^{\infty} \alpha_l z^l$ and its $(k + j, n)$-transformations F_{k+j}^n for $j = 0, \ldots, n$. Then*

$$F_k^n + \sum_{j=1}^{n} (-1)^j \sigma_j F_{k+j}^n = \alpha_k.$$

Property 2.9 follows from the definition of complete homogeneous symmetric polynomials and only requires the observation that $\zeta_l(z_1, \ldots, z_m, 0, \ldots, 0) = \zeta_l(z_1, \ldots, z_m)$.

Property 2.10 follows from Property 2.8 and equality $\zeta_0 = 1$. In the following lemma, we show that the determinant of an alternant matrix can be written as the product of a Vandermonde determinant and a matrix composed from (k, n)-transformations of the functions $f_1(z), \ldots, f_n(z)$ that are used in the alternant matrix. The proof is given in Appendix A.3.

LEMMA 2.11. *Suppose that the functions $f_1(z), \ldots, f_n(z)$ are analytic in $\Delta_r = \{z \in \mathbb{C} \colon |z| < r\}$. Then, for $z_k \in \Delta_r$, $k = 1, \ldots n$,*

$$\det \begin{pmatrix} f_1(z_1) & f_2(z_1) & \ldots & f_n(z_1) \\ f_1(z_2) & f_2(z_2) & \ldots & f_n(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(z_n) & f_2(z_n) & \ldots & f_n(z_n) \end{pmatrix} = V(z_1, \ldots, z_n) \det \Phi(z_1, \ldots, z_n), \tag{2.6}$$

where $V(z_1, \ldots, z_n)$ is the Vandermonde determinant and

$$\Phi(z_1, \ldots, z_n) = \begin{pmatrix} F^n_{1,0} & F^n_{2,0} & \cdots & F^n_{n,0} \\ F^n_{1,1} & F^n_{2,1} & \cdots & F^n_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ F^n_{1,n-1} & F^n_{2,n-1} & \cdots & F^n_{n,n-1} \end{pmatrix}. \tag{2.7}$$

Remark 2.12 (symmetry and singularity). Lemma 2.11 is an important result for our analysis in sections 3 and 4. Note that the matrix $\Phi(z_1, \ldots, z_n)$ in (2.7) consists of symmetric functions of $z_1, \ldots, z_n$. It is, in general, nonsingular for $z_k = z_j$, which will allow us in section 4 to give proofs using induction in the number of variables, n. Moreover, $\det \Phi(z_1, \ldots, z_n)$ is not identically equal to 0 even when $z_1 = \cdots = z_n$, provided that the functions $f_1(z), \ldots, f_n(z)$ are analytic and linearly independent. The reason for this is that the function $\det \Phi(z, \ldots, z)$ is up to a nonzero constant equal to the Wronskian (2.1) of the functions $f_1(z), \ldots, f_n(z)$ which is not identically equal to 0; see Property 2.6. To relate $\det \Phi(z, \ldots, z)$ with the Wronskian, it is sufficient to note that

$$F^n_{j,k}(z, \ldots, z) = \sum_{l=k}^{n-1} \frac{(n-1-k)!}{l!(l-k)!(n-1-l)!} z^{l-k} f_j^{(l)}(z)$$

for $k = 0, \ldots, n-1$. The last equation follows from

$$\zeta_m(\underbrace{z, \ldots, z}_{n}) = \binom{n+m-1}{m} z^m = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} z^{m+n-1}$$

$$= \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z^{n-1-k} z^{m+k}) = \frac{1}{(n-1)!} \sum_{l=k}^{n-1} \binom{n-1}{l} \frac{d^{n-1-l}}{dz^{n-1-l}} z^{n-1-k} \frac{d^l}{dz^l} z^{m+k}$$

$$= \sum_{l=k}^{n-1} \frac{(n-1-k)!}{l!(l-k)!(n-1-l)!} z^{l-k} \frac{d^l}{dz^l} z^{m+k}$$

for $k = 0, \ldots, n-1$.

2.5. Roots and contour integrals. In this subsection, we provide a way of computing the values of symmetric polynomials at special points. Consider the analytic function $D(z)$. Suppose $1, \hat{z}_1, \ldots, \hat{z}_n$ are the only roots of the equation

$$D(z) = 0 \tag{2.8}$$

in the closed unit disk $\bar{\Delta}_1 = \{z \in \mathbb{C}\colon |z| \leqslant 1\}$. Then it is possible to compute $\zeta_m(\hat{z}_1, \ldots, \hat{z}_n)$ without finding the roots. The first step is to represent the complete homogeneous symmetric polynomials in terms of the elementary symmetric polynomials; see Property 2.8. Then we recursively use Newton's formula (see [17]),

$$k\sigma_k = \sum_{j=1}^{k} (-1)^{j-1} \sigma_{k-j} \eta_j, \quad k = 1, \ldots, n, \tag{2.9}$$

to find the elementary symmetric polynomials in terms of the power sums $\eta_j = \eta_j(z_1, \ldots, z_n) = \sum_{k=1}^{n} z_k^j$, $j = 1, \ldots, n$. The power sums, in turn, are found using Cauchy's residue theorem; see [15]. Namely,

$$\eta_j(\hat{z}_1, \ldots, \hat{z}_n) + 1 = \frac{1}{2\pi \mathbf{i}} \oint_{\mathrm{S}_{1+\varepsilon}} \frac{D'(z)}{D(z)} z^j \, dz, \tag{2.10}$$

where $\varepsilon > 0$ is defined so that there are no roots of (2.8) with $1 < |z| \leqslant 1 + \varepsilon$, and $\mathrm{S}_r = \{z \in \mathbb{C}\colon |z| = r\}$.

Remark 2.13 (zero at 1). If (2.8) has no zero at 1, one needs to change the left-hand side of (2.10) to just $\eta_j(\hat{z}_1, \ldots, \hat{z}_n)$. We explicitly consider the case that 1 is a root since the denominators of the pgfs analyzed in section 3 have a zero at 1.

Remark 2.14 (functions F_k^n and contour integrals). If the function $f(z)$ is a polynomial, then $F_k^n(\hat{z}_1, \ldots, \hat{z}_n)$ is a finite sum of the complete homogeneous symmetric polynomials and can be found using (2.10), (2.9), and (2.4), hence, without explicitly computing the roots. The application of the Cauchy residue theorem (see (2.10)) is a crucial step from the root-finding approach to the contour-integral approach. If the function $f(z)$ is not a polynomial, one can truncate the infinite summation in $F_k^n(\hat{z}_1, \ldots, \hat{z}_n)$ using the following bound:

$$\left| F_k^n(\hat{z}_1, \ldots, \hat{z}_n) - \sum_{l=0}^{M} \alpha_{l+k} \zeta_l(\hat{z}_1, \ldots, \hat{z}_n) \right| \leqslant C r^k \binom{M+n}{n-1} \frac{(qr)^{M+1}}{(1-qr)^n}, \tag{2.11}$$

where $|\hat{z}_k| \leqslant q$ for $k = 1, \ldots, n$, $|\alpha_l| \leqslant C r^l$ for $l = 0, 1, \ldots,$ $f(z) = \sum_{l=0}^{\infty} \alpha_l z^l$, and $qr < 1$; see the proof in Appendix A.4. In this way, the determinant of the matrix $\Phi(z_1, \ldots, z_n)$ in (2.7) can be found without computing the roots.

Remark 2.15 (functions F_k^n and minimal polynomials). For the sake of completeness, we also give an alternative way of finding F_k^n at $(\hat{z}_1, \ldots, \hat{z}_n)$ without truncation (2.11). Let $Q(z) = \prod_{k=1}^{n} (z - \hat{z}_k)$ be the minimal polynomial, i.e., the polynomial of the lowest degree, with roots $\hat{z}_1, \ldots, \hat{z}_n$. Due to (2.3),

$$Q(z) = \sum_{j=0}^{n} (-1)^j \sigma_j(\hat{z}_1, \ldots, \hat{z}_n) z^{n-j},$$

which means that this polynomial can be found without computing the roots; see (2.9) and (2.10). Then (see [4])

$$\zeta_m(\hat{z}_1, \ldots, \hat{z}_n) = \sum_{k=1}^{n} \frac{\hat{z}_k^{n+m-1}}{Q'(\hat{z}_k)}.$$

By applying the Cauchy residue theorem, we obtain

$$\zeta_m(\hat{z}_1, \dots, \hat{z}_n) = \frac{1}{2\pi\mathbf{i}} \oint_{\mathrm{S}_1} \frac{z^{n+m-1}}{Q(z)} dz.$$

Hence, we find

$$F_k^n(\hat{z}_1, \dots, \hat{z}_n) = \frac{1}{2\pi\mathbf{i}} \oint_{\mathrm{S}_1} \frac{f(z) - \sum_{l=0}^{k-1} \alpha_l z^l}{Q(z)} z^{n-k-1} dz.$$

Using this formula for each entry of the matrix (2.7), i.e., n^2 times, can be computationally demanding. Still, it can be useful in some special cases. For example, if $f_1(z) = f_0(z) z^k$, then $F_{1,m}^n = F_{0,m+k}^n$, which reduces the number of required computations, when k is small; see such examples in section 5.

Remark 2.16 (additive functions and contour integrals). Suppose that the function $f(z_1, \dots, z_n)$ is an additive function, i.e., $f(z_1, \dots, z_n) = \sum_{k=1}^n g(z_k)$. If the function $g(z)$ is analytic in $\Delta_{1+\epsilon} \setminus \{1\}$, then the value of $f(\hat{z}_1, \dots, \hat{z}_n)$ is given by one contour integral:

$$f(\hat{z}_1, \dots, \hat{z}_n) = \frac{1}{2\pi\mathbf{i}} \oint_{\mathrm{S}_{1+\varepsilon}} \frac{D'(z)}{D(z)} g(z) dz - r_1,$$

where $\varepsilon < \epsilon$ is defined as in (2.10), and r_1 is the residue of the function $D'(z)g(z)/D(z)$ at 1.

3. General case. In this section, we consider pgfs of the form (1.1). Recall that such a pgf has a rational form:

$$X(z) = \frac{\sum_{j=0}^n x_j f_j(z)}{D(z)} (z-1), \tag{1.1 revisited}$$

where coefficients x_j can be found using (1.4), $f_j(z)$ are analytic functions, and $D(z)$ is an analytic function with $n+1$ zeros inside the unit disk including 1, which we denote by $\hat{z}_0 = 1, \hat{z}_1, \dots, \hat{z}_n$. The goal of this section is twofold. First, we represent the pgf $X(z)$ as a symmetric function of the roots and provide an alternative way to compute the unknowns $x_0, \dots, x_n$; see Theorems 3.6 and 3.7 below. These results allow us to find $X(z)$ and $x_0, \dots, x_n$ without computing the roots. Second, we compare the numerical solutions of the classical system (1.4) and of the system proposed in Theorem 3.7.

Remark 3.1 (alternative representation). In the representation of the pgf, the term $(z-1)$ is usually included in the functions $f_j(z)$. One can also rewrite (1.1) as

$$X(z) = \frac{\sum_{j=0}^n x_j f_j(z)}{\tilde{D}(z)},$$

where $\tilde{D}(z) = D(z)/(z-1)$ is a function with n zeros, $\hat{z}_1, \dots, \hat{z}_n$, inside the unit disk. Since $z = 1$ is a zero of $D(z)$, if the function $D(z)$ is analytic in the open disk $\Delta_r = \{z \colon |z| < r\}$ for some $r > 1$, then so is $\tilde{D}(z)$.

Example 3.2 (general bulk-service queue). Examples of systems with a pgf of the form (1.1) are the bulk-service queue (see [3]), the multiserver queue (see [12]), and certain traffic-light queues (see [19] and [18]). Each of these examples can be represented as a more general discrete-time bulk-service queue with arrivals that depend on the queue length prior to service. This queuing system is defined as follows. Let $n+1$ be the server capacity, i.e., the maximum number of customers that can be served simultaneously during a time slot. Suppose there are m customers in the queue at the start of a time slot. Then, during this time slot, $\min(m, n+1)$ customers are served and there are Y_m arrivals with pgf $Y_m(z)$, where $Y_m = Y$ is independent of m for $m \geqslant n+1$ and, otherwise, may depend on m. Let $X(z)$ be the pgf of the queue length at the start of a time slot in steady state, let x_j be the probability of having j customers in the queue at the start of a time slot, and let $Y(z)$ be the pgf of Y. In steady state, the pgf of the queue length at the start of the next time slot coincides with $X(z)$. Thus,

$$X(z) = \frac{X(z) - \sum_{j=0}^{n} x_j z^j}{z^{n+1}} Y(z) + \sum_{j=0}^{n} x_j Y_j(z), \tag{3.1}$$

which can be rewritten in a fractional form as (1.1) by isolating $X(z)$:

$$X(z) = \frac{\sum_{j=0}^{n} x_j (z^{n+1} Y_j(z) - z^j Y(z))}{z^{n+1} - Y(z)}, \tag{3.2}$$

where $f_j(z) = (z^{n+1} Y_j(z) - z^j Y(z))/(z-1)$ and $D(z) = z^{n+1} - Y(z)$.

3.1. Pgf and unknowns as symmetric functions of roots. As we showed in the introduction, the unknowns in the numerator can be found using the equation

$$M(1, \hat{z}_1, \dots, \hat{z}_n)(x_0, \dots, x_n)^T = (D'(1), 0, \dots, 0)^T, \tag{1.4 revisited}$$

where

$$M(z, z_1, \dots, z_n) = \begin{pmatrix} f_0(z) & f_1(z) & \dots & f_n(z) \\ f_0(z_1) & f_1(z_1) & \dots & f_n(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(z_n) & f_1(z_n) & \dots & f_n(z_n) \end{pmatrix}. \tag{1.5 revisited}$$

Note that system (1.4) has the following properties. In $M(z, z_1, \dots, z_n)$ only the first row depends on z, and on the right-hand side of (1.4), only the first element of the vector is nonzero. In the following lemma, we provide the solution of a system of linear equations with these two properties. Application of this result to (1.4) gives a representation of the numerator in (1.1) in terms of the matrix $M(z, z_1, \dots, z_n)$. As a by-product of this representation, we obtain the following interesting result: it is not necessary to know x_j, $j = 0, \dots, n$, to evaluate the numerator of (1.1).

LEMMA 3.3. *Suppose the matrix*

$$K(z) = \begin{pmatrix} f_0(z) & f_1(z) & \dots & f_n(z) \\ \alpha_{10} & \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n0} & \alpha_{n1} & \dots & \alpha_{nn} \end{pmatrix}$$

is nonsingular at $z = 1$, and the vector $(x_0, \dots, x_n)$ is a solution of the equation

$$K(1)(x_0, \dots, x_n)^T = (a, 0, \dots, 0)^T; \tag{3.3}$$

then

$$\sum_{j=0}^{n} x_j f_j(z) = a\,\frac{\det K(z)}{\det K(1)} \tag{3.4}$$

and

$$x_j = (-1)^j a\,\frac{\det K_j}{\det K(1)}, \tag{3.5}$$

where K_j is the matrix $K(z)$ without the first row and the $(j+1)$st column.

Proof. Since the matrix $K(1)$ is nonsingular, (3.3) has a unique solution (3.5), which is found by Cramer's rule. Using the Laplace expansion for the first row of matrix $K(z)$, we readily obtain that (3.5) gives (3.4), which concludes the proof. ∎

COROLLARY 3.4. *Suppose the matrix $M(1, \hat{z}_1, \dots, \hat{z}_n)$ is nonsingular, and the vector $(x_0, \dots, x_n)$ is a solution of* (1.4)*; then*

$$\sum_{j=0}^{n} x_j f_j(z) = D'(1)\frac{\det M(z, \hat{z}_1, \dots, \hat{z}_n)}{\det M(1, \hat{z}_1, \dots, \hat{z}_n)}. \tag{3.6}$$

Remark 3.5 (on symmetry in (3.6)). The determinant $\det M(z, z_1, \dots, z_n)$ is a skew-symmetric function of the roots. Hence, the right-hand side of (3.6), which is equal to the numerator of (1.1), is a symmetric function. However, the form (3.6) does not show how to find the value of the numerator without finding the roots $\hat{z}_1, \dots, \hat{z}_n$. Therefore, we need an equivalent representation; see also Remark 2.5.

Consider the matrix

$$\bar{M}(z, z_1, \dots, z_n) = \begin{pmatrix} f_0(z) & f_1(z) & \dots & f_n(z) \\ F^n_{0,0} & F^n_{1,0} & \dots & F^n_{n,0} \\ \vdots & \vdots & \ddots & \vdots \\ F^n_{0,n-1} & F^n_{1,n-1} & \dots & F^n_{n,n-1} \end{pmatrix}, \tag{3.7}$$

where $F^n_{j,k}$ is the (k,n)-transformation of the function $f_j(z)$, defined in (2.5). From Lemma 2.11, we find that $\det M(z, z_1, \dots, z_n) = V(z_1, \dots, z_n)h(z, z_1, \dots, z_n)$, where $V(z_1, \dots, z_n)$ is the Vandermonde determinant and

$$h(z, z_1, \dots, z_n) = \det \bar{M}(z, z_1, \dots, z_n). \tag{3.8}$$

In particular,

$$\frac{\det M(z, z_1, \dots, z_n)}{\det M(1, z_1, \dots, z_n)} = \frac{h(z, z_1, \dots, z_n)}{h(1, z_1, \dots, z_n)}$$

is a symmetric function of the roots. Therefore, from Corollary 3.4, we obtain the following theorem.

THEOREM 3.6. *Suppose the matrix $M(1, \hat{z}_1, \dots, \hat{z}_n)$ is nonsingular, and the vector $(x_0, \dots, x_n)$ is a solution of* (1.4)*; then*

$$\sum_{j=0}^{n} x_j f_j(z) = D'(1)\frac{h(z, \hat{z}_1, \dots, \hat{z}_n)}{h(1, \hat{z}_1, \dots, \hat{z}_n)}$$

and

$$X(z) = \frac{D'(1)h(z, \hat{z}_1, \dots, \hat{z}_n)}{D(z)h(1, \hat{z}_1, \dots, \hat{z}_n)}(z-1). \tag{3.9}$$

Using Remark 2.14, we can find $h(z, \hat{z}_1, \ldots, \hat{z}_n)$ without knowing the roots $\hat{z}_1, \ldots, \hat{z}_n$ by computing n contour integrals. Thus, Theorem 3.6 provides a way of calculating the pgf $X(z)$ at an arbitrary point z without computing roots and the unknowns $x_0, \ldots, x_n$. Moreover, the analytic form of the representation (3.9) may be used directly to obtain further results. For example, the mean $X'(1)$ of the corresponding random variable can be found as follows:

$$\begin{aligned} X'(1) &= D'(1) \left(\frac{d}{dz} \frac{z-1}{D(z)} \right) \Big|_{z=1} + D'(1) \left(\frac{z-1}{D(z)} \right) \Big|_{z=1} \frac{h'(1, \hat{z}_1, \ldots, \hat{z}_n)}{h(1, \hat{z}_1, \ldots, \hat{z}_n)} \\ &= D'(1) \left(\frac{D(z) - (z-1)D'(z)}{D^2(z)} \right) \Big|_{z=1} + \frac{h'(1, \hat{z}_1, \ldots, \hat{z}_n)}{h(1, \hat{z}_1, \ldots, \hat{z}_n)} \\ &= -\frac{D''(1)}{2D'(1)} + \frac{h'(1, \hat{z}_1, \ldots, \hat{z}_n)}{h(1, \hat{z}_1, \ldots, \hat{z}_n)}, \end{aligned}$$

where we applied L'Hôpital's rule to find the values at $z = 1$, and, for simplicity, used $h'(1, \hat{z}_1, \ldots, \hat{z}_n)$ instead of $(\partial/\partial z\, h(z, \hat{z}_1, \ldots, \hat{z}_n))|_{z=1}$. In section 4, we consider a special case, in which $X'(1)$ may be found using a single contour integral.

To analyze the corresponding system, it can be important to compute the coefficients x_j for $j = 0, \ldots, n$. The following theorem shows how to find them using the matrix $\bar{M}(1, \hat{z}_1, \ldots, \hat{z}_n)$.

THEOREM 3.7. *Suppose the matrix* $M(1, \hat{z}_1, \ldots, \hat{z}_n)$ *is nonsingular, and the vector* $(x_0, \ldots, x_n)$ *is a solution of the equation*

$$\bar{M}(1, \hat{z}_1, \ldots, \hat{z}_n)(x_0, \ldots, x_n)^T = (D'(1), 0, \ldots, 0)^T; \tag{3.10}$$

then the vector $(x_0, \ldots, x_n)$ *is also the unique solution of* (1.4).

This theorem follows by applying Lemma 2.11 to the solution (3.5) of the system in (1.4) and Lemma 3.3 to (3.10).

Remark 3.8 (advantages of using matrix $\bar{M}(z, \hat{z}_1, \ldots, \hat{z}_n)$). Theorems 3.6 and 3.7 provide a reliable way to use the roots $\hat{z}_j$. In the classical approach, one needs to solve (1.4), which involves matrix $M(1, \hat{z}_1, \ldots, \hat{z}_n)$. This matrix becomes singular if two roots coincide. Thus, it is close to singular (in terms of a matrix norm) when the distance between a pair of roots is small, which renders system (1.4) ill-conditioned and may lead to serious numerical errors. This problem can occur when there are many roots; see subsection 3.2 below. In contrast, the matrix $\bar{M}(z, \hat{z}_1, \ldots, \hat{z}_n)$ is, in general, nonsingular even if all roots coincide; see Remark 2.12. Therefore, using (3.10) yields more reliable results than system (1.4); see subsection 3.2 for more details.

3.2. Numerical examples. In this subsection, we compare different numerical approaches to finding the unknowns x_j; for the source code, see [1]. The classical approach consists of two steps: finding the zeros of the denominator $D(z)$ and solving a system of linear equations; see (1.4). In general, there is no explicit formula for the zeros. Thus, the existing zero-finding algorithms are often iterative, meaning the precision of the found zeros depends on the number of iterations and the rate of convergence. Numerical errors that arise in this first step affect the solution of (1.4). When the matrix $M(1, \hat{z}_1, \ldots, z_n)$ is close to singular (see Remark 3.8), even a small error in the entries of the matrix may result in a huge error in the resulting coefficients $x_0, \ldots, x_n$. The other two approaches are based on (3.10) and differ in the way the matrix $\bar{M}(1, \hat{z}_1, \ldots, z_n)$ is computed: using the roots or using the contour integrals. We refer to these methods as the *modified root-finding approach* and the *contour-integral approach*, respectively.

We illustrate the numerical problems of the standard approach using the classical bulk-service queue; see [3], as an example. This queuing system is a special case of Example 3.2 for which $Y_m(z) = Y(z)$ for all m. As before, the size of the service batches is fixed and equals $n+1$. We consider binomial arrivals with pgf $Y(z) = (\lambda z+1-\lambda)^{2(n+1)}$, where $\lambda < 1/2$ to guarantee stability of the system. For such arrivals, $f_j(z) = (Y_j(z)z^{n+1} - z^j Y(z))/(z-1) = (z^j + \cdots + z^n)Y(z)$ and the denominator is equal to

$$D(z) = z^{n+1} - (\lambda z + 1 - \lambda)^{2(n+1)}. \tag{3.11}$$

Therefore, the pgf $X(z)$ of the queue length is given by

$$X(z) = \frac{\sum_{j=0}^{n} x_j (z^j + \cdots + z^n)}{z^{n+1} - (\lambda z + 1 - \lambda)^{2(n+1)}} (\lambda z + 1 - \lambda)^{2(n+1)} (z-1). \tag{3.12}$$

In subsections 3.2.1 and 3.2.2, we consider both steps of the classical approach of computing the unknowns $x_0, \ldots, x_n$.

3.2.1. Finding the zeros. In this subsection, we explain two ways of finding the zeros of the denominator (3.11). The first one is based on the expression for the zeros obtained in [12] under special conditions, such as $Y(z)$ to be zero-free in a certain region. For binomial arrivals, these conditions hold, and, in our case, we obtain

$$\hat{z}_k = \sum_{l=1}^{\infty} \frac{1}{l} \lambda^{l+1} (1-\lambda)^{l-1} \binom{2l}{l-1} w_k^l, \tag{3.13}$$

where $w_k = e^{2\pi \mathbf{i} k/(n+1)}$, $k = 0, \ldots, n$. For numerical purposes, one needs to truncate the series on the right-hand side of (3.13), resulting in an approximation:

$$\hat{z}_k \approx \hat{z}_{k,N} = \sum_{l=1}^{N} \frac{1}{l} \lambda^{l+1} (1-\lambda)^{l-1} \binom{2l}{l-1} w_k^l. \tag{3.14}$$

We use $N = N(\delta) = \min\{M \colon |1 - \hat{z}_{0,M}| < \delta\}$. Observe that this choice guarantees the precision of the roots, i.e., $|\hat{z}_k - \hat{z}_{k,N(\delta)}| \leqslant |\hat{z}_0 - \hat{z}_{0,N(\delta)}| < \delta$ for $k = 0, \ldots, n$, since the coefficients $\lambda^{l+1}(1-\lambda)^{l-1}\binom{2l}{l-1}/l$ are nonnegative. However, it may be computationally expensive to compute $N(\delta)$ when the rate of convergence in (3.13) is slow.

The second way provides an explicit formula for the zeros by exploiting the special form of (3.11). To find the zeros of the denominator, we separate the equation

$$z^{n+1} = (\lambda z + 1 - \lambda)^{2(n+1)}$$

into $n+1$ quadratic equations

$$z = w_k (\lambda z + 1 - \lambda)^2$$

for $k = 0, \ldots, n$. Solving for z yields

$$z = \frac{1 - 2w_k \lambda (1-\lambda) \pm \sqrt{1 - 4 w_k \lambda (1-\lambda)}}{2 w_k \lambda^2}. \tag{3.15}$$

Using (3.15) for $k = 0, \ldots, n$, we obtain $2(n+1)$ zeros of the denominator, from which we pick those that are inside the unit disk.

3.2.2. Solving the linear system. In this subsection, we compare the numerical solutions of (1.4) and (3.10). As a benchmark, we use the explicit solution of these systems, derived below.

Note that the functions $f_j(z)$ in the numerator of $X(z)$ have a common factor $Y(z)$. To simplify the computations, we will consider functions $\tilde{f}_j(z) = f_j(z)/Y(z) = z^j + \cdots + z^n$ instead. Such a change corresponds to a pgf of the queue length just after the start of the service, i.e., before the arrivals during the current time slot. Note that this change does not affect the solutions of (1.4) and (3.10).

For the classical bulk-service queue, one can find the solution of (1.4) explicitly in terms of the symmetrical functions of the zeros, which is the case for all pgfs that have the factorization property; see section 4 for details. First, note that the numerator of (3.12) is a polynomial of degree n and vanishes at $z = \hat{z}_k$ for $k = 1, \dots, n$. Observe also that $\hat{z}_k \neq \hat{z}_j$ for $k \neq j$. Therefore, from (2.3), we obtain

$$\sum_{l=0}^{n} x_l (z^l + \cdots + z^n) = \left(\sum_{l=0}^{n} x_l \right) \sum_{j=0}^{n} (-1)^{n-j} \sigma_{n-j}(\hat{z}_1, \dots, \hat{z}_n) z^j.$$

Comparing the coefficients at z^j on both sides of the equation, we find

$$\sum_{l=0}^{j} x_l = (-1)^{n-j} \sigma_{n-j}(\hat{z}_1, \dots, \hat{z}_n) \left(\sum_{l=0}^{n} x_l \right),$$

or, equivalently,

$$x_j = (-1)^{n-j} (\sigma_{n-j}(\hat{z}_1, \dots, \hat{z}_n) + \sigma_{n-j+1}(\hat{z}_1, \dots, \hat{z}_n)) \left(\sum_{l=0}^{n} x_l \right) \tag{3.16}$$

for $j = 0, \dots, n$, where, by definition, $\sigma_{n+1}(\hat{z}_1, \dots, \hat{z}_n) = 0$; see subsection 2.3. Second, since $\sum_{j=0}^{n} x_j f_j(1) = D'(1)$ (see (1.3)), we obtain

$$\sum_{j=0}^{n} x_j (n - j + 1) = \left(\sum_{l=0}^{n} x_l \right) \sum_{j=0}^{n} (-1)^{n-j} \sigma_{n-j}(\hat{z}_1, \dots, \hat{z}_n) = (n+1)(1 - 2\lambda),$$

which, together with (3.16), gives

$$x_j = (n+1)(1 - 2\lambda) \frac{(-1)^{n-j} (\sigma_{n-j}(\hat{z}_1, \dots, \hat{z}_n) + \sigma_{n-j+1}(\hat{z}_1, \dots, \hat{z}_n))}{\sum_{l=0}^{n} (-1)^{n-l} \sigma_{n-l}(\hat{z}_1, \dots, \hat{z}_n)}. \tag{3.17}$$

We use (3.17) with the zeros computed by (3.15) as a benchmark for the numerical solutions of (1.4) and (3.10). We consider the maximum of the absolute error, i.e., $\max_{j=0,\dots,n} |x_j - \tilde{x}_j|$, where $\tilde{x}_j$ is computed using either the classical approach, the modified root-finding approach, or the contour-integral approach. For $\lambda = 0.3$, 0.4, 0.45, we plot the results depending on the batch size $n+1$; see Figure 1. For the classical and modified root-finding approaches we compute the zeros by (3.14) with the truncation bound $N = N(\delta)$, $\delta = 10^{-10}$. The contour integral is computed using *Python* method *scipy.integrate.quad*. This method accepts an optional argument of the desired absolute accuracy, for which we also use $\delta = 10^{-10}$.

As can be seen from Figure 1, the classical approach of finding the probabilities $x_0, \dots, x_n$ leads to serious errors for large n. For example, for $\lambda = 0.45$ and $n \geqslant 27$,

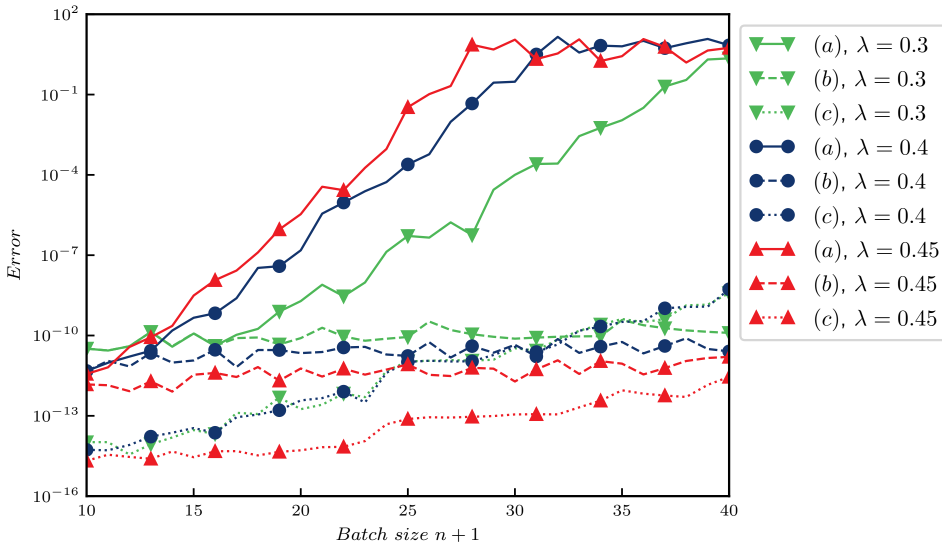

FIG. 1. *The absolute error of the numerical solutions for* (a) *the classical approach,* (b) *the modified root-finding approach, and* (c) *the contour-integral approach.*

even though the roots are computed with an error of less than δ, the error in the probabilities is about $10^{11}\delta$. Using the matrix $\bar{M}(1, \hat{z}_1, \ldots, \hat{z}_n)$ leads to errors of order δ or less. For small values of n, using contour integrals gives better results than using roots in (3.10), but for large n the results are similar. Also both approaches give better results for higher values of λ. We conclude that using (3.10) provides more reliable results than the classical system (1.4).

4. Factorization of the pgf. In this section, we give necessary and sufficient conditions for the pgf $X(z)$ of the form (1.1) to have some special properties. The first property is the *factorization property*, i.e., that the numerator of the pgf (1.1) can be represented as a product, where each of the terms depends on not more than one root:

$$X(z) = D'(1)\frac{g(z)\prod_{k=1}^{n} g(z, \hat{z}_k)}{D(z)}(z-1) \tag{4.1}$$

for some functions $g(z)$ and $g(z, w)$. This representation of the sum as a product is analogous to Vieta's formulas; see (2.3). For an example of the numerator that cannot be represented as such a product, consider three functions $f_0(z) = 1$, $f_1(z) = z$, and $f_2(z) = z^3$. Then $\sum_{j=0}^{2} x_j f_j(z)$ is a polynomial of degree 3 with roots z_1 and z_2. Thus, for some w, it is equal, up to a constant, to $(z - z_1)(z - z_2)(z - w)$. As the coefficient at z^2 is zero, we have $w = -z_1 - z_2$, which makes the product form (4.1) impossible.

The second property is the *additive-mean property*; i.e., $X'(1)$, which represents the mean of the corresponding random variable, is an additive function of the roots. One can easily see that the factorization property implies the additive-mean property. Indeed, from (4.1), L'Hôpital's rule gives us

$$1 = X(1) = g(1) \prod_{k=1}^{n} g(1, \hat{z}_k), \tag{4.2}$$

and the mean of the corresponding random variable is given by a symmetric additive function of the roots:

$$X'(1) = -\frac{D''(1)}{2D'(1)} + \frac{g'(1)}{g(1)} + \sum_{k=1}^{n} \frac{\partial}{\partial z} \frac{g(z,\hat{z}_k)}{g(1,\hat{z}_k)}\bigg|_{z=1}. \tag{4.3}$$

To find the above equation, take the derivative of (4.1) and use equality (4.2). An additive form, as in (4.3), implies that under some conditions, the mean value can be found using one contour integral; see Remark 2.16. Note that for certain functions $g(z,w)$, it is possible to represent the pgf as an exponent of a contour integral; see [7].

The following theorem gives the relation between factorization and additive-mean properties in terms of the function $h(z, z_1, \ldots, z_n)$, and Corollary 4.3 summarizes the necessary and sufficient conditions for these properties.

THEOREM 4.1. *Suppose the functions $f_j(z)$ are analytic in $\Delta_1 = \{z\colon |z| < 1\}$, the matrix $M(z, z_1, \ldots, z_n)$ is defined by* (1.5), *and the function $h(z, z_1, \ldots, z_n)$ is given by* (3.8). *Consider the following conditions:*

(a) *There exist a nonsingular matrix*

$$A = \begin{pmatrix} a_{0,0} & \cdots & a_{0,n} \\ \vdots & \ddots & \vdots \\ a_{n,0} & \cdots & a_{n,n} \end{pmatrix},$$

an analytic function $B(z)$ in Δ_1, and a nonconstant meromorphic function $C(z)$ in Δ_1 such that $f_j(z) = \sum_{k=0}^{n} a_{j,k}\tilde{f}_k(z)$, where

$$\tilde{f}_k(z) = B(z)C(z)^k. \tag{4.4}$$

(b) *There exist meromorphic functions $g(z)$ and $g(z,w)$ such that*

$$\frac{h(z, z_1, \ldots, z_n)}{h(1, z_1, \ldots, z_n)} = g(z) \prod_{k=1}^{n} g(z, z_k). \tag{4.5}$$

(c) *There exist a constant c and a meromorphic function $f(z)$ such that*

$$\frac{\partial}{\partial z} \frac{h(z, z_1, \ldots, z_n)}{h(1, z_1, \ldots, z_n)}\bigg|_{z=1} = c + \sum_{k=1}^{n} f(z_k). \tag{4.6}$$

If

$$(*) \qquad \text{the matrix } M(1, \hat{z}_1, \ldots, \hat{z}_n) \text{ is nonsingular for some } \hat{z}_1, \ldots, \hat{z}_n,$$

then conditions (a) *and* (b) *are equivalent and* (c) *follows from them. If, moreover,*

$$(**) \qquad \text{there exist } 1 \leqslant j < k \leqslant n \text{ such that } f_j'(1)f_k(1) \neq f_k'(1)f_j(1),$$

then all three conditions are equivalent. Also, if $(**)$ *does not hold, then* (c) *is satisfied for a constant function $f(z)$. Furthermore, if conditions* (a) *and* $(*)$ *hold, then conditions* (b) *and* (c) *hold for*

$$g(z) = \frac{B(z)}{B(1)}, \qquad g(z,w) = \frac{C(z) - C(w)}{C(1) - C(w)}, \tag{4.7}$$

$$c = \frac{B'(1)}{B(1)}, \qquad f(w) = \frac{C'(1)}{C(1) - C(w)}. \tag{4.8}$$

The proof of the theorem is given in the following subsections, where we prove the implications (a) $\Rightarrow$ (b) (under $(*)$), (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (a) (under conditions $(*)$ and $(**)$), and (b) $\Rightarrow$ (a) (under $(*)$). Afterwards, in subsection 4.5, we provide a way to check condition (a) for an arbitrary set of functions $f_0(z), \dots, f_n(z)$.

Remark 4.2 (condition $(*)$). The nonsingularity condition $(*)$ means that equation (1.4) is well-defined at least for one choice of $\hat{z}_1, \dots, \hat{z}_n$, which is a natural requirement for the analysis of (1.1). Since $f_j(1) \neq 0$ for some $j = 0, \dots, n$, condition $(*)$ is equivalent to linear independence between the functions $f_0(z), \dots, f_n(z)$; see Property 2.7 for $\Lambda_a = M(1, \hat{z}_1, \dots, \hat{z}_n)$.

COROLLARY 4.3. *Suppose the functions $f_j(z)$ are analytic in Δ_1 and linearly independent. Then*

- *the pgf* (1.1) *has the factorization property if and only if the functions $f_j(z)$, up to a linear transformation, form a geometric sequence* (4.4)*;*
- *the pgf* (1.1) *has the additive-mean property if and only if the functions $f_j(z)$ either satisfy $f_j'(1)f_k(1) = f_k'(1)f_j(1)$ for all $j, k = 0, \dots, n$, or, up to a linear transformation, form a geometric sequence* (4.4).

Remark 4.4 (comparison with [18]). Theorem 4.1 was partially proven for a specific function $C(z)$ in [18]. There the focus was on proving (4.6) for certain systems such as the bulk-service queue and the fixed-cycle traffic-light queue. This result allows us to use contour integrals for finding the average queue length. Theorem 4.1 generalizes the result of [18] and describes all systems for which (4.6) applies. However, it does not imply that these are the only queuing systems for which the mean value can be found using one contour integral. In this paper, we have considered a special class of the pgf; see (1.1). If one is able to find the pgf in a factorized form, e.g., as in (4.1), then the mean will be an additive function of the roots. For example, this result can be applied to the $G_D/G_D/1$ queue considered in [21].

Remark 4.5 (degenerate case). Note that from definition (3.8) of the function $h(z, z_1, \dots, z_n)$, it follows that if condition $(**)$ does not hold, then (4.6) holds for $f(z) = 0$ and $c = f_0'(1)/f_0(1)$. In Example 4.6, we give an example of a queuing system without condition $(**)$.

Example 4.6 (degenerate case). Consider a special bulk-service queue with vacations depending on the queue size. The arrivals are Poisson with rate 1. The size of the batch is 3 and the service time is deterministic and equal to some d such that $2 < d < 3$. If the server visits the queue and finds at least three customers, it immediately starts serving the first three customers. If upon a visit the server finds the queue with j customers, $j < 3$, it serves them instantly and takes a vacation of deterministic time v_j with

$$v_0 = -1 + \sqrt{d^2 - 4d + 7}, \qquad v_1 = -1 + \sqrt{d^2 - 3d + 3}, \qquad v_2 = d - 2.$$

Note that for $d > 2$, this time v_j is positive for $j = 0, 1, 2$. It is possible to find the pgf $X(z)$ of the queue length at the times when the server visits the queue, i.e., after a service or a vacation,

$$X(z) = \frac{\sum_{j=0}^{2} x_j \hat{f}_j(z)}{z^3 - e^{d(z-1)}},$$

where x_j is the probability of finding j customers in the queue upon a visit, and $\hat{f}_j(z) = (z-1)f_j(z) = e^{v_j(z-1)}z^3 - z^j e^{d(z-1)}$. One can check that $f_j'(1) = \hat{f}_j''(1)/2 = 2f_j(1) = \hat{f}_j'(1)$ for all $j = 0, 1, 2$, which means that $(**)$ does not hold, and, therefore, that (c) holds for $f(z) = 0$. Thus, the mean queue length upon the server arrival is independent of the roots of the characteristic equation $z^3 = e^{d(z-1)}$:

$$X'(1) = \left(\frac{z-1}{z^3 - e^{d(z-1)}} \right)' \Bigg|_{z=1} (3-d) + \frac{\sum_{j=0}^{2} x_j f_j'(1)}{\sum_{j=0}^{2} x_j f_j(1)} = \frac{d^2 - 4d + 6}{6 - 2d}.$$

4.1. Proof of Theorem 4.1: (a) $\Rightarrow$ (b). Consider the matrix $M(z, z_1, \ldots, z_n)$ at some point $(z, z_1, \ldots, z_n)$ such that the matrix is nonsingular. Given (a), the matrix $M(z, z_1, \ldots, z_n)$ is a linear transformation of an alternant matrix with the functions $\tilde{f}_k(z) = B(z)C(z)^k$:

$$M(z, z_1, \ldots, z_n)(A^T)^{-1} = \begin{pmatrix} B(z) & \ldots & B(z)C(z)^n \\ \vdots & \ddots & \vdots \\ B(z_n) & \ldots & B(z_n)C(z_n)^n \end{pmatrix},$$

which is similar to a Vandermonde matrix. Hence, its determinant is equal to

$$\det(M(z, z_1, \ldots, z_n)(A^T)^{-1}) = B(z) \prod_{k=1}^{n} B(z_k) V_C(z, \ldots, z_n),$$

where $V_C(z, \ldots, z_n) = V(C(z), C(z_1), \ldots, C(z_n))$ is the Vandermonde determinant for variables $C(z), C(z_1), \ldots, C(z_n)$. Therefore, given the fact that the matrices A and $M(1, z_1, \ldots, z_n)$ are nonsingular, we obtain

$$\frac{h(z, z_1, \ldots, z_n)}{h(1, z_1, \ldots, z_n)} = \frac{\det(M(z, z_1, \ldots, z_n)(A^T)^{-1})}{\det(M(1, z_1, \ldots, z_n)(A^T)^{-1})} = \frac{B(z)}{B(1)} \prod_{k=1}^{n} \frac{C(z) - C(z_k)}{C(1) - C(z_k)}. \tag{4.9}$$

This is exactly (4.5) with the functions $g(z)$ and $g(z, w)$ defined as in (4.7). Note that due to the continuity of the functions on the left-hand side and the right-hand side of (4.9) in their support, the equality holds also at points where the matrix $M(1, z_1, \ldots, z_n)$ is singular, but $h(1, z_1, \ldots, z_n) \neq 0$; see also Remarks 2.5 and 2.12.

4.2. Proof of Theorem 4.1: (b) $\Rightarrow$ (c). This implication does not require $(*)$ or $(**)$. Note that from (4.5) it follows that $g(1) \prod_{k=1}^{n} g(1, z_k) = 1$. Hence,

$$\frac{\partial}{\partial z} \frac{h(z, z_1, \ldots, z_n)}{h(1, z_1, \ldots, z_n)} \Bigg|_{z=1} = \frac{\partial}{\partial z} \left(\frac{g(z)}{g(1)} \prod_{k=1}^{n} \frac{g(z, z_k)}{g(1, z_k)} \right) \Bigg|_{z=1} = \frac{g'(1)}{g(1)} + \sum_{k=1}^{n} \frac{\partial}{\partial z} \frac{g(z, z_k)}{g(1, z_k)} \Bigg|_{z=1}.$$

Thus, one can choose $c = g'(1)/g(1)$ and $f(w) = \partial/\partial z \, g(z, w)/g(1, w)|_{z=1}$. If the functions $g(z)$ and $g(z, w)$ are defined as in (4.7), then c and $f(w)$ are defined as in (4.8).

4.3. Proof of Theorem 4.1: (c) $\Rightarrow$ (a). This is the most involved part of the proof. Here, we use several results, given as lemmas, that are proved separately in the appendix. First, we consider a linear transformation of the functions $f_j(z)$, given by the following lemma; see proof in Appendix A.5.

LEMMA 4.7. *If the functions $f_j(z)$ are analytic in Δ_1 and linearly independent, then there exist a point $z^* \in \Delta_1$ and functions $\tilde{f}_k(z)$ such that $f_j(z) = \sum_{k=0}^{n} \tilde{a}_{j,k} \tilde{f}_k(z)$ and*

$$\tilde{f}_k(z) = (z - z^*)^k + o((z - z^*)^k) \text{ as } z \to z^*. \tag{4.10}$$

Moreover, z^* can be any point in Δ_1 except a finite set of points, and the function $\tilde{f}_{n-1}(z)$ can be chosen such that $\tilde{f}_{n-1}(1) \neq 0$.

In our case, we can apply Lemma 4.7, because the matrix $M(1, \hat{z}_1, \dots, \hat{z}_n)$ is nonsingular, and, therefore, the functions $f_0(z), \dots, f_n(z)$ are linearly independent. Since z^* can be any point in the unit disk except a finite number of points, we can assume, without loss of generality, that $z^* = 0$, $a_{j,k} = \delta_{jk}$, and $\tilde{f}_{n-1}(1) \neq 0$, where δ_{jk} is the Kronecker delta.

Now, suppose that (4.6) holds. We will focus on determining $f_{k+1}(z)/f_k(z)$. For this, we will consider the cases where $z_{k+1} = \cdots = z_n = 0$ for $k = 1, \dots, n$. The following lemma gives the value of $h(z, z_1, \dots, z_n)/h(1, z_1, \dots, z_n)$ for each k. The proof is given in Appendix A.6.

Lemma 4.8. *For $k > 0$,*

$$\frac{h(z, z_1, \dots, z_k, 0, \dots, 0)}{h(1, z_1, \dots, z_k, 0, \dots, 0)} = \frac{\det \Lambda_k(z, z_1, \dots, z_k)}{\det \Lambda_k(1, z_1, \dots, z_k)}, \tag{4.11}$$

where

$$\Lambda_k(z, z_1, \dots, z_k) = \begin{pmatrix} f_{n-k}(z) & \dots & f_n(z) \\ f_{n-k}(z_1) & \dots & f_n(z_1) \\ \vdots & \ddots & \vdots \\ f_{n-k}(z_k) & \dots & f_n(z_k) \end{pmatrix}.$$

To find the function $f(z)$ in (4.6), we consider $k = 1$. Using Lemma 4.8, we find

$$\frac{h(z, z_1, 0, \dots, 0)}{h(1, z_1, 0, \dots, 0)} = \frac{f_n(z_1)f_{n-1}(z) - f_n(z)f_{n-1}(z_1)}{f_n(z_1)f_{n-1}(1) - f_n(1)f_{n-1}(z_1)}. \tag{4.12}$$

Let $C(z) = f_n(z)/f_{n-1}(z)$. Note that the function $C(z)$ is not a constant since the functions $f_n(z)$ and $f_{n-1}(z)$ are linearly independent. Using the function $C(z)$, we can rewrite (4.12) as

$$\frac{h(z, z_1, 0, \dots, 0)}{h(1, z_1, 0, \dots, 0)} = \frac{f_{n-1}(z)}{f_{n-1}(1)} \frac{C(z) - C(z_1)}{C(1) - C(z_1)}. \tag{4.13}$$

Note that $C(1)$ and the right-hand side of (4.13) are well-defined since $f_{n-1}(1) = \tilde{f}_{n-1}(1) \neq 0$; see the choice of z^*. Taking the derivative gives us

$$\frac{\partial}{\partial z} \frac{h(z, z_1, 0, \dots, 0)}{h(1, z_1, 0, \dots, 0)}\bigg|_{z=1} = \frac{f'_{n-1}(1)}{f_{n-1}(1)} + \frac{C'(1)}{C(1) - C(z_1)}.$$

Equation (4.6) defines the constant c and the function $f(w)$ up to a constant, i.e., constant c can be arbitrarily chosen. Therefore, we redefine $f(w)$ as $C'(1)/\,(C(1)-C(w))$. This leads to $c = f'_{n-1}(1)/f_{n-1}(1) - (n-1)C'(1)/C(1)$ since $f(0) = C'(1)/C(1)$. Here, we used the fact that $C(0) = \lim_{z\to 0} f_n(z)/f_{n-1}(z) = \lim_{z\to 0} z^n/z^{n-1} = 0$.

Now, it is left to prove that condition (a) follows from the equation

$$\frac{\partial}{\partial z} \frac{h(z, z_1, \dots, z_n)}{h(1, z_1, \dots, z_n)}\bigg|_{z=1} = \frac{f'_{n-1}(1)}{f_{n-1}(1)} - (n-1)\frac{C'(1)}{C(1)} + \sum_{k=1}^{n} \frac{C'(1)}{C(1) - C(z_k)} \tag{4.14}$$

with $B(z) = f_n(z)/C(z)^n$. Recursive application of the following lemma together with a linear transformation of the functions $f_j(z)$ concludes the proof of implication (c) $\Rightarrow$ (a). The proof of Lemma 4.9 is given in Appendix A.7.

LEMMA 4.9. *Consider $k \geqslant 2$. Suppose that* $(**)$ *is satisfied and* (4.14) *holds for* $z_{k+1} = \cdots = z_n = 0$. *Suppose also that* $f_{j+1}(z)/f_j(z) = C(z)$ *for* $j = n-k+1, \ldots, n-1$. *Then there exist coefficients* β_j, $j = 0, \ldots, k$, *such that*

$$f_{n-k}(z) = \beta_0 f_{n-k+1}(z)/C(z) + \sum_{j=1}^{k} \beta_j f_{n-k+j}(z). \tag{4.15}$$

Remark 4.10 (importance of linear transformation). It would be sufficient to prove that $f_{j+1}(z) = C(z) f_j(z)$ for $j = 0, \ldots, n-2$. However, it may be not true. For example, the functions $1+z$, z, and z^2 satisfy conditions (a)–(c) and (4.10), but $z/(1+z) \neq z = z^2/z$.

4.4. Proof of Theorem 4.1: (b) $\Rightarrow$ (a). We prove (b) $\Rightarrow$ (a) under $(*)$ similarly to (c) $\Rightarrow$ (a) under $(*)$ and $(**)$. We proceed from (4.13) and define $g(z, w)$ as in (4.7). Note that equality (4.13) does not require condition $(**)$, which we used only in Lemma 4.9. Using (4.13) and the definition of $g(z, w)$, we find that

$$g(z) = \frac{f_{n-1}(z)(C(1))^{n-1}}{f_{n-1}(1)(C(z))^{n-1}}.$$

Thus, from (4.5), we obtain

$$\frac{h(z, z_1, \ldots, z_n)}{h(1, z_1, \ldots, z_n)} = \frac{f_{n-1}(z)(C(1))^{n-1}}{f_{n-1}(1)(C(z))^{n-1}} \prod_{k=1}^{n} \frac{C(z) - C(z_k)}{C(1) - C(z_k)}. \tag{4.16}$$

Similarly to the case (c) $\Rightarrow$ (a), we conclude the proof by recursively applying the following lemma together with a linear transformation of the functions $f_k(z)$. The proof of Lemma 4.11 is similar to that of Lemma 4.9 and is given in Appendix A.8.

LEMMA 4.11. *Consider $k \geqslant 2$. Suppose that* (4.16) *holds for* $z_{k+1} = \cdots = z_n = 0$, *and that* $f_{j+1}(z)/f_j(z) = C(z)$ *for* $j = n-k+1, \ldots, n-1$. *Then there exist coefficients* β_j, $j = 0, \ldots, k$, *such that* (4.15) *holds.*

4.5. Factorization property and linear transformations. In this subsection, we show how to check whether the pgf (1.1) has the factorization property and, consequently, also the additive-mean property. Assuming condition (a) of Theorem 4.1, we give an algorithm for finding the required matrix A and the functions $B(z)$ and $C(z)$. Afterwards, one only needs to check that the equality (4.4) holds.

Consider linearly independent functions $f_0(z), \ldots, f_n(z)$. Without loss of generality, we can assume that their Wronskian $\det W(z)$, defined in (2.1), is not equal to 0 at $z = 0$. Then, as shown in the proof of Lemma 4.7 in Appendix A.5, matrix $W(0)$ gives such a linear transformation of the functions $f_j(z)$, $j = 0, \ldots, n$, that the resulting functions $\tilde{f}_j(z)$ satisfy (4.10) for $z^* = 0$; see (A.2). From the proof of the implication (b) $\Rightarrow$ (a) it follows that if the functions $\tilde{f}_j(z)$ satisfy condition (a) and property (4.10) for $z^* = 0$, then one can choose $C(z) = \tilde{f}_n(z)/\tilde{f}_{n-1}(z)$, and there exist such coefficients $\beta_{j,k}$, $j = 0, \ldots, n-2$, $k = j, \ldots, n$, that

$$\tilde{f}_j(z) = \beta_{j,j} \frac{z}{C(z)} \frac{\tilde{f}_{j+1}(z)}{z} + \beta_{j,j+1} \tilde{f}_{j+1}(z) + \cdots + \beta_{j,n} \tilde{f}_n(z); \tag{4.17}$$

see Lemma 4.11. Here the functions $z/C(z)$ and $\tilde{f}_{j+1}(z)/z$ are both analytic in a neighborhood of 0. To find coefficients $\beta_{j,k}$, consider the Taylor expansion of both sides of (4.17) at $z = 0$. Since the functions $\tilde{f}_j(z)$ satisfy property (4.10) for $z^* = 0$, the first j terms in the expansions are equal to 0. The next $n+1-j$ terms give a linear system of equations for $\beta_{j,k}$, $k = j, \ldots, n$, with a unitriangular matrix. Plugging the solution of this system back in (4.17), one gets a functional equality for each j if and only if condition (a) of Theorem 4.1 holds.

5. Applications. The results of this paper may be applied to many queuing systems. Example 3.2 describes a generic system where the customers are served in batches when the queue is long. Note that the batch size may be random, and only its maximum size is fixed. In case the queue is short, the arrival and the service processes can be arbitrary and can even depend on the queue length. Such a batching service occurs in numerous applications. For example, in traffic models, an intersection can serve a number of vehicles during a green period, but when there is no queue, the vehicles may proceed without stopping with a higher speed than those that need to wait. As a result, more vehicles may pass the intersection; see [19], [18]. For the multiserver queues $M/D/s$ and $Geo/D/s$, the queue length at times md, where d is the deterministic length of service, $m \in \mathbb{N}$, behaves as the classical bulk-service queue length just after service with Poisson or binomial arrivals, respectively; see [12], [14]. In many manufacturing systems, production at each stage takes place in batches, due to often large setup or changeover times to prepare for a particular product type. Also the transportation between stages of production or facilities is done in batches to use the transportation means economically. In these systems, the service only starts if at least a minimum load or number of products is available; see [11]. In subsection 3.2, we considered a special case of a classical bulk-service queue. This system has the factorization property, which allowed us to give the explicit solution (3.17). However, not every general bulk-service queue has this property; see subsection 5.1.

The roots of a characteristic equation may appear in other cases as well. For example, it often happens if the distributions of the involved random variables have a specific form. For a queuing example, we refer the reader to [21], where a $G_D/G_D/1$ queue is analyzed for the case when either interarrival or service times have a rational pgf. In subsections 5.2 and 5.3, we give two nonqueuing examples.

5.1. A general bulk-service queue. Consider a general bulk-service queue introduced in Example 3.2. Recall that $f_j(z) = (z^{n+1}Y_j(z) - z^j Y(z))/(z-1)$, where $Y_j(z)$ (resp., $Y(z)$) is the pgf of the number of arrivals if there are $j < n+1$ (resp., at least $n+1$) customers in the queue prior to service, and the service capacity is $n+1$.

First, we consider the case when $Y(0) \neq 0$. Then the functions $f_j(z)/Y(0)$ satisfy property (4.10) for $z^* = 0$. Therefore, the Wronskian (2.1) of the functions $f_j(z)$ is not equal to zero at $z = 0$, which means that the functions $f_j(z)$ are linearly independent.

For this bulk-service queue, (4.17) can be rewritten as

$$
\begin{aligned}
(5.1)\quad Y_j(z)z^{n+1} = z^j Y(z) &+ \beta_{j,j}\frac{Y_{n-1}(z)z^2 - Y(z)}{Y_n(z)z - Y(z)}(Y_{j+1}(z)z^n - z^j Y(z)) \\
&+ \beta_{j,j+1}(Y_{j+1}(z)z^{n+1} - z^{j+1}Y(z)) + \ldots + \beta_{j,n}(Y_n(z)z^{n+1} - z^n Y(z)).
\end{aligned}
$$

Note that the system for coefficients $\beta_{j,k}$ is independent of $Y_j(z)$. Thus, $Y_j(z)$ is completely defined by the functions $Y_{j+1}(z), \ldots, Y_n(z)$ and $Y(z)$. If (5.1) holds for $j = 0, \ldots, n-2$, we get that functions $Y_0(z), \ldots, Y_{n-2}(z)$ are fully determined by $Y_{n-1}(z)$, $Y_n(z)$, and $Y(z)$. Thus, a general bulk-service queue with the factorization property can be defined by the value of n and three functions. Note, however, that not all combinations of pgfs $Y_{n-1}(z)$, $Y_n(z)$, and $Y(z)$ result in feasible functions $Y_j(z)$, meaning there is no such bulk-service queue with the factorization property and such pgfs $Y_j(z)$.

Now, suppose $Y(0) = 0$. In this case, $z = 0$ is a zero of the denominator and of each function $f_j(z)$. Thus, the matrix (1.5) has at least one row of zeros, and the solution of (1.4) is not well-defined. To resolve this problem, consider $X(0) = x_0$, which is the probability of having an empty queue at the start of a time slot. Plugging $z = 0$ in (3.1), we obtain a balance equation $x_0(1 - Y_0(0)) = \sum_{j=1}^{n} x_j Y_j(0)$, which is nontrivial for x_0 if $Y_0(0) \neq 1$. Note that if $Y_0(0) = 1$, the queue always stays empty if it becomes empty once. Plugging the value of x_0 into the numerator of $X(z)$ and dividing both the numerator and the denominator by z, we obtain another representation of $X(z)$ as (1.1) with only n functions. This procedure can be repeated in case $Y'(0) = \cdots = Y^{(k)}(0) = 0$ for some $k \geqslant 1$. Afterwards, one can use (4.17) to check whether the pgf has the factorization property.

5.2. Item-degradation model. In condition-based maintenance, the maintenance decisions are based on the system degradation, e.g., wear, fatigue, and corrosion; see [2]. The degradation is monitored systematically using an appropriate sensors technology that reveals the item condition. Consider an item that is subject to discrete-value random degradation, which is independent of the current condition of the item. The item is maintained according to the control limit policy (see [5]), as follows. There is a preventive degradation threshold, y_p (in units of degradation), upon which and beyond the item receives preventive maintenance. If the failure degradation threshold, y_c ($\geqslant y_p$), is reached, the item has failed and must be repaired. Preventive maintenance and failure repair take one unit of time and bring the item back to zero degradation value (as-good-as-new state). The maintenance and the repair action costs may depend on the degradation value at which these actions are taken.

The item's degradation evolution over time can be modeled as a discrete-time Markov chain with infinite state space $\{0\} \cup \mathbb{N}$, where state k corresponds to k units of degradation. From the state $k < y_p$, the Markov chain jumps to state $k + j$ with probability p_j, $j \geqslant 0$. For the state $k \geqslant y_p$, the only transition is to state 0 with probability 1. Let x_k be the steady-state probability for state k, and let $X(z)$ be the corresponding pgf. The pgf of the distribution of p_j is denoted by $F(z)$. Then the balance equations are given by

$$\begin{aligned} x_0 &= p_0 x_0 + \sum_{j=0}^{\infty} x_{y_p+j}, \\ x_k &= \sum_{j=0}^{\min\{k, y_p-1\}} p_{k-j} x_j \qquad\qquad \text{for } k \geqslant 1. \end{aligned}$$

This system can be written in terms of pgfs as follows:

$$X(z) = X(z)F(z) + \sum_{j=0}^{\infty} x_{y_p+j}(1 - z^{y_p+j}F(z)). \tag{5.2}$$

Now suppose that the distribution of the degradation random variable has a geometric tail starting at $n+1$ for some $n \geqslant 0$, i.e., for some $\rho < 1$, $p_{n+j+1} = \rho^j p_{n+1}$ for $j \geqslant 0$ and $p_{n+1} \neq \rho p_n$. Then, from the balance equations, we obtain $x_{y_p+n+j} = \rho^j x_{y_p+n}$, $j \geqslant 0$, which means that the distribution of the item's degradation also has a geometric tail. Thus, (5.2) can be rewritten as

$$X(z) = \frac{\sum_{j=0}^{n-1} x_{y_p+j}(1 - z^{y_p+j}F(z))(1-\rho z) + x_{y_p+n}\left(\frac{1-\rho z}{1-\rho} - z^{y_p+n}F(z)\right)}{(1-F(z))(1-\rho z)}. \tag{5.3}$$

There are $n+1$ unknowns in the numerator of the pgf, namely, $x_{y_p}, \dots, x_{y_p+n}$. The denominator $D(z) = (1-F(z))(1-\rho z) = 1 - \rho z - \sum_{j=0}^{n} p_j z^j (1-\rho z) - p_{n+1}z^{n+1}$ is a polynomial of power $n+1$ and, therefore, has $n+1$ zeros, $\hat{z}_0, \dots, \hat{z}_n$, in the complex plain, including $\hat{z}_0 = 1$. Note that $D(1/\rho) \neq 0$, which means that $\hat{z}_j \neq 1/\rho$, $j = 0, \dots, n$. The form (5.3) suggests that $x_m = c\rho^m + \sum_{j=1}^{n} c_j/\hat{z}_j^m$ for large m. Since $x_{m+1} = \rho x_m$ for large m, we get that $c_j = 0$, which means that the numerator of (5.3) is zero at $\hat{z}_j$, $j = 1, \dots, n$. Thus, we obtain a linear system of equations of type (1.4).

Note that $D(z)/(z-1)$ gives, up to a constant coefficient, the minimal polynomial for $\hat{z}_1, \dots, \hat{z}_n$, and, therefore, the values of the elementary symmetric polynomials σ_m; see (2.2) and (2.3). To simplify the computations, one can substitute $F(z)$ by 1 in the functions of the numerator, which results in the same values of $f_j(\hat{z}_k)$, but simplifies (k,n)-transforms, given in (2.5), of the functions. In this case, $f_j(z)$, $j = 0, \dots, n$, are given by a linear combination of the functions 1, z, and $z^{y_p}, \dots, z^{y_p+n}$. Note that the (k,n)-transform of z^{y_p+j} coincides with $(k+1,n)$-transform of z^{y_p+1}. Thus, the number of required computations is linear in n.

5.3. Renewal risk model. Consider a discrete-time Sparre Andersen risk process, defined as follows. Suppose there is an initial surplus $u \in \{0\} \cup \mathbb{N}$ with a constant premium rate 1. The claims arrive according to a renewal process with interarrival times $W_i \in \mathbb{N}$ with the pgf $k(z)$ and claim sizes $X_i \in \mathbb{N}$ with the pgf $p(z)$, $i = 1, 2, \dots$. It is assumed that $\mathbb{E}(X_i) < \mathbb{E}(W_i)$. Let $N(n) = \max\{k\colon W_1 + W_2 + \cdots + W_k \leqslant n\}$ be the number of claims up to time n. Then the assets at time n are given by

$$U(n) = u + n - \sum_{i=1}^{N(n)} X_i, \qquad n = 1, 2, \dots;$$

see [16]. Let $T = \min\{n \in \mathbb{N}\colon U(n) < 0\}$ be the time of ruin. Suppose there is a nonnegative penalty function $w(x,y)$ for the surplus, x, just before ruin and the deficit, y, at ruin, meaning $U(T-1) = x$, $|U(T)| = y$. Let $v \in (0,1)$ be the discount factor over one time interval. Consider the expected discounted penalty (Gerber–Shiu) function:

$$\phi(u) = \mathbb{E}(v^T w(U(T-1), |U(T)|)\, \mathbb{1}\{T < \infty\} | U(0) = u),$$

where $\mathbb{1}\{A\}$ is the indicator function of the event A. Let $\hat{\phi}(z) = \sum_{u=0}^{\infty} z^u \phi(u)$ be the generating function of $\phi(u)$.

Suppose that the interarrival times W_i have a discrete K_n distribution, i.e., $k(z) = k_1(z)/k_2(z)$, where $k_1(z)$ is a polynomial of power not more than n, $k_2(z) = \prod_{j=1}^{n}(1 - q_j z)$ for some $0 < q_j < 1$. This class of distributions includes (shifted) geometric distribution, negative binomial distribution, and convolution of several geometric distributions. In this case (see [16]), $\hat{\phi}(z)$ can be written in a fractional form:

$$\hat{\phi}(z) = \frac{\omega(z) k_1(v/z) z^n + Q_{n-1}(z)}{(1/k(v/z) - p(z)) k_1(v/z) z^n},$$

where $\omega(z) = \sum_{y=1}^{\infty} z^y \sum_{x=y+1}^{\infty} w(y-1, x-y) \mathbb{P}(X_1 = x)$, and $Q_{n-1}(z)$ is an unknown polynomial of power $n-1$. Note that $k_1(v/z)z^n$ is a polynomial of power at most n, and, therefore, $\omega(z)k_1(v/z)z^n$ is an analytic function. In [16], it is shown, that $1/k(v/z) - p(z)$ has n zeros $\hat{z}_1, \ldots, \hat{z}_n$ in the unit circle with $0 < |z_k| < 1$, and that they also should be zeros of the numerator. This gives a linear system of equations of type (1.2), with $f_0(z) = \omega(z)k_1(v/z)z^n$, $f_j(z) = z^{j-1}$ for $j = 1, \ldots, n$, and fixed $x_0 = 1$. The only difference with the system (1.4), studied in this paper, is in the normalization equation, i.e., $x_0 = 1$ instead of (1.3). Thus, the solution will differ from the solution of (1.4) by a constant multiplier, which can be found using (3.5) for $j = 0$. In this way, we obtain

$$\hat{\phi}(z) = \frac{\det \bar{M}(z, \hat{z}_1, \ldots, \hat{z}_n)}{D(z) \det \bar{M}_0(\hat{z}_1, \ldots, \hat{z}_n)},$$

where $D(z) = (1/k(v/z) - p(z))k_1(v/z)z^n$, and $\bar{M}_0$ is the matrix $\bar{M}(z, \hat{z}_1, \ldots, \hat{z}_n)$, defined in (3.7), without the first row and column. We note that the function $\hat{\phi}(z)$ can be used, for example, to obtain the ruin probability; see [16] for the details.

6. Conclusions. In this paper, we have analyzed a class of pgfs that often occurs in the analysis of queuing systems and other stochastic models. The pgf contains several unknowns that can be found using the roots of the characteristic equation. We have given an explicit matrix representation of the pgf in terms of the roots (see Theorem 3.6), where the matrix entries are symmetric functions of the roots. The same matrix can be used to obtain the unknowns; see Theorem 3.7. Our representation allows one to use the roots even if they are close to each other or coincide without encountering the corresponding numerical problems. Moreover, it is possible to find the pgf and the unknowns using contour integrals instead of computing the roots, which can further improve the accuracy.

We have studied the cases where the pgf has a special product form, and where the mean value is an additive function of the roots. We have shown that these properties are equivalent under a nondegeneracy condition and have given a necessary and sufficient condition for them; see Theorem 4.1. For systems with these properties, both the pgf at a point and the mean may be found using one contour integral. If the nondegeneracy condition does not hold, the mean is independent of the roots.

One of the directions for further research is to generalize our results to other queuing systems that involve roots of a certain equation but do not have a rational pgf such as the one studied in this paper. For example, this may happen when a system is analyzed using Laplace–Stieltjes transforms (see, e.g., [9]) or matrices (see, e.g., [8]). Since the roots are usually numbered in an arbitrary order (except for $z_0 = 1$), the dependency of the unknowns on these roots should be symmetric. For this reason, we believe that the ideas of this paper can be used in the analysis of such systems, and, therefore, the unknowns can be expressed in terms of contour integrals. Another research direction would be to describe all symmetric functions that can be represented as contour integrals. Also, it would be interesting to find such model conditions that guarantee the special product form of the pgf studied in Theorem 4.1. Finally, our closed-form integral results might be used to obtain structural results for the considered models.

Appendix A. Proofs of the auxiliary results. In this appendix, we give the proofs of the auxiliary results: Properties 2.7 and 2.8, Lemmas 2.11, 4.7–4.9, and 4.11, and bound (2.11).

A.1. Proof of Property 2.7. In this subsection, we prove Property 2.7. We use induction by n. Consider $n = 1$. Suppose the matrix $\left(\begin{smallmatrix} a_0 & a_1 \\ f_0(z_1) & f_1(z_1) \end{smallmatrix}\right)$ is singular for all

z_1. Then $a_0 f_1(z_1) = a_1 f_0(z_1)$ for all z_1. If either a_0 or a_1 is nonzero, the functions are linearly dependent. Suppose we proved the statement for $n-1$. Using the Laplace expansion for the second row of Λ_a, we find $0 = \det \Lambda_a = \sum_{j=0}^{n} (-1)^{j+1} \det \Lambda_{a,j} f_j(z_1)$, where $\Lambda_{a,j}$ is the matrix Λ_a without the second row and the $(j+1)$st column. If $\det \Lambda_{a,j}$, as a function of $z_2, \ldots, z_n$, is not identically 0 for some j, then the functions $f_0(z_1), \ldots, f_n(z_1)$ are linearly dependent. Now suppose that $\det \Lambda_{a,j}$ is identically 0 for all $j = 0, \ldots, n$. Without loss of generality, we can assume $a_1 \neq 0$. Applying the induction hypothesis to the $n \times n$ matrix $\Lambda_{a,0}$, we find that the functions $f_1(z), \ldots, f_n(z)$ are linearly dependent, and so are the functions $f_0(z), \ldots, f_n(z)$.

A.2. Proof of Property 2.8. In this subsection, we prove Property 2.8. Recall that we need to prove that $\sum_{j=0}^{n} (-1)^j \sigma_j \zeta_{m-j} = 0$ for any $m > 0$. The proof will be done using generating functions. First, we note that

$$\sum_{m=0}^{\infty} \zeta_m z^m = \prod_{k=1}^{n} \sum_{l=0}^{\infty} (z_k z)^l = \prod_{k=1}^{n} \frac{1}{1 - z_k z}.$$

The above equality holds for sufficiently small z, i.e., for $|z| < \min_{k=1,\ldots,n-1} 1/|z_k|$. Second, from (2.3), we get

$$\prod_{k=1}^{n} (1 - z_k z) = z^n \prod_{k=1}^{n} (z^{-1} - z_k) = \sum_{j=0}^{n} (-1)^j \sigma_j \, z^j.$$

Hence,

$$1 = \prod_{k=1}^{n} \frac{1 - z_k z}{1 - z_k z} = \sum_{m=0}^{\infty} \zeta_m z^m \sum_{j=0}^{n} (-1)^j \sigma_j \, z^j.$$

Note that the last equation is an equality of two analytic functions. Thus, the coefficients at powers of z should coincide. Result (2.4) follows from considering the coefficient at z^m for $m > 0$.

A.3. Proof of Lemma 2.11. In this subsection, we prove Lemma 2.11. We use the first Jacobi–Trudi formula (see [4]), which can be written as

$$\text{(A.1)} \qquad \det \begin{pmatrix} z_1^{m_1} & \ldots & z_1^{m_n} \\ \vdots & \ddots & \vdots \\ z_n^{m_1} & \ldots & z_n^{m_n} \end{pmatrix} = V(z_1, \ldots, z_n) \det \begin{pmatrix} \zeta_{m_1} & \cdots & \zeta_{m_n} \\ \vdots & \ddots & \vdots \\ \zeta_{m_1-n+1} & \cdots & \zeta_{m_n-n+1} \end{pmatrix}.$$

It is used for the Schur polynomials, for which $m_1 > \cdots > m_n$. However, the result is general. In particular, a permutation of rows gives the result for any $m_1, \ldots, m_n$ such that $m_k \neq m_j$ for any $k \neq j$. Note also that if $m_k = m_j$ for $k \neq j$, then both sides of (A.1) are equal to 0. Lemma 2.11 follows from (A.1) by summing it for all possible combinations $(m_1, \ldots, m_n)$ with coefficients $\prod_{j=1}^{n} \alpha_{j,m_j}$. Note also that (A.1) is a special case of Lemma 2.11.

A.4. Proof of bound (2.11). In this subsection, we prove bound (2.11). Suppose $|\hat{z}_k| \leqslant q$ for $k = 1, \ldots, n$, and $|\alpha_l| \leqslant Cr^l$ for $l = 0, 1, \ldots$, where $f(z) = \sum_{l=0}^{\infty} \alpha_l z^k$. The latter condition holds if the function $f(z)$ is analytic in a disk with radius greater than $1/r$. Suppose also that $qr < 1$. Then we can give the following bound:

$$
\begin{aligned}
\left| F_k^n(\hat z_1,\dots,\hat z_n) - \sum_{l=0}^{M} \alpha_{k+l}\zeta_l(\hat z_1,\dots,\hat z_n)\right| &= \left|\sum_{l=M+1}^{\infty} \alpha_{l+k}\zeta_l(\hat z_1,\dots,\hat z_n)\right| \\
&\leqslant \sum_{l=M+1}^{\infty} |\alpha_{l+k}||\zeta_l(\hat z_1,\dots,\hat z_n)| \leqslant Cr^k \sum_{l=M+1}^{\infty} r^l \zeta_l(|\hat z_1|,\dots,|\hat z_n|) \\
&\leqslant Cr^k \sum_{l=M+1}^{\infty} r^l\zeta_l(q,\dots,q) \leqslant Cr^k \sum_{l=M+1}^{\infty}\binom{l+n-1}{l}(qr)^l \\
&\stackrel{!}{=} Cr^k\binom{M+n}{M}\frac{\sum_{l=1}^{n}(-1)^{l+1}\binom{n}{l}\frac{l}{M+l}(qr)^{M+l}}{(1-qr)^n} \\
&= Cr^k\binom{M+n}{M}\frac{n\sum_{l=0}^{n-1}(-1)^{l}\binom{n-1}{l}\frac{1}{M+l+1}(qr)^{M+l+1}}{(1-qr)^n} \\
&= Cr^k\binom{M+n}{n}\frac{n\int_0^{qr}x^M(1-x)^{n-1}dx}{(1-qr)^n} \leqslant Cr^k\binom{M+n}{n}\frac{n\int_0^{qr}x^M dx}{(1-qr)^n} \\
&= Cr^k\binom{M+n}{n}\frac{n(qr)^{M+1}}{(M+1)(1-qr)^n} = Cr^k\binom{M+n}{n-1}\frac{(qr)^{M+1}}{(1-qr)^n}.
\end{aligned}
$$

Equality $\stackrel{!}{=}$ can be proven using induction by M as follows. Consider $S_{n,M} = S_{n,M}(x) = \sum_{l=M+1}^{\infty}\binom{l+n-1}{l}x^l$. Observe that $\binom{l+n-1}{l}$ is equal to the number of ways to put l objects into n boxes. Therefore, for $M=0$, we get

$$
S_{n,0} = \left(\sum_{j=0}^{\infty} x^j\right)^n - 1 = \frac{1}{(1-x)^n} - 1 = \frac{\sum_{l=1}^{n}(-1)^{l+1}\binom{n}{l}x^l}{(1-x)^n}.
$$

Now suppose we have proved the statement for $S_{n,M-1}$. Consider $S_{n,M}$:

$$
\begin{aligned}
S_{n,M}(1-x)^n &= \left(S_{n,M-1} - \binom{M+n-1}{M}x^M\right)(1-x)^n \\
&= \binom{M+n-1}{M-1}\sum_{l=1}^{n}(-1)^{l+1}\binom{n}{l}\frac{l}{M+l-1}x^{M+l-1} - \binom{M+n-1}{M}x^M(1-x)^n \\
&= \binom{M+n-1}{M-1}\sum_{l=0}^{n-1}(-1)^l\binom{n}{l+1}\frac{l+1}{M+l}x^{M+l} - \binom{M+n-1}{M}\sum_{l=0}^{n}(-1)^l\binom{n}{l}x^{M+l} \\
&= \sum_{l=0}^{n}(-1)^l\left[\frac{(M+n-1)!(n-l)}{(M-1)!l!(n-l)!(M+l)} - \frac{(M+n-1)!n}{M!l!(n-l)!}\right]x^{M+l} \\
&= \sum_{l=0}^{n}(-1)^l\frac{(M+n-1)!}{(M-1)!l!(n-l)!}\left(\frac{n-l}{M+l} - \frac{n}{M}\right)x^{M+l} \\
&= \sum_{l=0}^{n}(-1)^{l+1}\frac{(M+n)!}{M!l!(n-l)!}\frac{l}{M+l}x^{M+l} = \binom{M+n}{M}\sum_{l=1}^{n}(-1)^{l+1}\binom{n}{l}\frac{l}{M+l}x^{M+l}.
\end{aligned}
$$

A.5. Proof of Lemma 4.7. In this subsection, we prove Lemma 4.7. Recall that we need to prove that there exists a point $z^* \in \Delta_1$ such that the functions

$f_j(z)$ after a linear transformation give the functions $\tilde{f}_k(z)$ that are locally equal to $(z-z^*)^k + o((z-z^*)^k)$.

Functions $f_j(z)$, $j = 0, \dots, n$, are analytic and linearly independent. Therefore, the Wronskian $\det W(z)$ of functions $f_0(z), \dots, f_n(z)$ is not identically 0; see Property 2.6. Note that $\det W(z)$ is an analytic function in Δ_1 and, therefore, it has not more than a finite number of zeros in Δ_1. Let $z^* \in \Delta_1$ be such that $\det W(z^*) \neq 0$. Define the functions $\tilde{f}_0(z), \dots, \tilde{f}_n(z)$ by the following linear combination:

$$\text{(A.2)}\quad \begin{pmatrix} \tilde{f}_0(z)/0! & \tilde{f}_1(z)/1! & \dots & \tilde{f}_n(z)/n! \end{pmatrix} = \begin{pmatrix} f_0(z) & f_1(z) & \dots & f_n(z) \end{pmatrix} (W(z^*))^{-1}.$$

Consider the matrix $\tilde{W}(z)$ for the functions $\tilde{f}_0(z)/0!, \dots, \tilde{f}_n(z)/n!$ defined as

$$\tilde{W}(z) = \begin{pmatrix} \tilde{f}_0(z) & \tilde{f}_1(z) & \dots & \tilde{f}_n(z)/n! \\ \tilde{f}_0'(z) & \tilde{f}_1'(z) & \dots & \tilde{f}_n'(z)/n! \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{f}_0^{(n)}(z) & \tilde{f}_1^{(n)}(z) & \dots & \tilde{f}_n^{(n)}(z)/n! \end{pmatrix}.$$

At point z^*, we get that $\tilde{W}(z^*) = W(z^*)(W(z^*))^{-1}$ becomes the identity matrix. Thus, $\tilde{f}_k^{(l)}(z^*) = 0$ for $l < k$ and $\tilde{f}_k^{(k)}(z^*)/k! = 1$, which means that the functions $\tilde{f}_0(z), \dots, \tilde{f}_n(z)$ satisfy (4.10).

Remark A.1 (nonzero value at 1). Given condition $(*)$ (see Theorem 4.1), one can choose z^* and the functions $\tilde{f}_k(z)$ such that for a particular index k, $\tilde{f}_k(1) \neq 0$. To see this, let us first consider the case $k = n$. Since $f_j(1) \neq 0$ for at least one j, we can consider a linear transformation of the functions $f_k(z)$ such that $f_0(1) \neq 0$, and $f_k(1) = 0$ for all $k \neq 0$. Then $\tilde{f}_n(1) \neq 0$ if and only if the coefficient of $f_0(z)$ in the definition of $\tilde{f}_n(z)$ (see (A.2)) is nonzero. This coefficient, up to multiplication by $\det W(z^*)$ and a sign, is equal to the Wronskian for the functions $f_1(z), \dots, f_n(z)$ at point z^*, which is nonzero for all possible z^* except a finite set. Note that if $\tilde{f}_n(1) \neq 0$, then either $\tilde{f}_k(1) \neq 0$ or $\tilde{f}_k(1) + \tilde{f}_n(1) \neq 0$. Therefore, the result for all k follows from the fact that $\tilde{f}_k(z) + \tilde{f}_n(z)$ satisfies (4.10) for $i = k$.

A.6. Proof of Lemma 4.8. In this subsection, we prove Lemma 4.8. We need to prove (4.11), i.e.,

$$\frac{h(z, z_1, \dots, z_k, 0, \dots, 0)}{h(1, z_1, \dots, z_k, 0, \dots, 0)} = \frac{\det \Lambda_k(z, z_1, \dots, z_k)}{\det \Lambda_k(1, z_1, \dots, z_k)},$$

where $\Lambda_k(z, z_1, \dots, z_k)$ is an alternant matrix constructed using the functions $f_{n-k}(z)$, $\dots, f_n(z)$ and the points $z, z_1, \dots, z_k$.

Recall that the function $h(z, z_1, \dots, z_k, 0, \dots, 0)$ is the determinant of the matrix $\bar{M}(z, z_1, \dots, z_k, 0, \dots, 0)$, whose entries are $F^n_{j,m}(z_1, \dots, z_k, 0, \dots, 0)$. From Property 2.9, we get $F^n_{j,m}(z_1, \dots, z_k, 0, \dots, 0) = F^k_{j,m}(z_1, \dots, z_k)$. Now, according to Property 2.10, $F^k_{j,m} + \sum_{l=1}^{k} (-1)^l \sigma_l(z_1, \dots, z_k) F^k_{j,m+l} = \alpha_{j,m}$, where $f_j(z) = \sum_{l=0}^{\infty} \alpha_{j,l} z^l$. Thus, after a linear transformation, we get that the matrix $\bar{M}(z, z_1, \dots, z_k, 0, \dots, 0)$

changes to

$$\begin{pmatrix} f_0(z) & \dots & f_n(z) \\ \alpha_{0,0} & \dots & \alpha_{n,0} \\ \vdots & \ddots & \vdots \\ \alpha_{0,n-k-1} & \dots & \alpha_{n,n-k-1} \\ F^k_{0,n-k} & \dots & F^k_{n,n-k} \\ \vdots & \ddots & \vdots \\ F^k_{0,n-1} & \dots & F^k_{n,n-1} \end{pmatrix}. \tag{A.3}$$

Here, we added the $(m+l+2)$nd row multiplied by $(-1)^l\sigma_l(z_1,\dots,z_k)$ to the $(m+2)$nd row for $l = 1,\dots,k$ and $m = 0,\dots,n-k-1$. Now, note that $\alpha_{j,k} = 0$ for $j > k$ and $\alpha_{j,j} = 1$, which means that the matrix in (A.3) is equal to

$$\begin{pmatrix} f_0(z) & \dots & f_{n-k-1}(z) & f_{n-k}(z) & \dots & f_n(z) \\ 1 & \dots & 0 & 0 & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{0,n-k-1} & \dots & 1 & 0 & \dots & 0 \\ F^k_{0,n-k} & \dots & F^k_{n-k-1,n-k} & F^k_{n-k,n-k} & \dots & F^k_{n,n-k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ F^k_{0,n-1} & \dots & F^k_{n-k-1,n-1} & F^k_{n-k,n-1} & \dots & F^k_{n,n-1} \end{pmatrix}.$$

Hence, the determinant is equal to

$$h(z,z_1,\dots,z_k,0,\dots,0) = (-1)^{n-k}\det\begin{pmatrix} f_{n-k}(z) & \dots & f_n(z) \\ F^k_{n-k,n-k} & \dots & F^k_{n,n-k} \\ \vdots & \ddots & \vdots \\ F^k_{n-k,n-1} & \dots & F^k_{n,n-1} \end{pmatrix}. \tag{A.4}$$

At this moment, we can use Lemma 2.11 to find that up to multiplication by the Vandermonde determinant $V(z_1,\dots,z_k)$, the determinant of the matrix on the right-hand side of (A.4) is equal to the determinant of an almost alternant matrix

$$V(z_1,\dots,z_k)\det\begin{pmatrix} f_{n-k}(z) & \dots & f_n(z) \\ F^k_{n-k,n-k} & \dots & F^k_{n,n-k} \\ \vdots & \ddots & \vdots \\ F^k_{n-k,n-1} & \dots & F^k_{n,n-1} \end{pmatrix} = \det\begin{pmatrix} f_{n-k}(z) & \dots & f_n(z) \\ \frac{f_{n-k}(z_1)}{z_1^{n-k}} & \dots & \frac{f_n(z_1)}{z_1^{n-k}} \\ \vdots & \ddots & \vdots \\ \frac{f_{n-k}(z_k)}{z_k^{n-k}} & \dots & \frac{f_n(z_k)}{z_k^{n-k}} \end{pmatrix}, \tag{A.5}$$

which is not identically equal to 0 due to the linear independence of the functions $f_{n-k}(z),\dots,f_n(z)$. Here, we used the fact that $f_j(z)$, $j = n-k,\dots,n$, satisfies (4.10) and, therefore, has first $n-k-1$ coefficients in the Taylor expansion equal to 0. Hence, the (m,k)-transformation of the function $f_j(z)/z^{n-k} = \sum_{l=n-k}^{\infty}\alpha_{j,l}z^{l-n+k}$ is equal to $F^k_{j,n-k+m} = \sum_{l=n-k+m}^{\infty}\alpha_{j,l}\zeta_{l-n+k-m}$. Combining (A.4) and (A.5) gives (4.11), thereby concluding the proof.

A.7. Proof of Lemma 4.9. In this subsection, we prove Lemma 4.9. In this lemma, we assume that (4.14) holds for $z_{k+1} = \dots = z_n = 0$, i.e.,

$$\frac{\partial}{\partial z}\frac{h(z,z_1,\dots,z_k,0,\dots,0)}{h(1,z_1,\dots,z_k,0,\dots,0)}\bigg|_{z=1} = \frac{f'_{n-1}(1)}{f_{n-1}(1)} - (k-1)\frac{C'(1)}{C(1)} + \sum_{j=1}^{k}\frac{C'(1)}{C(1)-C(z_j)},$$

and that $f_{j+1}(z) = C(z)f_j(z)$ for $j = n-k+1,\dots,n-1$. We need to prove that the function $f_{n-k}(z)$ is equal (up to a linear combination of the functions $f_{n-k+1}(z),\dots,f_n(z)$) to $\beta_0 f_{n-k+1}/C(z)$.

First, we apply Lemma 4.8 and get

$$\frac{h(z,z_1,\dots,z_k,0,\dots,0)}{h(1,z_1,\dots,z_k,0,\dots,0)} = \frac{f_{n-k+1}(z)\det M_G(z,z_1,\dots,z_k)}{f_{n-k+1}(1)\det M_G(1,z_1,\dots,z_k)}, \tag{A.6}$$

where

$$M_G(z,z_1,\dots,z_k) = \begin{pmatrix} \frac{1}{G(z)} & 1 & \dots & C(z)^{k-1} \\ \frac{1}{G(z_1)} & 1 & \dots & C(z_1)^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{G(z_k)} & 1 & \dots & C(z_k)^{k-1} \end{pmatrix}, \tag{A.7}$$

and $G(z) = f_{n-k+1}(z)/f_{n-k}(z)$.

Second, we find the derivative of (A.6) at 1:

$$\frac{\partial}{\partial z}\frac{h(z,z_1,\dots,z_k,0,\dots,0)}{h(1,z_1,\dots,z_k,0,\dots,0)}\bigg|_{z=1} = \frac{f'_{n-k+1}(1)}{f_{n-k+1}(1)} + \frac{\partial}{\partial z}\frac{\det M_G(z,z_1,\dots,z_k)}{\det M_G(1,z_1,\dots,z_k)}\bigg|_{z=1}. \tag{A.8}$$

Note that $f_{n-1}(z) = f_{n-k+1}(z)C(z)^{k-2}$. Thus,

$$\frac{f'_{n-1}(1)}{f_{n-1}(1)} = \frac{f'_{n-k+1}(1)}{f_{n-k+1}(1)} + (k-2)\frac{C'(1)}{C(1)}. \tag{A.9}$$

Combining (4.14), (A.8), and (A.9), we get that

$$\frac{\partial}{\partial z}\frac{\det M_G(z,z_1,\dots,z_k)}{\det M_G(1,z_1,\dots,z_k)}\bigg|_{z=1} = -\frac{C'(1)}{C(1)} + \sum_{l=1}^{k}\frac{C'(1)}{C(1)-C(z_l)}. \tag{A.10}$$

Third, we prove that the functions $C(z_1)/G(z_1), 1,\dots,C(z_1)^k$ are linearly dependent if $C'(1)\neq 0$, which we show later. Let $\mu_j(z)$ be the determinant of the matrix $M_G(z,z_1,\dots,z_k)$ without the second row and $(j+1)$st column, multiplied by $(-1)^j$. Note that $\mu_j(z)$ does not depend on z_1. Fix any $z_2,\dots,z_k$ such that $\mu_0(1)\neq 0$, which is possible since the function $C(z)$ is not constant. By multiplying both sides of (A.10) by $-\det M_G(1,z_1,\dots,z_k)$, we get

$$\begin{aligned} &\frac{1}{G(z_1)}\mu'_0(1) + \sum_{j=0}^{k-1}\mu'_{j+1}(1)C(z_1)^j = -\frac{1}{G(z_1)}\mu_0(1)\frac{C'(1)}{C(1)} \\ &+\frac{1}{G(z_1)}\mu_0(1)\sum_{l=1}^{k}\frac{C'(1)}{C(1)-C(z_l)} + \sum_{j=0}^{k-1}\mu_{j+1}(1)C(z_1)^j\left(-\frac{C'(1)}{C(1)} + \sum_{l=1}^{k}\frac{C'(1)}{C(1)-C(z_l)}\right). \end{aligned} \tag{A.11}$$

Note that $\mu_0(z) = V(C(z),C(z_2),\dots,C(z_k))$. Hence, $\mu'_0(1) = \mu_0(1)\sum_{l=2}^{k}\frac{C'(1)}{C(1)-C(z_l)}$. Therefore, we can rewrite (A.11) as

$$\begin{aligned} &\frac{C(z_1)}{G(z_1)}\mu_0(1)\frac{C'(1)}{C(1)(C(1)-C(z_1))} \\ &\quad = \sum_{j=0}^{k-1}C(z_1)^j\left(\mu'_{j+1}(1) + \mu_{j+1}(1)\frac{C'(1)}{C(1)} - \mu_{j+1}(1)\sum_{l=1}^{k}\frac{C'(1)}{C(1)-C(z_l)}\right). \end{aligned} \tag{A.12}$$

Note that multiplying both sides by $C(1) - C(z_1)$ will give a linear dependency between the functions $C(z_1)/G(z_1)$, $1, \dots, C(z_1)^k$, in which the coefficient for $C(z_1)/G(z_1)$ is nonzero (if $C'(1) \neq 0$). To get (4.15), one needs to multiply both sides of (A.12) by $f_{n-k+1}(z_1)\, C(1)\, (C(1) - C(z_1))/(C(z_1)\, \mu_0(1)\, C'(1))$.

Finally, we prove that $C'(1) \neq 0$. Suppose $C'(1) = 0$. We will prove that this contradicts $(**)$. Note that $c = f'_{n-1}(1)/f_{n-1}(1)$ and, due to (4.14), $h'(1, z_1, \dots, z_n) = c\, h(1, z_1, \dots, z_n)$. This means that the matrix

$$\begin{pmatrix} f'_0(1) - c\, f_0(1) & \dots & f'_n(1) - c\, f_n(1) \\ f_0(z_1) & \dots & f_n(z_1) \\ \vdots & \ddots & \vdots \\ f_0(z_n) & \dots & f_n(z_n) \end{pmatrix}$$

is singular for all $z_1, \dots, z_n$. Since the functions $f_0(z), \dots, f_n(z)$ are linearly independent, we get that $f'_i(1) = c f_i(1)$ for $i = 0, \dots, n$, which contradicts $(**)$.

A.8. Proof of Lemma 4.11. In this subsection, we prove Lemma 4.11 similarly to Lemma 4.9. Suppose that (4.16) holds for $z_{k+1} = \dots = z_n = 0$, i.e.,

$$\frac{h(z, z_1, \dots, z_k, 0, \dots, 0)}{h(1, z_1, \dots, z_k, 0, \dots, 0)} = \frac{f_{n-1}(z)(C(1))^{k-1}}{f_{n-1}(1)(C(z))^{k-1}} \prod_{j=1}^{k} \frac{C(z) - C(z_j)}{C(1) - C(z_j)},$$

and that $f_{j+1}(z) = C(z) f_j(z)$ for $j = n - k + 1, \dots, n - 1$. We need to prove that the function $f_{n-k}(z)$ is equal (up to a linear combination of the functions $f_{n-k+1}(z), \dots, f_n(z)$) to $\beta_0 f_{n-k+1}/C(z)$.

Using (A.6) and $f_{n-1}(z) = f_{n-k+1}(z) C(z)^{k-2}$, we obtain

$$\frac{\det M_G(z, z_1, \dots, z_k)}{\det M_G(1, z_1, \dots, z_k)} = \frac{C(1)}{C(z)} \prod_{j=1}^{k} \frac{C(z) - C(z_j)}{C(1) - C(z_j)}, \tag{A.13}$$

where the matrix $M_G(z, z_1, \dots, z_k)$ is defined in (A.7).

From (A.13), we prove that the functions $C(z)/G(z)$, $1, \dots, C(z)^k$ are linearly dependent. Let μ_j be the determinant of the matrix $M_G(z, z_1, \dots, z_k)$ without the *first* row and $(j + 1)$st column, multiplied by $(-1)^j$. Fix any $z_1, \dots, z_k$ such that $\mu_0 \neq 0$, which is possible since the function $C(z)$ is not constant. By multiplying both sides of (A.13) by $\det M_G(1, z_1, \dots, z_k) C(z)$, we obtain

$$\frac{C(z)}{G(z)} \mu_0 + \sum_{l=1}^{k} \mu_l C(z)^l = C(1) \left(\frac{C(1)}{G(1)} \mu_0 + \sum_{l=1}^{k} \mu_l C(1)^l \right) \prod_{j=1}^{k} \frac{C(z) - C(z_j)}{C(1) - C(z_j)}.$$

Note that the product on the right-hand side can be rewritten as a linear combination of the functions $1, \dots, C(z)^k$. This observation concludes the proof.

Remark A.2 (comparison to the proof of Lemma 4.9). In the proof in Appendix A.7, we expand an alternant matrix over the row that depends on z_1. In this case, the result requires an additional condition $C'(1) \neq 0$, which follows from $(**)$. In this subsection, we can expand a similar matrix over the first row that depends on z (in the proof of Lemma 4.9 this is a constant row, and we cannot use such an expansion). For this reason, we do not need the extra condition $(**)$.

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