

Fluid dynamical systems as Hamiltonian boundary control systems

A.J. van der Schaft¹

B.M. Maschke²

Abstract

It is shown how the geometric framework for distributed-parameter port-controlled Hamiltonian systems as recently provided in [11, 12] can be adapted to formulate ideal isentropic compressible fluids with non-zero energy flow through the boundary of the spatial domain as Hamiltonian boundary control systems. The key ingredient is the modification of the Stokes-Dirac structure introduced in [11] to a Dirac structure defined on the space of mass density 3-forms and velocity 1-forms, incorporating three-dimensional convection. Some initial steps towards stabilization of these boundary control systems, based on the generation of Casimir functions for the closed-loop Hamiltonian system, are discussed.

1 Introduction

In recent publications [10, 22, 4, 20, 21] a systematic framework has been provided for the geometric modelling of network models of lumped-parameter physical systems as port-controlled Hamiltonian (PCH) systems (with or without dissipation). The key notion in this framework is that of a power-conserving interconnection, formalized by the geometric concept of a Dirac structure. Furthermore ([23, 4, 21, 17, 19]) it has been shown how by interconnection with a controller system that is itself a PCH system, the system may be stabilized at a desired set-point by generating Casimir functions (conserved quantities) determined by the closed-loop interconnection structure, thus effectively shaping the energy of the system.

Recently [11, 12] we have started to expand this research program on finite-dimensional PCH systems to the distributed parameter case. However, a fundamental difficulty which arises is the treatment of *boundary conditions*. Indeed, from a control and interconnection point of view it is essential to describe a distributed parameter system with varying boundary conditions inducing *energy exchange through the boundary*, since in many applications the interaction with the environment (e.g. actuation or measurement) will actually take place through the boundary of the system. On the other hand, the treatment of distributed parameter Hamiltonian systems in the literature ([14, 8, 9, 15, 13, 1]) seems mostly focussed on systems

with infinite spatial domain, where the variables go to zero for the spatial variables tending to infinity, or on systems with boundary conditions such that the energy exchange through the boundary is *zero*. In [11, 12] we have proposed a framework to overcome this fundamental problem, by defining a *Dirac structure* on certain spaces of differential forms on the spatial domain and its boundary, based on the use of Stokes' theorem. This framework has been successfully applied to the port-controlled Hamiltonian formulation of e.g. the telegraph equations and Maxwell's equations.

In the present paper we extend and generalize this differential-geometric framework to the Eulerian description of 3-dimensional ideal isentropic fluids (see Section 2). The basic set up is to represent the mass density as a 3-form and the Eulerian velocity as a 1-form (see also [8, 9] for a similar point of view), and to define a *modified Stokes-Dirac structure* on the space of these state variables according to mass and momentum balance ("modified" because of an additional term arising from 3-dimensional convection). For zero-boundary conditions our formulation reduces to the elegant Poisson bracket formulation given before in [14, 8, 9, 13]. The resulting infinite-dimensional system with boundary variables can be interpreted as a (nonlinear) *boundary control system* in the sense of e.g. [7].

The identification of the underlying Hamiltonian structure of fluid dynamics has proved to be instrumental in deriving all sorts of results on integrability, existence of soliton solutions, stability, reduction, etc., and in *unifying* existing results, see e.g. [6, 1, 13]. We believe it will also be a fruitful starting point for the control of such systems. In Section 3 we shall already provide some initial ideas how the theory of interconnection and energy-shaping as developed for finite-dimensional port-controlled Hamiltonian systems might be extended to the fluid dynamics case.

2 Geometric boundary control formulation of fluid dynamics

2.1 Introduction

An ideal compressible isentropic fluid in three dimensions is described by the equations (in vector calculus notation with $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$)

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v) \quad (1)$$

¹Faculty of Mathematical Sciences, University of Twente, PO Box 217, 7500 AE Enschede, The Netherlands

²Lab. d'Automatique et de Genie des Procédes, Université Claude Bernard Lyon-1, F-69622 Villeurbanne, Cedex, France

$$\frac{\partial v}{\partial t} = -v \cdot \nabla v - \frac{1}{\rho} \nabla p \quad (2)$$

Here $\rho(x, t)$ denotes the mass density at the spatial position $x \in \mathbb{R}^3$ at time t , and $v(x, t)$ is the Eulerian velocity, that is, the velocity of the fluid at the (fixed) spatial position x at time t . Furthermore, $p(x, t)$ is the pressure, which is derivable from an internal energy density $U(\rho)$ as

$$p(x, t) = \rho^2(x, t) \frac{\partial U}{\partial \rho}(\rho(x, t)) \quad (3)$$

Both equations (1) and (2) are conservation laws, expressing respectively mass-balance and momentum-balance, and more generally can be expressed in an integral form. Indeed, let W be any fixed 3-dimensional subdomain of some given domain $\mathcal{D} \subset \mathbb{R}^3$, filled with the fluid. Then (1) expresses that the change of mass inside W is equal to minus the mass flow through the boundary of W , while (2) corresponds to Newton's second law.

It can be readily checked that the total stored energy in W (with dV the standard volume element in \mathbb{R}^3)

$$H_W := \int_W \left(\frac{1}{2} \rho \|v\|^2 + \rho U(\rho) \right) dV \quad (4)$$

satisfies the balance equation

$$\frac{d}{dt} H_W = - \int_{\partial W} \left[\frac{1}{2} \rho \|v\|^2 + h(\rho) \right] \rho v \cdot n dA \quad (5)$$

(with dA denoting the standard area element) where n is the outward normal vector to the boundary ∂W , and $h(\rho) := U(\rho) + \rho \frac{\partial U}{\partial \rho}(\rho)$ is the enthalpy. Alternatively, using (3), the energy balance (5) can be rewritten in "convective form" as

$$\begin{aligned} \frac{d}{dt} H_W &= - \int_{\partial W} \left[\frac{1}{2} \rho \|v\|^2 + \rho U(\rho) \right] v \cdot n dA \\ &\quad - \int_{\partial W} p v \cdot n dA \end{aligned} \quad (6)$$

It immediately follows that if v is such that $v \cdot n = 0$ at the boundary ∂W (no fluid flow through the boundary), then the total energy H_W is conserved. In fact, not only the energy H_W is conserved in this case, but the dynamical equations (1), (2) of the fluid on W can be formulated as an infinite-dimensional Hamiltonian system on the infinite-dimensional space of mass densities ρ and Eulerian velocities v on W . This is done via the introduction of an infinite-dimensional Poisson bracket, see e.g. [14, 8, 9, 6, 13] for clear expositions and further ramifications. From a control point of view, however, we would like to consider the fluid dynamical system as a boundary control system, with time-varying boundary conditions different from $v \cdot n|_{\partial W} = 0$, since the interaction of the system with its environment will often take place through the boundary.

2.2 Stokes-Dirac structure

The basic concept we need is that of a Dirac structure, as introduced by Courant [4] and Dorfman [7] as a generalization of symplectic and Poisson structures, and employed in e.g. [22, 4, 21] as the geometric notion formalizing general power-conserving interconnections.

Definition 2.1 Let V be a linear space (possibly infinite-dimensional). There exists on $V \times V^*$ the canonically defined symmetric bilinear form

$$\langle\langle (f_1, e_1), (f_2, e_2) \rangle\rangle := \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle \quad (7)$$

with $f_i \in V$, $e_i \in V^*$, $i = 1, 2$, and $\langle | \rangle$ denoting the duality product between V and its dual space V^* . A constant Dirac structure on V is a linear subspace $D \subset V \times V^*$ such that

$$D = D^\perp \quad (8)$$

where \perp denotes orthogonal complement with respect to the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$.

Let now $(f, e) \in D = D^\perp$. Then as an immediate consequence of (7)

$$0 = \langle\langle (f, e), (f, e) \rangle\rangle = 2 \langle e | f \rangle \quad (9)$$

Thus for all $(f, e) \in D$ we obtain $\langle e | f \rangle = 0$, expressing power conservation with respect to the dual power variables $f \in V$ and $e \in V$. (In an electrical context the components of f and e will denote pairs of currents and voltages, while in a mechanical context they will be pairs of generalized velocities and forces.)

The Stokes-Dirac structure corresponding to 3-dimensional fluid dynamics is now defined as follows. Let $W \subset \mathcal{D} \subset \mathbb{R}^3$ be a 3-dimensional manifold with smooth 2-dimensional boundary ∂W . Let $\Omega^k(W)$ denote the space of differential k -forms on W , $k = 0, 1, 2, 3$, and let $\Omega^k(\partial W)$ denote the k -forms on ∂W , $k = 0, 1, 2$. We identify the mass density ρ with a 3-form on W (see e.g. [8, 9]), that is, with an element in $\Omega^3(W)$. Furthermore, we identify the Eulerian velocity v with a 1-form v , that is, with an element of $\Omega^1(W)$; see later on for some additional motivation. This leads to the consideration of the (linear) space of energy variables

$$X := \Omega^3(W) \times \Omega^1(W) \quad (10)$$

Next we consider the boundary external variables (or boundary input and output variables). First we consider the space $\Omega^0(\partial W)$ of 0-forms, that is, the functions on ∂W . They will represent the "stagnation pressure divided by ρ " at the boundary. Secondly, we consider the space $\Omega^2(\partial W)$ of 2-forms on ∂W , representing the "boundary mass flow". Thus we consider the space of boundary variables

$$\Omega^0(\partial W) \times \Omega^2(\partial W) \quad (11)$$

Note that (see also [8, 9]) there is a pairing (\cdot, \cdot) between $\Omega^0(\partial W)$ and $\Omega^2(\partial W)$, given by

$$(f, \alpha) := \int_{\partial W} f \alpha, \quad f \in \Omega^0(\partial W), \alpha \in \Omega^2(\partial W) \quad (12)$$

This pairing is weakly non-degenerate, that is, if $(f, \alpha) = 0$ for all $\alpha \in \Omega^2(\partial W)$ then $f = 0$, and if $(f, \alpha) = 0$ for all f ,

then $\alpha = 0$. Thus we can regard $\Omega^0(\partial W)$ as a dual space of $\Omega^2(\partial W)$, that is,

$$\Omega^0(\partial W) = (\Omega^2(\partial W))^* \quad (13)$$

(Note that in this way $\Omega^0(\partial W)$ is a subspace of the functional analytic dual of $\Omega^2(\partial W)$.) The pairing (12) will represent the power flowing into the system through the boundary ∂W . In a similar way we define

$$\begin{aligned} (\Omega^3(W))^* &= \Omega^0(W) \\ (\Omega^1(W))^* &= \Omega^2(W) \end{aligned} \quad (14)$$

using the weakly non-degenerate pairing

$$(\alpha, \beta) = \int_W \alpha \wedge \beta \quad (15)$$

with $\alpha \in \Omega^0(W)$, $\beta \in \Omega^3(W)$, respectively $\alpha \in \Omega^2(W)$, $\beta \in \Omega^1(W)$.

Theorem 2.2 (Stokes-Dirac structure) Let $W \subset \mathbb{R}^3$ be a 3-dimensional manifold with boundary ∂W . Consider $V := X \times \Omega^0(\partial W) = \Omega^3(W) \times \Omega^1(W) \times \Omega^0(\partial W)$, and $V^* = \Omega^0(W) \times \Omega^2(W) \times \Omega^2(\partial W)$, together with the bilinear form induced by the pairing (12) and (15)

$$\begin{aligned} &\ll (f_\rho^1, f_\nu^1, f_b^1, e_\rho^1, e_\nu^1, e_b^1), (f_\rho^2, f_\nu^2, f_b^2, e_\rho^2, e_\nu^2, e_b^2) \gg \\ &:= \int_W (e_\rho^1 \wedge f_\rho^2 + e_\nu^1 \wedge f_\nu^2 + e_b^1 \wedge f_b^2 + e_\rho^2 \wedge f_\rho^1 \\ &+ \int_{\partial W} (e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1) \end{aligned} \quad (16)$$

where

$$\begin{aligned} f_\rho^i &\in \Omega^3(W), f_\nu^i \in \Omega^1(W), f_b^i \in \Omega^0(\partial W) \\ e_\rho^i &\in \Omega^0(W), e_\nu^i \in \Omega^2(W), e_b^i \in \Omega^2(\partial W) \end{aligned} \quad (17)$$

Then $D \subset V \times V^*$ defined as

$$\begin{aligned} D &= \{(f_\rho, f_\nu, f_b, e_\rho, e_\nu, e_b) \in V \times V^* \mid \\ &f_\rho = de_\nu, f_\nu = de_\rho, f_b = e_{\rho|\partial W}, e_b = -e_{\nu|\partial W}\} \end{aligned} \quad (18)$$

where d is the exterior derivative (mapping k -forms into $(k+1)$ -forms), and where $\lfloor_{\partial W}$ denotes the restriction of k -forms on W to k -forms on the boundary ∂W , is a Dirac structure with respect to the bilinear form \ll, \gg defined in (16).

Proof This can be proved along the same lines as in [11], making use of Stokes' theorem $\int_W d\alpha = \int_{\partial W} \alpha$ for any 2-form α . (In [11] the "symmetric" case was considered with $V = \Omega^2(W) \times \Omega^2(W) \times \Omega^1(\partial W)$ on a 3-dimensional domain $W \subset \mathbb{R}^3$, which turns out to be the appropriate setting for Maxwell's equations.) \square

2.3 The Hamiltonian formulation

The idea is now to regard the Stokes-Dirac structure of Theorem 2.2 as the power-conserving interconnection relating the boundary external variables f_b, e_b to the internal variables $f_\rho, f_\nu, e_\rho, e_\nu$. Furthermore, following the framework in [11, 12] the internal variables f_ρ, f_ν are equated with (minus) the time-derivatives $\frac{\partial \rho}{\partial t}, \frac{\partial \nu}{\partial t}$ of the energy variables ρ, ν , while the internal variables e_ρ, e_ν are equated with the co-energy variables $\delta_\rho H, \delta_\nu H$. However, contrary to the case of the telegrapher's equations or Maxwell's equations as treated in [11, 12], we still need to introduce an additional term to the Stokes-Dirac structure given above, which is due to the 3-dimensional geometry associated with convection. The problem thus concerns the geometric formulation of the term $v \cdot \nabla v$ in (2).

From a general differential-geometric point of view this can be done as follows. Let \langle, \rangle be any Riemannian metric on W , with ∇ denoting its unique symmetric covariant derivative. (If \langle, \rangle is the Euclidean metric then ∇ is just the ordinary derivative operator $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ as above.) Let u be a vector field on W , and let u^b denote the corresponding 1-form, defined as $u^b = i_u \langle, \rangle$ ("index raising" via the metric). Then the following formula holds, relating the covariant derivative to the Lie-derivative:

$$L_u u^b = (\nabla_u u)^b + \frac{1}{2} d \langle u, u \rangle \quad (19)$$

(see for a proof [1, p. 202]). By Cartan's magical formula

$$L_u u^b = i_u du^b + di_u u^b = i_u du^b + d \langle u, u \rangle \quad (20)$$

we therefore obtain

$$(\nabla_u u)^b = i_u du^b + \frac{1}{2} d \langle u, u \rangle, \quad (21)$$

which is the coordinate-free analog of the classical vector calculus formula (using the Euclidean metric)

$$u \cdot \nabla u = \text{curl } u \times u + \frac{1}{2} \nabla |u|^2 \quad (22)$$

Let us now consider v in (2) to be a 1-form. By "index lowering" with respect to the Riemannian metric the 1-form v defines a vector field v^h (such that $(v^h)^b = v$). Hence, we may represent (2) as

$$\frac{\partial v}{\partial t} = -i_{v^h} dv - d \left(\frac{1}{2} \langle v^h, v^h \rangle \right) - \frac{1}{\bar{\rho}} dp \quad (23)$$

with $\bar{\rho}$ the mass density function, formally defined as $\bar{\rho} = * \rho$, with $*$ denoting the Hodge star operator determined by \langle, \rangle ; converting the 3-form ρ into the 0-form (function) $* \rho$. Furthermore, by (3) it follows that

$$\frac{1}{\bar{\rho}} dp = d(U(\bar{\rho}) + \bar{\rho} \frac{\partial U}{\partial \bar{\rho}}(\bar{\rho})) = d \left(\frac{\partial}{\partial \bar{\rho}} (\bar{\rho} U(\bar{\rho})) \right) (= d(h(\bar{\rho}))) \quad (24)$$

Hence we may rewrite (23) as

$$\frac{\partial v}{\partial t} = -i_v \cdot dv - d \left(\frac{\partial}{\partial \bar{\rho}} \left[\frac{1}{2} \bar{\rho} \langle v^h, v^h \rangle + \bar{\rho} U(\bar{\rho}) \right] \right) \quad (25)$$

where in the second term on the right-hand side we recognize (see (4)) the total energy density.

Finally, consider the total energy H_W given in (4) which formally can be rewritten as a function of the 3-form ρ and the 1-form v as

$$H_W = \int_W \left[\frac{1}{2} \langle v^h, v^h \rangle + U(*\rho) \right] \rho \quad (26)$$

The partial derivative $\delta_\rho H_W$ is an element of $(\Omega^3(W))^*$, and thus can be identified with an element of $\Omega^0(W)$ (namely, with the function $\frac{\partial}{\partial \bar{\rho}} \left[\frac{1}{2} \bar{\rho} \langle v^h, v^h \rangle + \bar{\rho} U(\bar{\rho}) \right] = \frac{1}{2} \langle v^h, v^h \rangle + h(*\rho)$ in (25)), while the other partial derivative $\delta_v H_W$ is an element of $(\Omega^1(W))^*$, and thus can be equated with an element of $\Omega^2(W)$ (in fact, with the 2-form $i_v \cdot \rho$). It also follows immediately that $\delta_\rho H_W$ and $\delta_v H_W$ only depend on the *energy density* (the integrand in (4) or (26)), and thus we simply write $\delta_\rho H$ and $\delta_v H$. Finally, we note the equality (most easily checked in a basis)

$$i_v \cdot dv = \frac{1}{*\rho} * ((*dv) \wedge (*\delta_v H)) \quad (27)$$

with dv , $\delta_v H$ denoting 2-forms, and $*$ the Hodge star operator converting 2-forms into 1-forms.

Summarizing, we can rewrite (1) into the following form

$$\frac{\partial \rho}{\partial t} = -d(\delta_v H) \quad (28)$$

$$\frac{\partial v}{\partial t} = -d(\delta_\rho H) - \frac{1}{*\rho} * ((*dv) \wedge (*\delta_v H)) \quad (29)$$

Comparing with the Stokes-Dirac structure given in Theorem 2.2 we notice the additional term in the right-hand side of (29). This is incorporated into the following definition of a *modified Stokes-Dirac structure*

Proposition 2.3 (Modified Stokes-Dirac structure) Consider the same setting as in Theorem 2.2. Then $D^m \subset V \times V^*$ defined as

$$\begin{aligned} D^m &= \{(f_\rho, f_v, f_b, e_\rho, e_v, e_b) \in V \times V^*\} \\ f_\rho &= de_v, f_v = de_\rho + \frac{1}{*\rho} * ((*dv) \wedge (*e_v)), \\ f_b &= e_\rho|_{\partial W}, e_b = -e_v|_{\partial W} \end{aligned} \quad (30)$$

is a Dirac structure.

Proof This is based on the fact that $e_v^2 * ((*dv) \wedge (*e_v^1))$ is skew-symmetric in $e_v^1, e_v^2 \in \Omega^2(W)$, and hence does not contribute to the bilinear form (16). (In fact, in vector calculus notation $e_v^2 * ((*dv) \wedge (*e_v^1)) = (*dv) \cdot (e_v^1 \times e_v^2)$.) \square

Remark 2.4 Note however that D as given in (30) is not anymore a constant Dirac structure, since it depends on the energy variables ρ and v .

Remark 2.5 For a 1- or 2-dimensional fluid the extra term in (30) is automatically zero. Furthermore, if in the three-dimensional case the 2-form $dv(t)$ happens to be zero at a certain time-instant t_0 (irrotational flow), then it is zero for all time $t \geq t_0$. Hence also in this case the extra term in the modified Stokes-Dirac structure D^m vanishes, and the port-controlled Hamiltonian system describing the Euler equations reduces to the standard distributed-parameter port-controlled Hamiltonian system given in [11, 12].

As announced before, the dynamics corresponding to the modified Stokes-Dirac structure (30) and the Hamiltonian (4) is now defined by setting

$$\begin{aligned} f_\rho &= -\frac{\partial \rho}{\partial t}, e_\rho = \delta_\rho H \\ f_v &= -\frac{\partial v}{\partial t}, e_v = \delta_v H \end{aligned} \quad (31)$$

leading immediately to the port-controlled Hamiltonian system whose dynamics is given by (28), (29), with boundary external variables

$$\begin{aligned} f_b &= \delta_\rho H|_{\partial W} (= [\frac{1}{2} \langle v^h, v^h \rangle + h(*\rho)]|_{\partial W}) \\ e_b &= -\delta_v H|_{\partial W} (= -i_v \cdot \rho|_{\partial W}) \end{aligned} \quad (32)$$

The resulting system can be regarded as a *boundary control system* in the sense of e.g. [7]. Indeed, we can either regard f_b as the *boundary control* variable (with e_b being the *boundary output*), or the other way around.

Energy exchange through the boundary is not the only way a distributed-parameter system may interact with its environment. Instead of boundary external variables we may also incorporate *distributed* external variables, leading to *distributed* control problems; see [11] for some developments. Also, energy *dissipation* can be incorporated in the framework by *terminating* some of the ports (boundary or distributed) by a resistive relation (given by a Rayleigh dissipation functional). In this way we can represent the Navier-Stokes equations.

2.4 Energy-balance

It immediately follows from the power-conservation property (9) of any Dirac structure that the modified Stokes-Dirac structure D^m defined in Proposition 2.3 has the property

$$\int_W (e_\rho \wedge f_\rho + e_v \wedge f_v) + \int_{\partial W} e_b \wedge f_b = 0 \quad (33)$$

Hence by substituting (31) we immediately obtain

$$\frac{d}{dt} H_W = \int_{\partial W} e_b \wedge f_b = - \int_{\partial W} \delta_v H \wedge \delta_\rho H \quad (34)$$

where $\delta_\rho H = \frac{1}{2} \langle v^b, v^b \rangle + h(*\rho)$ is a function, and $\delta_v H$ is the 2-form $i_{v^a} \rho$. This is exactly the coordinate-free version of (5). The 2-form $\delta_v H$ represents the mass-flow and $\delta_\rho H$ is the stagnation pressure divided by ρ . Note that alternatively we can write

$$\begin{aligned} \int_{\partial W} \delta_\rho H \wedge \delta_v H &= \int_{\partial W} i_{v^a} \left[\frac{1}{2} \langle v^b, v^b \rangle + \rho + U(*\rho) \rho \right] \\ &\quad + \int_{\partial W} i_{v^a} (*p) \end{aligned} \quad (35)$$

where $*p$ is the pressure 3-form $h(*\rho)\rho - U(*\rho)\rho$. This is the coordinate-free version of (6).

3 Conservation laws and passivity-based control of fluid dynamical systems

The Energy-Casimir method has proved to be a very valuable tool in the stability analysis of fluid dynamical systems (and Hamiltonian systems in general); see e.g. [6, 1, 13] for further information. The basic idea is to determine the conserved quantities or *Casimir functions* of the system, and to consider as candidate Lyapunov functions the Hamiltonian function (the energy) plus a suitable Casimir function. The idea of using the Energy-Casimir method for *stabilization* of finite-dimensional Hamiltonian control systems was explored in e.g. [2, 23, 4, 17, 18, 21, 19]. In particular, in [23, 4, 17, 18, 21, 19] it has been shown how by power-conserving *interconnection* with a Hamiltonian controller system Casimir functions for the closed-loop system can be *generated*. Underlying this construction is the fact (see [21]) that any power-conserving interconnection of Dirac structures defines another Dirac structure, and thus the closed-loop system is again Hamiltonian. Then the Energy-Casimir method can be applied to the closed-loop system (with Hamiltonian being the energy of the Hamiltonian plant system together with the energy of the Hamiltonian controller system). Furthermore, it has been shown ([17, 18, 19]) how this approach relates to the energy-shaping and interconnection-damping-assignment methods of passivity-based control, which have proved to be quite powerful for the control of (electro-)mechanical systems, see e.g. [16, 17, 18, 21, 19].

The extension of these ideas to fluid dynamical control systems can be approached as follows. From the Dirac structure given in Proposition 2.3 one infers conservation laws of the Hamiltonian boundary control system. A physically obvious conservation law corresponds to the total mass $\int_W \rho$. Indeed, one immediately verifies

$$\frac{d}{dt} \int_W \rho = - \int_W d(\delta_v H) = - \int_{\partial W} \delta_v H = \int_{\partial W} e_b \quad (36)$$

(which is nothing else than the mass-balance (1)). Then consider an additional controller system, also of port-controlled Hamiltonian form, but with internally *distributed*

control u_c and output y_c

$$\begin{aligned} \frac{\partial x_c}{\partial t} &= u_c \\ y_c &= \delta_{x_c} H_c \end{aligned} \quad (37)$$

with x_c a 2-form on ∂W , and $H_c = \int_{\partial W} \mathcal{H}_c(x_c)$ the controller Hamiltonian for a certain density 2-form $\mathcal{H}_c(x_c)$. Interconnect this controller to the fluid dynamic system via the power-conserving interconnection

$$\begin{aligned} u_c &= e_b \\ f_b &= -y_c \end{aligned} \quad (38)$$

(note that y_c is a function on ∂W). Then the closed-loop system is again a Hamiltonian system with total Hamiltonian $H_W + H_c$. Furthermore, because of (36), the function

$$\int_W \rho - \int_{\partial W} x_c \quad (39)$$

is a Casimir function (conserved quantity). Therefore, by the Energy-Casimir method, any other function

$$V := H_W + H_c + P \left(\int_W \rho - \int_{\partial W} x_c \right) \quad (40)$$

with $P: \mathbb{R} \rightarrow \mathbb{R}$ still to be assigned, can be used as an energy function for the closed-loop system, and therefore as a candidate Lyapunov function. Its potential for the control of fluid dynamical systems has to be investigated.

The next conservation law to be considered derives from the *helicity* of the fluid, defined as

$$\int_W v \wedge dv \quad (41)$$

This quantity measures the “knottedness” of the fluid, see e.g. [1]. Time-differentiation of (41) yields

$$\begin{aligned} \frac{d}{dt} \int_W v \wedge dv &= \int_W \left(\frac{\partial v}{\partial t} \wedge dv + v \wedge d \frac{\partial v}{\partial t} \right) = \\ &= - \int_W d(\delta_\rho H) \wedge dv = - \int_W d(\delta_\rho H \wedge dv) \\ &= - \int_{\partial W} \delta_\rho H \wedge dv = - \int_{\partial W} f_b \wedge dv \end{aligned} \quad (42)$$

showing the boundary variable f_b which can be interconnected to a controller Hamiltonian system as before, leading again to new candidate Lyapunov functions.

4 Conclusions

We have shown how 3-dimensional ideal isentropic fluids can be modelled as a Hamiltonian boundary control system, using the notion of a Stokes-Dirac structure. Among others, this opens up the way for the application of passivity-based control techniques, which have been proven to be very effective for the control of lumped parameter physical systems modelled as port-controlled Hamiltonian systems.

References

- [1] V. I. Arnold, B.A. Khesin, *Topological Methods in Hydrodynamics*, Springer Verlag, Applied Mathematical Sciences 125, New York, 1998.
- [2] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden, G. Sanchez de Alvarez, Stabilization of rigid body dynamics by internal and external torques, *Automatica*, 28, pp. 745–756, 1992.
- [3] T. Courant, Dirac manifolds, *Trans. American Math. Soc.*, 319, pp. 631–661, 1990.
- [4] M. Dalsmo & A.J. van der Schaft, "On representations and integrability of mathematical structures in energy-conserving physical systems", *SIAM J. Control and Optimization*, 37, pp. 54–91, 1999.
- [5] I. Dorfman, *Dirac Structures and Integrability of Nonlinear Evolution Equations*, John Wiley, Chichester, 1993.
- [6] D.D. Holm, J.E. Marsden, T.E. Ratiu, A. Weinstein, "Non-linear stability of fluid and plasma equilibria", *Phys.Rep.*, 123, pp. 1–116, 1985.
- [7] H.O. Fattorini, "Boundary control systems", *SIAM J. Control*, 6, pp. 349–385 1968.
- [8] J.E. Marsden, T. Ratiu, A. Weinstein, "Semidirect products and reduction in mechanics", *Trans. American Math. Society*, 281, pp. 147–177, 1984.
- [9] J.E. Marsden, T. Ratiu, A. Weinstein, "Reduction and Hamiltonian structures on duals of semidirect product Lie algebras", *AMS Contemporary Mathematics*, 28, pp. 55–100, 1984.
- [10] B.M. Maschke, A.J. van der Schaft & P.C. Breedveld, "An intrinsic Hamiltonian formulation of network dynamics: Non-standard Poisson structures and gyrators", *J. Franklin Inst.*, 329, pp. 923–966, 1992.
- [11] B.M. Maschke, A.J. van der Schaft, "Port controlled Hamiltonian representation of distributed parameter systems", Proc. IFAC Workshop on Lagrangian and Hamiltonian methods for nonlinear control, Princeton University, Editors N.E. Leonard, R. Ortega, pp.28–38, 2000.
- [12] B.M. Maschke, A.J. van der Schaft, "Hamiltonian representation of distributed parameter systems with boundary energy flow", *Nonlinear Control in the Year 2000*. Eds. A. Isidori, F. Lamnabhi-Lagarigue, W. Respondek. Springer-Verlag, pp. 137–142, 2000.
- [13] P.J. Morrison, "Hamiltonian description of the ideal fluid", *Rev. Mod. Phys.*, 70, pp. 467–521, 1998.
- [14] P.J. Morrison & J.M. Greene, "Noncanonical hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics", *Phys. Rev. Letters*, 45, pp. 790–794, 1980.
- [15] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, second edition, 1993.
- [16] R. Ortega, A. Loria, P.J. Nicklasson & H. Sira-Ramirez, *Passivity-based Control of Euler-Lagrange Systems*, Springer-Verlag, London, 1998.
- [17] R. Ortega, A.J. van der Schaft, B.M. Maschke & G. Escobar, "Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems", 1999, submitted for publication.
- [18] R. Ortega, A.J. van der Schaft, B.M. Maschke, G. Escobar, "Energy-shaping of port-controlled Hamiltonian systems by interconnection", Proc. 38th IEEE Conf. on Decision and Control, Phoenix, AZ, pp. 1646–1651, December 1999.
- [19] R. Ortega, A.J. van der Schaft, I. Mareels, & B.M. Maschke, "Putting energy back in control", *Control Systems Magazine*, 21, pp. 18–33, 2001.
- [20] A.J. van der Schaft, "Interconnection and geometry", in *The Mathematics of Systems and Control, From Intelligent Control to Behavioral Systems* (eds. J.W. Polderman, H.L. Trentelman), Groningen, 1999.
- [21] A.J. van der Schaft, *L₂-Gain and Passivity Techniques in Nonlinear Control*, 2nd revised and enlarged edition, Springer-Verlag, Springer Communications and Control Engineering series, p. xvi+249, London, 2000 (first edition Lect. Notes in Control and Inf. Sciences, vol. 218, Springer-Verlag, Berlin, 1996).
- [22] A.J. van der Schaft & B.M. Maschke, "The Hamiltonian formulation of energy conserving physical systems with external ports", *Archiv für Elektronik und Übertragungstechnik*, 49, pp. 362–371, 1995.
- [23] S. Stramigioli, B.M. Maschke & A.J. van der Schaft, "Passive output feedback and port interconnection", in Proc. 4th IFAC NOLCOS, Enschede, pp. 613–618, 1998.