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interarrival-time distributions

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A note on the $GI/GI/\infty$ system with identical service and interarrival-time distributions

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Abstract. We study the stationary distribution of the number of busy servers in a $GI/GI/\infty$ system in which the service-time distribution is identical to the interarrival-time distribution, and obtain several representations for the variance. As a result we can verify an expression for the variance, conjectured by Rajaratnam and Takawira (*IEEE Trans. Vehicular Technol.* **50** (2001) 817-834), when the common distribution of interarrival and service times is a gamma distribution.

Keywords and phrases: infinite-server system, busy-server distribution, gamma distribution, Mellin transform

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1 Introduction

We are interested in the steady-state distribution – and in particular the first two moments – of the number of busy servers in a $GI/GI/\infty$ system in which the service time distribution is identical to the interarrival-time distribution. This particular model comes about when an infinite pool of customers has to be served by a tandem service system consisting of a finite group of servers (of size N , say) followed by an infinite-server group. If service times at all servers are independent and identically distributed random variables, then the arrival process at the second group is simply a superposition of N independent renewal processes, each with an interarrival-time distribution equal to the service-time distribution. The number of busy servers in the infinite-server system is therefore the sum of N independent and identically distributed random variables, each representing the number of busy servers in the infinite-server group when $N = 1$. It is the stationary distribution of the latter random variable which is our concern in this note.

The model of a tandem system consisting of a finite-server group followed by an infinite-server group, but with a Poisson arrival process to the first group, has been proposed by Rajaratnam and Takawira in [4] and earlier papers (see the references in [4]) in the performance analysis of cellular mobile networks. In this setting it is of interest to know the stationary distribution – and in particular the mean and variance – of the number of busy servers in the infinite-server group, but this distribution seems difficult to obtain. Rajaratnam and Takawira [4] therefore resort to an approximate analysis which involves the model studied in this note.

We reveal in the next section a remarkably simple expression for $M(t)$, the *time-dependent* mean number of busy servers in the $GI/GI/\infty$ system, when interarrival and service times are identically distributed. This result enables us to find V , the variance of the *stationary* number of busy servers, by exploiting a classic result of Takács [5]. Some special cases are considered in Section 3. In particular, we will verify an expression for V which was conjectured by Rajaratnam and Takawira [4] in the case that interarrival and service times have identical gamma distributions.

We finally note that an interesting representation for V has been given by Yamazaki et al. [6], but it does not seem to lead to the explicit expression of Section 2.

2 The number of busy servers

We start off with some general notation and terminology. Let us consider a $GI/GI/\infty$ system in which the interarrival times have a common distribution function F , and the service times have a common distribution function H . Both

F and H are supposed to have a finite first moment, so that the arrival and service rates

$$\lambda \equiv \left(\int_0^\infty t dF(t) \right)^{-1} \quad \text{and} \quad \mu \equiv \left(\int_0^\infty t dH(t) \right)^{-1},$$

respectively, are positive. We assume that the system starts empty at time 0 and that the time until the first arrival has distribution function G . By m we denote the renewal function associated with F , that is,

$$m(t) \equiv \sum_{n=1}^{\infty} F^{n*}(t), \quad t \geq 0,$$

where F^{n*} stands for the n -fold convolution of F . By $X(t)$ and X we denote the number of busy servers at time t and in steady state, respectively, and we let $B_n(t)$ and B_n be their respective n th binomial moments, that is,

$$B_n(t) \equiv \sum_{k=n}^{\infty} \binom{k}{n} \Pr\{X(t) = k\} \quad \text{and} \quad B_n \equiv \sum_{k=n}^{\infty} \binom{k}{n} \Pr\{X = k\}, \quad n \geq 1.$$

The next theorem summarizes some classic results of Takács' [5] on these binomial moments. (For generalizations of Takács' results we refer to Liu et al. [3] and Ayhan et al. [2], and the references there.)

Theorem 1 (i) *The binomial moments $B_n(t)$ exist for all n and $t \geq 0$, and, if $G = F$, satisfy the recurrence relation*

$$B_n(t) = \int_0^t B_{n-1}(t-u)(1-H(t-u))dm(u), \quad t \geq 0, \quad n = 1, 2, \dots, \quad (1)$$

where $B_0(t) = 1$.

(ii) *If F is not a lattice distribution, then there exists a unique steady-state distribution with binomial moments B_n satisfying*

$$B_n = \lambda \int_0^\infty B_{n-1}(t)(1-H(t))dt, \quad n = 1, 2, \dots, \quad (2)$$

where $B_n(t)$, $n = 0, 1, \dots$, are the time-dependent binomial moments in the case $G = F$.

As announced, we wish to obtain the mean M and variance V of the stationary number of busy servers in the case that $F = H$. To this end we first observe a surprisingly simple corollary to Theorem 1 concerning $M(t)$, the mean number of busy servers at time t .

Corollary 2 *If $G = F = H$ then $M(t) = H(t)$ for all $t \geq 0$.*

Proof From Theorem 1 (i) we see that

$$M(t) = B_1(t) = \int_0^t (1 - H(t - u)) dm(u) = m(t) - H * m(t).$$

But since $m(t) \equiv \sum_{n=1}^{\infty} F^{n*}(t) = \sum_{n=1}^{\infty} H^{n*}(t)$, the result follows immediately. \square

Remark A more direct (and perhaps more appealing) argument leads to a generalization of Corollary 2 in which we do not require $G = F$. Indeed, let T_i be the arrival time of the i th customer, and S_i his service time. Then we can write

$$X(t) = \sum_{i=1}^{\infty} I_{[T_i, T_i + S_i)}(t), \quad t \geq 0,$$

where I_A denotes the indicator function of a set A . Taking expectations on both sides we get

$$M(t) = \sum_{i=1}^{\infty} \Pr\{T_i \leq t < T_i + S_i\} = \sum_{i=1}^{\infty} \Pr\{T_i \leq t < T_{i+1}\} = \Pr\{T_1 \leq t\},$$

since $T_{i+1} - T_i$ and S_i are independent and identically distributed, and also independent of T_i . So we actually have $M(t) = G(t)$ for all $t \geq 0$.

If $F = H$ then Theorem 1 (ii) (or Little's formula) tells us $M = B_1 = 1$, since $\lambda^{-1} = \mu^{-1} = \int_0^{\infty} (1 - H(t)) dt$. Moreover, Corollary 2 tells us that $B_1(t) = M(t) = H(t)$, which upon substitution in (2) gives us B_2 and hence $V = 2B_2 + M - M^2 = 2B_2$. Summarizing we have the following.

Theorem 3 *If $F = H$ and H is not a lattice distribution, then the mean M and variance V of the stationary number of busy servers in the system $GI/GI/\infty$ are given by*

$$M = 1 \quad \text{and} \quad V = 2\mu \int_0^{\infty} H(t)(1 - H(t)) dt. \quad (3)$$

We easily see that changing the unit of time does not affect the value of V (which is obvious also on physical grounds). We also note that

$$0 < V < 2\mu \int_0^{\infty} (1 - H(t)) dt = 2.$$

The special case of a gamma distribution (discussed in Section 3.2) may be used to show that both lower and upper bound can be approached arbitrarily close by choosing the parameter c in (8) sufficiently large and small, respectively.

It is interesting to observe that V can be represented as

$$V = \mu \int_0^{\infty} \int_0^{\infty} |t_1 - t_2| dH(t_1) dH(t_2), \quad (4)$$

so that V/μ may be interpreted as the expected absolute value of the difference of two service times.

If H has a continuous density h on $(0, \infty)$, with $h(t) = \mathcal{O}(t^{a-1})$ for some $a > 0$ and $t \downarrow 0$, and $h(t) = \mathcal{O}(t^{-b-2})$ for some $b > 0$ and $t \rightarrow \infty$, we may also express V in terms of the Mellin transform of h , given by

$$M(h, z) \equiv \int_0^\infty t^{z-1} h(t) dt, \quad 1 - a < \Re(z) < 2 + b.$$

To do so we use the general Parseval relation for Mellin transforms (and integration by parts), and end up with the contour-integral representation

$$V = \frac{\mu}{\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \frac{M(h, z + 1)M(h, 2 - z)}{z(z - 1)} dz, \quad -\min(a, b) < \kappa < 0. \quad (5)$$

3 Special cases

In this section we will look more closely at two special cases, namely H is a mixture of a degenerate and an exponential distribution, and H is a gamma distribution. Since V is independent of the unit of time it is no restriction of generality to assume $\mu = 1$ in the remainder of this section.

3.1 Mixtures of degenerate and exponential distributions

Let us assume that the interarrival and service times in the $GI/GI/\infty$ system are all drawn from a mixture of a degenerate and an exponential distribution with means 1, that is, for some p , $0 \leq p < 1$,

$$H(t) = F(t) = pI_{[1, \infty)}(t) + (1 - p)(1 - e^{-t}), \quad t \geq 0. \quad (6)$$

Substitution of this distribution function in (3) readily yields

$$V \equiv V(p) = (1 - p)(1 + (4/e - 1)p). \quad (7)$$

Rajaratnam and Takawira [4] observed that $V(0) = 1$ and $V(1) = 0$, and found via simulation that $V(1/2) \approx 5/8 = 0.6250$ (in reality, $V(1/2) = 1/4 + 1/e \approx 0.6179$). Hence, they proposed the quadratic interpolation formula

$$V(p) \approx (1 - p)(1 + p/2),$$

which is pretty close to (7) since $4/e - 1 \approx 0.4715$.

3.2 Gamma distributions

Now suppose that the interarrival and service times in the $GI/GI/\infty$ system are drawn from a common gamma distribution with mean 1, that is,

$$F(t) = H(t) = \frac{1}{\Gamma(c)} \int_0^t c(cu)^{c-1} e^{-cu} du, \quad t \geq 0, \quad (8)$$

where c is some positive constant and Γ is the gamma function

$$\Gamma(a) \equiv \int_0^\infty u^{a-1} e^{-u} du, \quad a > 0.$$

Analysing the $E_c/E_c/\infty$ queue by standard Markovian techniques, Rajaratnam and Takawira [4] found that for small integral values of c the variance V of the stationary number of busy servers is given by

$$V \equiv V(c) = 2^{1-2c} \frac{\Gamma(2c+1)}{\Gamma(c+1)^2}, \quad (9)$$

and they conjectured the validity of this expression for arbitrary $c > 0$. Before proving this in two different ways, we note that by the duplication formula for the gamma function (see [1, (6.1.18)]) $V(c)$ may also be written as

$$V(c) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(c + \frac{1}{2})}{\Gamma(c+1)}. \quad (10)$$

As a consequence, by [1, (6.1.47)],

$$V(c) \sim \frac{2}{\sqrt{\pi c}} \left\{ 1 - \frac{1}{8}c^{-1} + \dots \right\} \quad \text{as } c \rightarrow \infty.$$

Moreover $V(c)$ is a decreasing function of c by the log-convexity of the gamma function on $(0, \infty)$. In fact, letting $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ (the *digamma function*), we have

$$\frac{d}{dc} \log V(c) = \psi(c + \frac{1}{2}) - \psi(c+1) < 0,$$

since, by [1, (6.4.10)],

$$\psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2} > 0, \quad z \notin \{0, -1, -2, \dots\}.$$

Theorem 4 *The variance of the stationary number of busy servers in the system $GI/GI/\infty$ when interarrival and service times have a common distribution function (8) is given by (9), or (10), for all $c > 0$.*

Proof Elementary substitution of (8) in (3), followed by appropriate changes of variables and a change in the order of integration, leads to

$$\begin{aligned}
V(c) &= \frac{2}{(\Gamma(c))^2} \int_0^\infty \left(\int_0^t c(cu)^{c-1} e^{-cu} du \right) \left(\int_t^\infty c(cv)^{c-1} e^{-cv} dv \right) dt \\
&= \frac{2}{(\Gamma(c))^2} \int_0^\infty \left(\int_0^1 (ct)^c x^{c-1} e^{-cxt} dx \right) \left(\int_1^\infty (ct)^c y^{c-1} e^{-c yt} dy \right) dt \\
&= \frac{2}{(\Gamma(c))^2} \int_0^1 \int_1^\infty (xy)^{c-1} \int_0^\infty (ct)^{2c} e^{-c(x+y)t} dt dy dx \\
&= \frac{2}{c(\Gamma(c))^2} \int_0^1 \int_1^\infty \frac{(xy)^{c-1}}{(x+y)^{2c+1}} \int_0^\infty u^{2c} e^{-u} du dy dx \\
&= \frac{2\Gamma(2c+1)}{(\Gamma(c+1))^2} \int_0^1 \int_1^\infty \frac{c(xy)^{c-1}}{(x+y)^{2c+1}} dy dx,
\end{aligned}$$

where we have used $\Gamma(c+1) = c\Gamma(c)$. Finally, substitution of $y = u/v$ and $x = 1/v$ and another change in the order of integration gives us

$$\begin{aligned}
\int_0^1 \int_1^\infty \frac{c(xy)^{c-1}}{(x+y)^{2c+1}} dy dx &= \int_1^\infty \int_v^\infty \frac{c u^{c-1}}{(1+u)^{2c+1}} du dv \\
&= \int_1^\infty \int_1^u \frac{c u^{c-1}}{(1+u)^{2c+1}} dv du = \int_1^\infty \frac{c u^{c-1}(u-1)}{(1+u)^{2c+1}} du \\
&= \int_1^\infty d \left(\frac{-u^c}{(1+u)^{2c}} \right) = \frac{1}{2^{2c}},
\end{aligned} \tag{11}$$

as required. \square

Remark The integral in (11) is closely related to a particular value of a hypergeometric function. In fact, by [1, (15.3.1) and (15.1.21)] we have

$$\begin{aligned}
\int_1^\infty \frac{c u^{c-1}(u-1)}{(1+u)^{2c+1}} du &= \int_0^1 \frac{ct^{c-1}(t-1)}{(1+t)^{2c+1}} dt \\
&= \frac{\Gamma(c+1)\Gamma(2)}{\Gamma(c+2)} F(2c+1, c; c+2; -1) = 2^{-2c},
\end{aligned}$$

as before.

Second proof of Theorem 4 We apply (5) with $h(t) = c^t t^{c-1} e^{-ct} / \Gamma(c)$ and $b = \infty$ to obtain

$$M(h, z) = \frac{c^z}{\Gamma(c)} \int_0^\infty t^{z+c-2} e^{-ct} dt = c^{1-z} \frac{\Gamma(z+c-1)}{\Gamma(c)}, \quad \Re(z) > 1-c,$$

and

$$V(c) = \frac{c^{-1}}{\Gamma(c)^2 \pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\Gamma(z+c)\Gamma(1-z+c)}{z(z-1)} dz, \quad -c < \kappa < 0.$$

By replacing $z(z-1)$ with $\Gamma(2-z)/\Gamma(-z)$, followed by a substitution of $z = -c-s$, we get (up to a factor) a Mellin-Barnes integral for the hypergeometric function value mentioned in the above Remark. Actually, by [1, (15.3.2) and (15.1.21)] it follows that

$$\begin{aligned} V(c) &= \frac{c^{-1}}{\Gamma(c)^2 \pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\Gamma(s+2c+1)\Gamma(s+c)\Gamma(-s)}{\Gamma(s+c+2)} ds \\ &= 2 \frac{\Gamma(2c+1)}{\Gamma(c+2)\Gamma(c+1)} F(2c+1, c; c+2; -1) = 2^{1-2c} \frac{\Gamma(2c+1)}{\Gamma(c+1)^2}, \end{aligned}$$

as before. \square

We finally note that for integral values of c a third proof of (9) may be based on the interpretation (4) of $V(c)$ as the expected absolute value of the difference of two service times. Namely, imagine two service times S_1 and S_2 , each consisting of c exponentially distributed phases of mean $1/c$, starting at time 0, and a counter going up (or down) one unit each time a phase of S_1 (or S_2) elapses, until one of the service times ends. If we denote the state of the counter after the n th count by X_n , and let

$$N \equiv \min\{n : |X_{c+n}| = c - n\},$$

then, for $n = 1, 2, \dots, N$, X_n is distributed as $\sum_{k=1}^n Y_k$, where Y_1, Y_2, \dots are independent random variables taking the values $+1$ and -1 with equal probabilities. Moreover, at the time of the N th count either S_1 or S_2 has ended, and the remaining part of the surviving service time consists of $c - N$ exponentially distributed phases with means $1/c$. Consequently,

$$V(c) = \mathbb{E}|S_1 - S_2| = (c - \mathbb{E}N)/c. \quad (12)$$

A combinatorial argument shows that N has a truncated negative binomial distribution

$$\Pr\{N = n\} = 2 \binom{c+n-1}{n} \left(\frac{1}{2}\right)^{c+n}, \quad n = 0, 1, \dots, c-1,$$

with first moment

$$\mathbb{E}N = c - c \binom{2c}{c} \left(\frac{1}{2}\right)^{2c-1},$$

which, together with (12), gives us (9) again.

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References

- [1] M. Abramowitz and I.A. Stegun (eds), *Handbook of Mathematical Functions*. Dover, New York, 1972.
- [2] H. Ayhan, J. Limon-Robles and M.A. Wortman, On the time-dependent occupancy and backlog distributions for the $GI/G/\infty$ queue. *J. Appl. Probab.* **36** (1999) 558-569.
- [3] L. Liu, B.R.K. Kashyap and J.G.C. Templeton, On the $GI^X/G/\infty$ system. *J. Appl. Probab.* **27** (1990) 671-683.
- [4] M. Rajaratnam and F. Takawira, Handoff traffic characterization in cellular networks under nonclassical arrivals and service time distributions. *IEEE Trans. Vehicular Technol.* **50** (2001) 954-970.
- [5] L. Takács, On a coincidence problem in telephone traffic. *Acta Math. Acad. Sci. Hung.* **9** (1958) 45-80.
- [6] G. Yamazaki, K. Sigman and M. Miyazawa, Moments in infinite channel queues. *Comput. Math. Appl.* **24** (1992) 1-6.