

# BLOCK BOOTSTRAPPING THE EMPIRICAL DISTANCE COVARIANCE

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ABSTRACT. We prove the validity of a non-overlapping block bootstrap for the empirical distance covariance under the assumption of strictly stationary and absolutely regular sample data. From this, we develop a test for independence of two strictly stationary and absolutely regular processes. In proving our results, we derive explicit bounds on the expected Wasserstein distance between an empirical measure and its limit for strictly stationary and strongly mixing sample sequences.

## 1. INTRODUCTION

Consider  $Z = (Z_1, \dots, Z_n)$  as the initial segment of a strictly stationary stochastic process  $(Z_k)_{k \in \mathbb{N}}$ , where  $Z_k = (X_k, Y_k)$  for all  $k \in \mathbb{N}$ . For a generic random vector  $(X, Y)$  with the same joint distribution as any of the random vectors  $(X_k, Y_k)$ , our goal is to test the hypothesis that  $(X_k)_{k \in \mathbb{N}}$  and  $(Y_k)_{k \in \mathbb{N}}$  are independent against the alternative that  $(X_k)_{k \in \mathbb{N}}$  and  $(Y_k)_{k \in \mathbb{N}}$  are dependent. Note that independence of the processes implies independence of the coordinates, i.e., under the hypothesis  $X$  and  $Y$  are independent.

**1.1. Distance Covariance.** In a series of papers, Székely et al. (2007) and Székely and Rizzo (2009, 2012, 2013, 2014) introduced distance covariance and distance correlation as measures of the degree of dependence between two random vectors  $X$  and  $Y$  with values in  $\mathbb{R}^{\ell_1}$  and  $\mathbb{R}^{\ell_2}$ , respectively. They define the distance covariance of  $X$  and  $Y$  as

$$(1) \quad \mathcal{V}^2(X, Y; w) := \int_{\mathbb{R}^{\ell_1}} \int_{\mathbb{R}^{\ell_2}} |\varphi_{X,Y}(s, t) - \varphi_X(s)\varphi_Y(t)|^2 w(s, t) \, ds dt,$$

where  $w$  is a positive weight function. For a variety of reasons, usually the weight function

$$w(s, t) = \frac{c_{\ell_1, \ell_2}}{\|s\|_2^{1+\ell_1} \|t\|_2^{1+\ell_2}}$$

with some constant  $c_{\ell_1, \ell_2} > 0$  is considered.

Lyons (2013) extended the theory of distance covariance from Euclidean spaces to general separable, metric spaces  $\mathcal{X}$  and  $\mathcal{Y}$ . Given that  $\theta$  denotes the simultaneous distribution of  $(X, Y) \in \mathcal{X} \times \mathcal{Y}$  with marginal distributions  $\mu$  on  $\mathcal{X}$  and  $\nu$  on  $\mathcal{Y}$ , Lyons (2013) defined the distance covariance by

$$\text{dcov}(\theta) := \int \delta_\theta((x, y), (x', y')) \, d\theta^2((x, y), (x', y')),$$

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where

$$\begin{aligned}\delta_\theta((x, y), (x', y')) &:= d_\mu(x, x')d_\nu(y, y'), \\ d_\mu(x, x') &:= d(x, x') - a_\mu(x) - a_\mu(x') + D(\mu), \\ d_\nu(y, y') &:= d(y, y') - a_\nu(y) - a_\nu(y') + D(\nu),\end{aligned}$$

and

$$\begin{aligned}a_\mu(x) &:= \int d(x, x') \, d\mu(x'), & a_\nu(y) &:= \int d(y, y') \, d\nu(y'), \\ D(\mu) &:= \int d(x, x') \, d\mu^2(x, x'), & D(\nu) &:= \int d(y, y') \, d\nu^2(y, y'),\end{aligned}$$

with  $d$  denoting either the metric on  $\mathcal{X}$  or on  $\mathcal{Y}$ . (For notational convenience, we suppress the indices indicating the affiliation to  $\mathcal{X}$  or  $\mathcal{Y}$ .) Metric spaces that satisfy  $\text{dcov}(\theta) = 0$  if, and only if,  $\theta = \mu \otimes \nu$  are characterised in Lyons (2013) as those of strong negative type. These spaces allow for the existence of a mapping  $\phi : \mathcal{X} \rightarrow H$ , where  $H$  is a Hilbert space, such that

$$d(x, x') = \|\phi(x) - \phi(x')\|_H^2 \text{ for all } x, x' \in \mathcal{X}.$$

It additionally holds that  $D(\mu_1 - \mu_2) = 0$  if, and only if,  $\mu_1 = \mu_2$  for all probability measures  $\mu_1, \mu_2$  with finite first moments.

The definition of distance covariance of two random vectors  $X$  and  $Y$  with values in the Euclidean spaces  $\mathbb{R}^{\ell_1}$  and  $\mathbb{R}^{\ell_2}$  as introduced in Székely et al. (2007) and defined by (1) follows from this definition as a special case by choosing the embedding

$$\begin{aligned}\phi : \mathbb{R}^d &\rightarrow L_2(w_d) := \left\{ f : \mathbb{R}^d \rightarrow \mathbb{C} \mid \int |f|^2 w_d \lambda^d < \infty \right\}, \\ x &\mapsto \frac{1}{\sqrt{2}} (1 - \exp(i\langle \cdot, x \rangle))\end{aligned}$$

with  $w_d(s) = \Gamma\left(\frac{d+1}{2}\right) \pi^{-\frac{d+1}{2}} \|s\|_2^{-(d+1)}$ . This embedding is referred to as the *Fourier embedding* in Lyons (2013). One can show that  $\text{dcov}(\theta) = \mathcal{V}^2(X, Y; w)$ ; for more details, cf. Lyons (2013).

With respect to the goal of testing the hypothesis that  $X$  and  $Y$  are independent against the alternative that  $X$  and  $Y$  are dependent, Székely et al. (2007) propose to use an empirical version of  $\mathcal{V}^2(X, Y, w)$  as test statistic. For this, they define the empirical distance covariance as

$$\mathcal{V}_n^2(X, Y; w) := \int_{\mathbb{R}^{\ell_2}} \int_{\mathbb{R}^{\ell_1}} \left| \varphi_{X,Y}^{(n)}(s, t) - \varphi_X^{(n)}(s) \varphi_Y^{(n)}(t) \right|^2 w(s, t) \, ds dt,$$

where

$$\varphi_{X,Y}^{(n)}(s, t) := \frac{1}{n} \sum_{j=1}^n e^{i\langle s, X_j \rangle + i\langle t, Y_j \rangle}$$

denotes the joint empirical characteristic function of  $X, Y$ , and

$$\varphi_X^{(n)}(s) := \frac{1}{n} \sum_{j=1}^n e^{i\langle s, X_j \rangle}, \quad \varphi_Y^{(n)}(t) := \frac{1}{n} \sum_{j=1}^n e^{i\langle t, Y_j \rangle}$$

correspond to the empirical characteristic functions of  $X$  and  $Y$ .

Based on the Fourier embedding given in Lyons (2013), the empirical distance covariance  $\mathcal{V}_n^2(X, Y; w)$  corresponds to  $\text{dcov}(\theta_n)$ , where  $\theta_n$  denotes the empirical measure of the observations  $Z_1, \dots, Z_n$  and can be expressed as a  $V$ -statistic with kernel

$$h'(z_1, \dots, z_6) := f(x_1, x_2, x_3, x_4) f(y_1, y_2, y_5, y_6),$$

where

$$f(x_1, \dots, x_4) := \|x_1 - x_2\|_2 - \|x_1 - x_3\|_2 - \|x_2 - x_4\|_2 + \|x_3 - x_4\|_2.$$

More precisely, this means that

$$\begin{aligned} \text{dcov}(\theta_n) &= V_h(Z_1, \dots, Z_n) = \frac{1}{n^6} \sum_{i_1=1}^n \cdots \sum_{i_6=1}^n h'(Z_{i_1}, Z_{i_2}, \dots, Z_{i_6}) \\ (2) \qquad \qquad &= \frac{1}{n^6} \sum_{i_1=1}^n \cdots \sum_{i_6=1}^n h(Z_{i_1}, Z_{i_2}, \dots, Z_{i_6}), \end{aligned}$$

where  $h$  denotes the symmetrization of  $h'$ .

**1.2. Bootstrap.** In order to base a test decision for the testing problem

$$H_0 : (X_k)_{k \in \mathbb{N}} \text{ and } (Y_k)_{k \in \mathbb{N}} \text{ are independent}$$

on a computation of the empirical distance covariance  $\text{dcov}(\theta_n)$ , we have to set critical values, i.e., we have to determine quantiles of the distribution of  $\text{dcov}(\theta_n)$  under the hypothesis  $H_0$ . In general, these quantiles can be approximated by quantiles of the distance covariance's asymptotic distribution. Given some technical conditions, Theorem 2 in Kroll (2021) states that, if  $\text{dcov}(\theta) = 0$ , i.e., under the hypothesis  $H_0$ ,

$$n \text{dcov}(\theta_n) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \sum_{k=1}^{\infty} \lambda_k \zeta_k^2,$$

where  $\lambda_k, k \in \mathbb{N}$ , are unknown parameters, the eigenvalues of certain integral operators, and  $\zeta_k, k \in \mathbb{N}$ , are Gaussian random variables, whose covariances are determined by the dependence structure of the observations  $Z_1, \dots, Z_n$  and the  $\lambda_k$ -matching eigenfunctions  $\varphi_k, k \in \mathbb{N}$ . Unfortunately, the parameters  $\lambda_k, k \in \mathbb{N}$ , as well as the auto-covariance function of  $(\zeta_k)_{k \in \mathbb{N}}$ , are unknown. As a result, the distribution of the limit is not available.

In this section, we therefore advocate the use of a bootstrap-procedure for approximating the distribution of  $\text{dcov}(\theta_n)$ . The bootstrap procedure has to be designed in such a way that it mimics the behaviour of the test statistic under the hypothesis  $H_0$ , i.e., for a time series  $(Z_k)_{k \in \mathbb{N}} = (X_k, Y_k)_{k \in \mathbb{N}}$ , with  $(X_k)_{k \in \mathbb{N}}$  and  $(Y_k)_{k \in \mathbb{N}}$  independent, given both: data generated according to the model assumptions under the hypothesis as well as data generated according to the model assumptions under the alternative  $H_1$ . To not destroy the dependence structure of the individual time series  $(X_k)_{k \in \mathbb{N}}$ , and  $(Y_k)_{k \in \mathbb{N}}$ , it seems reasonable to consider blocks of observations.

Therefore, we define non-overlapping blocks of growing length  $d = d(n)$ ,

$$(3) \qquad B_{X,k} := (X_{(k-1)d+1}, \dots, X_{kd}), \quad B_{Y,k} := (Y_{(k-1)d+1}, \dots, Y_{kd}),$$

for  $k = 1, \dots, N := \frac{n}{d}$ . To mimic independence of  $(X_k)_{k \in \mathbb{N}}$  and  $(Y_k)_{k \in \mathbb{N}}$ , it seems reasonable to sample blocks of the two time series independently. For this purpose, we draw  $N$  times with replacement from

$$\mathcal{B}_X := \{B_{X,1}, \dots, B_{X,N}\}$$

and, independently, from

$$\mathcal{B}_Y := \{B_{Y,1}, \dots, B_{Y,N}\}$$

for every bootstrap repetition. We then denote the corresponding bootstrap blocks by  $B_{X,1}^*, B_{X,2}^*, \dots, B_{X,N}^*$  and  $B_{Y,1}^*, B_{Y,2}^*, \dots, B_{Y,N}^*$ , and we define our bootstrap observations by

$$(4) \quad B_k^* := (Z_{(k-1)d+1}^*, \dots, Z_{kd}^*) := \begin{pmatrix} B_{X,k}^* \\ B_{Y,k}^* \end{pmatrix}, k = 1, \dots, N.$$

We then use identity (2) on these bootstrap observations  $Z_1^*, \dots, Z_n^*$ , i.e., we define the bootstrapped empirical distance covariance as

$$V^* := \text{dcov}(\theta_n^*),$$

where  $\theta_n^*$  is the empirical measure of our bootstrap samples  $Z_1^*, \dots, Z_n^*$ .

**1.3. Weak Dependence Conditions and Wasserstein Distance.** Let us briefly recall from Bradley (2007) three basic mixing conditions, since these notions will be central to the rest of this work. For any two  $\sigma$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ , let

$$\begin{aligned} \alpha(\mathcal{A}, \mathcal{B}) &:= \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|, \\ \beta(\mathcal{A}, \mathcal{B}) &:= \sup \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^J |\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)|, \\ \phi(\mathcal{A}, \mathcal{B}) &:= \sup_{A \in \mathcal{A}, B \in \mathcal{B}, \mathbb{P}(A) > 0} |\mathbb{P}(B|A) - \mathbb{P}(B)|, \end{aligned}$$

where in the second line the supremum is taken over all finite partitions  $A_1, \dots, A_I \in \mathcal{A}$  and  $B_1, \dots, B_J \in \mathcal{B}$ . Now, a strictly stationary process  $(U_k)_{k \in \mathbb{N}}$  is called strongly mixing (or  $\alpha$ -mixing) if

$$\alpha(n) := \sup_{l \in \mathbb{N}} \alpha(\sigma(U_1, \dots, U_l), \sigma(U_{l+n}, U_{l+n+1}, \dots)) \xrightarrow{n \rightarrow \infty} 0.$$

It is called absolutely regular (or  $\beta$ -mixing) if

$$\beta(n) := \sup_{l \in \mathbb{N}} \beta(\sigma(U_1, \dots, U_l), \sigma(U_{l+n}, U_{l+n+1}, \dots)) \xrightarrow{n \rightarrow \infty} 0.$$

It is called  $\phi$ -mixing if

$$\phi(n) := \sup_{l \in \mathbb{N}} \phi(\sigma(U_1, \dots, U_l), \sigma(U_{l+n}, U_{l+n+1}, \dots)) \xrightarrow{n \rightarrow \infty} 0.$$

We will mostly concern ourselves with absolutely regular processes. One can show that every  $\phi$ -mixing process is absolutely regular, and every absolutely regular process is strongly mixing (Proposition 3.11, Bradley (2007)).

One example of absolutely regular processes are ARMA time series. More precisely, let  $p, q \in \mathbb{N}$  and let  $(U_k)_{k \in \mathbb{N}}$  be a strictly stationary solution of the ARMA( $p, q$ )-model, i.e.,

$$U_k = \sum_{i=1}^p \varphi_i U_{k-i} + \sum_{i=1}^q \psi_i \varepsilon_{k-i} + \varepsilon_k,$$

where  $(\varepsilon_k)_{k \in \mathbb{N}}$  is a centred iid process, and we assume  $\varphi : u \mapsto 1 - \sum_{i=1}^p \varphi_i u^i$  and  $\psi : u \mapsto 1 + \sum_{i=1}^q \psi_i u^i$  to have no common roots and  $\varphi$  to have no roots on the complex unit sphere. Suppose further that no root of  $\varphi$  lies within the closed unit disk and that  $\varepsilon_1$  has a Lebesgue density. Then  $(U_k)_{k \in \mathbb{N}}$  is absolutely regular with  $\beta(n) = \mathcal{O}(\rho^n)$  for some  $0 < \rho < 1$  (Theorem 1, Mokkadem (1988)).

Finally, recall that for some fixed value  $p \geq 1$ , the Wasserstein distance  $d_p(\eta, \xi)$  between any two distributions  $\eta$  and  $\xi$  is given by

$$d_p^p(\eta, \xi) := \inf \left\{ \int \|s - s'\|_2^p d\gamma(s, s') \right\},$$

where the infimum is taken over all measures  $\gamma$  with marginal distributions  $\eta$  and  $\xi$ , provided this quantity exists. It is a well-known fact that convergence in  $d_p$  is equivalent to weak convergence and convergence of the  $p$ -th means; cf. Villani (2009), Theorem 6.9.

For any two random variables  $U, V$ , by abuse of notation we sometimes write  $d_p(U, V)$  for  $d_p(\mathcal{L}(U), \mathcal{L}(V))$ , where  $\mathcal{L}(U)$  denotes the distribution of  $U$ .

**1.4. Summary of Main Results.** In Theorem 1 we show that, under certain technical assumptions, most importantly absolute regularity,  $nV^*$  has the same limiting distribution in probability as  $\text{ndcov}(\theta_n)$  has under the hypothesis  $H_0$ . From this it follows directly that determining the upper  $\alpha$ -quantile of  $nV^*$  and rejecting  $H_0$  if  $\text{ndcov}(\theta_n)$  exceeds this quantile yields a test with asymptotic level  $\alpha$ . Furthermore, because  $\theta \neq \mu \otimes \nu$  implies that  $\text{ndcov}(\theta_n) \xrightarrow[n \rightarrow \infty]{a.s.} \infty$  (Theorem 1 in Kroll (2021)), this test is consistent against every alternative in which  $X$  and  $Y$  are not independent. Thus, the only alternatives against which our test is not consistent are those in which the marginals  $X_k$  and  $Y_k$  are independent for every  $k \in \mathbb{N}$  but the entire processes  $(X_k)_{k \in \mathbb{N}}$  and  $(Y_k)_{k \in \mathbb{N}}$  are not independent. However, by bootstrapping vectors of observations, our test can be adapted to be consistent even against these pathological cases. Corollary 1 and the remark thereafter formalise this notion.

Theorem 2 extends existing results concerning the limiting distribution of second-order degenerate  $V$ -statistics to a certain kind of triangular array, namely those which arise from sectioning a stationary and absolutely regular sample sequence into independent blocks of length  $d = d(n)$ . We show that this kind of partitioning does not change the limiting distribution of the  $V$ -statistic. We show this in terms of convergence in the Wasserstein distance  $d_1$ , which is a stronger result than weak convergence.

Theorem 3 is a similar result for the empirical distance covariance (which is a  $V$ -statistic of order 6). Again, we show convergence in  $d_1$ .

In Theorem 4 we give an explicit bound on  $\mathbb{E}d_p^p(\xi_n, \xi)$ , where  $\xi$  is some measure on  $\mathbb{R}^d$  and  $\xi_n$  is the empirical measure of a stationary and strongly mixing process with marginal distribution  $\xi$ . It is known that for growing dimension  $d = d(n)$ , this expected value does not necessarily converge to 0, since the number of observations required for an approximation of fixed precision grows with  $d$  – this phenomenon is sometimes known as the curse of dimensionality. Since the upper bound in Theorem 4 is explicit, it allows us to find a rate of growth for  $d = d(n)$  that still results in  $\mathbb{E}d_p^p(\xi_n, \xi)$  converging to 0. Previous results of this type, such as those in Dereich et al. (2013), required iid data as opposed to our weaker assumption of stationary and strongly mixing sample data.

Finally, Corollary 2 is a handy consequence of Theorem 4 for the special case where  $\xi$  is the distribution of the first  $d'$  observations of a strictly stationary process.

## 2. MAIN RESULTS

The central result of this article is the validity of our proposed bootstrap method, stated in Theorem 1. More precisely, we prove convergence of the bootstrap in the Wasserstein distance. However, the results of this Section can also serve as a blueprint for proving bootstrap procedures for different statistics.

For stating the main results of this work, it is useful to introduce some shorthand notation. For any measure  $\xi$  on  $\mathbb{R}^\ell$ , we define the following assumption.

**Assumption 1.** *There exists a constant  $M > 0$  such that  $\xi(F) \leq M \cdot \text{vol}(F)$  for all hypercubes  $F$ , where  $\text{vol}$  denotes the volume with respect to the Lebesgue measure.*

We note that Assumption 1 is fulfilled if  $\xi$  has a bounded Lebesgue density.

For a sequence of measures  $(\xi_d)_{d \in \mathbb{N}}$  on  $\mathbb{R}^{\ell_d}$ , we define

**Assumption 2.** *Every  $\xi_d$  fulfills Assumption 1, and the corresponding sequence of bounding constants  $M(d)$  grows at most exponentially in  $d$ .*

We can now state our main result.

**Theorem 1.** *Suppose that  $X_1$  and  $Y_1$  both have finite  $(4 + \delta)$ -th moments for some  $\delta > 0$ , and that the process  $(Z_k)_{k \in \mathbb{N}}$  is strictly stationary and absolutely regular with  $\beta(n) = \mathcal{O}(n^{-r})$  for some  $r > 18$ . Furthermore, suppose that the joint distributions of the vectors  $(Z_1, \dots, Z_d)$  fulfill Assumption 2, and that  $d = \log(n)^\gamma$  for some  $0 < \gamma < 1/2$ .*

*Then it holds that*

$$d_1(\zeta, nV^*) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0,$$

where  $\zeta$  is the weak limit of  $n \cdot \text{dcov}(\theta_n)$  under the hypothesis

$$H_0 : (X_k)_{k \in \mathbb{N}} \text{ and } (Y_k)_{k \in \mathbb{N}} \text{ are independent.}$$

*Remark 1.* Theorem 1 also holds if the joint distributions of the vectors  $(Z_1, \dots, Z_d)$  do not fulfill Assumption 2, if we instead assume the sample generating process  $(Z_k)_{k \in \mathbb{N}}$  to be  $\phi$ -mixing.

*Remark 2.* One can also use this bootstrap procedure to approximate the limiting distribution of different  $V$ -statistics of absolutely regular sample data. E.g., the empirical Pearson covariance of  $X$  and  $Y$  can be expressed as a  $V$ -statistic with kernel function

$$(z, z') \mapsto (x - x')(y - y').$$

This kernel function is not degenerate, and so one has to use a different normalising factor (namely,  $\sqrt{n}$ ); thus adapted, the bootstrap procedure still yields the appropriate limiting distribution (in this case, a Gaussian distribution).

**Corollary 1.** *Consider a test which rejects  $H_0$  if  $\text{ndcov}(\theta_n) > c_\alpha^*$ , where  $c_\alpha^*$  is the upper  $\alpha$ -quantile of  $nV^*$ . Then this test has asymptotic level  $\alpha$ , and the only alternatives against which it is not consistent are those for which  $X_k$  and  $Y_k$  are independent for every  $k \in \mathbb{N}$ , but the entire processes  $(X_k)_{k \in \mathbb{N}}$  and  $(Y_k)_{k \in \mathbb{N}}$  are not independent.*

*Proof.* The fact that our test has asymptotic level  $\alpha$  follows directly from Theorem 1. Now, if  $X$  and  $Y$  are not independent, it follows that

$$\text{dcov}(\theta_n) \xrightarrow[n \rightarrow \infty]{a.s.} \text{dcov}(\theta) > 0,$$

by Theorem 1 in Kroll (2021), and thus  $\text{ndcov}(\theta_n) > c_\alpha^*$  for large  $n$  almost surely.  $\square$

*Remark 3.* The trouble with the pathological alternatives for which  $X_1$  and  $Y_1$  are independent but the entire processes  $(X_k)_{k \in \mathbb{N}}$  and  $(Y_k)_{k \in \mathbb{N}}$  are not lies in the fact that here,  $\text{ndcov}(\theta_n)$  does converge to a non-degenerate limiting distribution  $\zeta'$ . However, because  $\zeta'$  depends on the covariance structure of the process  $(Z_k)_{k \in \mathbb{N}}$ , it will not be identical to the limiting distribution  $\zeta$  of our bootstrapped  $V$ -statistic  $nV^*$ . Thus, the probability of rejecting  $H_0$  will converge to  $\mathbb{P}(\zeta' > c_\alpha^*) < 1$ .

We can, however, address issue by ‘vectorising’ the original sequence  $(Z_k)_{k \in \mathbb{N}}$ . Assume that  $X_1$  and  $Y_1$  are independent but  $(X_k)_{k \in \mathbb{N}}$  and  $(Y_k)_{k \in \mathbb{N}}$  are not. Then there has to be

some  $J \in \mathbb{N}$  such that  $(X_1, \dots, X_J)$  and  $(Y_1, \dots, Y_J)$  are not independent. Therefore, if we bootstrap the vectorised sequence  $\left(Z_k^{(v)}\right)_{k \in \mathbb{N}}$  given by

$$Z_k^{(v)} := (Z_{(k-1)J+1}, \dots, Z_{kJ}),$$

we will almost surely reject  $H_0$  for large  $n$ .

For a given strictly stationary sequence  $(U_k)_{k \in \mathbb{N}}$ , we construct a triangular array  $\tilde{U}$  by taking the  $n$ -th row as  $N = N(n)$  iid copies of  $(U_1, \dots, U_d)$ ,  $d = d(n)$  such that  $n/(Nd) \rightarrow 1$  as  $n \rightarrow \infty$ . If we then are interested in the limiting distributions of  $V$ -statistics with sample data  $U_1, \dots, U_n$ , compared to those with sample data  $\tilde{U}_{1,n}, \dots, \tilde{U}_{Nd,n}$ , we can use the following theorem, the first part of which is a stronger version of Theorem 2 in Kroll (2021).

**Theorem 2.** (i) Let  $\mathcal{U}$  be a  $\sigma$ -compact metrisable topological space,  $(U_k)_{k \in \mathbb{N}}$  a strictly stationary sequence of  $\mathcal{U}$ -valued random variables with marginal distribution  $\mathcal{L}(U_1) = \xi$ . Consider a continuous, symmetric, degenerate and positive semidefinite kernel  $g : \mathcal{U}^2 \rightarrow \mathbb{R}$  with finite  $(2 + \varepsilon)$ -moments with respect to  $\xi^2$  and finite  $(1 + \frac{\varepsilon}{2})$ -moments on the diagonal, i.e.  $\mathbb{E}|g(U_1, U_1)|^{1+\varepsilon/2} < \infty$ . Furthermore, let the sequence  $(U_k)_{k \in \mathbb{N}}$  satisfy an  $\alpha$ -mixing condition such that  $\alpha(n) = O(n^{-r})$  for some  $r > 1 + 2\varepsilon^{-1}$ . Then, with  $V = V_g(U_1, \dots, U_n)$ ,

$$d_1 \left( nV, \sum_{k=1}^{\infty} \lambda_k \zeta_k^2 \right) \xrightarrow[n \rightarrow \infty]{} 0,$$

where  $(\lambda_k, \varphi_k)$  are pairs of the non-negative eigenvalues and matching eigenfunctions of the integral operator

$$f \mapsto \int g(\cdot, z) f(z) \, d\xi(z),$$

and  $(\zeta_k)_{k \in \mathbb{N}}$  is a sequence of centred Gaussian random variables whose covariance structure is given by

$$(5) \quad \text{Cov}(\zeta_i, \zeta_j) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{s,t=1}^n \text{Cov}(\varphi_i(U_s), \varphi_j(U_t)).$$

(ii) Suppose the conditions of (i) are satisfied. Furthermore, assume  $g$  to have finite  $(3 + \varepsilon)$ -moments and  $\alpha(n) = O(n^{-r})$  with  $r > 2 + 6\varepsilon^{-1}$ . Then, with  $\tilde{V} = V_g(\tilde{U}_{1,n}, \dots, \tilde{U}_{Nd,n})$ , if  $d \rightarrow \infty$  for  $n \rightarrow \infty$ ,

$$d_1 \left( n\tilde{V}, \sum_{k=1}^{\infty} \lambda_k \zeta_k^2 \right) \xrightarrow[n \rightarrow \infty]{} 0,$$

where  $\lambda_k$  and  $\zeta_k$  are the objects from (i).

Theorem 2 is a very general result. A more explicit version concerning the empirical distance covariance is the following. While it is used mainly in the proof of Theorem 1, it is also of interest on its own since it provides a stronger convergence result for the empirical distance covariance than the usual weak limit theorems.

Construct the triangular array  $\tilde{Z}$  from the strictly stationary sequence  $(Z_k)_{k \in \mathbb{N}}$  as detailed above, and let  $\tilde{\theta}_n$  be the empirical measure of  $\tilde{Z}_{1,n}, \dots, \tilde{Z}_{n,n}$ .

**Theorem 3.** If  $X$  and  $Y$  are independent and have finite  $(2 + \varepsilon)$ -moments for some  $\varepsilon > 0$ , and the sequence  $(Z_k)_{k \in \mathbb{N}}$  is absolutely regular with mixing coefficients  $\beta(n) = O(n^{-r})$  for some  $r > 6(1 + 2\varepsilon^{-1})$ ,  $d = d(n) \rightarrow \infty$  for  $n \rightarrow \infty$  and  $d^3 = o(n)$ , it holds that

$$d_1(\text{ndcov}(\tilde{\theta}_n), \zeta) \xrightarrow[n \rightarrow \infty]{} 0,$$

and

$$d_1(\text{ndcov}(\tilde{\theta}_n), \zeta) \xrightarrow[n \rightarrow \infty]{} 0,$$

where  $\zeta := \sum_{k=1}^{\infty} \lambda_k \zeta_k^2$  with  $\zeta_k$  being centred Gaussian random variables whose covariance function is determined by the dependence structure of the sequence  $(Z_k)_{k \in \mathbb{N}}$ , and the parameters  $\lambda_k > 0$  are determined by the underlying distribution  $\theta$ .

*Remark.* Theorem 3 also holds if  $X$  and  $Y$  take values in separable metric spaces  $\mathcal{X}$  and  $\mathcal{Y}$  of negative type if  $\mathcal{X} \times \mathcal{Y}$  is  $\sigma$ -compact.

The next theorem is concerned with bounding the Wasserstein distance between an empirical measure of strongly mixing sample data and its marginal distribution. Prior results, e.g. those in Dereich et al. (2013), require iid sample data.

As stated in Section 1.4, Theorem 4 provides a valuable tool in determining a rate of growth of  $d = d(n)$  slow enough that it does not hinder the convergence of  $\xi_n$  to  $\xi$ . Traditional Glivenko-Cantelli type results usually break down when one allows for a growing dimension since they implicitly depend on this dimension for their asymptotic bounds.

Corollary 2 is a ready to use consequence of Theorem 4 which may prove useful when one is dealing with segments of growing length of stationary processes.

**Theorem 4.** *Let  $d \in \mathbb{N}$  and  $1 \leq p \leq d/2$  and  $q > p$  be fixed. Let  $\xi$  be a probability measure on  $\mathbb{R}^d$  with finite  $q$ -moments that fulfills Assumption 1. Then, for any  $n, K \in \mathbb{N}$ , it holds that*

$$\mathbb{E}d_p^p(\xi_n, \xi) \leq 3^{p-1} \left\{ 2^p \left( \xi(U_K(0)^C)^{\frac{q-p}{q}} m_q^{p/q} + \xi(U_K(0)^C) K^p \right) + K^{d/2} \cdot \mathfrak{M}^p \right\},$$

where  $U_K(0)$  is the open sphere centred around the origin with radius  $K$ ,  $m_q$  is the  $q$ -th moment of  $\xi$ ,

$$\mathfrak{M}^p = c_0 n^{-\frac{p-2}{2d}} 2^{3d/2-p} \mathfrak{d}^p \left( \frac{1 + M^{\frac{d/2-p}{d}}}{1 - 2^{p-d/2}} + \frac{1}{1 - 2^{-p}} + 4M^{\frac{1}{d}} \right),$$

for some uniform constant  $c_0$ ,  $\mathfrak{d} := \sup\{\|u_1 - u_2\|_2 \mid u_1, u_2 \in [0, 1]^d\}$ , and  $\xi_n$  is the empirical measure of a strictly stationary and  $\alpha$ -mixing process  $(U_i)_{i \in \mathbb{N}}$  with marginal distribution  $\xi$  and  $\alpha(n) \leq f(n) = \mathcal{O}(n^{-r_0})$  for some function  $f$  and some constant  $r_0 > 1$ . The constant  $c_0$  only depends on  $f$  and  $r_0$ .

*Remark.* (i) If  $\xi$  is the measure of a random vector  $(U'_1, \dots, U'_{d'})$  with equal marginal distributions, then  $m_q$  can be bounded by  $(d')^{q/2} \cdot \|U'_1\|_{L_q}^q$ .

(ii)  $\xi$  having finite  $q$ -moments implies that  $\xi(U_K(0)^C) = o(K^{-q})$ .

(iii) For a useful bound, we want to choose  $K = n^{\delta(p-2)/(d^2)}$  for some  $0 < \delta < 1$ . This ensures that  $K^{d/2} \cdot \mathfrak{M}^p$  will still converge to 0 for appropriate  $d$ .

(iv) The observations above, combined with the fact that  $d$  needs to be of order  $\log(n)^\gamma$  with  $0 < \gamma < 1/2$  for  $\mathfrak{M}$  to converge to 0, yield a useful bound for  $\mathbb{E}d_p^p(\xi_n, \xi)$  which allows for changing  $d$ .

**Corollary 2.** *Suppose that the assumptions of Theorem 4 are satisfied, and that  $\xi$  is the measure of a random vector  $(U'_1, \dots, U'_{d'})$  with equal marginal distributions (such as the first  $d'$  observations of a stationary process) with finite  $q$ -moments. Then it holds that*

$$\begin{aligned} \mathbb{E}d_p^p(\xi, \xi_n) &\leq 6^p c_0 2^{3d/2-p} \mathfrak{d}^p n^{-\frac{p-2}{4d}} \left( \frac{1 + M^{\frac{d/2-p}{d}}}{1 - 2^{p-d/2}} + \frac{1}{1 - 2^{-p}} + 4M^{\frac{1}{d}} \right) \\ &\quad + 6^p 2c' \cdot d^{1+q/2} n^{(p-2)(p-q)/(2d^2)}, \end{aligned}$$



for all  $n \geq n_0$ , where  $m'_q$  is the  $q$ -th moment of  $U'_1$ . The threshold  $n_0$  and the constant  $c'$  only depend on  $\mathcal{L}(U'_1)$ , and are therefore independent of  $d$  and  $d'$ . The constant  $c_0$  only depends on  $f$  and  $r_0$ .

**2.1. The Condition of Bounded Densities.** In our results we have made use of the assumption that the distributions of the random vectors  $Z := (Z_1, \dots, Z_d)$  fulfill Assumption 2, which in particular holds if the random vectors  $Z$  have Lebesgue densities which are bounded exponentially in  $d$ . We will show that this assumption is fulfilled most importantly by Gaussian processes, but also by elements of the more general class of elliptical distributions, i.e. distributions whose Lebesgue densities are of the form

$$f(x) = c \cdot g \left( (x - \mathbb{E}Z)^T \Sigma^{-1} (x - \mathbb{E}Z) \right),$$

where  $c$  is some normalisation constant. Here we assume that the random vectors are centred for the sake of simplicity. Some analysis shows that

$$\int g(x^T \Sigma^{-1} x) \, dx = \int g \left( \left\| \sqrt{\Sigma^{-1}} x \right\|_2^2 \right) \, dx = \frac{1}{\sqrt{|\det(\Sigma^{-1})|}} \int g(\|x\|_2^2) \, dx,$$

so in order for the density to integrate to a total measure of one, the normalisation constant  $c$  must be given by

$$c = \sqrt{|\det(\Sigma^{-1})|} \left( \int g(\|x\|_2^2) \, dx \right)^{-1} = \sqrt{|\det(\Sigma^{-1})|} \cdot \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \left( \int_0^\infty t^{d-1} g(t^2) \, dt \right)^{-1},$$

where in the last equality we have used identity 3.3.2.1, Chapter 5, in Prudnikov et al. (1986). The function  $g$  is known, and so the factor

$$(6) \quad \Gamma\left(\frac{d}{2}\right) \left( \int_0^\infty t^{d-1} g(t^2) \, dt \right)^{-1}$$

can be evaluated.

The most important members of this class of distributions are the Gaussian distributions. Here, the function  $g$  is given by  $g(t) = \exp(-t/2)$ , and so the product in (6) is equal to 1. This gives us the bound of  $\det(\Sigma^{-1/2}) \pi^{-d/2}$  for the multivariate Gaussian density. The following lemma therefore implies that the condition of bounded densities is fulfilled by any process whose first  $d$  observations are elliptically distributed with a sufficiently well-behaved function  $g$ , most importantly Gaussian processes.

**Lemma 1.** *Let  $\Sigma$  be the covariance matrix of the first  $d$  observations  $U_1, \dots, U_d$  of a strictly stationary process  $(U_k)_{k \in \mathbb{N}}$  such that  $\sum_{h=0}^\infty |\text{Cov}(U_1, U_{1+h})| < \infty$ . Then it holds that*

$$\det(\Sigma^{-1}) \leq K^d,$$

where  $K > 0$  is a constant independent of  $d$ .

*Proof.* Since the covariances are absolutely summable, there exists a spectral density  $f$  such that

$$\Sigma_{jk} = \text{Cov}(U_j, U_k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda) \exp(-i(j-k)\lambda) \, d\lambda.$$

The determinants of such Toeplitz forms are well studied. In particular, Szegő's limit theorem (cf. Grenander and Szegő (1984)) gives us

$$\lim_{d \rightarrow \infty} \det(\Sigma)^{\frac{1}{d}} = \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\lambda) \, d\lambda \right),$$

and thus

$$\det(\Sigma^{-1}) \leq \exp\left(-\frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\lambda) \, d\lambda\right)^d$$

for sufficiently large  $d$ . □

As a final remark, we note that many densities, while not elliptical, will be of the form

$$f(x) = \sqrt{\det(\Sigma^{-1})} \cdot \tilde{c} \cdot g\left(\sqrt{\Sigma^{-1}}(x - \mathbb{E}Z)\right),$$

due to the fact that the density is more spread out if  $\Sigma$  has a large determinant and more concentrated if its determinant is small. Thus, if the function  $g$  and the remaining normalising constant  $\tilde{c}$  are known or can be bounded in some way, we can once again apply the above lemma.

### 3. PROOF OUTLINES

**3.1. Proof Outline for Theorem 1.** Before we begin our sketch of proof for Theorem 1, let us briefly recall some general  $V$ -statistic theory. The Hoeffding decomposition of a symmetric  $V$ -statistic with kernel  $g$  of order  $m$  and with respect to a measure  $\xi$  is given by

$$V_n(g; \xi) = \sum_{i=0}^m \binom{m}{i} V_n^{(i)}(g; \xi),$$

where  $V_n^{(i)}(g; \xi)$  is the  $V$ -Statistic with respect to the kernel

$$g_i(z_1, \dots, z_i; \xi) = \sum_{k=0}^i \binom{i}{k} (-1)^{i-k} \mathbf{g}_k(z_1, \dots, z_k; \xi)$$

with

$$\mathbf{g}_k(z_1, \dots, z_k; \xi) = \int \cdots \int g(z_1, \dots, z_m) \, d\xi(z_{k+1}) \cdots d\xi(z_m);$$

see Denker and Keller (1983). Standard theory implies that if the  $V$ -statistic is degenerate, i.e. if  $g_1(z, \xi) = 0$  for almost all  $z$ , then, under some technical assumptions, the limiting behaviour of the entire  $V$ -statistic is determined by that of  $V_n^{(2)}(g; \xi)$ .

It should be pointed out that, as remarked in Dehling and Mikosch (1994), and first noted in Bretagnolle (1983), a bootstrap attempt such as we propose usually does not work for degenerate  $U$ -statistics. Instead, a consideration of the second-order term in the statistic's Hoeffding decomposition results in the appropriate Bootstrap statistic; see also Arcones and Giné (1992). However, it can be easily checked that our specific kernel  $h$  has the special property that it is degenerate with respect to any product measure. It is this property that ensures the validity of our bootstrap procedure, despite the fact that we are not bootstrapping the second-order term in the statistic's Hoeffding decomposition.

We will now present the proof outline for Theorem 1. Consider that any two distinct blocks  $B_i^*$  and  $B_j^*$  of our bootstrap sequence correspond to concatenations  $(B_{X,k_1}, B_{Y,l_1})$  and  $(B_{X,k_2}, B_{Y,l_2})$  of blocks from our sample sequences  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$ . Roughly speaking, for large  $n$ , these four sample blocks will be far apart from each other with high probability, and because we assume  $(Z_k)_{k \in \mathbb{N}}$  to be absolutely regular, this implies that they will be “almost” independent. The sequence of bootstrap samples  $Z_1^*, \dots, Z_n^*$  can therefore be expected to behave similarly to a collection of random variables  $\tilde{Z}_{1,n}, \dots, \tilde{Z}_{n,n}$  consisting of  $N$  iid random vectors of length  $d$  with marginal distribution

$$\mathcal{L}(X_1, \dots, X_d) \otimes \mathcal{L}(Y_1, \dots, Y_d) \stackrel{H_0}{=} \mathcal{L}(Z_1, \dots, Z_d).$$

Denote by  $\tilde{V}$  the empirical distance covariance of  $\tilde{Z}_{1,n}, \dots, \tilde{Z}_{n,n}$ , then a simple application of the triangle inequality yields

$$d_1(\zeta, nV^*) \leq \overline{d}_1(\zeta, n\tilde{V}) + d_1(n\tilde{V}, nV^*).$$

By Theorem 3, we have

$$d_1(\zeta, n\tilde{V}) \xrightarrow[n \rightarrow \infty]{} 0.$$

It remains to control the distance  $d_1(n\tilde{V}, nV^*)$ . By the standard theory for  $V$ -statistics, we can instead investigate the simpler object

$$d_1\left(n\tilde{V}_n^{(2)}(h; \mu \otimes \nu), nV_n^{*(2)}(h; \mu_n \otimes \nu_n)\right).$$

(Technically, one has to show that the second-order Hoeffding terms not only determine the weak limits of the original  $V$ -statistics, but also the limits with respect to the Wasserstein distance. We do this in Appendix A.)

To specify our intuitive reasoning above, note that  $\text{dcov}(\theta_n)$  can be expressed as a ‘ $V$ -statistic’ with kernel

$$(7) \quad H(B_1, \dots, B_6) := \frac{1}{d^6} \sum_{1 \leq i_1, \dots, i_6 \leq d} h(B_{1,i_1}, \dots, B_{6,i_6}),$$

where  $B_i := (Z_{(i-1)d+1}, \dots, Z_{id})$ ,  $1 \leq i \leq N := n/d$ , denotes the blocks of length  $d = d(n)$  and  $B_{j,i_j}$  denotes the  $i_j$ -th coordinate of  $B_j$ . More precisely, this means that

$$(8) \quad \text{dcov}(\theta_n) = V_H(B_1, \dots, B_N) = \frac{1}{N^6} \sum_{i_1=1}^N \dots \sum_{i_6=1}^N H(B_{i_1}, B_{i_2}, \dots, B_{i_6}).$$

(Technically, this is not a well-defined  $V$ -statistic, since the kernel depends on  $d$ , i.e., it changes with  $n$ .)

The bootstrapped empirical distance covariance  $V^*$  and the statistic  $\tilde{V}$  allow for analogous representations, e.g.

$$(9) \quad V^* = V_H(B_1^*, \dots, B_N^*) = \frac{1}{N^6} \sum_{i_1=1}^N \dots \sum_{i_6=1}^N H(B_{i_1}^*, B_{i_2}^*, \dots, B_{i_6}^*).$$

Furthermore, we can establish a link between the Hoeffding decompositions of the  $V$ -statistic with kernel  $h$  and that of the blockwise ‘ $V$ -statistic’ with kernel  $H$ . For this, let

$$\begin{aligned} F^{(X)} &:= F_d^{(X)} := \mathcal{L}(X_1, \dots, X_d), \\ F^{(Y)} &:= F_d^{(Y)} := \mathcal{L}(Y_1, \dots, Y_d), \end{aligned}$$

and let  $F_N^{(X)}$  and  $F_N^{(Y)}$  be their empirical versions, i.e. the empirical measures of  $B_{X,1}, \dots, B_{X,N}$  and  $B_{Y,1}, \dots, B_{Y,N}$ , respectively. We show in Appendix A that

$$(10) \quad \begin{aligned} \tilde{V}_n^{(2)}(h, \mu \otimes \nu) &= \tilde{V}_N^{(2)}\left(H, F^{(X)} \otimes F^{(Y)}\right), \\ V_n^{*(2)}(h, \mu_n \otimes \nu_n) &= V_N^{*(2)}\left(H, F_N^{(X)} \otimes F_N^{(Y)}\right). \end{aligned}$$

Conditionally on  $Z_1, \dots, Z_n$ , both objects on the right-hand side are  $V$ -statistics of iid sample data. To control their Wasserstein distance, we will use the following proposition.

**Proposition 1.** *Let  $U_i \stackrel{iid}{\sim} \eta$ ,  $V_i \stackrel{iid}{\sim} \xi$ . Then there exists a constant  $C$  such that*

$$d_2 \left( \mathcal{L}_\eta \left( \frac{1}{n} \sum_{1 \leq i, j \leq n} H_2(U_i, U_j; \eta) \right), \mathcal{L}_\xi \left( \frac{1}{n} \sum_{1 \leq i, j \leq n} H_2(V_i, V_j; \xi) \right) \right) \leq C \cdot d_4(\eta, \xi).$$

*Remark.* By the notation  $\mathcal{L}_\eta$ , and  $\mathcal{L}_\xi$ , we indicate that the random variables  $(U_i)_{i \geq 1}$ , and  $(V_i)_{i \geq 1}$ , have distribution  $\eta$ , and  $\xi$  respectively. Moreover, in the present context both of them are iid processes. Strictly speaking, we could use the same symbols for the random variables in both cases, i.e. also  $\mathcal{L}_\xi(\frac{1}{n} \sum_{1 \leq i \neq j \leq n} g_2(U_i, U_j; \xi))$ , since the symbol  $\mathcal{L}_\xi$  specifies the distribution of the process.

This allows us to bound the Wasserstein distance of the  $V$ -statistics of interest by the Wasserstein distance of the marginal distributions of the underlying iid processes, i.e. by  $d_4 \left( F^{(X)} \otimes F^{(Y)}, F_N^{(X)} \otimes F_N^{(Y)} \right)$ . Furthermore, we show in Appendix A that for some constant  $c$ ,

$$d_4 \left( F^{(X)} \otimes F^{(Y)}, F_N^{(X)} \otimes F_N^{(Y)} \right) \leq c \cdot \left\{ d_4 \left( F^{(X)}, F_N^{(X)} \right) + d_4 \left( F^{(Y)}, F_N^{(Y)} \right) \right\},$$

and thus we have reduced the problem to that of bounding the Wasserstein distance between the empirical measure of an absolutely regular process and its marginal distribution. To achieve this, we use Corollary 2.

**3.2. Proof Outline for Theorems 2 and 3.** Recall that convergence in  $d_p$  is equivalent to weak convergence and convergence of the  $p$ -th moments. The weak convergence in part (i) of Theorem 2 is merely Theorem 2 in Kroll (2021). Part (ii) of Theorem 2 can be shown much in the same way with standard arguments. It then remains to show convergence of the first moments. Again, this is done using standard arguments.

For Theorem 3, we use the fact that the empirical distance covariance is a degenerate  $V$ -statistic. As we have seen in Section 3.1, the limiting behaviour of such  $V$ -statistics is, under certain assumptions, determined by the second-order term of their Hoeffding decomposition. More precisely we show in Appendix C that, under certain assumptions,

$$\left| nV - 15nV_n^{(2)} \right| \xrightarrow[n \rightarrow \infty]{(2)} 0.$$

It then remains to show that the second-order term of the Hoeffding decomposition of the empirical distance covariance fulfills the assumptions of Theorem 2.

**3.3. Proof Outline for Theorem 4 and Corollary 2.** A similar result of this type was previously developed by Dereich et al. (2013) for iid sample generating processes. In generalising the authors' result to strictly stationary and  $\alpha$ -mixing processes, we use some of the same general ideas that form the basis for their proof. Let us briefly recall the basic definitions from Dereich et al. (2013).

Consider the  $d$ -dimensional hypercube  $[0, 1]^d$ . For any given  $l \in \mathbb{N}_0$ , we can partition  $[0, 1]^d$  into  $2^{dl}$  translations of  $2^{-l}[0, 1]^d$ . Denote by  $\mathcal{P}_l$  the collection of all these translations. On the union  $\mathcal{P} := \bigcup_{l=0}^{\infty} \mathcal{P}_l$ , we can define a tree structure as follows: For any given set  $C \in \mathcal{P}_l$ ,  $l \in \mathbb{N}$ , the father of  $C$  is the unique set  $F \in \mathcal{P}_{l-1}$  with  $C \subseteq F$ . We adopt the shorthand notation  $C \leftarrow F$  for  $C$  is a child of  $F$ . Furthermore, we define  $\mathfrak{d} := \sup\{\|z_1 - z_2\|_2 \mid z_1, z_2 \in [0, 1]^d\}$ .

We also recall Lemma 2 from Dereich et al. (2013), which forms the basis for their proofs (and that we will use in the same manner). Because it is central to their proofs as well as ours, we include it here for the sake of clarity.

**Lemma 2** (Dereich, Scheutzow, Schottstedt). *Let  $\eta$  and  $\xi$  be two probability measures on  $[0, 1]^d$ , with the property that  $\eta(C) > 0$  if  $\xi(C) > 0$  for all  $C \in \mathcal{P}$ . It then holds that*

$$d_p^p(\eta, \xi) \leq \frac{1}{2} \mathfrak{d}^p \sum_{l=0}^{\infty} 2^{-pl} \sum_{F \in \mathcal{P}_l} \sum_{C \leftarrow F} \left| \xi(C) - \xi(F) \frac{\eta(C)}{\eta(F)} \right|.$$

Because we are not working under an iid assumption, we cannot use the same methods as Dereich et al. (2013) to further bound in expectation the object in Lemma 2, which will be used for the distance  $d_p^p(\xi, \xi_n)$ . Note that

$$\begin{aligned} \left| \xi_n(C) - \xi_n(F) \frac{\xi(C)}{\xi(F)} \right| &= \left| \xi_n(C) - \xi(C) + \xi(C) \xi_n(F) \frac{\xi(C)}{\xi(F)} \right| \\ &\leq |\xi_n(C) - \xi(C)| + \frac{\xi(C)}{\xi(F)} |\xi_n(F) - \xi(F)|. \end{aligned}$$

Ignoring the factor  $\xi(C)/\xi(F)$  for a moment, this means that finding a sufficient bound for the distances in the series in Lemma 2 can be reduced to finding a bound for  $\mathbb{E}|\xi_n(F) - \xi(F)|$ , where  $F$  is now a generic set in  $\mathcal{P}$ . Necessarily, such a bound will depend on  $\xi(F)$ . We are able to derive a bound for sufficiently small sets  $F$  (small in the sense of having a small measure), and have to somehow bound the number of sets which have too big a measure for our bound to apply. This explains the necessity for Assumption 1: Because the sets in  $\mathcal{P}_l$  have small Lebesgue measure for large  $l$ , Assumption 1 allows us to determine the maximum  $l$  such that a set that has a too large volume with respect to  $\xi$  could still be an element of  $\mathcal{P}_l$ .

From this, we can prove Theorem 4. Corollary 2 is a direct consequence of Theorem 4 and the remarks thereafter.

We briefly mention that one can derive similar bounds for the Wasserstein distance between an empirical measure and its marginal distribution without need for Assumption 1 if one instead assumes the underlying sample process to be  $\phi$ -mixing. This is because under  $\phi$ -mixing one is able to find a smaller bound for  $\mathbb{E}|\xi_n(F) - \xi(F)|$ . Explicit proofs of this are contained in Appendix B.

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#### APPENDIX A. PROOF OF THEOREM 1 AND RELATED RESULTS

For a given kernel function  $g$ , let us introduce the metric

$$d_{p,g}^p(\eta, \xi) := \inf \left\{ \int |g(u_1, \dots, u_m) - g(v_1, \dots, v_m)|^p d\gamma^m((u_1, v_1), \dots, (u_m, v_m)) \right\},$$

where the infimum is taken over all distributions  $\gamma$  with marginals  $\eta$  and  $\xi$ .

We will prove Proposition 1 in two steps: First, we will bound the Wasserstein distance between the  $V$ -statistics in terms of  $d_{2,H}$ ; this is achieved by Theorem 5. Next, in Lemma 3, we show that this bound can in turn be bounded in terms of  $d_4$ .

Theorem 5 is an extension of the results by Dehling and Mikosch (1994). It has been stated for  $U$ -statistics of kernel functions of arbitrary degree as Lemma 5.1 by Dehling et al. (2020), but without an explicit proof.

**Theorem 5.** Let  $U_i \stackrel{iid}{\sim} \eta$ ,  $V_i \stackrel{iid}{\sim} \xi$  and  $g$  be a square-integrable kernel function of order 6. Then there exists a constant  $C$  such that

$$\begin{aligned} & d_2 \left( \mathcal{L}_\eta \left( \frac{1}{n} \sum_{1 \leq i, j \leq n} g_2(U_i, U_j; \eta) \right), \mathcal{L}_\xi \left( \frac{1}{n} \sum_{1 \leq i, j \leq n} g_2(V_i, V_j; \xi) \right) \right) \\ & \leq C \cdot \left\{ d_{2,g}(\eta, \xi) + \mathbb{E} \left[ (g(U_1, U_1, U_3, \dots, U_6) - g(V_1, V_1, V_3, \dots, V_6))^2 \right] \right\}. \end{aligned}$$

*Proof.* Let  $\varepsilon > 0$  be arbitrary. By definition of  $d_{2,g}(\eta, \xi)$ , we can find a distribution  $\gamma$  with marginals  $\eta$  and  $\xi$ , such that for the process of i.i.d. pairs  $(U_i, V_i)_{i \in \mathbb{N}}$  with marginal distribution  $\gamma$ , we have

$$\mathbb{E} \left[ (g(U_1, \dots, U_6) - g(V_1, \dots, V_6))^2 \right] \leq d_{2,g}(\eta, \xi) + \varepsilon.$$

Thus, the Wasserstein distance of the two distributions  $\mathcal{L}_\eta \left( \frac{1}{n} \sum_{1 \leq i, j \leq n} g_2(U_i, U_j; \eta) \right)$  and  $\mathcal{L}_\xi \left( \frac{1}{n} \sum_{1 \leq i, j \leq n} g_2(V_i, V_j; \xi) \right)$  is bounded by the  $L_2$ -distance between the random variables  $\frac{1}{n} \sum_{1 \leq i, j \leq n} g_2(U_i, U_j; \eta)$  and  $\frac{1}{n} \sum_{1 \leq i, j \leq n} g_2(V_i, V_j; \xi)$ , where  $(U_i, V_i)$  have joint distribution  $\gamma$ . We thus obtain

$$\begin{aligned} & d_2^2 \left( \mathcal{L}_\eta \left( \frac{1}{n} \sum_{1 \leq i, j \leq n} g_2(U_i, U_j; \eta) \right), \mathcal{L}_\xi \left( \frac{1}{n} \sum_{1 \leq i, j \leq n} g_2(V_i, V_j; \xi) \right) \right) \\ & \leq \mathbb{E} \left[ \left( \frac{1}{n} \sum_{1 \leq i, j \leq n} g_2(U_i, U_j; \eta) - \frac{1}{n} \sum_{1 \leq i, j \leq n} g_2(V_i, V_j; \xi) \right)^2 \right] \\ (11) \quad & \leq 2\mathbb{E} \left[ \left( \frac{1}{n} \sum_{1 \leq i \neq j \leq n} g_2(U_i, U_j; \eta) - \frac{1}{n} \sum_{1 \leq i \neq j \leq n} g_2(V_i, V_j; \xi) \right)^2 \right] \\ & \quad + 2\mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^n g_2(U_i, U_i; \eta) - \frac{1}{n} \sum_{i=1}^n g_2(V_i, V_i; \xi) \right)^2 \right] \\ & =: 2I_1 + 2I_2. \end{aligned}$$

Let us first consider  $I_1$ . We have

$$\begin{aligned} & I_1 = \mathbb{E} \left[ \left( \frac{1}{n} \sum_{1 \leq i \neq j \leq n} g_2(U_i, U_j; \eta) - \frac{1}{n} \sum_{1 \leq i \neq j \leq n} g_2(V_i, V_j; \xi) \right)^2 \right] \\ (12) \quad & = \frac{1}{n^2} \mathbb{E} \left[ \left( \sum_{1 \leq i \neq j \leq n} (g_2(U_i, U_j; \eta) - g_2(V_i, V_j; \xi)) \right)^2 \right] \\ & = \frac{n(n-1)}{n^2} \mathbb{E} \left[ (g_2(U_1, U_2; \eta) - g_2(V_1, V_2; \xi))^2 \right] \\ & \leq \mathbb{E} \left[ (g_2(U_1, U_2; \eta) - g_2(V_1, V_2; \xi))^2 \right] \end{aligned}$$

where in the last step, we have used the fact that by degeneracy of the kernels  $g_2$ , the summands  $g_2(U_i, U_j; \eta) - g_2(V_i, V_j; \xi)$  are uncorrelated. To verify this claim, we consider,

e.g., the terms  $g_2(U_1, U_2; \eta) - g_2(V_1, V_2; \xi)$  and  $g_2(U_1, U_3; \eta) - g_2(V_1, V_3; \xi)$ , and show that they are uncorrelated. It holds that

$$\begin{aligned} & \mathbb{E}[(g_2(U_1, U_2; \eta) - g_2(V_1, V_2; \xi))(g_2(U_1, U_3; \eta) - g_2(V_1, V_3; \xi))] \\ &= \iiint (g_2(u_1, u_2; \eta) - g_2(v_1, v_2; \xi))(g_2(u_1, u_3; \eta) - g_2(v_1, v_3; \xi)) \\ & \quad d\gamma(u_1, v_1)d\gamma(u_2, v_2)d\gamma(u_3, v_3) \\ &= \iint \left( \int (g_2(u_1, u_2; \eta) - g_2(v_1, v_2; \xi)) d\gamma(u_2, v_2) \right) \\ & \quad \cdot (g_2(u_1, u_3; \eta) - g_2(v_1, v_3; \xi)) d\gamma(u_1, v_1)d\gamma(u_3, v_3) = 0, \end{aligned}$$

since

$$\begin{aligned} & \int g_2(u_1, u_2; \eta) - g_2(v_1, v_2; \xi) d\gamma(u_2, v_2) \\ &= \int g_2(u_1, u_2; \eta) d\eta(u_2) - \int g_2(v_1, v_2; \xi) d\xi(v_2) = 0. \end{aligned}$$

It remains to show that the right-hand side of (12) can be bounded by a constant times  $d_{2,g}(\eta, \xi)$ .

Recall the definition of  $g_2(u, v; \eta)$ ,

$$\begin{aligned} & g_2(u, v; \eta) \\ &= \int \cdots \int g(u, v, u_3, u_4, u_5, u_6) d\eta(u_3)d\eta(u_4)d\eta(u_5)d\eta(u_6) \\ & \quad - \int \cdots \int g(u, u_2, u_3, u_4, u_5, u_6) d\eta(u_2)d\eta(u_3)d\eta(u_4)d\eta(u_5)d\eta(u_6) \\ & \quad - \int \cdots \int g(u_1, v, u_3, u_4, u_5, u_6) d\eta(u_1)d\eta(u_3)d\eta(u_4)d\eta(u_5)d\eta(u_6) \\ & \quad + \int \cdots \int g(u_1, u_2, u_3, u_4, u_5, u_6) d\eta(u_1)d\eta(u_2)d\eta(u_3)d\eta(u_4)d\eta(u_5)d\eta(u_6). \end{aligned}$$

Thus we obtain from Minkowski's inequality

$$\begin{aligned} & \left\{ \mathbb{E} \left[ (g_2(U_1, U_2; \eta) - g_2(V_1, V_2; \xi))^2 \right] \right\}^{1/2} \\ & \leq \left\{ \mathbb{E} \left[ \left( \int \cdots \int g(U_1, U_2, u_3, u_4, u_5, u_6) d\eta(u_3)d\eta(u_4)d\eta(u_5)d\eta(u_6) \right. \right. \right. \\ & \quad \left. \left. \left. - \int \cdots \int g(V_1, V_2, v_3, v_4, v_5, v_6) d\xi(v_3)d\xi(v_4)d\xi(v_5)d\xi(v_6) \right)^2 \right] \right\}^{1/2} \\ (13) \quad & + 2 \left\{ \mathbb{E} \left[ \left( \int \cdots \int g(U_1, u_2, u_3, u_4, u_5, u_6) d\eta(u_2)d\eta(u_3)d\eta(u_4)d\eta(u_5)d\eta(u_6) \right. \right. \right. \\ & \quad \left. \left. \left. - \int \cdots \int g(V_1, v_2, v_3, v_4, v_5, v_6) d\xi(v_2)d\xi(v_3)d\xi(v_4)d\xi(v_5)d\xi(v_6) \right)^2 \right] \right\}^{1/2} \\ & + |\mathbb{E}g(U_1, \dots, U_6) - \mathbb{E}g(V_1, \dots, V_6)|. \end{aligned}$$



We can now bound each of the terms on the right-hand side using Hölder's inequality. E.g., we obtain for the first term

$$\begin{aligned}
& \mathbb{E} \left[ \left( \int \cdots \int g(U_1, U_2, u_3, u_4, u_5, u_6) \, d\eta(u_3) d\eta(u_4) d\eta(u_5) d\eta(u_6) \right. \right. \\
& \quad \left. \left. - \int \cdots \int g(V_1, V_2, v_3, v_4, v_5, v_6) \, d\xi(v_3) d\xi(v_4) d\xi(v_5) d\xi(v_6) \right)^2 \right] \\
&= \iint \left( \int \cdots \int g(u_1, u_2, u_3, u_4, u_5, u_6) \, d\eta(u_3) d\eta(u_4) d\eta(u_5) d\eta(u_6) \right. \\
(14) \quad & \left. - \int \cdots \int g(v_1, v_2, v_3, v_4, v_5, v_6) \, d\xi(v_3) d\xi(v_4) d\xi(v_5) d\xi(v_6) \right)^2 \, d\gamma(u_1, v_1) d\gamma(u_2, v_2) \\
&= \iint \left( \int \cdots \int g(u_1, u_2, u_3, u_4, u_5, u_6) - g(v_1, v_2, v_3, v_4, v_5, v_6) \right. \\
& \quad \left. d\gamma(u_3, v_3) d\gamma(u_4, v_4) d\gamma(u_5, v_5) d\gamma(u_6, v_6) \right)^2 \, d\gamma(u_1, v_1) d\gamma(u_2, v_2) \\
&\leq \mathbb{E} \left[ (g(U_1, \dots, U_6) - g(V_1, \dots, V_6))^2 \right] \\
&\leq d_{2,g}(\eta, \xi) + \varepsilon,
\end{aligned}$$

and  $\varepsilon > 0$  is arbitrary.

Let us now turn to  $I_2$ . We immediately get

$$\begin{aligned}
I_2 &= \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^n g_2(U_i, U_i; \eta) - \frac{1}{n} \sum_{i=1}^n g_2(V_i, V_i; \xi) \right)^2 \right] \\
&\leq \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \{g_2(U_i, U_i; \eta) - g_2(V_i, V_i; \xi)\}^2 \right] \\
&= \mathbb{E} \left[ (g_2(U_1, U_1; \eta) - g_2(V_1, V_1; \xi))^2 \right].
\end{aligned}$$

We can now proceed as in (13) and (14) and obtain a bound in terms of

$$\mathbb{E} \left[ (g(U_1, U_1, U_3, \dots, U_6) - g(V_1, V_1, V_3, \dots, V_6))^2 \right].$$

Thus, the statement of the theorem follows from (11).  $\square$

We will now prove that the bound from Theorem 5 can in turn be bounded in terms of  $d_p(\eta, \xi)$ . This can be understood as a type of Lipschitz continuity of the kernel  $g$ . Because not all kernels  $g$  have this property, we only show it for the kernel to which we will eventually apply this result, namely the function  $H$  as defined in (7).

**Lemma 3.** *Let  $r, s \geq 2$  be given, such that  $\frac{1}{r} + \frac{1}{s} = \frac{1}{2}$ , and let  $U_i \stackrel{iid}{\sim} \eta$  and  $V_i \stackrel{iid}{\sim} \xi$ . Then it holds that*

$$\begin{aligned}
& d_{2,H}(\eta, \xi) + \|H(U_1, U_1, U_3, \dots, U_6) - H(V_1, V_1, V_3, \dots, V_6)\|_{L_2} \\
& \leq 64 \cdot d_r(\eta, \xi) (\|U_1\|_{L_s} + \|V_1\|_{L_s}),
\end{aligned}$$

for all measures  $\eta$  and  $\xi$  on  $\mathbb{R}^{(\ell_1 + \ell_2)d}$ . In particular, choosing  $r = s = 4$  yields the bound

$$64 \cdot d_4(\eta, \xi) (\|U_1\|_{L_4} + \|V_1\|_{L_4}).$$

*Remark 4.* (i) Implicitly, we assume that the measures  $\eta$  and  $\xi$  have finite  $r$ -th moments.

(ii) Lemma 3 also holds for  $r = 2$  and  $s = \infty$ . In this case, i.e. for measure with bounded support in  $[-K, K]^{(\ell_1 + \ell_2)d}$ , we obtain

$$d_{s,H}(\eta, \xi) \leq 64 \cdot K \cdot d_2(\eta, \xi).$$

*Proof of Lemma 3:* We choose  $\mathbb{R}^{(\ell_1 + \ell_2)d}$ -valued random vectors  $\mathbf{W}, \mathbf{W}'$  with distributions  $\eta$  and  $\xi$ , respectively, satisfying

$$\mathbb{E}\|\mathbf{W} - \mathbf{W}'\|_2^r = d_r^r(\eta, \xi).$$

We denote the coordinates of  $\mathbf{W}$  by  $W_1, \dots, W_d$ , noting that the  $W_i$ 's are  $\mathbb{R}^{(\ell_1 + \ell_2)d}$ -valued random variables whose projections into  $\mathbb{R}^{\ell_1}$  and  $\mathbb{R}^{\ell_2}$  we again denote by  $U_i$  and  $V_i$ , respectively. Thus, we have

$$\begin{aligned} \mathbf{W} &= (U_1, V_1, \dots, U_d, V_d) \\ \mathbf{W}' &= (U'_1, V'_1, \dots, U'_d, V'_d). \end{aligned}$$

Let  $(\mathbf{W}_1, \mathbf{W}'_1), \dots, (\mathbf{W}_6, \mathbf{W}'_6)$  be independent copies of  $(\mathbf{W}, \mathbf{W}')$ . By definition of  $d_{2,H}(\eta, \xi)$ , we obtain

$$\begin{aligned} d_{2,H}(\eta, \xi) &\leq \|H(\mathbf{W}_1, \dots, \mathbf{W}_6) - H(\mathbf{W}'_1, \dots, \mathbf{W}'_6)\|_{L_2} \\ &\leq \frac{1}{d^6} \sum_{1 \leq i_1, \dots, i_6 \leq d} \|h(W_{1,i_1}, \dots, W_{6,i_6}) - h(W'_{1,i_1}, \dots, W'_{6,i_6})\|_{L_2} \\ &\leq \frac{1}{d^6 6!} \sum_{1 \leq i_1, \dots, i_6 \leq d} \sum_{\sigma \in \mathfrak{S}_6} \|h'(W_{\sigma(1), i_{\sigma(1)}}, \dots, W_{\sigma(6), i_{\sigma(6)}}) - h'(W'_{\sigma(1), i_{\sigma(1)}}, \dots, W'_{\sigma(6), i_{\sigma(6)}})\|_{L_2}. \end{aligned}$$

Each of the summands on the right-hand side is of the type

$$(15) \quad \|h'(W_{i_1, j_1}, \dots, W_{i_6, j_6}) - h'(W'_{i_1, j_1}, \dots, W'_{i_6, j_6})\|_{L_2},$$

where  $(i_1, \dots, i_6)$  is a permutation of  $(1, \dots, 6)$ , and where  $1 \leq j_1, \dots, j_6 \leq d$ . We will give upper bounds for each of these summands in the more general case that  $1 \leq i_1, \dots, i_6 \leq 6$ . Note that

$$\begin{aligned} &h'(W_{i_1, j_1}, \dots, W_{i_6, j_6}) - h'(W'_{i_1, j_1}, \dots, W'_{i_6, j_6}) \\ &= f(U_{i_1, j_1}, U_{i_2, j_2}, U_{i_3, j_3}, U_{i_4, j_4}) f(V_{i_1, j_1}, V_{i_2, j_2}, V_{i_5, j_5}, V_{i_6, j_6}) \\ &\quad - f(U'_{i_1, j_1}, U'_{i_2, j_2}, U'_{i_3, j_3}, U'_{i_4, j_4}) f(V'_{i_1, j_1}, V'_{i_2, j_2}, V'_{i_5, j_5}, V'_{i_6, j_6}) \\ &= f(U_{i_1, j_1}, U_{i_2, j_2}, U_{i_3, j_3}, U_{i_4, j_4}) (f(V_{i_1, j_1}, V_{i_2, j_2}, V_{i_5, j_5}, V_{i_6, j_6}) - f(V'_{i_1, j_1}, V'_{i_2, j_2}, V'_{i_5, j_5}, V'_{i_6, j_6})) \\ &\quad + f(V'_{i_1, j_1}, V'_{i_2, j_2}, V'_{i_5, j_5}, V'_{i_6, j_6}) (f(U_{i_1, j_1}, U_{i_2, j_2}, U_{i_3, j_3}, U_{i_4, j_4}) - f(U'_{i_1, j_1}, U'_{i_2, j_2}, U'_{i_3, j_3}, U'_{i_4, j_4})). \end{aligned}$$

We can now apply Hölder's inequality, and obtain

$$(16) \quad \begin{aligned} &\|f(U_{i_1, j_1}, U_{i_2, j_2}, U_{i_3, j_3}, U_{i_4, j_4}) \\ &\quad \cdot (f(V_{i_1, j_1}, V_{i_2, j_2}, V_{i_5, j_5}, V_{i_6, j_6}) - f(V'_{i_1, j_1}, V'_{i_2, j_2}, V'_{i_5, j_5}, V'_{i_6, j_6}))\|_{L_2} \\ &\leq \|f(U_{i_1, j_1}, U_{i_2, j_2}, U_{i_3, j_3}, U_{i_4, j_4})\|_{L_s} \\ &\quad \cdot \|f(V_{i_1, j_1}, V_{i_2, j_2}, V_{i_5, j_5}, V_{i_6, j_6}) - f(V'_{i_1, j_1}, V'_{i_2, j_2}, V'_{i_5, j_5}, V'_{i_6, j_6})\|_{L_r}. \end{aligned}$$

Note that by the triangle inequality it holds that

$$\begin{aligned} &|f(V_{i_1, j_1}, V_{i_2, j_2}, V_{i_3, j_3}, V_{i_4, j_4}) - f(V'_{i_1, j_1}, V'_{i_2, j_2}, V'_{i_3, j_3}, V'_{i_4, j_4})| \\ &\leq 2 (\|V_{i_1, j_1} - V'_{i_1, j_1}\|_2 + \|V_{i_2, j_2} - V'_{i_2, j_2}\|_2 + \|V_{i_3, j_3} - V'_{i_3, j_3}\|_2 + \|V_{i_4, j_4} - V'_{i_4, j_4}\|_2) \\ &\leq 8 \|\mathbf{W} - \mathbf{W}'\|_2, \end{aligned}$$

and thus

$$\|f(V_{i_1, j_1}, V_{i_2, j_2}, V_{i_3, j_3}, V_{i_4, j_4}) - f(V'_{i_1, j_1}, V'_{i_2, j_2}, V'_{i_3, j_3}, V'_{i_4, j_4})\|_{L_r} \leq 8\|\mathbf{W} - \mathbf{W}'\|_{L_r}.$$

Similarly, one shows that

$$\|f(U_{i_1, j_1}, U_{i_2, j_2}, U_{i_3, j_3}, U_{i_4, j_4})\|_{L_s} \leq 4\|U_{i_1, j_1}\|_{L_s} = 4\left(\int \|x\|_2^s d\eta(x)\right)^{1/s}.$$

Combining the above inequalities, we can bound (16) by

$$32\left(\int \|x\|_2^s d\eta(x)\right)^{1/s} d_r(\eta, \xi),$$

and thus

$$d_{2,H}(\eta, \xi) \leq 32d_r(\eta, \xi) \left\{ \left(\int \|x\|_2^s d\eta(x)\right)^{1/s} + \left(\int \|y\|_2^s d\xi(y)\right)^{1/s} \right\}.$$

To bound  $\|H(U_1, U_1, U_3, \dots, U_6) - H(V_1, V_1, V_3, \dots, V_6)\|_{L_2}$ , we again need to find a bound for (15), with the difference being that  $(i_1, \dots, i_6)$  is now not a permutation of  $(1, \dots, 6)$ , but rather a permutation of  $(1, 1, 3, \dots, 6)$ . But because we have deduced the bounds above not only in the case where  $(i_1, \dots, i_6)$  is a permutation of  $(1, \dots, 6)$  but rather any collection of indices between 1 and 6, we arrive at the same bound as before, i.e.

$$\begin{aligned} & \|H(U_1, U_1, U_3, \dots, U_6) - H(V_1, V_1, V_3, \dots, V_6)\|_{L_2} \\ & \leq 32d_r(\eta, \xi) \left\{ \left(\int \|x\|_2^s d\eta(x)\right)^{1/s} + \left(\int \|y\|_2^s d\xi(y)\right)^{1/s} \right\}. \end{aligned}$$

This proves the lemma.  $\square$

Proposition 1 now follows from Theorem 5 and Lemma 3.

**Lemma 4.** *Let  $\eta_i$  and  $\xi_i$  be measures with finite  $p$ -moments on  $\mathbb{R}^{\ell_i}$ ,  $i = 1, 2$ . Then,*

$$d_p^p(\eta_1 \otimes \eta_2, \xi_1 \otimes \xi_2) \leq \max\{1, 2^{p/2-1}\} (d_p^p(\eta_1, \xi_1) + d_p^p(\eta_2, \xi_2)).$$

*Proof.* For any  $z, z' \in \mathbb{R}^{\ell_1 + \ell_2}$ , it holds that

$$\|z - z'\|_2^p \leq \max\{1, 2^{p/2-1}\} (\|x - x'\|_2^p + \|y - y'\|_2^p),$$

where  $x$  (or  $x'$ ) and  $y$  (or  $y'$ ) denote the corresponding projections on  $\mathbb{R}^{\ell_1}$  and  $\mathbb{R}^{\ell_2}$ , respectively. Let  $\Gamma$  denote the set of all couplings of  $\eta_1 \otimes \eta_2$  and  $\xi_1 \otimes \xi_2$ , and  $\Gamma_i$  the set of all couplings of  $\eta_i$  and  $\xi_i$ ,  $i = 1, 2$ .

For any given measure  $\gamma$ , we say that two given projections  $\pi_1$  and  $\pi_2$  are independent in  $\gamma$ , if the pushforward of  $\gamma$  under  $(\pi_1, \pi_2)$  is equal to the product measure of the individual pushforwards, i.e.

$$\gamma^{(\pi_1, \pi_2)} = \gamma^{\pi_1} \otimes \gamma^{\pi_2}.$$

Any  $\gamma \in \Gamma$  operates on  $\mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2} \times \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2}$ , which we will associate with the four projections  $\pi_i$  and  $\tau_i$ ,  $i = 1, 2$ , such that  $\gamma^{\pi_i} = \eta_i$  and  $\gamma^{\tau_i} = \xi_i$  for  $i = 1, 2$ . Because  $\Gamma$  is the set of all couplings of  $\eta_1 \otimes \eta_2$  and  $\xi_1 \otimes \xi_2$ , it holds that  $\pi_1$  and  $\pi_2$  are independent in  $\gamma$ , for any  $\gamma \in \Gamma$ . The same is true for  $\tau_1$  and  $\tau_2$ . Finally, let  $\Gamma'$  be the subset of all measures  $\gamma \in \Gamma$ , such that  $\pi_1$  and  $\tau_2$  are independent in  $\gamma$  and  $\pi_2$  and  $\tau_1$  are independent in  $\gamma$ . For such  $\gamma \in \Gamma'$ , the only pairs projections that are not independent in  $\gamma$  are  $\pi_1$  and  $\tau_1$ , as well as  $\pi_2$  and  $\tau_2$ . This means that

$$\Gamma' = \{\gamma_1 \otimes \gamma_2 \mid \gamma_i \in \Gamma_i, i = 1, 2\}.$$

Therefore it holds that

$$\begin{aligned}
d_p^p(\eta_1 \otimes \eta_2, \xi_1 \otimes \xi_2) &= \inf_{\gamma \in \Gamma} \int \|z - z'\|_2^p \, d\gamma(z, z') \\
&\leq \max \left\{ 1, 2^{p/2-1} \right\} \inf_{\gamma \in \Gamma} \left\{ \int \|x - x'\|_2^p \, d\gamma(z, z') + \int \|y - y'\|_2^p \, d\gamma(z, z') \right\} \\
&\leq \max \left\{ 1, 2^{p/2-1} \right\} \inf_{\gamma \in \Gamma'} \left\{ \int \|x - x'\|_2^p \, d\gamma(z, z') + \int \|y - y'\|_2^p \, d\gamma(z, z') \right\} \\
&= \max \left\{ 1, 2^{p/2-1} \right\} \left( \inf_{\gamma_1 \in \Gamma_1} \int \|x - x'\|_2^p \, d\gamma_1(x, x') + \inf_{\gamma_2 \in \Gamma_2} \int \|y - y'\|_2^p \, d\gamma_2(y, y') \right) \\
&= \max \left\{ 1, 2^{p/2-1} \right\} (d_p^p(\eta_1, \xi_1) + d_p^p(\eta_2, \xi_2)).
\end{aligned}$$

□

**Lemma 5.** *Under the hypothesis the following two identities hold for any  $r \in \{0, 1, \dots, 6\}$ :*

$$\begin{aligned}
H_r \left( B_1, \dots, B_r; F^{(X)} \otimes F^{(Y)} \right) &= d^{-r} \sum_{1 \leq i_1, \dots, i_r \leq d} h_r(B_{1, i_1}, \dots, B_{r, i_r}; \theta), \\
H_r \left( B_1, \dots, B_r; F_N^{(X)} \otimes F_N^{(Y)} \right) &= d^{-r} \sum_{1 \leq i_1, \dots, i_r \leq d} h_r(B_{1, i_1}, \dots, B_{r, i_r}; \mu_n \otimes \nu_n),
\end{aligned}$$

where  $B_{i,j}$  denotes the  $j$ -th element from the  $i$ -th block.

*Proof.* We prove the claim for  $r = 2$ , noting that the other cases can be proven analogously. Let  $F := F^{(X)} \otimes F^{(Y)}$ . Using the linearity of the integral and the definition of  $H$ , we get that

$$\begin{aligned}
H_2(B_1, B_2; F) &= \frac{1}{d^6} \sum_{1 \leq i_1, \dots, i_6 \leq d} \left\{ \int h(B_{1, i_1}, B_{2, i_2}, B'_{3, i_3}, \dots, B'_{6, i_6}) \, dF^4(B'_3, \dots, B'_6) \right. \\
&\quad - \int h(B_{1, i_1}, B'_{2, i_2}, \dots, B'_{6, i_6}) \, dF^5(B'_2, \dots, B'_6) \\
&\quad - \int h(B_{2, i_1}, B'_{2, i_2}, \dots, B'_{6, i_6}) \, dF^5(B'_2, \dots, B'_6) \\
&\quad \left. + \int h(B'_{1, i_1}, \dots, B'_{6, i_6}) \, dF^6(B'_1, \dots, B'_6) \right\} \\
&= \frac{1}{d^6} \sum_{1 \leq i_1, \dots, i_6 \leq d} \left\{ \int h(B_{1, i_1}, B_{2, i_2}, z_3, \dots, z_6) \, d\theta^4(z_3, \dots, z_6) \right. \\
&\quad - \int h(B_{1, i_1}, z_2, \dots, z_6) \, d\theta^5(z_2, \dots, z_6) \\
&\quad - \int h(B_{2, i_1}, z_2, \dots, z_6) \, d\theta^5(z_2, \dots, z_6) \\
&\quad \left. + \int h(z_1, \dots, z_6) \, d\theta^6(z_1, \dots, z_6) \right\}.
\end{aligned}$$

Note that the summands in the last sum are equal to  $h_2(B_{1, i_1}, B_{2, i_2}; \theta)$ , and thus the first identity is proven. What we have used here is the fact that in every summand, each block  $B'_j$  has an effect only through a single coordinate  $B'_{j, i_j}$ , which has marginal distribution  $\mu \otimes \nu = \theta$ . Because there is exactly one such coordinate per block in every summand, we need not worry about the dependence between the coordinates.

The second identity can be shown analogously, noting that taking the mean over  $i_3, \dots, i_6$  gives us as marginals  $\mu_n \otimes \nu_n$ .  $\square$

The following lemma is a slight extension of Lemma 3 by Arcones (1998), where it is stated for  $U$ -statistics. Since we derive the result for  $V$ -statistics, we include a full proof for the sake of readability.

**Lemma 6.** *Let  $(U_i)_{i \in \mathbb{N}}$  be a strictly stationary and absolutely regular process and  $g$  be a symmetric and degenerate kernel of order  $m$ , i.e.,  $\mathbb{E}g(U_1, u_2, \dots, u_m) = 0$  almost surely. Furthermore, assume that for some  $p > 2$  it holds that*

$$\|g(U_{i_1}, \dots, U_{i_m})\|_{L_p} < M$$

for some  $M$  uniform in  $i_1, \dots, i_m$ . Then it holds that

$$\mathbb{E} \left[ \left( \sum_{1 \leq i_1, \dots, i_m \leq n} g(U_{i_1}, \dots, U_{i_m}) \right)^2 \right] \leq 8M^2 m(2m)! \cdot n^m \left\{ 1 + \sum_{d=1}^n d^{m-1} \beta(d)^{(p-2)/p} \right\}.$$

*Proof.* Lemma 3 in Arcones (1998) gives a proof in the case of  $U$ -statistics. By their method of proof we consider

$$\begin{aligned} & \mathbb{E} \left[ \left( \sum_{1 \leq i_1, \dots, i_m \leq n} g(U_{i_1}, \dots, U_{i_m}) \right)^2 \right] \\ &= \sum_{1 \leq i_1, \dots, i_{2m} \leq n} \mathbb{E}[g(U_{i_1}, \dots, U_{i_m})g(U_{i_{m+1}}, \dots, U_{i_{2m}})] \\ &= \sum_{\sigma \in \mathfrak{S}_{2m}} \sum_{1 \leq i_1 \leq \dots \leq i_{2m} \leq n} \mathbb{E}[g(U_{i_{\sigma(1)}}, \dots, U_{i_{\sigma(m)}})g(U_{i_{\sigma(m+1)}}, \dots, U_{i_{\sigma(2m)}})]. \end{aligned}$$

Using the definition

$$I_l := \{(i_1, \dots, i_{2m}) \mid 1 \leq i_1 \leq \dots \leq i_{2m} \leq n \wedge \#\{i_1, \dots, i_{2m}\} = l\},$$

we can write the sum above as

$$(17) \quad \sum_{l=1}^{2m} \sum_{\sigma \in \mathfrak{S}_{2m}} \sum_{(i_1, \dots, i_{2m}) \in I_l} \mathbb{E}[g(U_{\sigma(i_1)}, \dots, U_{\sigma(i_m)})g(U_{\sigma(i_{m+1})}, \dots, U_{\sigma(i_{2m})})].$$

We will bound this object using different methods, depending on  $l$ .

By using the Cauchy-Schwarz-inequality and the monotony of the  $L_p$ -norm, the expected value appearing in (17) can be bounded by  $M^2$ . Furthermore, if  $1 \leq l \leq m$ ,  $\#I_l$  can be bounded by  $n^m$ . Thus,

$$(18) \quad \begin{aligned} & \sum_{l=1}^m \sum_{\sigma \in \mathfrak{S}_{2m}} \sum_{(i_1, \dots, i_{2m}) \in I_l} \mathbb{E}[g(U_{\sigma(i_1)}, \dots, U_{\sigma(i_m)})g(U_{\sigma(i_{m+1})}, \dots, U_{\sigma(i_{2m})})] \\ & \leq M^2 m(2m)! \cdot n^m. \end{aligned}$$

Next, suppose that  $(i_1, \dots, i_{2m}) \in I_l$  for some  $m < l \leq 2m$ . Define

$$j_k := \begin{cases} i_2 - i_1 & \text{for } k = 1 \\ \min\{i_{2k-1} - i_{2k-2}, i_{2k} - i_{2k-1}\} & \text{for } 2 \leq k \leq m-1 \\ i_{2m} - i_{2m-1} & \text{for } k = m \end{cases}$$

We want to make use of Lemma 2 in Arcones (1998), which allows us to compare the expected value  $\mathbb{E}[g(U_{i_{\sigma(1)}}, \dots, U_{i_{\sigma(m)}})g(U_{i_{\sigma(m+1)}}, \dots, U_{i_{\sigma(2m)}})]$  with the integral

$$\int g(u_{\sigma(1)}, \dots, u_{\sigma(m)})g(u_{\sigma(m+1)}, \dots, u_{\sigma(2m)}) \\ d(\mathcal{L}(U_{i_1}, \dots, U_{i_s}) \otimes \mathcal{L}(U_{i_{s+1}}, \dots, U_{i_t}) \otimes \mathcal{L}(U_{i_{t+1}}, \dots, U_{i_{2m}}))(u_1, \dots, u_{2m}),$$

where  $s$  and  $t$  are chosen such that  $i_s < i_{s+1} \leq i_t < i_{t+1}$ , i.e., we separate the original sequence into three independent blocks. Because of the degeneracy of  $g$  and the specific choice of our blocks, this will allow us to bound  $\mathbb{E}[g(U_{i_{\sigma(1)}}, \dots, U_{i_{\sigma(m)}})g(U_{i_{\sigma(m+1)}}, \dots, U_{i_{\sigma(2m)}})]$  itself.

More precisely, in this situation, we want to determine a  $K$  such that  $j_K = \max\{j_1, \dots, j_m\}$  and then separate the original sequence  $U_{i_1}, \dots, U_{i_{2m}}$  into three blocks,  $\{U_{i_1}, \dots, U_{i_{2K-2}}\}$ ,  $\{U_{i_{2K-1}}\}$  and  $\{U_{i_{2K}}, \dots, U_{i_{2m}}\}$ . If the maximum is achieved at  $j_1$  or  $j_m$ , we instead separate the sequence into two blocks,  $\{U_{i_1}\}$  and  $\{U_{i_2}, \dots, U_{i_{2m}}\}$ , or  $\{U_{i_1}, \dots, U_{i_{2m-1}}\}$  and  $\{U_{i_{2m}}\}$ , respectively.

However, for this method to work, we need to ensure that  $i_{2K-1}$  is distinct from both  $i_{2K-2}$  and  $i_{2K}$  if  $2 \leq K \leq m-1$ ; that  $i_1$  is distinct from  $i_2$  if  $K=1$ ; or that  $i_{2m-1}$  is distinct from  $i_{2m}$  if  $K=m$ . Equivalently, one needs to show that  $j_K > 0$ .

Because  $j_K = \max\{j_1, \dots, j_m\}$ , it suffices to show that under our assumptions, there is some  $1 \leq k \leq m$  such that  $j_k > 0$ . Suppose that this is not true, i.e.,  $j_k = 0$  for all  $1 \leq k \leq m$ . This implies

$$\begin{aligned} i_1 &= i_2, \\ i_{2k-1} &= i_{2k-2} \vee i_{2k} = i_{2k-1} \quad \text{for } 2 \leq k \leq m-1, \\ i_{2m} &= i_{2m-1}. \end{aligned}$$

Under this set of constraints,  $\#\{i_1, \dots, i_{2m}\}$  is maximised if for any  $2 \leq k \leq m-1$  either  $i_{2k-1} = i_{2k-2}$  or  $i_{2k} = i_{2k-1}$  hold, but not both, and  $i_{2m-1} \neq i_{2m-2}$ . In this case,  $\#\{i_1, \dots, i_{2m}\} = m$ . This can be seen via induction over  $m$ . Therefore,  $j_K = 0$  implies  $\#\{i_1, \dots, i_{2m}\} \leq m$ , but we have assumed that  $(i_1, \dots, i_{2m}) \in I_l$  for some  $m < l \leq 2m$ , and so  $j_K > 0$ .

Let us now formalise the ‘comparison’ of integrals described above. Suppose that  $j_K = \max\{j_1, \dots, j_m\}$ , then by Lemma 2 in Arcones (1998), the degeneracy of  $g$  and the observations above, it holds that

$$|\mathbb{E}[g(U_{i_{\sigma(1)}}, \dots, U_{i_{\sigma(m)}})g(U_{i_{\sigma(m+1)}}, \dots, U_{i_{\sigma(2m)}})]| \leq 8M^2\beta(j_K)^{(p-2)/p}.$$

Now if  $K=1$  and  $j_K = j_1 = d$ , then there are  $n-d \leq n$  possible values for  $i_1$  which also determine  $i_2$ . Next, because  $j_1 \geq j_2, \dots, j_m$ , it holds that

$$j_2 = \min\{i_3 - i_2, i_4 - i_3\} \leq d.$$

If  $i_4 - i_3 \leq i_3 - i_2$ , then  $i_4 - i_3 \leq d$  and thus there are at most  $d$  possible values for  $i_4$  and at most  $n$  possible values for  $i_3$  (this is a somewhat trivial bound, as there are at most  $n$  possible values for any index. The important part is that we can bound the number of possible values for one of the two indices, in this case  $i_4$ , by something other than  $n$ ). Conversely, if  $i_3 - i_2 \leq i_4 - i_3$ , then there are at most  $d$  possible values for  $i_3$  and at most  $n$  possible values for  $i_4$ . In general, it holds for any  $2 \leq k \leq m$  that there are at most  $nd$  possible values for  $(i_{2k-1}, i_{2k})$  and thus the number of all ordered collections of indices  $(i_1, \dots, i_{2m})$  such that  $j_1 \geq j_2, \dots, j_m$  is bounded by  $n^m d^{m-1}$ . The same method can be employed for any

$1 \leq K \leq m$ , and thus we obtain that

$$(19) \quad \begin{aligned} & \sum_{l=m+1}^{2m} \sum_{\sigma \in \mathfrak{S}_{2m}} \sum_{(i_1, \dots, i_{2m}) \in I_l} \mathbb{E}[g(U_{\sigma(i_1)}, \dots, U_{\sigma(i_m)})g(U_{\sigma(i_{m+1})}, \dots, U_{\sigma(i_{2m})})] \\ & \leq 8M^2 m(2m)! \cdot n^m \sum_{d=0}^n d^{m-1} \beta(d)^{(p-2)/p}. \end{aligned}$$

Combining (18) and (19), we can bound (17) by

$$8M^2 m(2m)! \cdot n^m \left\{ 1 + \sum_{d=1}^n d^{m-1} \beta(d)^{(p-2)/p} \right\},$$

which proves the lemma.  $\square$

**Lemma 7.** *Let  $Z_1$  have finite  $(2 + \delta)$ -moments for some  $\delta > 0$  and suppose that  $d^3 = o(n)$ . Under the hypothesis, it holds that*

$$\mathbb{E} \left[ \left( n\tilde{V} - 15n\tilde{V}_n^{(2)}(h; \theta) \right)^2 \right] \xrightarrow{n \rightarrow \infty} 0,$$

and

$$\mathbb{E} \left[ \left( nV^* - 15nV_n^{*(2)}(h; \mu_n \otimes \nu_n) \right)^2 \right] \xrightarrow{n \rightarrow \infty} 0.$$

*Proof.* We first consider

$$\tilde{V}_n^{(r)}(h; \theta) = \tilde{V}_N^{(r)} \left( H; F^{(X)} \otimes F^{(Y)} \right),$$

where the equality holds due to Lemma 5. The  $n$ -th row of the triangular array  $\tilde{Z}$  can be taken as the first  $N$  observations of an iid process  $(\tilde{B}_k)_{k \in \mathbb{N}}$  with marginal distribution  $\mathcal{L}(Z_1, \dots, Z_d)$ . By Lemma 6, we only need to find a uniform bound for

$$\mathbb{E} \left[ H \left( \tilde{B}_{i_1}, \dots, \tilde{B}_{i_6} \right)^{2+\delta} \right].$$

This is possible if we can instead find a uniform bound for

$$\mathbb{E} \left[ h \left( \tilde{Z}_{i_1}, \dots, \tilde{Z}_{i_6} \right)^{2+\delta} \right],$$

which in turn follows if we can find a finite bound for

$$\mathbb{E} \left[ f \left( \tilde{X}_{i_1}, \dots, \tilde{X}_{i_4} \right)^{2+\delta} \right]$$

for any  $1 \leq i_1, \dots, i_4 \leq n$ . Recall the definition of  $f$  from Section 1. By the triangle inequality, it holds that

$$(20) \quad f(x_1, \dots, x_4) \leq 2\|x_2 - x_3\|_2 \wedge 2\|x_1 - x_4\|_2$$

for all  $x_1, \dots, x_4$ . The uniform bound then follows because  $Z_1$  has finite  $(2 + \delta)$ -moments by assumption.

Thus, by Lemma 6, it holds for all  $2 \leq r \leq 6$  that

$$\mathbb{E} \left[ \tilde{V}_n^{(r)}(h; \theta)^2 \right] = \mathcal{O}(N^{-r}),$$

with the constant involved being independent of  $n$ . Therefore,

$$\begin{aligned} \mathbb{E} \left[ \left( n\tilde{V} - 15n\tilde{V}_n^{(2)}(h; \theta) \right)^2 \right] &= n^2 \mathbb{E} \left[ \left( \sum_{i=3}^6 \binom{6}{i} V_n^{(i)}(h; \theta) \right)^2 \right] \\ &= n^2 \mathcal{O}(N^{-3}) = \mathcal{O}\left(\frac{d^3}{n}\right), \end{aligned}$$

and the bound on the right-hand side is  $o(1)$  by assumption.

Now let us consider

$$V_n^{*(r)}(h; \mu_n \otimes \nu_n) = V_N^{*(r)}\left(H; F_N^{(X)} \otimes F_N^{(Y)}\right).$$

After conditioning on  $Z_1, \dots, Z_n$ , we can once more use Lemma 6. However, we now need to consider the expected values conditional on  $Z_1, \dots, Z_n$ . Let  $j_1, \dots, j_6$  be any collection of indices. Then,

$$\begin{aligned} (21) \quad & \mathbb{E} \left[ H(B_{j_1}^*, \dots, B_{j_6}^*)^{2+\delta} \mid Z_1, \dots, Z_n \right] \\ &= \mathbb{E} \left[ \left( d^{-6} \sum_{1 \leq i_1, \dots, i_6 \leq d} h(B_{j_1, i_1}^*, \dots, B_{j_6, i_6}^*) \right)^{2+\delta} \mid Z_1, \dots, Z_n \right] \\ &\leq d^{-6} \sum_{1 \leq i_1, \dots, i_6 \leq d} \mathbb{E} \left[ h(B_{j_1, i_1}^*, \dots, B_{j_6, i_6}^*)^{2+\delta} \mid Z_1, \dots, Z_n \right] \end{aligned}$$

Applying the Jensen inequality once more shows that the  $(2 + \delta)$ -th power of  $h$ , which by definition is the symmetrisation of  $h'$ , is bounded by the symmetrisation of the  $(2 + \delta)$ -th powers of  $h'$ . This only changes the order of the arguments of  $h'$ , so we will have proven our claim if we can show that

$$(22) \quad d^{-6} \sum_{1 \leq i_1, \dots, i_6 \leq d} \mathbb{E} \left[ h'(B_{j_1, i_1}^*, \dots, B_{j_6, i_6}^*)^{2+\delta} \mid Z_1, \dots, Z_n \right]$$

is uniformly bounded for any collection of indices  $1 \leq j_1, \dots, j_6 \leq N$ . Using (20), it holds for any  $z_1, \dots, z_6$  that

$$|h'(z_1, \dots, z_6)| \leq 4 \left( \sum_{k=1}^6 \|x_k\|_2 \right) \left( \sum_{k=1}^6 \|y_k\|_2 \right),$$



Now, because the bootstrapped  $X$ -blocks and  $Y$ -blocks, denoted for the moment by  $B_k^{*(X)}$  and  $B_k^{*(Y)}$ , form two independent iid sequences, (22) can be bounded by

$$\begin{aligned}
& 4^{2+\delta} d^{-6} \sum_{1 \leq i_1, \dots, i_6 \leq d} \left\{ \mathbb{E} \left[ \left( \sum_{k=1}^6 \|B_{j_k, i_k}^{*(X)}\|_2 \right)^{2+\delta} \mid X_1, \dots, X_n \right] \right. \\
& \quad \left. \cdot \mathbb{E} \left[ \left( \sum_{k=1}^6 \|B_{j_k, i_k}^{*(Y)}\|_2 \right)^{2+\delta} \mid Y_1, \dots, Y_n \right] \right\} \\
& \leq 4^{2+\delta} 6^{2(1+\delta)} d^{-6} \sum_{1 \leq i_1, \dots, i_6 \leq d} \left\{ \mathbb{E} \left[ \sum_{k=1}^6 \|B_{1, i_k}^{*(X)}\|_2^{2+\delta} \mid X_1, \dots, X_n \right] \right. \\
& \quad \left. \cdot \mathbb{E} \left[ \sum_{k=1}^6 \|B_{1, i_k}^{*(Y)}\|_2^{2+\delta} \mid Y_1, \dots, Y_n \right] \right\} \\
& = 4^{2+\delta} 6^{2(1+\delta)} \sum_{k,l=1}^6 d^{-6} \sum_{1 \leq i_1, \dots, i_6 \leq d} \left\{ \mathbb{E} \left[ \|B_{1, i_k}^{*(X)}\|_2^{2+\delta} \mid X_1, \dots, X_n \right] \right. \\
& \quad \left. \cdot \mathbb{E} \left[ \|B_{1, i_l}^{*(Y)}\|_2^{2+\delta} \mid Y_1, \dots, Y_n \right] \right\} \\
& = 4^{2+\delta} 6^{2(1+\delta)} d^{-2} \sum_{1 \leq i_1, i_2 \leq d} \mathbb{E} \left[ \|B_{1, i_1}^{*(X)}\|_2^{2+\delta} \mid X_1, \dots, X_n \right] \mathbb{E} \left[ \|B_{1, i_2}^{*(Y)}\|_2^{2+\delta} \mid Y_1, \dots, Y_n \right] \\
& = 4^{2+\delta} 6^{2(1+\delta)} \left( (Nd)^{-1} \sum_{k=1}^N \sum_{i=1}^d \|X_{(k-1)d+i}\|_2^{2+\delta} \right) \left( (Nd)^{-1} \sum_{k=1}^N \sum_{i=1}^d \|Y_{(k-1)d+i}\|_2^{2+\delta} \right) \\
& = 4^{2+\delta} 6^{2(1+\delta)} \int \|x\|_2^{2+\delta} d\mu_n(x) \int \|y\|_2^{2+\delta} d\nu_n(y) \\
& \xrightarrow[n \rightarrow \infty]{a.s.} 4^{2+\delta} 6^{2(1+\delta)} \int \|x\|_2^{2+\delta} d\mu(x) \int \|y\|_2^{2+\delta} d\nu(y)
\end{aligned}$$

by Birkhoff's Ergodic Theorem. The integrals in the last line are finite because  $X_1$  and  $Y_1$  have finite  $(2 + \delta)$ -moments by assumption. From this it follows that we can almost surely bound (22) and therefore (21) uniformly in  $j_1, \dots, j_6$  and for almost all  $n$ , and with a sufficiently large constant even for all  $n$ . Therefore, by Lemma 6,

$$\mathbb{E} \left[ V_n^{*(r)}(h; \mu_n \otimes \nu_n)^2 \mid Z_1, \dots, Z_n \right] = \mathcal{O}(N^{-r})$$

and

$$\begin{aligned}
\mathbb{E} \left[ V_n^{*(r)}(h; \mu_n \otimes \nu_n)^2 \right] &= \mathbb{E} \left[ \mathbb{E} \left[ V_n^{*(r)}(h; \mu_n \otimes \nu_n)^2 \mid Z_1, \dots, Z_n \right] \right] \\
&= \mathcal{O}(N^{-r})
\end{aligned}$$

for all  $2 \leq r \leq 6$ , with the constant involved being independent of  $n$ . From here we can proceed as before, concluding the proof.  $\square$

The previous results have given us enough tools to derive a bound in terms of Wasserstein distances of the block distributions. We will need to show that these converge in some sense to 0. This is achieved by the next lemma, which is simply an application of the results of Appendix B to the situation at hand.

**Lemma 8.** *Suppose that the conditions of Corollary 2 are fulfilled with the bound  $M = M(d)$  growing at most exponentially in  $d$ . If  $d = \log(n)^\gamma$  for some  $0 < \gamma < 1/2$  and  $Z_1$  has finite  $q$ -moments for some  $q > p > 2$ , then, for any  $s > 0$ , we have that*

$$d_p^p \left( F_N^{(X)}, F^{(X)} \right) = o_{\mathbb{P}} \left( d^{-s} \right),$$

and

$$d_p^p \left( F_N^{(Y)}, F^{(Y)} \right) = o_{\mathbb{P}} \left( d^{-s} \right).$$

*Proof.* We prove the first claim. The second claim can be proven in the same way, substituting  $\ell_2$  for  $\ell_1$ .

For any  $\varepsilon > 0$ , the Markov inequality and Corollary 2 give us

$$\begin{aligned} \mathbb{P} \left( d_p^p \left( F_N^{(X)}, F^{(X)} \right) \geq \varepsilon d^{-s} \right) &\leq \frac{d^s}{\varepsilon} \mathbb{E} d_p^p \left( F_N^{(X)}, F^{(X)} \right) \\ (23) \quad &\leq \frac{d^s}{\varepsilon} 6^p c_0 2^{3(\ell_1 d)/2-p} \mathfrak{D}^p N^{-\frac{p-2}{4(\ell_1 d)}} \left( \frac{1 + M^{\frac{(\ell_1 d)/2-p}{(\ell_1 d)}}}{1 - 2^{p-(\ell_1 d)/2}} + \frac{1}{1 - 2^{-p}} + 4M^{\frac{1}{(\ell_1 d)}} \right) \\ &\quad + \frac{d^s}{\varepsilon} 6^p 2c' \cdot (\ell_1 d)^{1+q/2} N^{(p-2)(p-q)/(2(\ell_1 d)^2)} \end{aligned}$$

where  $\mathfrak{D} := \sup\{\|z_1 - z_2\|_2 \mid z_1, z_2 \in [0, 1]^{\ell_1 d}\}$ .

We first consider the term

$$(24) \quad \frac{d^s}{\varepsilon} 6^p c_0 2^{3(\ell_1 d)/2-p} \mathfrak{D}^p N^{-\frac{p-2}{4(\ell_1 d)}} \left( \frac{1 + M^{\frac{(\ell_1 d)/2-p}{(\ell_1 d)}}}{1 - 2^{p-(\ell_1 d)/2}} + \frac{1}{1 - 2^{-p}} + 4M^{\frac{1}{(\ell_1 d)}} \right).$$

Because  $M$  grows at most exponentially in  $d$ , we can bound  $M^d$  by  $2^{\rho d}$  for some  $\rho > 0$ . Without loss of generality, assume that  $\rho > \ell_1$ . Using this in (24) results in a bound whose asymptotic behaviour is determined by

$$(25) \quad N^{-\frac{p-2}{4(\ell_1 d)}} 2^{\frac{(3\ell_1 + \rho)d}{2}} d^s \mathfrak{D}^p \leq N^{-\frac{p-2}{4(\ell_1 d)}} 2^{3\rho d},$$

where the inequality holds for sufficiently large  $n$ , because  $d^s \mathfrak{D}^p$  is polynomial in growth (and because  $\rho > \ell_1$ ). Observe that, due to our choice of  $d$ ,

$$N^{\frac{1}{d}} = \exp(\log(N))^{\frac{1}{\log(n)^\gamma}} = \exp\left(\frac{\log(N)}{\log(n)^\gamma}\right) = \exp\left(\log(n)^{1-\gamma} - \frac{\log(d)}{\log(n)^\gamma}\right),$$

and  $2^d = \exp(\log(2) \cdot \log(n)^\gamma)$ . Thus, the righthand side of (25) is equal to

$$(26) \quad \exp\left(3\rho \log(2) \cdot \log(n)^\gamma - \frac{p-2}{4\ell_1} \left\{ \log(n)^{1-\gamma} - \frac{\log(d)}{\log(n)^\gamma} \right\}\right).$$

Because  $0 < \gamma < 1/2$ , it holds that

$$3\rho \log(2) \cdot \log(n)^\gamma - \frac{p-2}{4\ell_1} \log(n)^{1-\gamma} \xrightarrow[n \rightarrow \infty]{} -\infty,$$

and

$$\frac{\log(d)}{\log(n)^\gamma} = \frac{\log(d)}{d} \xrightarrow[n \rightarrow \infty]{} 0.$$

This implies that (26) converges to 0 as  $n \rightarrow \infty$ , which in turn implies that the righthand side of (25) and thus (24) converge to 0.

Let us now consider the term

$$(27) \quad \frac{d^s}{\varepsilon} 6^p 2c' \cdot (\ell_1 d)^{1+q/2} N^{(p-2)(p-q)/(2(\ell_1 d)^2)},$$

which, writing  $\alpha := (p-2)(p-q)/(2\ell_1^2)$ , is equal to

$$\frac{6^p 2c' \ell_1^{1+q/2}}{\varepsilon} \exp\left(\left(1 + \frac{q}{2} + s\right) \log(d) + \alpha \log(n)^{1-2\gamma} - \alpha \frac{\log(d)}{d}\right).$$

The asymptotic behaviour of this expression is determined by

$$\begin{aligned} & \exp\left(\left(1 + \frac{q}{2} + s\right) \log(d) + \alpha \log(n)^{1-2\gamma}\right) \\ &= \exp\left(\gamma\left(1 + \frac{q}{2} + s\right) \log(\log(n)) + \alpha \log(n)^{1-2\gamma}\right) \xrightarrow[n \rightarrow \infty]{} 0, \end{aligned}$$

where the convergence holds because  $0 < \gamma < 1/2$  and  $\alpha < 0$ . Thus, (27) converges to 0, which together with the convergence of (24) implies that the entire bound (23) converges to 0, proving the lemma.  $\square$

We are now in a position to prove the validity of our bootstrap method.

*Proof of Theorem 1.* The triangle inequality gives us

$$d_1(\zeta, nV^*) \leq d_1(\zeta, n\tilde{V}) + d_1(n\tilde{V}, nV^*),$$

By Theorem 3, the first summand converges to 0 as  $n \rightarrow \infty$ . It remains to examine the second summand.

By Lemma 7, it is sufficient to consider the  $V$ -statistics with kernels  $\bar{h}_2(z, z'; \theta)$  and  $\bar{h}_2(z, z'; \mu_n \otimes \nu_n)$ , which, due to equal Lemma 5, are equal to the  $V$ -statistics with kernels  $H_2(B, B'; F^{(X)} \otimes F^{(Y)})$  and  $H_2(B, B'; F_N^{(X)} \otimes F_N^{(Y)})$ . Thus, we consider the distance

$$\begin{aligned} (28) \quad & d_1\left(\frac{Nd}{N^2} \sum_{1 \leq i_1, i_2 \leq N} H_2(\tilde{B}_{i_1}, \tilde{B}_{i_2}; F^{(X)} \otimes F^{(Y)}), \frac{Nd}{N^2} \sum_{1 \leq i_1, i_2 \leq N} H_2(B_{i_1}^*, B_{i_2}^*; F_N^{(X)} \otimes F_N^{(Y)})\right) \\ & \leq d \cdot d_2\left(\frac{1}{N} \sum_{1 \leq i_1, i_2 \leq N} H_2(\tilde{B}_{i_1}, \tilde{B}_{i_2}; F^{(X)} \otimes F^{(Y)}), \frac{1}{N} \sum_{1 \leq i_1, i_2 \leq N} H_2(B_{i_1}^*, B_{i_2}^*; F_N^{(X)} \otimes F_N^{(Y)})\right) \\ & \leq \text{const} \cdot d \cdot d_4\left(F_N^{(X)} \otimes F_N^{(Y)}, F^{(X)} \otimes F^{(Y)}\right) \\ & \leq \text{const} \cdot d \cdot \left\{d_4\left(F_N^{(X)}, F^{(X)}\right) + d_4\left(F_N^{(Y)}, F^{(Y)}\right)\right\}, \end{aligned}$$

for some constant uniform in  $n$ , where in the last two lines we have used Proposition 1 and Lemma 4. The distances  $d_4\left(F_N^{(X)}, F^{(X)}\right)$  and  $d_4\left(F_N^{(Y)}, F^{(Y)}\right)$  are  $o_{\mathbb{P}}(d^{-1})$  due to Lemma 8, so the entire bound converges to 0 in probability.  $\square$

#### APPENDIX B. BOUNDING THE WASSERSTEIN DISTANCE BETWEEN THE EMPIRICAL MEASURE OF A STRONGLY MIXING PROCESS AND ITS MARGINAL DISTRIBUTION

In the proof of Theorem 1, we have used the auxiliary results developed in Appendix A to bound the Wasserstein distance between the empirical distance covariance and the bootstrap statistic in terms of the Wasserstein distances between the block distributions and their empirical analogues. In this section, we will develop general results to bound the expected Wasserstein distance between some measure  $\xi$  and its empirical analogue  $\xi_n$ , provided the underlying sample generating process is strictly stationary and  $\alpha$ -mixing. The central results of this section are Propositions 2 and 3, Theorem 4 and Corollary 2.

**Lemma 9.** *Let  $(U_i)_{i \in \mathbb{N}}$  be a strictly stationary and  $\alpha$ -mixing sequence of random variables with values in some separable metric space  $\mathcal{X}$  and marginal distribution  $\xi$ , and let  $\alpha(n) \leq f(n) = \mathcal{O}(n^{-r_0})$  for some  $r_0 > 1$ . Furthermore, let  $F \subseteq \mathcal{X}$  be a subset with  $\xi(F) \geq t_0$  for some  $t_0 > 0$ . Then there exists a constant  $c_0 > 2$ , dependent on  $r$  and  $f$ , but not on  $n$ ,  $F$  or  $t_0$ , such that*

$$\text{Var} \left( \sum_{i=1}^n \mathbf{1}_F(U_i) \right) \leq c_0 n t_0^{-1} \xi(F).$$

*If the sequence is  $\phi$ -mixing, we can drop the assumption  $\xi(F) \geq t_0$ . The assumptions concerning the growth rate of the mixing coefficients then apply to the  $\phi$ -mixing coefficients, and we have the inequality*

$$\text{Var} \left( \sum_{i=1}^n \mathbf{1}_F(U_i) \right) \leq c_0 n \xi(F).$$

*Proof.* We have

$$\begin{aligned} \text{Var} \left( \sum_{i=1}^n \mathbf{1}_F(U_i) \right) &= \left| \sum_{i=1}^n \text{Var}(\mathbf{1}_F(U_i)) + 2 \sum_{d=1}^{n-1} (n-d) \text{Cov}(\mathbf{1}_F(U_1), \mathbf{1}_F(U_{1+d})) \right| \\ &\leq \sum_{i=1}^n \text{Var}(\mathbf{1}_F(U_i)) + 4n \sum_{d=1}^{n-1} |\text{Cov}(\mathbf{1}_F(U_1), \mathbf{1}_F(U_{1+d}))|, \end{aligned}$$

and  $|\text{Cov}(\mathbf{1}_F(U_1), \mathbf{1}_F(U_{1+d}))| \leq \alpha(d) \leq t_0^{-1} \xi(F) \alpha(d)$ . By assumption there is some constant  $c = c(f) > 0$  such that  $\alpha(n) \leq f(n) \leq c n^{-r_0}$  for almost all  $n$ , and without loss of generality for all  $n$ . Therefore

$$\sum_{d=1}^{n-1} |\text{Cov}(\mathbf{1}_F(U_1), \mathbf{1}_F(U_{1+d}))| \leq c t_0^{-1} \xi(F) \sum_{d=1}^{n-1} d^{-r_0} \leq c \zeta(r_0) t_0^{-1} \xi(F).$$

Furthermore,  $\text{Var}(\mathbf{1}_F(U_1)) = \xi(F) \xi(F^C) \leq \xi(F) \leq t_0^{-1} \xi(F)$ . Putting all of this together, we obtain

$$\text{Var} \left( \sum_{i=1}^n \mathbf{1}_F(U_i) \right) \leq t_0^{-1} n \xi(F) + 64 n c \zeta(r_0) t_0^{-1} \xi(F) = (1 + 64 \zeta(r_0)) \cdot n t_0^{-1} \xi(F).$$

For  $\phi$ -mixing random variables, we can bound the covariance directly using Theorem 3.9 in Bradley (2007), obtaining

$$|\text{Cov}(\mathbf{1}_F(U_1), \mathbf{1}_F(U_{1+d}))| \leq 2\phi(d) \|\mathbf{1}_F(U_1)\|_{L_1} \|\mathbf{1}_F(U_1)\|_{L_\infty} = 2\phi(d) \xi(F),$$

and the rest of the proof proceeds analogously.  $\square$

*Remark.* We assume that  $\alpha$  is bounded by some function  $f$  to exclude the possibility of any implicit influence the dimension might have on the mixing coefficients. Obviously,  $f$  must not depend on the dimension.

**Lemma 10.** *Let  $(U_i)_{i \in \mathbb{N}}$  be a random process that fulfills the assumptions of Lemma 9,  $C$  a Borel set and  $r > 1$ . Then, setting  $\zeta(t) := \sqrt{t} \wedge t$  for  $t \geq 0$ , the following inequality holds:*

$$\mathbb{E}|n \xi_n(C) - n \xi(C)| \leq \zeta_{n,r}(\xi(C)),$$

where

$$\zeta_{n,r}(\xi(C)) := \begin{cases} c_0 \zeta(n \xi(C)) & \text{for } \xi(C) \leq n^{-1} \\ c_0 n^{1/2-1/(2r)} (n \xi(C))^{1/2} & \text{for } n^{-1} < \xi(C) \leq n^{-1/2} \\ c_0 n^{1/4} \zeta(n \xi(C)) & \text{for } \xi(C) > n^{-1/2} \end{cases}$$

*Proof.* We use the triangle inequality to obtain

$$\mathbb{E}|n\xi_n(C) - n\xi(C)| \leq 2n\xi(C) \leq c_0n\xi(C).$$

If  $\xi(C) \leq n^{-1}$ , the right-hand side is equal to  $c_0\zeta(n\xi(C))$ .

If  $n^{-1} < \xi(C) \leq n^{-1/2}$ , then for any  $r > 1$ ,

$$\begin{aligned} \mathbb{E}|n\xi_n(C) - n\xi(C)| &\leq c_0n\xi(C) = c_0(n^r\xi(C)^r)^{\frac{1}{r}} \\ &\leq c_0(n^{r-(r-1)/2}\xi(C))^{\frac{1}{r}} \\ &= c_0(n\xi(C))^{\frac{1}{r}}n^{(r-(r-1)/2-1)/r} \\ &= c_0(n\xi(C))^{\frac{1}{r}}n^{1/2-1/(2r)}. \end{aligned}$$

For  $\xi(C) > n^{-1/2}$  we employ Lemma 9 and the Cauchy-Schwarz inequality to obtain

$$\mathbb{E}|n\xi_n(C) - n\xi(C)| \leq \sqrt{\text{Var}(n\xi_n(C))} \leq \sqrt{c_0n^{3/2}\xi(C)} = n^{\frac{1}{4}}\zeta(c_0n\xi(C)),$$

where the last equality holds because of  $\xi(C) \geq n^{-1/2} \geq n^{-1}$ .  $\square$

**Lemma 11.** *Let  $(U_i)_{i \in \mathbb{N}}$  be a random process that fulfills the assumptions of Lemma 9 in the  $\phi$ -mixing version, and let  $C$  be any Borel set. Then, with the same function  $\zeta$  as in Lemma 10, the following inequality holds:*

$$\mathbb{E}|n\xi_n(C) - n\xi(C)| \leq \zeta(2c_0n\xi(C)).$$

*Proof.* If  $\xi(C) \geq n^{-1}$ , the Jensen inequality yields

$$\mathbb{E}|n\xi_n(C) - n\xi(C)| \leq \sqrt{\text{Var}(n\xi_n(C))} \leq \sqrt{c_0n\xi(C)} = \zeta(c_0n\xi(C)),$$

where we have made use of Lemma 9 in the second inequality.

Suppose now that  $\xi(C) \leq n^{-1}$ . The function  $\zeta$  is bijective on  $\mathbb{R}_{\geq 0}$ . We therefore have the existence of an inverse function  $\zeta^{-1}$ , given by  $\zeta^{-1}(t) = t^2 \vee t$ . Applying the Jensen inequality gives us

$$\begin{aligned} \zeta^{-1}(\mathbb{E}|n\xi_n(C) - n\xi(C)|) &\leq \mathbb{E}[\zeta^{-1}(|n\xi_n(C) - n\xi(C)|)] \\ &\leq \mathbb{E}[\mathbf{1}_{(0,1)}(|n\xi_n(C) - n\xi(C)|)|n\xi_n(C) - n\xi(C)|] \\ &\quad + \text{Var}(n\xi_n(C)), \end{aligned}$$

where in the last inequality we first partition the domain of integration into  $(0, 1)$  and  $[1, \infty)$  and then bound the integral on  $[1, \infty)$  by the integral on  $\mathbb{R}_{\geq 0}$ .

We now want to examine the first expected value. For this we define the sets

$$\begin{aligned} M &:= \{|n\xi_n(C) - n\xi(C)| < 1\}, \\ M_0 &:= \{n\xi_n(C) = 0\}, \\ M_1 &:= \{n\xi_n(C) = 1\}. \end{aligned}$$

Due to  $n\xi(C) < 1$ , the sets  $M_0$  and  $M_1$  form a partition of  $M$ . Therefore,

$$\begin{aligned} &\mathbb{E}[\mathbf{1}_{(0,1)}(|n\xi_n(C) - n\xi(C)|)|n\xi_n(C) - n\xi(C)|] \\ &= n\xi(C)\mathbb{P}(M_0) + (1 - n\xi(C))\mathbb{P}(M_1) \\ &\leq n\xi(C) + \mathbb{P}(M_1). \end{aligned}$$

We can furthermore bound  $\mathbb{P}(M_1)$  by

$$\mathbb{P}(M_1) = \sum_{i=1}^n \mathbb{P}(U_i \in C \wedge U_j \notin C \quad \forall j \neq i) \leq \sum_{i=1}^n \mathbb{P}(U_i \in C) = n\xi(C),$$

and so we get

$$\mathbb{E} \left[ \mathbf{1}_{(0,1)} (|n\xi_n(C) - n\xi(C)|) |n\xi_n(C) - n\xi(C)| \right] \leq 2n\xi(C) \leq c_0 n\xi(C).$$

With the upper bounds obtained before, it follows that

$$\zeta^{-1}(\mathbb{E}|n\xi_n(C) - n\xi(C)|) \leq c_0 n\xi(C) + \text{Var}(n\xi_n(C)) \leq 2c_0 n\xi(C).$$

Together with the identity

$$\mathbb{E}|n\xi_n(C) - n\xi(C)| = \zeta \left( \zeta^{-1}(\mathbb{E}|n\xi_n(C) - n\xi(C)|) \right)$$

this proves the lemma, because  $\zeta$  is an isotone function.  $\square$

**Lemma 12.** *Let  $\xi$  be a probability measure on  $[0, 1]^d$ , and  $M > 0$  such that for all  $F \in \mathcal{P}_l$  it holds that  $\xi(F) \leq M2^{-dl}$ . This is the case if  $\xi$  has a bounded density function with respect to the Lebesgue measure. Then, for any  $r > 0$ , the following inequality holds:*

$$L_r = L_r(n) := \max \{ l \in \mathbb{N} \mid \exists F \in \mathcal{P}_l : \xi(F) > n^{-r} \} \leq \log_2 \left( 2M^{\frac{1}{d}} n^{\frac{r}{d}} \right)$$

*Proof.* Let  $F \in \mathcal{P}_l$ , then  $\xi(F) \leq M2^{-dl}$ . This gives us

$$L_1(M^{-1}2^{dl}) \leq l = \log_2 \left( M^{\frac{1}{d}} (M^{-1}2^{dl})^{\frac{1}{d}} \right)$$

for all  $l \in \mathbb{N}$ . Now fix some  $n \in \mathbb{N}$ . We set

$$l^* := \min \{ l \in \mathbb{N} \mid M^{-1}2^{d(l-1)} \leq n \leq M^{-1}2^{dl} \}.$$

Obviously it holds that  $2^d n \geq M^{-1}2^{dl^*}$ . The mappings  $t \mapsto \log_2 \left( (Mt)^{\frac{1}{d}} \right)$  and  $L_1$  are isotone, and therefore

$$L_1(n) \leq L_1 \left( M^{-1}2^{dl^*} \right) \leq \log_2 \left( M^{\frac{1}{d}} (M^{-1}2^{dl^*})^{\frac{1}{d}} \right) \leq \log_2 \left( M^{\frac{1}{d}} (2^d n)^{\frac{1}{d}} \right).$$

This proves the claim for  $r = 1$ . For arbitrary  $r > 0$ , notice that  $L_r(n) = L_1(n^r)$ .  $\square$

*Remark 5.* Note that if we are considering a hypercube with side length  $K$  instead of the unit cube, then we can apply Lemma 12 after  $\log_2(K)$  steps, and therefore the resulting bound will be

$$\log_2 \left( 2M^{\frac{1}{d}} n^{\frac{r}{d}} \right) + \log_2(K) = \log_2 \left( 2KM^{\frac{1}{d}} n^{\frac{r}{d}} \right).$$

The following proposition is a generalisation of Proposition 1 in Dereich et al. (2013).

**Proposition 2.** *Let  $d \in \mathbb{N}$  and  $1 \leq p < d/2$  be fixed. Let  $\xi$  be a probability measure on  $[0, 1]^d$  that fulfills the assumptions of Lemma 12. Then, for any  $n \in \mathbb{N}$ , it holds that*

$$\left( \mathbb{E} d_p^p(\xi_n, \xi) \right)^{\frac{1}{p}} \leq n^{-\frac{p-2}{2pd}} \left\{ c_0 2^{3d/2-p} \mathfrak{D}^p \left( \frac{1 + M^{\frac{d/2-p}{d}}}{1 - 2^{p-d/2}} + \frac{1}{1 - 2^{-p}} + 4M^{\frac{1}{d}} \right) \right\}^{\frac{1}{p}},$$

where  $\xi_n$  is the empirical measure of a strictly stationary and  $\alpha$ -mixing process  $(U_i)_{i \in \mathbb{N}}$  with marginal distribution  $\xi$  and  $\alpha(n) \leq f(n) = \mathcal{O}(n^{-r_0})$  for some function  $f$  and some constant  $r_0 > 1$ . The constant  $c_0$  only depends on  $f$  and  $r_0$ .

*Proof.* Lemma 2 gives us

$$(29) \quad d_p^p(\xi_n, \xi) \leq \frac{1}{2} \mathfrak{D}^p \sum_{l=0}^{\infty} 2^{-pl} \sum_{F \in \mathcal{P}_l} \sum_{C \leftarrow F} \left| \xi_n(C) - \xi_n(F) \frac{\xi(C)}{\xi(F)} \right|.$$

We consider the object

$$(30) \quad \mathbb{E} \left| \xi_n(C) - \xi_n(F) \frac{\xi(C)}{\xi(F)} \right|$$

An application of the triangle inequality lets us bound this expectation by

$$\mathbb{E} |\xi_n(C) - \xi(C)| + \xi(C) \mathbb{E} \left| 1 - \frac{\xi_n(F)}{\xi(F)} \right|.$$

Using Lemma 10, this gives us

$$\mathbb{E} |\xi_n(C) - \xi(C)| \leq n^{-1} \zeta_{n,r}(\xi(C)),$$

for any  $r > 1$ . In a similar way, we obtain

$$\begin{aligned} \xi(C) \mathbb{E} \left| 1 - \frac{\xi_n(F)}{\xi(F)} \right| &= \frac{\xi(C)}{\xi(F)} \mathbb{E} |\xi(F) - \xi_n(F)| \\ &\leq n^{-1} \frac{\xi(C)}{\xi(F)} \zeta_{n,r}(\xi(F)). \end{aligned}$$

Thus, (30) can be bounded by

$$(31) \quad \begin{aligned} \mathbb{E} \left| \xi_n(C) - \xi_n(F) \frac{\xi(C)}{\xi(F)} \right| &\leq n^{-1} \left\{ \zeta_{n,r}(\xi(C)) + \frac{\xi(C)}{\xi(F)} \zeta_{n,r}(\xi(F)) \right\} \\ &=: n^{-1} (R_1(C) + R_2(C, F)). \end{aligned}$$

We now wish to bound the sums further. For this, we will make use of the fact that only finitely many sets have measure greater than  $n^{-1/2}$ . Note that  $\sum_{C \leftarrow F} R_1(C)$  is equal to

$$(32) \quad \begin{aligned} &\sum_{C \leftarrow F: \xi(C) \leq n^{-1}} \zeta_{n,r}(\xi(C)) + \sum_{C \leftarrow F: n^{-1} < \xi(C) \leq n^{-1/2}} \zeta_{n,r}(\xi(C)) + \sum_{C \leftarrow F: \xi(C) > n^{-1/2}} \zeta_{n,r}(\xi(C)) \\ &= c_0 \left( \sum_{C \leftarrow F: \xi(C) \leq n^{-1}} \zeta(n\xi(C)) + n^{1/2-1/(2r)} \sum_{C \leftarrow F: n^{-1} < \xi(C) \leq n^{-1/2}} (n\xi(C))^{1/r} \right. \\ &\quad \left. + n^{1/4} \sum_{C \leftarrow F: \xi(C) > n^{-1/2}} \zeta(n\xi(C)) \right) \\ &\leq c_0 \left( \sum_{C \leftarrow F} \zeta(n\xi(C)) + n^{1/2-1/(2r)} \sum_{C \leftarrow F: \xi(C) > n^{-1}} (n\xi(C))^{1/r} + n^{1/4} \sum_{C \leftarrow F: \xi(C) > n^{-1/2}} \zeta(n\xi(C)) \right). \end{aligned}$$

The Jensen inequality implies that

$$(33) \quad \begin{aligned} \sum_{C \leftarrow F} \zeta(n\xi(C)) &= 2^d 2^{-d} \sum_{C \leftarrow F} \zeta(n\xi(C)) \\ &\leq 2^d \zeta \left( 2^{-d} \sum_{C \leftarrow F} n\xi(C) \right) \\ &= 2^d \zeta(2^{-d} n\xi(F)). \end{aligned}$$

Recall that  $\#\mathcal{P}_l = 2^{dl}$ , and that every  $\mathcal{P}_l$  forms a partition of  $[0, 1]^d$ . This gives us

$$(34) \quad \sum_{F \in \mathcal{P}_l} \zeta(2^{-d}n\xi(F)) \leq 2^{dl} \zeta \left( 2^{-dl} \sum_{F \in \mathcal{P}_l} 2^{-d}n\xi(F) \right) \leq 2^{dl} \zeta(2^{-dl}n).$$

We now choose  $l^* := \lfloor \log_2 \left( n^{\frac{1}{d}} \right) \rfloor$ , with which we have

$$(35) \quad \begin{aligned} \sum_{l=0}^{\infty} 2^{(d-p)l} \zeta(2^{-dl}n) &= \sum_{l=0}^{l^*} 2^{(d/2-p)l} \sqrt{n} + \sum_{l>l^*} 2^{-pl}n \\ &\leq \sum_{k=0}^{\infty} 2^{(d/2-p)(l^*-k)} \sqrt{n} + 2^{-p(l^*+1)} \sum_{l=0}^{\infty} 2^{-pl}n \\ &= n^{\frac{d/2-p}{d}} \sqrt{n} \frac{1}{1-2^{p-d/2}} + n^{-\frac{p}{d}} n \frac{1}{1-2^{-p}} \\ &= n^{1-\frac{p}{d}} \left\{ \frac{1}{1-2^{p-d/2}} + \frac{1}{1-2^{-p}} \right\}. \end{aligned}$$

Notice that, in (32), the sum over all sets  $C \leftarrow F$  with measure greater than  $n^{-1/2}$  is empty if  $\xi(F) \leq n^{-1/2}$ . Thus,

$$(36) \quad \begin{aligned} &\sum_{F \in \mathcal{P}_l} n^{\frac{1}{4}} \sum_{C \leftarrow F: \xi(C) > n^{-1/2}} \zeta(n\xi(C)) \\ &\leq n^{\frac{1}{4}} \sum_{F \in \mathcal{P}_l: \xi(F) > n^{-1/2}} \sum_{C \leftarrow F} \zeta(n\xi(C)) \\ &\leq n^{\frac{1}{4}} \sum_{F \in \mathcal{P}_l: \xi(F) > n^{-1/2}} 2^d \zeta(2^{-d}n\xi(F)), \end{aligned}$$



where we have made use of (33) in the last step. Lemma 12 therefore implies

$$\begin{aligned}
& n^{\frac{1}{4}} \sum_{l=0}^{\infty} 2^{-pl} \sum_{F \in \mathcal{P}_l: \xi(F) > n^{-1/2}} \zeta(2^{-d} n \xi(F)) \\
& \leq n^{\frac{1}{4}} \sum_{l=0}^{L_{1/2}(n)} 2^{-pl} \sum_{F \in \mathcal{P}_l} \zeta(2^{-d} n \xi(F)) \\
& \leq n^{\frac{1}{4}} \sum_{l=0}^{L_{1/2}(n)} 2^{(d-p)l} \zeta(2^{-dl} n) \\
& \leq n^{\frac{1}{4}} \sum_{l=0}^{L_{1/2}(n)} 2^{(d-p)l} \sqrt{2^{-dl} n} \\
(37) \quad & = n^{\frac{3}{4}} \sum_{l=0}^{L_{1/2}(n)} 2^{(d/2-p)l} \\
& \leq n^{\frac{3}{4}} \sum_{k=0}^{\infty} 2^{(d/2-p)(L_{1/2}(n)-k)} \\
& = n^{\frac{3}{4}} \left( 2M^{\frac{1}{d}} n^{\frac{1}{2d}} \right)^{d/2-p} \sum_{k=0}^{\infty} 2^{-k(d/2-p)} \\
& = 2^{d/2-p} M^{\frac{d/2-p}{d}} n^{\frac{3}{4} + \frac{d}{4d} - \frac{p}{2d}} \frac{1}{1 - 2^{p-d/2}} \\
& = 2^{d/2-p} M^{\frac{d/2-p}{d}} n^{1 - \frac{p}{2d}} \frac{1}{1 - 2^{p-d/2}}.
\end{aligned}$$

We will now bound the sum over all sets  $C \leftarrow F$  with measure greater than  $n^{-1}$ . Once again, it suffices to consider only those sets whose parents  $F$  have measure greater than  $n^{-1}$ , and because the function  $t \mapsto t^{\frac{1}{r}}$  is concave, we obtain

$$\begin{aligned}
& \sum_{F \in \mathcal{P}_l} n^{1/2-1/(2r)} \sum_{C \leftarrow F: \xi(C) > n^{-1}} (n \xi(C))^{\frac{1}{r}} \\
(38) \quad & \leq n^{1/2-1/(2r)} \sum_{F \in \mathcal{P}_l: \xi(F) > n^{-1}} \sum_{C \leftarrow F} (n \xi(C))^{\frac{1}{r}} \\
& \leq n^{1/2-1/(2r)} \sum_{F \in \mathcal{P}_l: \xi(F) > n^{-1}} 2^d (2^{-d} n \xi(F))^{\frac{1}{r}},
\end{aligned}$$

and thus

$$\begin{aligned}
(39) \quad & n^{1/2-1/(2r)} \sum_{l=0}^{\infty} 2^{-pl} \sum_{F \in \mathcal{P}_l: \xi(F) > n^{-1}} (2^{-d} n \xi(F))^{\frac{1}{r}} \\
& \leq n^{1/2-1/(2r)} \sum_{l=0}^{L_1(n)} 2^{-pl} \sum_{F \in \mathcal{P}_l} (2^{-d} n \xi(F))^{\frac{1}{r}} \\
& \leq n^{1/2-1/(2r)} \sum_{l=0}^{L_1(n)} 2^{(d-p)l} (2^{-dl} n)^{\frac{1}{r}} \\
& = n^{1/2+1/(2r)} \sum_{l=0}^{L_1(n)} 2^{(d(1-1/r)-p)l}.
\end{aligned}$$

With the special choice  $r = d/(d-p)$  it holds that  $1/2+1/(2r) \leq 1-p/(2d)$  and  $1-1/r = p/d$ . This implies the bound

$$(40) \quad n^{1-\frac{p}{2d}} (L_1(n) + 1) \leq n^{1-\frac{p}{2d}} \log_2 \left( 4M^{\frac{1}{d}} n^{\frac{1}{d}} \right) \leq 4M^{\frac{1}{d}} n^{1-\frac{p-2}{2d}}.$$

The above equations (32) through (40) give us

$$\begin{aligned}
(41) \quad & \sum_{l=0}^{\infty} 2^{-pl} \sum_{F \in \mathcal{P}_l} \sum_{C \leftarrow F} n^{-1} R_1(C) \\
& \leq 2^d c_0 \left\{ n^{-\frac{p}{d}} \left( \frac{1}{1-2^{p-d/2}} + \frac{1}{1-2^{-p}} \right) + n^{-\frac{p}{2d}} 2^{d/2-p} M^{\frac{d/2-p}{d}} \frac{1}{1-2^{p-d/2}} + n^{-\frac{p-2}{2d}} 4M^{\frac{1}{d}} \right\} \\
& \leq 2^{3d/2-p} c_0 n^{-\frac{p-2}{2d}} \left\{ \frac{1+M^{\frac{d/2-p}{d}}}{1-2^{p-d/2}} + \frac{1}{1-2^{-p}} + 4M^{\frac{1}{d}} \right\}.
\end{aligned}$$

It now remains to bound the sums with the summands  $R_2(C, F)$ . Obviously, it holds that  $\sum_{C \leftarrow F} \xi(C)/\xi(F) = 1$ , and thus

$$(42) \quad \sum_{C \leftarrow F} R_2(C, F) = \sum_{C \leftarrow F} \frac{\xi(C)}{\xi(F)} \zeta_{n,r}(\xi(F)) = \zeta_{n,r}(\xi(F)),$$

and

$$\begin{aligned}
(43) \quad & \sum_{F \in \mathcal{P}_l} \zeta_{n,r}(\xi(F)) \\
& \leq c_0 \left\{ \sum_{F \in \mathcal{P}_l} \zeta(n\xi(F)) + n^{1/2-1/(2r)} \sum_{F \in \mathcal{P}_l: \xi(F) > n^{-1}} (n\xi(F))^{\frac{1}{r}} + n^{\frac{1}{4}} \sum_{F \in \mathcal{P}_l: \xi(F) > n^{-1}} \zeta(n\xi(F)) \right\}.
\end{aligned}$$

Notice that these sums are bounded by the last terms of (33), (36) and (38). Thus we again obtain the upper bound given in (41) for the sum

$$\sum_{l=0}^{\infty} \sum_{F \in \mathcal{P}_l} \sum_{C \leftarrow F} n^{-1} R_2(C, F).$$

Therefore, using (29), (31) and (41), it follows that

$$(44) \quad \mathbb{E}d_p^p(\xi_n, \xi) \leq c_0 2^{3d/2-p} \mathfrak{D}^p n^{-\frac{p-2}{2d}} \left\{ \frac{1 + M^{\frac{d/2-p}{d}}}{1 - 2^{p-d/2}} + \frac{1}{1 - 2^{-p}} + 4M^{\frac{1}{d}} \right\}.$$

□

**Corollary 3.** *Let  $d \in \mathbb{N}$  and  $1 \leq p < d/2$  be fixed. Let  $\xi$  be a probability measure on the open ball  $U_K(0)$  for some  $K > 0$  that fulfills Assumption 1. Then, for any  $n \in \mathbb{N}$  it holds that*

$$(\mathbb{E}d_p^p(\xi_n, \xi))^{\frac{1}{p}} \leq K^{d/(2p)} \mathfrak{M},$$

where  $\xi_n$  is the empirical measure of a strictly stationary and  $\alpha$ -mixing process  $(U_i)_{i \in \mathbb{N}}$  with marginal distribution  $\xi$  and  $\alpha(n) \leq f(n) = \mathcal{O}(n^{-r_0})$  for some function  $f$  and some constant  $r_0 > 1$ , and  $\mathfrak{M}$  is the bound from Proposition 2.

*Proof.* We define a new stationary and  $\alpha$ -mixing process  $(\tilde{U}_i)_{i \in \mathbb{N}}$  by

$$\tilde{U}_i := \frac{U_i}{2K} + \left( \frac{1}{2}, \dots, \frac{1}{2} \right).$$

Then  $\tilde{\xi} := \mathcal{L}(\tilde{U}_1)$  is a measure on  $[0, 1)^d$ , and  $\tilde{\xi}_n$ , defined as the empirical measure of  $\tilde{U}_1, \dots, \tilde{U}_n$ , fulfills the assumptions of Proposition 2. Furthermore, for any  $F \in \mathcal{P}_l$ , it holds that

$$\begin{aligned} \tilde{\xi}(F) &= \mathbb{P}(\tilde{U}_1 \in F) = \mathbb{P}(U_1 \in 2K \cdot F + (K, \dots, K)) \\ &\leq M \cdot \text{vol}(2K \cdot F + (K, \dots, K)) \\ &= (2K)^d M \cdot \text{vol}(F) \\ &=: \tilde{M} \cdot \text{vol}(F), \end{aligned}$$

because  $\xi$  fulfills Assumption 1. Therefore, by Proposition 2,

$$\begin{aligned} \mathbb{E}d_p^p(\tilde{\xi}_n, \tilde{\xi}) &\leq c_0 2^{3d/2-p} \mathfrak{D}^p n^{-\frac{p-2}{2d}} \left\{ \frac{1 + \tilde{M}^{\frac{d/2-p}{d}}}{1 - 2^{p-d/2}} + \frac{1}{1 - 2^{-p}} + 4\tilde{M}^{\frac{1}{d}} \right\} \\ &\leq (2K)^{d/2-p} \mathfrak{M}^p. \end{aligned}$$

Thus,

$$\mathbb{E}d_p^p(\xi_n, \xi) = (2K)^p \mathbb{E}d_p^p(\tilde{\xi}_n, \tilde{\xi}) \leq (2K)^{d/2} \mathfrak{M}^p.$$

□

**Proposition 3** (Version for  $\phi$ -Mixing). *Let  $d \in \mathbb{N}$  and  $1 \leq p < d/2$  be fixed. Let  $\xi$  be a probability measure on  $[0, 1)^d$ . Then, for any  $n \in \mathbb{N}$ , it holds that*

$$(\mathbb{E}d_p^p(\xi_n, \xi))^{\frac{1}{p}} \leq n^{-\frac{p}{d}} \left\{ c_0 2^{d+1} \mathfrak{D}^p \left( \frac{1}{1 - 2^{p-d/2}} + \frac{1}{1 - 2^{-p}} \right) \right\}^{\frac{1}{p}},$$

where  $\xi_n$  is the empirical measure of a strictly stationary and  $\phi$ -mixing process  $(U_i)_{i \in \mathbb{N}}$  with marginal distribution  $\xi$  and  $\phi(n) \leq f(n) = \mathcal{O}(n^{-r_0})$  for some function  $f$  and some constant  $r_0 > 1$ . The constant  $c_0$  only depends on  $f$  and  $r_0$ .

*Proof.* The proof will follow that of Proposition 2. The bound given in (29) remains valid, and similarly to (31) we can use Lemma 11 to obtain the bound

$$(45) \quad \begin{aligned} & \mathbb{E} \left| \xi_n(C) - \xi_n(F) \frac{\xi(C)}{\xi(F)} \right| \\ & \leq n^{-1} \left\{ \zeta(2c_0 n \xi(C)) + \frac{\xi(C)}{\xi(F)} \zeta(2c_0 n \xi(F)) \right\} \\ & =: n^{-1} (R_1(C) + R_2(C, F)). \end{aligned}$$

As in (33), it follows that

$$(46) \quad \sum_{C \leftarrow F} R_1(C) = \sum_{C \leftarrow F} \zeta(2c_0 n \xi(C)) \leq 2^d \zeta(2c_0 2^{-d} n \xi(F)),$$

and as in (34) that

$$(47) \quad \sum_{F \in \mathcal{P}_l} \zeta(2c_0 2^{-d} n \xi(F)) \leq 2^{dl} \zeta(2c_0 2^{-dl} n).$$

We now choose  $l^* := \lceil \log_2 \left( (2c_0 n)^{\frac{1}{d}} \right) \rceil$ , so that, as in (35),

$$(48) \quad \sum_{l=0}^{\infty} 2^{(d-p)l} \zeta(2c_0 2^{-dl} n) \leq 2c_0 n^{1-\frac{p}{d}} \left\{ \frac{1}{1-2^{p-d/2}} + \frac{1}{1-2^{-p}} \right\}.$$

This implies

$$(49) \quad \sum_{C \leftarrow F} R_1(C) \leq 2^{d+1} c_0 n^{1-\frac{p}{d}} \left\{ \frac{1}{1-2^{p-d/2}} + \frac{1}{1-2^{-p}} \right\},$$

and because the sum

$$\sum_{C \leftarrow F} \frac{\xi(C)}{\xi(F)} R_2(C, F) = \zeta(2c_0 n \xi(F))$$

is bounded by the last term in (46), we get with (29) and (45) through (49) that

$$\mathbb{E} d_p^p(\xi_n, \xi) \leq n^{-\frac{p}{d}} \left\{ c_0 2^{d+1} \mathfrak{d}^p \left( \frac{1}{1-2^{p-d/2}} + \frac{1}{1-2^{-p}} \right) \right\}.$$

□

*Remark.* In the  $\phi$ -mixing version, extending the support of  $\xi$  to  $U_K(0)$  only impacts the bound by a factor of  $2K$  instead of  $(2K)^{d/(2p)}$ . The proof is analogous to that of Corollary 3.

*Proof of Theorem 4.* Let  $n$  and  $K$  be arbitrary but fixed. Define the transformation

$$\varphi_K := \mathbf{1}_{U_K(0)} \cdot \text{id} + \mathbf{1}_{U_K(0)^c} \cdot z_K$$

for some arbitrary but fixed  $z_K \in \partial U_K(0)$ .  $\varphi_K$  maps all points outside of  $U_K(0)$  to the single point  $z_K$  and leaves the interior of  $U_K(0)$  unchanged. Now define the probability measures  $\xi^{(K)}$  and  $\xi_n^{(K)}$  by

$$\begin{aligned} \xi^{(K)}(A) &:= \xi(A \cap U_K(0)) + \mathbf{1}_A(z_K) \cdot \xi(U_K(0)^c), \\ \xi_n^{(K)}(A) &:= \xi_n(A \cap U_K(0)) + \mathbf{1}_A(z_K) \cdot \xi_n(U_K(0)^c). \end{aligned}$$

These measures are the pushforwards of  $\xi$  and  $\xi_n$ , respectively, under  $\varphi_K$ .  $\xi^{(K)}$  is the marginal distribution of the process  $(\varphi_K(U_i))_{i \in \mathbb{N}}$ , and  $\xi_n^{(K)}$  is the empirical measure of its first  $n$  observations. Because  $\varphi_K$  is measurable,  $(\varphi_K(U_i))_{i \in \mathbb{N}}$  inherits the  $\alpha$ -mixing property from  $(X_i)_{i \in \mathbb{N}}$ .

Heuristically, in transforming  $\xi$  to  $\xi^{(K)}$ , we have concentrated the probability of  $U_K(0)^C$  to one single point, namely  $z_K$ . For large  $K$ , this probability of the complement will be small, so we expect  $\xi^{(K)}$  to be close to  $\xi$ .

Let us now quantify this notion of closeness. Let  $\gamma$  be the pushforward of  $\xi$  under  $x \mapsto (x, \varphi_K(x))$ , and  $U := U_K(0)$ . Observe that

$$\begin{aligned}
d_p^p(\xi, \xi^{(K)}) &\leq \int \|x - y\|^p d\gamma(x, y) = \int \|x - \varphi_K(x)\|^p d\xi(x) \\
&= \int_U \|x - \varphi_K(x)\|^p d\xi(x) + \int_{U^C} \|x - \varphi_K(x)\|^p d\xi(x) \\
(50) \quad &\leq 2^{p-1} \left( \int_U \|x\|^p d\xi(x) + \int_{U^C} \|z_K\|^p d\xi(x) \right) \\
&\leq 2^{p-1} \left( \xi(U^C)^{\frac{q-p}{q}} m_q^{p/q} + \xi(U^C) K^p \right),
\end{aligned}$$

where in the last line we have used the Hölder inequality and the fact that  $z_K \in \partial U$ .

The same argument shows that

$$d_p^p(\xi_n, \xi_n^{(K)}) \leq 2^{p-1} \left( \xi_n(U^C)^{\frac{q-p}{q}} m_{n,q}^{p/q} + \xi_n(U^C) K^p \right),$$

where  $m_{n,q}$  is the  $q$ -th moment of  $\xi_n$ . Observe that, by the Hölder inequality,

$$\begin{aligned}
\mathbb{E} \left[ \xi_n(U^C)^{\frac{q-p}{q}} m_{n,q}^{p/q} \right] &\leq \left\| \xi_n(U^C)^{\frac{q-p}{q}} \right\|_{q/(q-p)} \left\| m_{n,q}^{p/q} \right\|_{q/p} \\
&= \xi(U^C)^{\frac{q-p}{q}} m_q^{p/q},
\end{aligned}$$

and thus

$$(51) \quad \mathbb{E} d_p^p(\xi_n, \xi_n^{(K)}) \leq 2^{p-1} \left( \xi(U^C)^{\frac{q-p}{q}} m_q^{p/q} + \xi(U^C) K^p \right),$$

since  $\xi_n$  is the empirical measure of a stationary process with marginal distribution  $\xi$ .

Furthermore, due to Corollary 3,

$$(52) \quad \mathbb{E} d_p^p(\xi_n^{(K)}, \xi^{(K)}) \leq K^{d/2} \cdot \mathfrak{M}^p.$$

We now use the triangle inequality and Jensen's inequality to obtain

$$\mathbb{E} d_p^p(\xi, \xi_n) \leq 3^{p-1} \left\{ d_p^p(\xi, \xi^{(K)}) + \mathbb{E} d_p^p(\xi_n^{(K)}, \xi^{(K)}) + \mathbb{E} d_p^p(\xi_n, \xi_n^{(K)}) \right\},$$

and thus, by (50) through (52),

$$\mathbb{E} d_p^p(\xi, \xi_n) \leq 3^{p-1} \left\{ 2^p \left( \xi(U^C)^{\frac{q-p}{q}} m_q^{p/q} + \xi(U^C) K^p \right) + K^{d/2} \cdot \mathfrak{M}^p \right\}.$$

□

*Proof of Corollary 2.* By Theorem 4, we get

$$\begin{aligned}
(53) \quad \mathbb{E} d_p^p(\xi, \xi_n) &\leq 3^{p-1} \left\{ 2^p \left( \xi(U_K(0)^C)^{\frac{q-p}{q}} m_q^{p/q} + \xi(U_K(0)^C) K^p \right) + K^{d/2} \cdot \mathfrak{M}^p \right\} \\
&\leq 6^p \left\{ \xi(U_K(0)^C)^{\frac{q-p}{q}} m_q^{p/q} + \xi(U_K(0)^C) K^p + K^{d/2} \cdot \mathfrak{M}^p \right\}.
\end{aligned}$$

Choose  $K = n^{(p-2)/(2d^2)}$ , then

$$(54) \quad K^{d/2} \cdot \mathfrak{M}^p = c_0 2^{3d/2-p} \mathfrak{D}^p n^{-\frac{p-2}{4d}} \left\{ \frac{1 + M^{\frac{d/2-p}{d}}}{1 - 2^{p-d/2}} + \frac{1}{1 - 2^{-p}} + 4M^{\frac{1}{d}} \right\}.$$

Writing  $m'_q$  for the  $q$ -th moment of  $U'_1$ , we can bound  $m_q$  by

$$m_q = \mathbb{E} [\|U_1\|_2^q] = \mathbb{E} \left[ \left( \sum_{i=1}^{d'} \|U'_i\|_2^2 \right)^{q/2} \right] \leq (d')^{q/2} m'_q,$$

where we have used the Jensen inequality and the stationarity of the process  $(U'_i)_{i \in \mathbb{N}}$ .

Now observe that

$$\infty > m'_q = \int_0^\infty \mathbb{P}(\|U'_1\|_2^q > K) \, dK,$$

which implies that  $\mathbb{P}(\|U'_1\|_2^q > K) = o(K^{-1})$ , with the constant involved depending only on  $\mathcal{L}(U'_1)$ . Furthermore,

$$\begin{aligned} \xi(U_K(0)^C) &= \mathbb{P} \left( \sum_{i=1}^{d'} \|U'_i\|_2^2 > K^2 \right) \\ &\leq \mathbb{P} \left( \exists 1 \leq i \leq d' : \|U'_i\|_2 > \frac{K}{\sqrt{d'}} \right) \\ &\leq d' \cdot \mathbb{P} \left( \|U'_1\|_2 > \frac{K}{\sqrt{d'}} \right) \\ &= d' \cdot \mathbb{P} \left( \|U'_1\|_2^q > \frac{K^q}{(d')^{q/2}} \right), \end{aligned}$$

and thus

$$\xi(U_K(0)^C) = d' \cdot o \left( \frac{K^{-q}}{(d')^{-q/2}} \right) = (d')^{1+q/2} \cdot o(K^{-q}),$$

with the constant involved depending only on  $\mathcal{L}(U'_1)$ . In other words, there is a constant  $c'$  and some  $n_0 \in \mathbb{N}$ , both depending only on  $\mathcal{L}(U'_1)$ , such that

$$\xi(U_K(0)^C) \leq c' \cdot (d')^{1+q/2} K^{-q}$$

for all  $n \geq n_0$  (recall that our special choice of  $K = K(n)$  depends on  $n$ ). Thus, for  $n \geq n_0$ ,

$$(55) \quad \begin{aligned} \xi(U_K(0)^C)^{\frac{q-p}{q}} m_p^{p/q} &\leq (c')^{\frac{q-p}{q}} \cdot (d')^{\frac{(q-p)(2+q)+pq}{2q}} K^{p-q} \\ &= (c')^{\frac{q-p}{q}} \cdot (d')^{1+q/2-p/q} K^{p-q} \\ &\leq c' \cdot (d')^{1+q/2} K^{p-q} \\ &\leq c' \cdot d^{1+q/2} n^{(p-2)(p-q)/(2d^2)}, \end{aligned}$$

because  $d' \leq d$ , and

$$(56) \quad \xi(U_K(0)^C) K^p \leq c' \cdot (d')^{1+q/2} K^{p-q} \leq c' \cdot d^{1+q/2} n^{(p-2)(p-q)/(2d^2)}.$$

Thus, by (53) and (54) through (56), we get

$$\begin{aligned} \mathbb{E}d_p^p(\xi, \xi_n) &\leq 6^p c_0 2^{3d/2-p} \mathfrak{D}^p n^{-\frac{p-2}{4d}} \left( \frac{1 + M^{\frac{d/2-p}{d}}}{1 - 2^{p-d/2}} + \frac{1}{1 - 2^{-p}} + 4M^{\frac{1}{d}} \right) \\ &\quad + 6^p 2c' \cdot d^{1+q/2} n^{(p-2)(p-q)/(2d^2)}. \end{aligned}$$

□

### APPENDIX C. ASYMPTOTIC BEHAVIOUR OF $V$ -STATISTICS OF INDEPENDENT BLOCKS DERIVED FROM A STATIONARY PROCESS

Recall that for a given strictly stationary sequence  $(U_k)_{k \in \mathbb{N}}$ , we construct a triangular array  $\tilde{U}$  by taking the  $n$ -th row as  $N = N(n)$  iid copies of  $(U_1, \dots, U_d)$ ,  $d = d(n)$  such that  $n/(Nd) \rightarrow 1$  as  $n \rightarrow \infty$ . In this section, we develop some asymptotic theory for  $V$ -statistics of such triangular arrays. The main results are Theorems 2 and 3.

As a starting point, we will take Theorem 2 from Kroll (2021) and generalise it to triangular arrays. We will have to assume stronger moment conditions, but the proofs will be mostly similar. For ease of readability, we include this theorem here. Note that this theorem is a weaker result than Theorem 2 (i) since it only gives us weak convergence instead of convergence in  $d_1$ .

**Theorem 6** (Kroll, 2020). *Let  $\mathcal{U}$  be a  $\sigma$ -compact metrisable topological space,  $(U_k)_{k \in \mathbb{N}}$  a strictly stationary sequence of  $\mathcal{U}$ -valued random variables with marginal distribution  $\mathcal{L}(U_1) = \xi$ . Consider a continuous, symmetric, degenerate and positive semidefinite kernel  $g : \mathcal{U}^2 \rightarrow \mathbb{R}$  with finite  $(2 + \varepsilon)$ -moments with respect to  $\theta^2$  and finite  $(1 + \frac{\varepsilon}{2})$ -moments on the diagonal, i.e.  $\mathbb{E}|g(U_1, U_1)|^{1+\varepsilon/2} < \infty$ . Furthermore, let the sequence  $(U_k)_{k \in \mathbb{N}}$  satisfy an  $\alpha$ -mixing condition such that  $\alpha(n) = O(n^{-r})$  for some  $r > 1 + 2\varepsilon^{-1}$ . Then, with  $V = V_g(U_1, \dots, U_n)$  denoting the  $V$ -statistics with kernel  $g$ ,*

$$nV \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \sum_{k=1}^{\infty} \lambda_k \zeta_k^2,$$

where  $(\lambda_k, \varphi_k)$  are pairs of the non-negative eigenvalues and matching eigenfunctions of the integral operator

$$f \mapsto \int g(\cdot, z) f(z) \, d\xi(z),$$

and  $(\zeta_k)_{k \in \mathbb{N}}$  is a sequence of centred Gaussian random variables whose covariance structure is given by

$$(57) \quad \text{Cov}(\zeta_i, \zeta_j) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{s,t=1}^n \text{Cov}(\varphi_i(U_s), \varphi_j(U_t)).$$

We will now first prove a generalisation for the triangular array  $\tilde{U}$ . The only real difference from the original proof of Theorem 6 (cf. Kroll (2021), there Theorem 2) lies in the CLT that we use.

**Theorem 7.** *Under the conditions of Theorem 2 (ii), it holds that*

$$n\tilde{V} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \zeta := \sum_{k=1}^{\infty} \lambda_k \zeta_k^2,$$

where  $\lambda_k$  and  $\zeta_k$  are the objects from Theorem 2 (i).

*Proof.* As shown in the proof of Theorem 6 (cf. Kroll (2021), there Theorem 2), we have

$$g(z, z') = \sum_{k=1}^{\infty} \lambda_k \varphi_k(z) \varphi_k(z')$$

for all  $z, z' \in \text{supp}(\xi)$ . The  $\varphi_k$  are centred and form an orthonormal basis of  $L_2(\xi)$ . Let  $c_1, \dots, c_K$  be any selection of real constants. We define the following objects:

$$\begin{aligned} \vartheta_{t,n} &:= \sum_{k=1}^K c_k \varphi_k(\tilde{U}_{t,n}), \\ \zeta_{n,k} &:= \frac{1}{\sqrt{Nd}} \sum_{t=1}^{Nd} \varphi_k(\tilde{U}_{t,n}), \\ S_{i,n} &:= \frac{1}{\sqrt{Nd}} \sum_{t=1}^i \vartheta_{t,n}, \\ V_{i,n} &:= \text{Var}(S_{i,n}). \end{aligned}$$

Note that  $NdV^{(K)} = \sum_{k=1}^K \lambda_k \zeta_{n,k}^2$  and  $\sum_{k=1}^K c_k \zeta_{n,k} = S_{Nd,n}$ . Using the Cramér-Wold-device, we show that  $(\zeta_{n,k})_{1 \leq k \leq K}$  converges in distribution to  $(\zeta_1, \dots, \zeta_K)$ .

First, because the dependence structure within each block is the same as in the original sequence  $(U_k)_{k \in \mathbb{N}}$ , and the blocks themselves are independent from each other, we have that

$$\begin{aligned} V_{i,n} &= \text{Var} \left( \frac{1}{\sqrt{Nd}} \sum_{t=1}^d \vartheta_{t,n} \right) + \text{Var} \left( \frac{1}{\sqrt{Nd}} \sum_{t=d+1}^{2d} \vartheta_{t,n} \right) + \dots + \text{Var} \left( \frac{1}{\sqrt{Nd}} \sum_{t=\lfloor i/d \rfloor d + 1}^i \vartheta_{t,n} \right) \\ &= \left\lfloor \frac{i}{d} \right\rfloor \text{Var} \left( \frac{1}{\sqrt{Nd}} \sum_{t=1}^d \vartheta_{t,n} \right) + \text{Var} \left( \frac{1}{\sqrt{Nd}} \sum_{t=\lfloor i/d \rfloor d + 1}^i \vartheta_{t,n} \right) \\ &= \left\lfloor \frac{i}{d} \right\rfloor O \left( \frac{1}{N} \right) + O \left( \frac{i \bmod d}{Nd} \right). \end{aligned}$$

The rate of growth in the last line follows exactly as in the proof of Theorem 6 (cf. Kroll (2021), there Theorem 2), since, as mentioned before, the dependence structure within each block is identical to that of the original sequence  $U$ . Therefore,  $V_{Nd,n}$  converges and

$$V_{i,n} = O(N) \Theta \left( \frac{1}{N} \right) + O \left( \frac{1}{N} \right) = O(1),$$

where the constants involved are independent of  $i$ . From this it follows that

$$(58) \quad \limsup_{n \rightarrow \infty} \max_{1 \leq i \leq Nd} \frac{V_{i,n}}{V_{Nd,n}} < \infty.$$

For  $V_{Nd,n}$  in particular, we get

$$(59) \quad V_{Nd,n} = N \text{Var} \left( \frac{1}{\sqrt{Nd}} \sum_{t=1}^d \vartheta_{t,n} \right) = \text{Var} \left( \frac{1}{\sqrt{d}} \sum_{t=1}^d \vartheta_{t,n} \right) \xrightarrow{n \rightarrow \infty} \sigma^2.$$

Defining the functions  $Q_{t,n}(u) := \inf\{t \mid \mathbb{P}((Nd)^{-1/2} \vartheta_{t,n} > t) \leq u\}$  and  $\alpha^{-1}(u) := \inf\{k \in \mathbb{N} \mid \alpha(k) \leq u\}$ , a simple application of Markov's inequality yields  $Q_{t,n} \leq \|(Nd)^{-1/2} \vartheta_{t,n}\|_{L_p} u^{-1/p}$



for any  $p \geq 1$ , and  $\alpha^{-1}(u) \leq 2c_0 u^{-1/r}$  for some constant  $c_0 > 0$ , since  $\alpha(n) = O(n^{-r})$ . Furthermore, note that  $\|\vartheta_{t,n}\|_{L_p}$  is independent of  $t$  and  $n$ . Therefore,

$$\int_0^1 \alpha^{-2} \left( \frac{u}{2} \right) Q_{t,n}^3(u) \, du \leq \text{const} \cdot (Nd)^{-\frac{3}{2}} \int_0^1 u^{-\frac{2}{r}-\frac{3}{p}} \, du = O\left((Nd)^{-\frac{3}{2}}\right),$$

where we have used the special choice  $p = 3 + \varepsilon$ . We get

$$\begin{aligned} V_{Nd,n}^{-\frac{3}{2}} \sum_{t=1}^{Nd} \int_0^1 \alpha^{-1} \left( \frac{u}{2} \right) Q_{t,n}^2(u) \min \left\{ \alpha^{-1} \left( \frac{u}{2} \right) Q_{t,n}(u), \sqrt{V_{Nd,n}} \right\} \, du \\ \leq V_{Nd,n}^{-\frac{3}{2}} Nd \int_0^1 \alpha^{-2} \left( \frac{u}{2} \right) Q_{t,n}^3(u) \, du = O\left(\frac{1}{\sqrt{Nd}}\right), \end{aligned}$$

and from this and (58), Corollary 1 in Rio (1995) gives us

$$S_{Nd,n} V_{Nd,n}^{-1} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1).$$

Note that the limit  $\sigma^2$  in (59) is exactly the variance of  $\sum_{k=1}^K c_k \zeta_k$ , where  $\zeta_k$  are centred Gaussian random variables with their covariance function given in (5), and thus

$$\sum_{k=1}^K c_k \zeta_{n,k} = S_{Nd,n} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2) = \mathcal{L} \left( \sum_{k=1}^K c_k \zeta_k \right).$$

By the Cramér-Wold-device it follows that  $(\zeta_{n,k})_{1 \leq k \leq K} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} (\zeta_k)_{1 \leq k \leq K}$ . The continuous mapping theorem then gives us

$$(60) \quad Nd \tilde{V}^{(K)} = \sum_{k=1}^K \lambda_k \zeta_{n,k}^2 \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \sum_{k=1}^K \lambda_k \zeta_k^2 =: \zeta^{(K)}.$$

As in the proof of Theorem 6 (cf. Kroll (2021), there Theorem 2), the summability of the eigenvalues  $\lambda_k$  gives us

$$(61) \quad \mathbb{E}|\zeta - \zeta^{(K)}| = \sum_{k>K} \lambda_k \xrightarrow[K \rightarrow \infty]{} 0.$$

Again, it remains to show that

$$(62) \quad \lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}|Nd \tilde{V} - Nd \tilde{V}^{(K)}| = 0.$$

This can be achieved in exactly the same way as in the original proof of Theorem 6. We give the basic idea:

Let  $H$  be the Hilbert-space of all real-valued sequences  $(a_k)_{k \in \mathbb{N}}$  such that  $\sum_k \lambda_k a_k^2 < \infty$ , with the inner product  $\langle (a_k), (b_k) \rangle_H := \sum_k \lambda_k a_k b_k$ . Furthermore, we write  $T_K(\tilde{U}_{t,n}) := (0^K, (\varphi_k(\tilde{U}_{t,n}))_{k>K})$ , where  $0^K$  is the  $K$ -dimensional zero vector. Then,

$$\mathbb{E}|Nd \tilde{V} - Nd \tilde{V}^{(K)}| = \frac{1}{Nd} \sum_{s,t=1}^{Nd} \text{Cov}(T_K(\tilde{U}_{s,n}), T_K(\tilde{U}_{t,n})).$$

Now, with the same argument regarding the block structure as before – the dependence is identical within any given block, and the blocks are independent from each other –, we can instead examine the sum

$$\frac{1}{Nd} \sum_{s,t=1}^{Nd} \text{Cov}(T_K(U_s), T_K(U_t)).$$

In the original proof of Theorem 6 (cf. Kroll (2021), Theorem 2 there), it is shown that

$$\lim_{K \rightarrow \infty} \limsup_{Nd \rightarrow \infty} \frac{1}{Nd} \sum_{s,t=1}^{Nd} \text{Cov}(T_K(U_s), T_K(U_t)) = 0.$$

This implies (62), because  $Nd \xrightarrow[n \rightarrow \infty]{} \infty$ .

By Theorem 3.2 in Billingsley (1999), equations (60), (61) and (62) imply  $Nd\tilde{V} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \zeta$ . Now, because  $n/(Nd) \rightarrow 1$  as  $n \rightarrow \infty$ , it follows that

$$n\tilde{V} = \frac{n}{Nd} Nd\tilde{V} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \zeta,$$

by Slutsky's Theorem.  $\square$

Theorems 6 and 7 together give us that both  $V$ -statistics converge weakly to the same limiting distribution  $\zeta$ .

**Corollary 4.** *The convergences in Theorems 6 and 7 also hold in the Wasserstein metric  $d_1$ .*

*Proof.* Convergence in  $d_p$  is equivalent to weak convergence and convergence of the  $p$ -th moments. We therefore only have to show convergence of the first moments, for which we will borrow concepts from the proof of Theorem 2 in Kroll (2021).

With  $H$  being the Hilbert space of all real-valued sequences  $(a_k)_{k \in \mathbb{N}}$  such that  $\sum_{k=1}^{\infty} \lambda_k a_k^2 < \infty$ , where  $\lambda_k$  are the eigenvalues from Theorem 6, we have that, for any  $M > 0$ ,

$$\begin{aligned} & \mathbb{E}[\mathbf{1}_{\{nV > M\}} |nV(U_1, \dots, U_n)|] \\ &= \mathbb{E} \left[ \mathbf{1}_{\{nV > M\}} \sum_{k=1}^{\infty} \lambda_k \left( \frac{1}{\sqrt{n}} \sum_{t=1}^n \varphi_k(U_t) \right)^2 \right] \\ &= \mathbb{E} \left[ \mathbf{1}_{\{nV > M\}} \left\langle \frac{1}{\sqrt{n}} \sum_{t=1}^n (\varphi_k(U_t))_{k \in \mathbb{N}}, \frac{1}{\sqrt{n}} \sum_{t=1}^n (\varphi_k(U_t))_{k \in \mathbb{N}} \right\rangle_H \right] \\ &= \frac{1}{n} \sum_{s,t=1}^n \text{Cov}(\mathbf{1}_{\{nV > M\}} T(U_s), \mathbf{1}_{\{nV > M\}} T(U_t)), \end{aligned}$$

where  $T(z) := (\varphi_k(z))_{k \in \mathbb{N}}$ . Observe that, for any  $p > 0$ ,

$$\begin{aligned} \|\mathbf{1}_{\{nV > M\}} T(U_1)\|_{L^p}^p &= \int_{\{nV > M\}} \|T(z)\|_H^p d\xi(z) = \int_{\{nV > M\}} \left( \sum_{k=1}^{\infty} \lambda_k \varphi_k(z)^2 \right)^{p/2} d\xi(z) \\ &= \int_{\{nV > M\}} g(z, z)^{p/2} d\xi(z) = \|\mathbf{1}_{\{nV > M\}} g(U_1, U_1)\|_{L^{p/2}}^{p/2}. \end{aligned}$$

Now, under the assumptions of Theorem 6, we have  $\alpha(n) = O(n^{-r})$ , where  $r > 1 + 2\varepsilon^{-1}$ . We can choose  $0 < \delta < \varepsilon$  such that  $r > 1 + 2\delta^{-1}$ . Lemma 2.2 in Dehling and Philipp (1982) gives us

$$|\text{Cov}(\mathbf{1}_{\{nV > M\}} T(U_s), \mathbf{1}_{\{nV > M\}} T(U_t))| \leq 15 \|\mathbf{1}_{\{nV > M\}} T(U_1)\|_{L^{2+\delta}}^2 \alpha(|s-t|)^{\delta/(2+\delta)}.$$

The  $(2+\delta)$ -norm is finite because  $g$  has finite  $(1+\varepsilon/2)$ -moments on the diagonal by assumption and  $\delta < \varepsilon$ . On the other hand, we chose  $\delta$  sufficiently large so that  $r > 1 + 2\delta^{-1}$ , which implies

that  $n^{-1} \sum_{s,t=1}^n \alpha(|s-t|)^{\delta/(2+\delta)}$  converges to a finite limit. Therefore,

$$\frac{1}{n} \sum_{s,t=1}^n \text{Cov}(\mathbf{1}_{\{nV>M\}} T(U_s), \mathbf{1}_{\{nV>M\}} T(U_t)) \leq \text{const} \cdot \|\mathbf{1}_{\{nV>M\}} T(U_1)\|_{L_{2+\delta}}^2,$$

where the constant is independent of  $M$  and  $n$ . Now, letting  $p := (2 + \varepsilon)/(2 + \delta)$  and  $q$  its Hölder-conjugate (i.e.,  $q := p/(p - 1)$ ), we have

$$\begin{aligned} \|\mathbf{1}_{\{nV>M\}} T(U_1)\|_{L_{2+\delta}}^2 &\leq \mathbb{P}(nV > M)^{\frac{1}{q(2+\delta)}} \|T(U_1)\|_{L_{2+\varepsilon}} \\ &\xrightarrow{n \rightarrow \infty} \mathbb{P}(\zeta > M)^{\frac{1}{q(2+\delta)}} \|T(U_1)\|_{L_{2+\varepsilon}} \end{aligned}$$

by the Portmanteau-Theorem. This gives us

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}[\mathbf{1}_{\{nV>M\}} |nV(U_1, \dots, U_n)|] = 0.$$

Theorem 3.5 in Billingsley (1999) therefore implies  $\mathbb{E}[nV(U_1, \dots, U_n)] \rightarrow \mathbb{E}\zeta$  as  $n \rightarrow \infty$ .

Under the assumptions of Theorem 2 (ii), we have that

$$\begin{aligned} &\frac{1}{Nd} \sum_{s,t=1}^{Nd} \text{Cov}(\mathbf{1}_{\{nV>M\}} T(\tilde{U}_{s,n}), \mathbf{1}_{\{nV>M\}} T(\tilde{U}_{t,n})) \\ &= \frac{1}{Nd} \sum_{k=0}^{N-1} \text{Var} \left( \sum_{t=kd+1}^{(k+1)d} \mathbf{1}_{\{nV>M\}} T(\tilde{U}_{t,n}) \right) \\ &= \frac{1}{d} \text{Var} \left( \sum_{t=1}^d \mathbf{1}_{\{nV>M\}} T(\tilde{U}_{t,n}) \right) \\ &= \frac{1}{d} \text{Var} \left( \sum_{t=1}^d \mathbf{1}_{\{nV>M\}} T(U_t) \right), \end{aligned}$$

and thus, the same observations as before show that

$$\mathbb{E}[nV(\tilde{U}_1, \dots, \tilde{U}_{Nd})] \xrightarrow{n \rightarrow \infty} \mathbb{E}\zeta.$$

□

Theorem 2 now follows from Theorems 6 and 7 and Corollary 4.

*Proof of Theorem 3.* In the proof of Theorem 3 in Kroll (2021), it is shown that  $h_2(z, z'; \theta)$  fulfills the assumptions of Theorem 2. Therefore,

$$\begin{aligned} d_1(nV_n^{(2)}(h; \theta), \zeta) &\xrightarrow{n \rightarrow \infty} 0, \\ d_1(n\tilde{V}_n^{(2)}(h; \theta), \zeta) &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

It remains to show that

$$(63) \quad \mathbb{E} \left[ \left( nV - 15nV_n^{(2)}(h; \theta) \right)^2 \right] \xrightarrow{n \rightarrow \infty} 0,$$

and

$$(64) \quad \mathbb{E} \left[ \left( n\tilde{V} - 15n\tilde{V}_n^{(2)}(h; \theta) \right)^2 \right] \xrightarrow{n \rightarrow \infty} 0.$$

(64) has already been shown in Lemma 7. It thus remains to show (63). This is done as in the proof of Lemma 7, using once again Lemma 6 and the specific growth rate of  $\beta(n)$ . □