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New circle coverings of an equilateral triangle

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Abstract

Recently, Melissen has determined the thinnest coverings of an equilateral triangle with 1, . . . , 6, and 10 equal circles; see [5, 6]. Here, we will determine the thinnest covering with nine circles, and give a new, short proof for the thinnest covering with ten circles. Furthermore, we provide thin coverings with up to eighteen circles for the remaining cases. These coverings are conjectured to be optimal.

1 Introduction

The recent years have brought many new results in the theory of circle coverings. In particular, many new thin and thinnest circle coverings have been found for the square and the rectangle; see [3, 6, 7, 8, 11]. In this article, we will describe new circle coverings of an equilateral triangle. The first results were obtained by Melissen. He determined the thinnest coverings of an equilateral triangle with n congruent circles for $n \leq 6$ [5] and for $n = 10$ [6]. In Section 2 we will give an optimality proof for a covering with nine circles. Also, we will give a proof for $n = 10$ that is much shorter and simpler than the proof given in [6]; see Section 3.

In addition to these exact results, we have tried to find thin coverings for some remaining cases, by the use of two complementing optimisation techniques. This

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effort resulted in coverings with up to eighteen circles that we conjecture to be optimal. The conjectures are presented in Section 4, while the methods used are described in Section 5.

The smallest common radius of n congruent closed circular discs that can cover a unilateral triangular region \mathcal{T} (the edges are of length 1) will be denoted by τ_n .

2 Nine discs

The covering with nine circles in Fig. 2i may at first glance look somewhat chaotic, but closer inspection reveals its beauty. The arrangement exhibits threefold symmetry and the circles are arranged as part of a hexagonal lattice. We will now prove that this configuration is indeed the thinnest possible. The radius of the circles

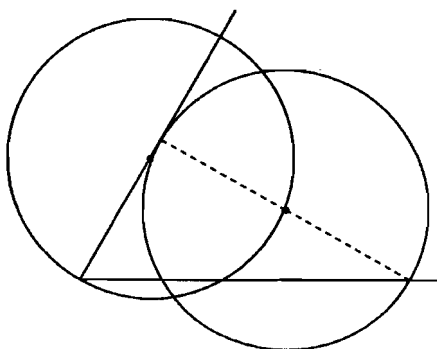


Figure 1: Maximum covering of the boundary with a double-edge circle and a vertex circle.

in Fig. 2i is $\tau_9 = 1/6$. It is easy to see that this radius is also minimal: Suppose that the triangle is covered by nine circles of radius $r \leq \tau_9$. As $6r \leq 1$, each vertex must be covered by its own circle. The six points on the edges at distance $1/3$ from the vertices, together with the centre of the triangle, must be covered by the six discs that do not cover a vertex, so two points must be in one circle, and $r = \tau_9$. First, we will show that there cannot be a circle that lies completely inside the triangle. If this were indeed the case, the boundary of the triangle would be covered by at most eight circles. This is

not possible if, apart from the three vertex circles, there are no “double-edge circles”, i.e., circles that cover parts of two edges, because one edge is then covered by only three circles. It is evident that the combination of a double-edge circle and its corresponding vertex circle can cover a length of at most $1/3$ of one edge and at most $2/\sqrt{27}$ of the other; see Fig. 1. This means that with one double-edge circle, one uncovered gap in an edge can be made smaller than $1/3$, so that it can be covered with just one circle. Unfortunately, the other two gaps still require two circles each, and since there are only three circles left, there must be another double-edge circle. The options with two and three double-edge circles can be excluded likewise, which shows that each circle must participate in the covering of the boundary.

If there are three double-edge circles, there are three gaps of length at least $1 - 4/\sqrt{27}$ in the boundary that must each be covered by one circle. Such a circle cannot cover points that are further than $\sqrt{18\sqrt{3}} - 30/9 + 1/6 = 0.2872063244\dots$ away from an edge. Fortunately, the centre of the triangle is at distance $\sqrt{3}/6 = 0.2886751345\dots$ from each edge, so it cannot be covered.

Table 1: Covering radius τ_n for a unilateral triangle.

n	Exact/conjecture	τ_n	Value
1	Exact	$1/\sqrt{3}$	=0.577350...
2	Exact	$1/2$	=0.500000...
3	Exact	$\sqrt{3}/6$	=0.288675...
4	Exact	$2 - \sqrt{3}$	=0.267949...
5	Exact	$1/4$	=0.250000...
6	Exact	$\sqrt{3}/9$	=0.192450...
7	Conjecture		0.185251...
8	Conjecture		0.176992...
9	Exact	$1/6$	=0.166666...
10	Exact	$\sqrt{3}/12$	=0.144337...
11	Conjecture		0.141054...
12	Conjecture		0.137323...
13	Conjecture		0.134021...
14	Conjecture		0.127516...
15	Conjecture	$\sqrt{3}/15$	=0.115470...
16	Conjecture		0.113712...
17	Conjecture		0.111394...
18	Conjecture	$\sqrt{21}/42$	=0.109108...

If there are two double-edge circles, there are two gaps adjacent to a vertex circle that has no corresponding double-edge circle. This leaves two possible situations.

1. One of these gaps has a length of at least $1/3$ and requires two circles. As there are at most three circles left for these two gaps, the number of circles per gap is fixed. One of the two circles that cover the same gap must also cover the centre of the triangle. This is possible in only one way: The vertex circle covers the maximum length $1/3$ of one edge, and the other circle covers the extreme point on the edge covered by the vertex circle and the centre of the triangle, which are $1/3$ apart. As the centre lies on the boundary of the circle, it must still be covered by another circle. This can only be the remaining circle that covers the same gap. The last gap is now too large to be covered by one circle.
2. The gap between the two vertices that have double-edge circles is at least $1/3$ and requires two circles. Let P be the point that lies at a distance which is

slightly larger than $1/3$ from the vertex with no double-edge circles on the bisector. It can be shown that this point cannot be covered.

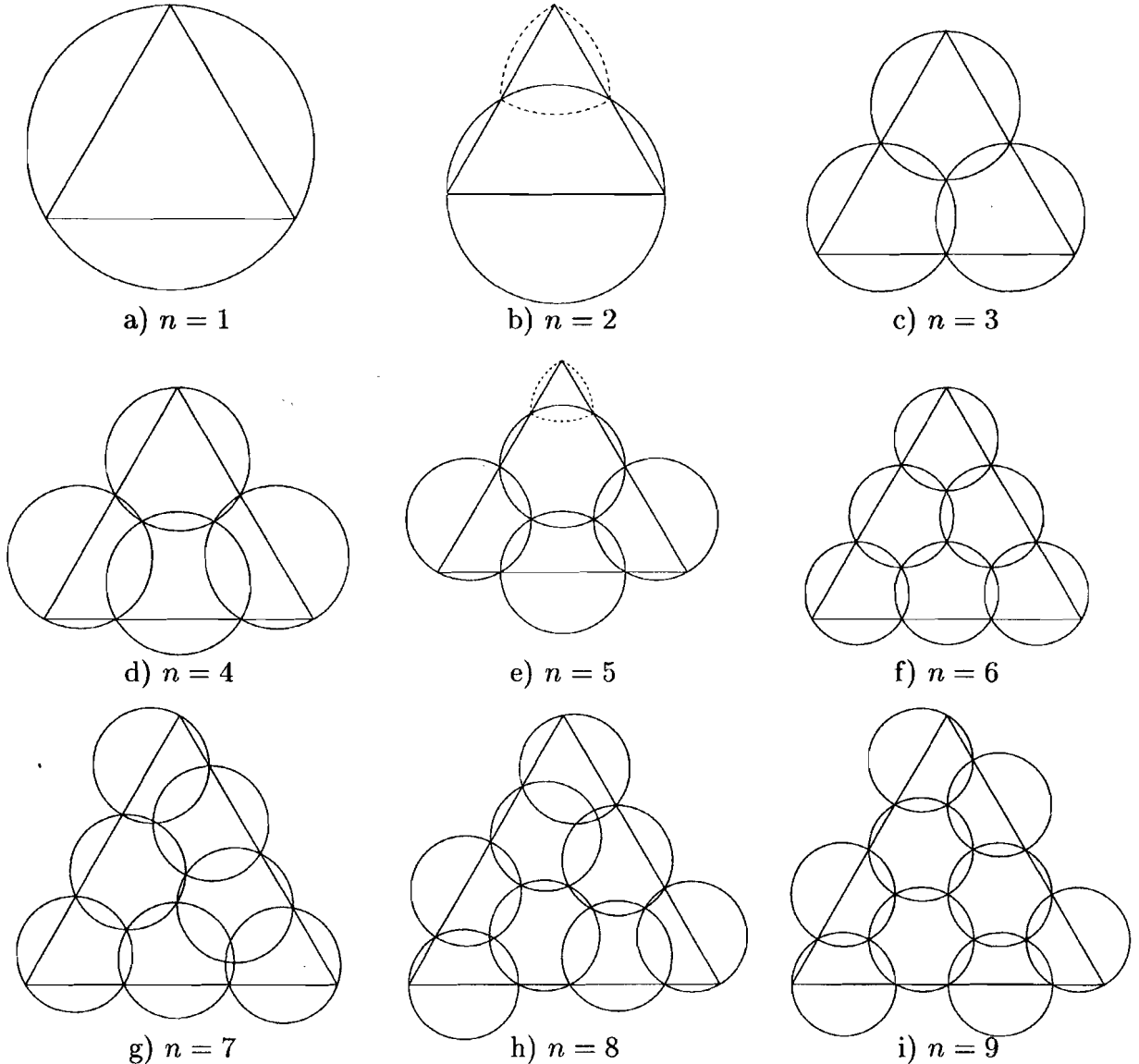


Figure 2: Optimal ($n \leq 6$, $n = 9$, and $n = 10$) and conjectured thinnest coverings of an equilateral triangle with circles. The dotted Reuleaux triangles in b) and e) delimit the possible positions of the centre of the remaining disc.

The case with one double-edge circle can be excluded likewise. The situation without double-edge circles is simple. The three segments of the boundary that are

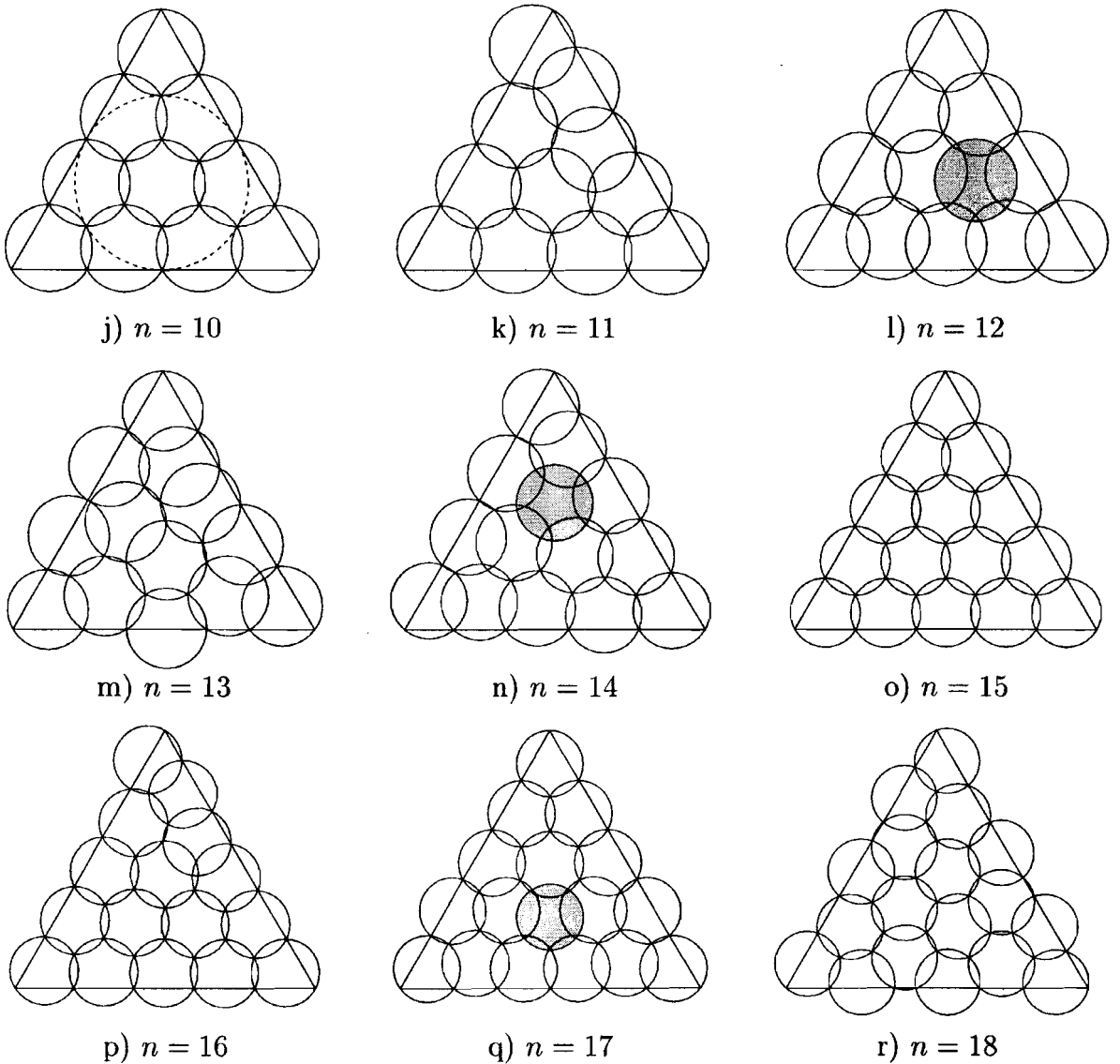


Figure 2: Continued. The shaded discs are not fixed.

not covered by the three vertex circles must each be covered by two circles. One of these circles must cover the centre, so it is fixed. The latter is also the case for one of the vertex circles. Then, there must be another circle that covers the centre, and with some combinatorial arguments, we finally arrive at either the configuration in Fig. 2i or its mirror image.

3 Ten circles

The thinnest covering of the triangle with ten circles is shown in Fig. 2j. A proof of its optimality is given in [6]. This proof is based on estimating the areas of Voronoi polygons. Here, we will present another, very simple proof. The key observation is that the three circles that cover the vertices of the triangle can cover at most three isolated points of the inscribed circle of the triangle. This means that this circle must be covered by the seven remaining circles. The thinnest covering of a circle with seven equal circles is well known [2]; it consists of one central circle, surrounded by six equally spaced circles. For the sake of completeness we give the argument here:

If the unit circle is to be covered with seven circles that have a diameter of at most 1, then the circle that covers the centre of the unit circle cannot participate in the covering of the boundary. The other six circles can cover the boundary only if their diameter is at least equal to 1. This uniquely fixes the thinnest covering, up to rotations.

Although there is still freedom of rotation around the centre of the circle, it is easy to see that there is essentially only one position in which the triangle can be covered completely with the circles that cover the inscribed circle.

In contrast to the proof in [6], our proof works only for $n = 10$ and cannot be adapted for $n = 6$.

4 Further conjectures

The conjectured optimal coverings for seven and eight circles are shown in Fig. 2g and h. If the configuration with seven circles is optimal, it would be the first to be nonsymmetric. An interesting aspect of this covering is that it can be thought of as being constructed from the covering with four circles to which an almost regular layer of three circles is added. In fact, there seems to be a range of coverings that is constructed in this way: $n = 4, 7, 11$, and 16. It may be conjectured that this is the optimal topology for all triangular numbers plus one. The thinnest covering found so far for eight circles is also strangely nonsymmetric; the same is true for the putatively best configurations with eleven, twelve, thirteen, fourteen, sixteen, and eighteen circles, shown in Fig. 2k-r. The conjectured configuration with twelve circles shows an interesting new phenomenon: One of the interior circles is not fixed by the other circles. In fact, it has some considerable freedom of movement. This indeterminacy apparently does not imply that the covering radius can be decreased further, as the radius is fixed by the other circles. The same happens for fourteen and seventeen circles. For $n = 15$, the conjectured thinnest covering is the lattice covering with a

covering radius of $\sqrt{3}/15 = 0.115470\dots$. This lattice packing is conjectured to yield the thinnest covering for the triangular numbers. The conjectured thinnest covering for $n = 17$ is symmetric, or can be chosen so by rearranging the loose circle.

The nice structure of the covering with nine circles suggests a generalisation for larger numbers of circles. The resulting covering with $n = 3k(k+1)/2$ ($k \geq 2$) circles has a covering radius of $1/(3k)$. Although this covering has the correct thinnest density asymptotically, it is not always the thinnest covering, because there is a thinner covering for $n = 18$. Fig 2r shows our conjecture for $n = 18$. At first sight, this configuration looks complicated, yet it has a threefold symmetry and a surprisingly simple expression for the covering radius: $1/\sqrt{84} = 0.109108\dots$. This is slightly better than $1/9$, the covering radius for the generalisation of the arrangement for $n = 9$. A remarkable feature of this covering is that it can be extended to a covering of the plane with twelve lattice arrangements of circles (or, equivalently, a covering with twelve equal circles of the fundamental parallelogram of this lattice, with periodic boundary conditions). The lattice vectors are $9/14(1, 0)$ and $9/14(1/2, \sqrt{3}/2)$, and the centres of the circles are of the form $(k/28, l\sqrt{3}/84)$, where $(k, l) = (7, 1), (15, 1), (2, 4), (11, 5), (6, 10), (12, 10), (11, 13), (20, 19), (7, 19), (15, 19), (11, 23),$ and $(20, 22)$. The density of this covering is $56\pi\sqrt{3}/243 = 1.253984\dots$, which is (of course) larger than the thinnest circle covering of the plane, which has a density of $2\pi\sqrt{3}/9 = 1.209199\dots$. This is slightly surprising in the sense that this last density corresponds to the inferior covering generalised from the arrangement with nine circles.

5 Generating the conjectures

The coverings presented in this paper were found by computer simulation. Simulated annealing was used to generate reasonable candidates for the thinnest coverings. The graphs corresponding to the coverings were then optimised, using a mechanical analogue of shrinking metal bars, which was also simulated on a computer.

First, a simulated annealing approach [1, 4] was used to generate coverings with a small covering radius. This optimisation method is particularly suited for this type of problem because there are lots of local optima. The algorithm is implemented as follows. A grid is placed over the triangle, and it is gradually refined during the optimisation process. We take all the assignments of the n circle centres to grid points as configurations. The cost function is the corresponding covering radius, i.e., the smallest number r such that the n circles with the above centres and with radius r cover the triangle. The algorithm starts off from an arbitrary initial configuration. In each iteration a new configuration is generated by slightly perturbing the current configuration. The difference in cost is compared with an acceptance criterion

which accepts all improvements but also admits, in a limited way, deteriorations in cost. Initially, this acceptance criterion accepts improvements, and deteriorations are also accepted with a high probability. As the optimisation process proceeds, the probability for accepting deteriorations is decreased. The process comes to a halt when — during a prescribed number of iterations — no further improvement of the best value occurs.

The covering radius r of a given configuration is determined as follows. Let P be the set of circle centres and consider the *Voronoi tessellation* [9] of the triangle \mathcal{T} , i.e., the partition of \mathcal{T} into polygonal cells $V(p)$, where $V(p)$ is the closure of the set of those points in \mathcal{T} that are closer to p than to any other centre in P . Let L denote the set of the three lines that define the boundary of \mathcal{T} , augmented with all the perpendicular bisectors of the line segments between any pair of centres. Let S denote the set of all intersections in \mathcal{T} of any pair of lines from L . The covering radius is then given by

$$r = \max_{p \in P} \max_{s \in S \cap V(p)} \|p - s\|.$$

Evidently, we may rewrite this in the computationally more manageable form

$$r = \max_{s \in S} \min_{p \in P} \|p - s\|.$$

In our program the latter formula is used only once, for the initial configuration. From then on, r is calculated incrementally. This algorithm has also been successfully applied to find new circle coverings of a rectangle with congruent circles [7, 8].

The previously described algorithm is not very suitable for obtaining high accuracy because it is computationally very expensive. To improve the coverings obtained by this first algorithm (or by any other guess of a good arrangement), we used a new version of the “cooling technique”. This method was developed by Tarnai and Gaspar, originally to find thin (locally optimal) circle coverings of a sphere [10] and later for a square [11]. The main idea of this method is that a locally optimal arrangement can be represented by a graph associated with an elastic bar-and-joint structure which will have tensile stresses in the bars (possibly some bars are free of stress) when the structure is uniformly cooled. The problem is that usually even the final topology of the graph is unknown, because three circles, which had only one common point in the starting arrangement, can later have a domain in common, or, on the contrary, three circles with a common domain can have only one point in common in the final configuration. We start with a bipartite graph where the vertices of the first kind are the centres of the circles, and each vertex of the second kind is one arbitrary point in the set of the common points (if it is not an empty set) of every triplet of circles. (Here, the edges of the triangles are considered as circles with infinitely large radius.) The edges of the graph are straight line segments,

drawn from each vertex of the second kind to the centres of the circles containing this vertex. The starting structure has bars (of equal length in the rest position, and equal rigidity) corresponding to the edges, the nodes of the second kind that lie on the side of the triangle will be supported by a roller and the nodes at the vertices of the triangle are supported by a fix joint. We determine the nearest equilibrium state of this structure by solving a system of nonlinear equations using the Newton-Raphson method. Usually, in the equilibrium state, some bars are in compression while others are in tension. We want to find a stress-free state, where all bars have equal length, so we try to decrease the temperature of the structure slowly (which amounts to decreasing the original lengths of the bars) until the absolute value of every stress decreases. If this cannot be accomplished we will change the normal rigidity of the bar in the largest compression to zero. This step changes the structure, so we have to recalculate the nearest equilibrium state. In every state we have to check whether the bars with zero rigidity are really shorter than the bars in the rest position. If this is not true, we have to put these bars back to the structure with the original rigidity. At the end of this procedure we arrive at a state where all bars are free of stress, and all bars with zero rigidity are shorter than the common bar length. If the temperature is decreased further, non-compression stresses will appear in bars. This configuration gives a locally optimal covering and the bar length corresponds to the radius of the circles.

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