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Simultaneous Inspection of Several Product Characteristics

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Abstract

In this paper we consider the inspection of a product of which several characteristics have to satisfy given specification limits. A problem which occurs in the inspection process quite often is that a measurement error occurs in measuring the characteristics. Therefore, it is common practice to inspect each characteristic by comparing its measurement to a test limit which is slightly more strict than the corresponding specification limit. An item then is accepted if for each characteristic the measurement conforms to the corresponding test limit. However, instead of inspecting an individual characteristic merely using its own measurement, it is (much) more efficient to use the measurements of the other characteristics as well, especially if some of the characteristics are highly correlated. In this paper it is shown how the measurements of all the characteristics can be used to test whether an item is conforming.

Keywords and phrases: measurement error, test limits, consumer risk, yield

1991 Mathematics Subject Classification: 62 H 20, 62 N 10

1 Introduction

In industry a lot of attention is devoted nowadays to quality improvement by means of reducing the process variability. Statistical tools that can be helpful to

achieve this are design of experiments (cf. Taguchi (1987)) and statistical process control (cf. Shewhart (1931) and Deming (1986)). Although it is very important to reduce the process variability and to keep the production process statistically under control, in quite a few production processes there is still need for inspection. For example, in many large scale production processes only very few items, i.e. only a few parts per million (ppm), are allowed to be nonconforming. Hence, if despite the application of design of experiments and statistical process control, still 1% of the produced items is nonconforming the specifications, while the consumer requires that no more than 10 ppm are nonconforming, then there is still need for 100% inspection, i.e. all produced items should be inspected.

In Albers, Kallenberg and Otten (1994a,b) the inspection is considered of one characteristic that has to conform a specification limit. As the measurement process used during inspection is typically not infallible, test limits are determined in such a way that only a fraction γ is accepted by the test limit, and is nonconforming the specification limit. With values of γ between 1 and 100 ppm, this results in a test limit which in most situations is slightly more strict than the specification limit.

Now suppose that not just one, but several characteristics of the same product have to satisfy given specification limits and that an item is nonconforming if there is at least one characteristic that is nonconforming its specification limit. In practice it is quite common to inspect each characteristic by comparing its measurement to a test limit, which because of a measurement error is slightly more strict than the corresponding specification limit. An item is then accepted if for each characteristic the measurement conforms to the corresponding test limit. However, inspection of a characteristic merely using the measurement of the characteristic itself, is not very efficient. Note that we have available the measurements of all characteristics that have to be inspected. Hence instead we can use for inspection these available measurements of the other characteristics as well, and accept an item if the measurements of the characteristics fall in some given test region. In this paper we determine a test region that results in an (almost) optimal yield, under the restriction that the consumer risk, the conditional probability that an item is nonconforming given that it is accepted by the test region, falls below a prescribed bound γ.

The outline of the paper is as follows. In section 2 we introduce the notation and the model assumptions. In section 3 we show what kind of test region should be used for the simultaneous inspection of several product characteristics. First we present the test region which under the restriction that the consumer risk is bounded by γ, results in an optimal yield. Due to its implicit form, this optimal test region is very difficult to evaluate and to interpret. Therefore, we also present a more simple test region, which results in a yield approximately equal to the optimal yield. Due to its more simple structure we can actually determine in section 4 the test limits that occur in this latter test region in case the parameters are known. First we present test limits which result in a consumer risk that falls

below the prescribed bound γ for most practical situations. Unfortunately, these test limits can be very conservative, i.e. the consumer risk can be much smaller than necessary, especially if the characteristics are highly correlated. Therefore, we also present an improvement of these test limits, which results in a consumer risk closer to the bound γ and a higher yield. In section 5 we present an example from semiconductor industry. This example clearly shows the benefits of the new procedure: the yield improves tremendously if an item is inspected by combining the measurements of all characteristics instead of inspecting each characteristic by means of its own measurement. In section 6 we consider the situation that parameters are unknown. We present estimators of the parameters involved and we present a modification of the test limits determined in section 4, to ensure that the expected consumer risk is approximately equal to the prescribed bound γ.

2 Notation and model assumptions

Suppose there are k characteristics that have to satisfy certain specification limits. To inspect an item we have measurements of all k characteristics. The vector of measurements is

$$\widetilde{\mathbf{X}} = \mathbf{X} + \mathbf{U},$$

where $\mathbf{X}$ is the vector of true values of the characteristics and $\mathbf{U}$ is the vector of measurement errors. We assume that $\mathbf{X}$ and $\mathbf{U}$ are independent and multivariate normally distributed: $\mathbf{X} \sim N_k(\mu_{\mathbf{X}}, \Sigma_{\mathbf{XX}})$ and $\mathbf{U} \sim N_k(\mathbf{0}, \Sigma_{\mathbf{UU}})$. Furthermore, in practice the measurement error of a characteristic is typically small compared to the total variation in the characteristic. Hence, we assume that $\sigma_{U_l}/\sigma_{X_l}$ is small for all l, where σ_{U_l} and σ_{X_l} are the diagonal elements of $\Sigma_{\mathbf{UU}}$ and $\Sigma_{\mathbf{XX}}$, respectively. In general the values of $\sigma_{U_l}/\sigma_{X_l}$ lie in $(0, 0.30]$. Since $\mathbf{X}$ and $\mathbf{U}$ are multivariate normally distributed, $\widetilde{\mathbf{X}}$ is multivariate normally distributed as well: $\widetilde{\mathbf{X}} \sim N_k(\mu_{\mathbf{X}}, \Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}})$, with $\Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}} = \Sigma_{\mathbf{XX}} + \Sigma_{\mathbf{UU}}$. Throughout the paper we assume that the covariance matrices $\Sigma_{\mathbf{XX}}$ and $\Sigma_{\mathbf{UU}}$ are positive definite.

Now let us consider the inspection process. A product is called nonconforming, if there is at least one characteristic which is nonconforming, and we call the l^{th} characteristic nonconforming if the true value X_l is larger than its specification limit s_l. Furthermore, we assume that large negative correlations between the true values of the characteristics (< -0.5) do not occur. If two characteristics are highly negatively correlated, then it is more likely that one of these has to fall below an upper specification limit and the other one has to be larger than a lower specification limit. In that case, by taking negative values of the characteristic which has to satisfy a lower specification limit, we are back in a situation that the characteristics are positively correlated and both have to fall below an upper specification limit. On the other hand, although the correlations between the

characteristics may become large, to avoid degeneration of the problem we do assume that extreme values over 0.99 do not occur.

The standardized specification limits $\bar{s}_l = (s_l - \mu_{X_l})/\sigma_{X_l}$ will never be very small. If they would be, the probability $\pi = P\big(\exists l : X_l > s_l\big)$ that an item is nonconforming, would become too large. For example, if $k = 1$ and $\bar{s}_1 = 0.8$, then $\pi = 0.21$, which is quite large. In a situation where π is large, one should aim at reducing the process variability before one starts final production and inspection, resulting in larger specification limits and a smaller π. In general, at the moment that inspection takes place, π will typically lie in (0,0.15].

3 Test region

3.1 Optimal test region

To test whether an item is nonconforming, measurements of all k characteristics are available. As we mentioned in the introduction, each characteristic can be inspected by comparing its measurement to a test limit. However, since the characteristics can be (highly) correlated, it is more efficient to inspect a characteristic using the measurements of all characteristics, instead of its own measurement only. In general, we can accept an item if $\widetilde{\mathbf{X}} \in T$, where $T \subset \mathbb{R}^k$ is some given test region. In this subsection we determine the optimal test region, i.e. the region that maximizes the yield, under the restriction that the consumer risk is bounded by γ.

The consumer risk, the conditional probability that an item is nonconforming given that it is accepted by the test region, is given by

$$CR = P\big(\exists l : X_l > s_l \big| \widetilde{\mathbf{X}} \in T\big).$$

Note that the restriction $CR \leq \gamma$ is equivalent with

$$P\big(\exists l : X_l > s_l, \widetilde{\mathbf{X}} \in T\big) - \gamma P\big(\widetilde{\mathbf{X}} \in T\big) \leq 0,$$

and the problem of finding the test region which under the restriction $CR \leq \gamma$, maximizes the yield, can be solved using the Neyman-Pearson lemma. Consider the problem of finding the function δ, with $\delta(\tilde{\mathbf{x}}) \in [0, 1]$, which subject to

$$\int \delta(\tilde{\mathbf{x}}) \big[P\big(\exists l : X_l > s_l \big| \widetilde{\mathbf{X}} = \tilde{\mathbf{x}}\big) - \gamma\big] f_{\widetilde{\mathbf{X}}}(\tilde{\mathbf{x}}) \mathrm{d}\tilde{\mathbf{x}} \leq 0,$$

maximizes

$$\int \delta(\tilde{\mathbf{x}}) f_{\widetilde{\mathbf{X}}}(\tilde{\mathbf{x}}) \mathrm{d}\tilde{\mathbf{x}}.$$

The Neyman-Pearson lemma states that the solution to this problem is given by

$$\delta(\tilde{\mathbf{x}}) = \begin{cases} 1 & \text{if } P\big(\exists l : X_l > s_l \big| \widetilde{\mathbf{X}} = \tilde{\mathbf{x}}\big) < c \\ 0 & \text{if } P\big(\exists l : X_l > s_l \big| \widetilde{\mathbf{X}} = \tilde{\mathbf{x}}\big) > c, \end{cases}$$

with c such that

$$\int \delta(\tilde{\mathbf{x}})\big[P\big(\exists l : X_l > s_l \big| \widetilde{\mathbf{X}} = \tilde{\mathbf{x}}\big) - \gamma\big] f_{\widetilde{\mathbf{X}}}(\tilde{\mathbf{x}}) \mathrm{d}\tilde{\mathbf{x}} = 0.$$

Hence, the test region that maximizes the yield under the restriction $CR \leq \gamma$ is given by

$$T^* = \Big\{\tilde{\mathbf{x}} \in \mathbb{R}^k \Big| P\big(\exists l : X_l > s_l \big| \widetilde{\mathbf{X}} = \tilde{\mathbf{x}}\big) < c\Big\}, \tag{3.1}$$

with c such that $CR^* = P\big(\exists l : X_l > s_l \big| \widetilde{\mathbf{X}} \in T^*\big) = \gamma$. Unfortunately, due to the multivariate probability involved in the test region T^*, it is not easily verified whether an item conforms to this test region. Therefore, in the next subsection, we present a test region for which it is much easier to check whether the vector of measurements of all characteristics falls in this test region, and which is approximately equal to the test region T^* from (3.1).

3.2 Approximation of the optimal test region

The distribution of X_l conditional on $\widetilde{\mathbf{X}}$ is given by

$$X_l \big| \widetilde{\mathbf{X}} \sim N\left(\mu_{X_l} + \Sigma_{X_l\mathbf{X}}^{\mathrm{T}} \Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}}^{-1} (\widetilde{\mathbf{X}} - \mu_{\mathbf{X}}), \sigma_{X_l}^2 - \Sigma_{X_l\mathbf{X}}^{\mathrm{T}} \Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}}^{-1} \Sigma_{X_l\mathbf{X}}\right), \tag{3.2}$$

where $\Sigma_{X_l\mathbf{X}} = \mathrm{Cov}(X_l, \mathbf{X})$, which is the l^{th} column of $\Sigma_{\mathbf{XX}}$. Since $\mathbf{X}$ and $\mathbf{U}$ are independent and multivariate normally distributed, for all $l = 1, \ldots, k$, we can write

$$\widetilde{\mathbf{X}} = \mathbf{a}_l + \mathbf{b}_l X_l + \mathbf{Z}_l, \tag{3.3}$$

with $\mathbf{Z}_l \sim N_k(\mathbf{0}, \Sigma_l)$ independent of X_l and

$$\mathbf{b}_l = \Sigma_{X_l\mathbf{X}}/\sigma_{X_l}^2,\ \mathbf{a}_l = \mu_{\mathbf{X}} - \mathbf{b}_l \mu_{X_l} \text{ and } \Sigma_l = \Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}} - \mathbf{b}_l \mathbf{b}_l^{\mathrm{T}} \sigma_{X_l}^2. \tag{3.4}$$

As $\Sigma_{\mathbf{XX}}$ and $\Sigma_{\mathbf{UU}}$ are positive definite, $\Sigma_l = \Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}} - \mathbf{b}_l \mathbf{b}_l^{\mathrm{T}} \sigma_{X_l}^2 = \Sigma_{\mathbf{XX}} - \Sigma_{X_l\mathbf{X}} \Sigma_{X_l\mathbf{X}}^{\mathrm{T}}/\sigma_{X_l}^2 + \Sigma_{\mathbf{UU}}$ is positive definite as well, and for all $l = 1, \ldots, k$, we can define the linear combination $\widetilde{Y}_l = \mathbf{b}_l^{\mathrm{T}} \Sigma_l^{-1} \widetilde{\mathbf{X}}$. Furthermore, define

$$\begin{aligned} &\alpha_l = \mathbf{b}_l^{\mathrm{T}} \Sigma_l^{-1} \mathbf{a}_l,\ \beta_l = \mathbf{b}_l^{\mathrm{T}} \Sigma_l^{-1} \mathbf{b}_l, \\ &Z_l = \mathbf{b}_l^{\mathrm{T}} \Sigma_l^{-1} \mathbf{Z}_l,\ \sigma_{Z_l}^2 = \mathrm{Var}(Z_l) = \mathbf{b}_l^{\mathrm{T}} \Sigma_l^{-1} \mathbf{b}_l,\ \sigma_l = \sigma_{Z_l}/(\beta_l \sigma_{X_l}), \end{aligned} \tag{3.5}$$

and note that $\sigma_{Z_l}^2 = \beta_l$. Then the linear combination is equal to $\widetilde{Y}_l = \alpha_l + \beta_l X_l + Z_l$, with $\mu_{\widetilde{Y}_l} = \mathrm{E}\widetilde{Y}_l = \alpha_l + \beta_l \mu_{X_l}$ and $\mathrm{Var}(\widetilde{Y}_l) = \mathbf{b}_l^{\mathrm{T}} \Sigma_l^{-1} \Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}} \Sigma_l^{-1} \mathbf{b}_l = \beta_l^2 \sigma_{X_l}^2 (1 + \sigma_l^2)$. Furthermore, we have

$$\begin{aligned} \mathbf{b}_l^{\mathrm{T}} \Sigma_l^{-1} &= \mathbf{b}_l^{\mathrm{T}} \Sigma_l^{-1} \Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}} \Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}}^{-1} = \mathbf{b}_l^{\mathrm{T}} \Sigma_l^{-1} (\mathbf{b}_l \mathbf{b}_l^{\mathrm{T}} \sigma_{X_l}^2 + \Sigma_l) \Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}}^{-1} \\ &= (\beta_l \sigma_{X_l}^2 + 1) \Sigma_{X_l\mathbf{X}}^{\mathrm{T}} \Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}}^{-1} / \sigma_{X_l}^2 = \beta_l (1 + \sigma_l^2) \Sigma_{X_l\mathbf{X}}^{\mathrm{T}} \Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}}^{-1} \end{aligned}$$

and (3.2) can be replaced by

$$X_l\big|\widetilde{\mathbf{X}} \sim N\left(\mu_{X_l} + (\widetilde{Y}_l - \mu_{\widetilde{Y}_l})/[\beta_l(1+\sigma_l^2)], 1/[\beta_l(1+\sigma_l^2)]\right). \tag{3.6}$$

Hence, with Φ the standard normal distribution function and $a = \Phi^{-1}(1-c)$, we have

$$\begin{aligned} &P\big(X_l > s_l\big|\widetilde{\mathbf{X}}\big) < c \Longleftrightarrow \\ &\widetilde{Y}_l < \mu_{\widetilde{Y}_l} + \beta_l(1+\sigma_l^2)(s_l - \mu_{X_l}) - a\sigma_{Z_l}\sqrt{1+\sigma_l^2}. \end{aligned} \tag{3.7}$$

Let T' be the region for which the last line of (3.7) holds true for all $l = 1, \ldots, k$, i.e.

$$T' = \Big\{\tilde{\mathbf{x}} \in \mathbb{R}^k \big| \mathbf{b}_l^{\mathrm{T}}\Sigma_l^{-1}\tilde{\mathbf{x}} < t_l' \;\forall l = 1, \ldots, k\Big\}, \tag{3.8}$$

where $t_l' = \mu_{\widetilde{Y}_l} + \beta_l(1+\sigma_l^2)(s_l - \mu_{X_l}) - a\sigma_{Z_l}\sqrt{1+\sigma_l^2}$. Note that if $\widetilde{\mathbf{X}} \notin T'$, i.e. if the last line of (3.7) is not true for some l, then $P\big(\exists l : X_l > s_l\big|\widetilde{\mathbf{X}}\big) \geq P\big(X_l > s_l\big|\widetilde{\mathbf{X}}\big) \geq c$ and $\widetilde{\mathbf{X}} \notin T^*$. This implies that if $\widetilde{\mathbf{X}}$ lies in the optimal test region T^*, then it will also lie in the region T'. Moreover, the test regions T^* and T' are approximately equal, i.e. the yield and the consumer risk corresponding to T^*, are approximately equal to the yield and consumer risk, respectively, corresponding to the test region T', which can be shown as follows.

First of all note that as $\Sigma_{X_l\mathbf{X}}^{\mathrm{T}}\Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}}^{-1}\widetilde{\mathbf{X}}$ has maximal correlation with X_l (see Anderson (1958)), $\mathbf{b}_l^{\mathrm{T}}\Sigma_l^{-1}\widetilde{\mathbf{X}} = \beta_l(1+\sigma_l^2)\Sigma_{X_l\mathbf{X}}^{\mathrm{T}}\Sigma_{\widetilde{\mathbf{X}}\widetilde{\mathbf{X}}}^{-1}\widetilde{\mathbf{X}}$ has maximal correlation with X_l as well. The correlation between X_l and $\widetilde{Y}_l$ (also known as the multiple correlation coefficient of X_l and $\widetilde{\mathbf{X}}$), equals

$$\rho_l = 1/\sqrt{1+\sigma_l^2}. \tag{3.9}$$

As the linear combination $\widetilde{Y}_l$ has maximal correlation with X_l, its correlation with X_l will be at least as large as the correlation of $\widetilde{X}_l$ with X_l, which is equal to $(1+\sigma_{U_l}^2/\sigma_{X_l}^2)^{-1/2}$. This implies that $\rho_l \geq (1+\sigma_{U_l}^2/\sigma_{X_l}^2)^{-1/2}$ and therefore $\sigma_l \leq \sigma_{U_l}/\sigma_{X_l}$. Note that as $\sigma_{U_l}/\sigma_{X_l}$ is assumed to be small, σ_l will be small as well, which will be used to show that the regions T^* and T' are approximately equal.

As we already mentioned, $\widetilde{\mathbf{X}} \in T^*$ implies $\widetilde{\mathbf{X}} \in T'$ and hence for the difference in yield between the two test regions T^* and T', we have

$$\begin{aligned} &P\big(\widetilde{\mathbf{X}} \in T'\big) - P\big(\widetilde{\mathbf{X}} \in T^*\big) = \\ &P\big(\widetilde{\mathbf{X}} \in T'\big) - P\big(\widetilde{\mathbf{X}} \in T^* \cap \widetilde{\mathbf{X}} \in T'\big) = P\big(\widetilde{\mathbf{X}} \in T' \cap \widetilde{\mathbf{X}} \notin T^*\big). \end{aligned}$$

Let

$$R\big(\widetilde{\mathbf{X}}\big) = P\big(X_l < s_l \,\forall\, l \,\big|\widetilde{\mathbf{X}}\big) - P\big(\widetilde{Y}_l < t_l' \,\forall\, l < k, X_k < s_k\big|\widetilde{\mathbf{X}}\big).$$

Note that if $\widetilde{\mathbf{X}} \in T'$, then $R(\widetilde{\mathbf{X}}) = P\big(X_l < s_l\,\forall\, l\big|\widetilde{\mathbf{X}}\big) - P\big(X_k < s_k\big|\widetilde{\mathbf{X}}\big)$. Furtermore, $\widetilde{\mathbf{X}} \notin T^*$ is equivalent with $P\big(X_l < s_l\,\forall\, l\big|\widetilde{\mathbf{X}}\big) = 1 - P\big(\exists l : X_l > s_l\big|\widetilde{\mathbf{X}}\big) < 1 - c$. Hence, if $\widetilde{\mathbf{X}} \in T'$, then $\widetilde{\mathbf{X}} \notin T^*$ is equivalent with $P\big(X_k < s_k\big|\widetilde{\mathbf{X}}\big) < 1 - c - R(\widetilde{\mathbf{X}})$, which in its turn is equivalent with $\widetilde{Y}_k > \mu_{\widetilde{Y}_k} + \beta_k(1+\sigma_k^2)(s_k - \mu_{X_k}) - \Phi^{-1}\big(1 - c - R(\widetilde{\mathbf{X}})\big)\sigma_{Z_k}\sqrt{1+\sigma_k^2}$. Hence for the difference in yield we have

$$
\begin{aligned}
&P\big(\widetilde{\mathbf{X}} \in T'\big) - P\big(\widetilde{\mathbf{X}} \in T^*\big) = P\big(\widetilde{\mathbf{X}} \in T' \cap \widetilde{\mathbf{X}} \notin T^*\big) \\
&= P\big(\widetilde{Y}_l < t'_l\ \forall l < k, \\
&\qquad \mu_{\widetilde{Y}_k} + \beta_k(1+\sigma_k^2)(s_k - \mu_{X_k}) - \Phi^{-1}\big(1 - c - R\big(\widetilde{\mathbf{X}}\big)\big)\sigma_{Z_k}\sqrt{1+\sigma_k^2} < \widetilde{Y}_k < t'_k\big) \\
&\le P\big(\mu_{\widetilde{Y}_k} + \beta_k(1+\sigma_k^2)(s_k - \mu_{X_k}) - \Phi^{-1}\big(1 - c - R\big(\widetilde{\mathbf{X}}\big)\big)\sigma_{Z_k}\sqrt{1+\sigma_k^2} < \widetilde{Y}_k < t'_k\big) \\
&= \Phi\left[\sqrt{1+\sigma_k^2}\bar{s}_k - \Phi^{-1}(1-c)\sigma_k\right] - \Phi\left[\sqrt{1+\sigma_k^2}\bar{s}_k - \Phi^{-1}(1 - c - R\big(\widetilde{\mathbf{X}}\big))\sigma_k\right].
\end{aligned}
$$

Note that as $\Sigma_{\mathbf{UU}} \to 0$, the random variable $X_l\big|\widetilde{\mathbf{X}}$ converges to X_l for all $l = 1, \ldots, k$. This implies that as $\Sigma_{\mathbf{UU}} \to 0$, the last line of (3.7) is asymptotically equivalent with $X_l < s_l$, and thus $R\big(\widetilde{\mathbf{X}}\big) \to 0$. Hence the difference in yield is $o(\sigma_k)$, as $\Sigma_{\mathbf{UU}} \to 0$. Similarly, one can verify that the difference in the consumer risk between the test regions T^* and T' is $o(\sigma_k)$, as $\Sigma_{\mathbf{UU}} \to 0$.

Although a is still implicitly defined, the test region T' is explicit in terms of the measurements, and hence the test region T' is much easier to interpret than the test region T^*. Using the test region T', one should accept an item if for all $l = 1, \ldots, k$, the linear combination $\widetilde{Y}_l = \mathbf{b}_l^{\mathrm{T}}\Sigma_l^{-1}\widetilde{\mathbf{X}}$ falls below the test limit $t'_l = \mu_{\widetilde{Y}_l} + \beta_l(1+\sigma_l^2)(s_l - \mu_{X_l}) - a\sigma_{Z_l}\sqrt{1+\sigma_l^2}$. However, instead of the test region T', we prefer to use the test region

$$
T = \Big\{\tilde{\mathbf{x}} \in \mathbb{R}^k \Big| \mathbf{b}_l^{\mathrm{T}}\Sigma_l^{-1}\tilde{\mathbf{x}} < \alpha_l + \beta_l s_l - a\sigma_{Z_l}\ \forall l = 1, \ldots, k\Big\}, \tag{3.10}
$$

which can be found by neglecting the terms σ_l^2 in the test limit t'_l for all $l = 1, \ldots, k$. It is clear that the test limits in T are even more simple than the test limits in T', while the difference in yield and consumer risk between T' and T, is only $o(\sum_l \sigma_l)$. This implies that the difference in yield and consumer risk between T^* and T also is $o(\sum_l \sigma_l)$. Besides the fact that the test limits are more simple, another advantage of using the test region T instead of T', is that the test limit in T for the l^{th} linear combination is similar to the test limit that occurs in Albers, Arts and Kallenberg (1998b), where one considers the inspection of one characteristic, by means of a linear combination of measurements of two or more correlated characteristics. Using the test region T instead of T', enables us to use these results on inspection of one characteristic, in case parameters are unknown.

Now that we have found a tractable test region which results in a yield and consumer risk approximately equal to the yield and consumer risk using T^*, the next step is to determine the test limits, or equivalently to determine a in (3.10), such that the consumer risk is approximately equal to γ.

4 Test limits if parameters are known

4.1 Test limits using an upper bound for the consumer risk

To inspect whether the characteristics satisfy the specification limits, i.e. whether $X_l < s_l$ for all $l = 1, \ldots, k$, we look at the linear combinations $\widetilde{Y}_1, \ldots, \widetilde{Y}_k$ and we accept an item as conforming if these linear combinations fall below the test limits $t_1, \ldots, t_k$, respectively, with $t_l = \alpha_l + \beta_l s_l - a\sigma_{Z_l}$ (see (3.10)). Since the linear combinations $\widetilde{Y}_1, \ldots, \widetilde{Y}_k$ are not perfectly correlated with $X_1, \ldots, X_k$, respectively, we will make errors in accepting or rejecting items. The consumer risk is the conditional probability that at least one characteristic is nonconforming, given that an item is accepted by the test limits. In formula, with $r = 1, \ldots, k$,

$$CR = P\big(\exists l : X_l > s_l \big| \widetilde{Y}_r < t_r \forall r\big). \tag{4.1}$$

Since the probability of making an error for two characteristics is much smaller than of making an error for one characteristic, we write

$$CR = \sum_{l=1}^{k} P\big(X_l > s_l \big| \widetilde{Y}_r < t_r \forall r\big) - R_1. \tag{4.2}$$

It is easily verified that the remainder $R_1 \geq 0$ and hence $\sum_l P\big(X_l > s_l \big| \widetilde{Y}_r < t_r \forall r\big)$ is an upper bound for CR. Furthermore, for the values of the correlations between the true values under consideration (i.e. $\rho(X_i, X_j) \leq 0.99$, $\forall i \neq j$), $\sum_l P\big(X_l > s_l \big| \widetilde{Y}_r < t_r \forall r\big)$ is a very accurate approximation of CR (cf. Arts (1998)).

Next we take a look at $P\big(X_l > s_l \big| \widetilde{Y}_r < t_r \forall r\big)$, which occurs on the right-hand side of (4.2). This is the probability that the l^{th} characteristic is nonconforming, given that an item is accepted by all the test limits. Note that to decide whether the l^{th} characteristic is conforming, the event $\widetilde{Y}_l < t_l$ will play the most important role, as of all linear combinations, $\widetilde{Y}_l$ has the largest correlation with X_l. If the correlations between the characteristics are small, then the correlations between $\widetilde{Y}_r$ and X_l will be small for all $r \neq l$, and neglecting the events $\widetilde{Y}_r < t_r$ for all $r \neq l$, will probably result in a good approximation of $P\big(X_l > s_l \big| \widetilde{Y}_r < t_r \forall r\big)$. More general, in theorem 4.1 we show that if the linear combinations are positively correlated, we find an upper bound for $P\big(X_l > s_l \big| \widetilde{Y}_r < t_r \forall r\big)$, if we ignore the events $\widetilde{Y}_r < t_r$ for all $r \neq l$. To prove this we make use of the following lemma:

Lemma 4.1. Let $\mathbf{V} = (V_1, \ldots, V_k)$ be multivariate normally distributed, with mean $\mathrm{E}\mathbf{V} = \mathbf{0}$, $\Sigma_{\mathbf{VV}}$ positive definite with diagonal elements equal to 1 and all other elements nonnegative, then if $w \geq q$,

$$P\big(V_r < v_r \forall r \neq l | V_l < w\big) \leq P\big(V_r < v_r \forall r \neq l | V_l < q\big) \quad \forall l = 1, \ldots, k. \tag{4.3}$$

Proof. First observe that $P\big(V_r < v_r \forall r \neq l | V_l = v_l\big)$ is decreasing in v_l if V_l is nonnegatively correlated with V_r for all $r \neq l$. Then we have, with ϕ the standard normal density function,

$$\begin{aligned}
P\big(V_r < v_r \forall r \neq l, q < V_l < w\big) &= \int_q^w P\big(V_r < v_r \forall r \neq l | V_l = v_l\big)\phi(v_l)\mathrm{d}v_l \\
&\leq \int_q^w P\big(V_r < v_r \forall r \neq l | V_l = q\big)\phi(v_l)\mathrm{d}v_l \\
&= P\big(q < V_l < w\big)P\big(V_r < v_r \forall r \neq l | V_l = q\big) \\
&= \frac{P\big(q < V_l < w\big)}{P\big(V_l < q\big)} \int_{-\infty}^q P\big(V_r < v_r \forall r \neq l | V_l = q\big)\phi(v_l)\mathrm{d}v_l \\
&\leq \frac{P\big(q < V_l < w\big)}{P\big(V_l < q\big)} \int_{-\infty}^q P\big(V_r < v_r \forall r \neq l | V_l = v_l\big)\phi(v_l)\mathrm{d}v_l \\
&= P\big(q < V_l < w\big)P\big(V_r < v_r \forall r \neq l | V_l < q\big).
\end{aligned}$$

Hence

$$P\big(V_r < v_r \forall r \neq l, V_l < w\big) \leq P\big(V_l < w\big)P\big(V_r < v_r \forall r \neq l | V_l < q\big),$$

and dividing by $P\big(V_l < w\big)$ we arrive at (4.3). □

Now we have the following theorem:

Theorem 4.1. If $\rho(\widetilde{Y}_r, \widetilde{Y}_q) \geq 0$, $\forall\ r \neq q = 1, \ldots, k$, then

$$P\big(X_l > s_l \big| \widetilde{Y}_r < t_r \forall r\big) \leq P\big(X_l > s_l \big| \widetilde{Y}_l < t_l\big). \tag{4.4}$$

Proof. Without loss of generality we assume that $\mathrm{E}X_l{=}0$ and $\mathrm{Var}(X_l){=}1$ for all l, so $\mathbf{a}_l = 0$ and $\mathbf{b}_l = \mathrm{Cov}(X_l, \mathbf{X})$, for all $l = 1, \ldots, k$ in (3.4). Define $\widetilde{\mathbf{Y}}_{-l}$ as the vector with elements $\widetilde{Y}_r, r \neq l$. Writing $\widetilde{Y}_r = \mathbf{b}_r^{\mathrm{T}}\Sigma_r^{\text{-}1}\widetilde{\mathbf{X}} = \mathbf{b}_r^{\mathrm{T}}\Sigma_r^{\text{-}1}(\mathbf{b}_l X_l + \mathbf{Z}_l)$ and noting that $\mathbf{b}_l^{\mathrm{T}}\Sigma_l^{\text{-}1}\mathbf{b}_l = \beta_l = \sigma^2_{Z_l}$ (see (3.5)), it is easily verified that $(\widetilde{\mathbf{Y}}_{-l}, X_l, \widetilde{Y}_l)$ is multivariate normally distributed with mean $\mathbf{0}$ and covariance matrix

$$\begin{bmatrix}
\mathrm{Var}(\widetilde{\mathbf{Y}}_{-l}) & \{_{\mathrm{c}}\mathbf{b}_r^{\mathrm{T}}\Sigma_r^{\text{-}1}\mathbf{b}_l\}_{r\neq l} & \{_{\mathrm{c}}\mathbf{b}_r^{\mathrm{T}}\Sigma_r^{\text{-}1}\mathbf{b}_l\}_{r\neq l}(1+\beta_l) \\
\{_{\mathrm{r}}\mathbf{b}_r^{\mathrm{T}}\Sigma_r^{\text{-}1}\mathbf{b}_l\}_{r\neq l} & 1 & \beta_l \\
\{_{\mathrm{r}}\mathbf{b}_r^{\mathrm{T}}\Sigma_r^{\text{-}1}\mathbf{b}_l\}_{r\neq l}(1+\beta_l) & \beta_l & \beta_l(1+\beta_l)
\end{bmatrix}, \tag{4.5}$$

where $\{_{\mathrm{c}}a_r\}_{r\neq l}$ and $\{_{\mathrm{r}}a_r\}_{r\neq l}$ are a column and row, respectively, with elements $a_r,\ r\neq l$. The distribution of $\widetilde{\mathbf{Y}}_{-l}$ conditioned on $X_l = x_l, \widetilde{Y}_l = \tilde{y}_l$, equals

$$\widetilde{\mathbf{Y}}_{-l}\big|X_l = x_l, \widetilde{Y}_l = \tilde{y}_l \sim N_{k-1}\left(\{_{\mathrm{c}}\mathbf{b}_r^{\mathrm{T}}\Sigma_r^{-1}\mathbf{b}_l\}_{r\neq l}\tilde{y}_l/\beta_l, \mathrm{Var}(\widetilde{\mathbf{Y}}_{-l}\big|\widetilde{Y}_l)\right),$$

which implies that

$$P\big(\widetilde{Y}_r < t_r \forall r\neq l\big|X_l = x_l, \widetilde{Y}_l = \tilde{y}_l\big) = P\big(\widetilde{Y}_r < t_r \forall r\neq l\big|\widetilde{Y}_l = \tilde{y}_l\big). \tag{4.6}$$

Consequently, with $f_{X_l,\widetilde{Y}_l}$ the joint density function of X_l and $\widetilde{Y}_l$, we have

$$\begin{aligned}
&P\big(X_l > s_l, \widetilde{Y}_r < t_r \forall r\big)\\
&= \int_{-\infty}^{t_l}\int_{s_l}^{\infty} P\big(\widetilde{Y}_r < t_r \forall r\neq l\big|X_l = x_l, \widetilde{Y}_l = \tilde{y}_l\big) f_{X_l,\widetilde{Y}_l}(x_l,\tilde{y}_l)\mathrm{d}x_l\mathrm{d}\tilde{y}_l\\
&= \int_{-\infty}^{t_l} P\big(\widetilde{Y}_r < t_r \forall r\neq l\big|\widetilde{Y}_l = \tilde{y}_l\big) P\big(X_l > s_l\big|\widetilde{Y}_l = \tilde{y}_l\big) f_{\widetilde{Y}_l}(\tilde{y}_l)d\tilde{y}_l\\
&= \int_{-\infty}^{t_l} P\big(\widetilde{Y}_r < t_r \forall r\neq l\big|\widetilde{Y}_l = \tilde{y}_l\big)\Big[1-\Phi\Big(\sqrt{1+\beta_l}s_l - \tilde{y}_l/\sqrt{1+\beta_l}\Big)\Big] f_{\widetilde{Y}_l}(\tilde{y}_l)\mathrm{d}\tilde{y}_l.
\end{aligned}$$

Using integration by parts and lemma 4.1, respectively, writing YD for the yield $P(\widetilde{Y}_r < t_r \forall r)$, we find that

$$\begin{aligned}
P\big(X_l > s_l, \widetilde{Y}_r < t_r \forall r\big) &= YD\Big[1-\Phi\Big(\sqrt{1+\beta_l}s_l - t_l/\sqrt{1+\beta_l}\Big)\Big]\\
&\quad -\frac{1}{\sqrt{1+\beta_l}}\int_{-\infty}^{t_l}\phi\Big(\sqrt{1+\beta_l}s_l - \tilde{y}_l/\sqrt{1+\beta_l}\Big)P\big(\widetilde{Y}_r < t_r \forall r\neq l, \widetilde{Y}_l < \tilde{y}_l\big)\mathrm{d}\tilde{y}_l\\
&\leq YD\Big[1-\Phi\Big(\sqrt{1+\beta_l}s_l - t_l/\sqrt{1+\beta_l}\Big)\Big]\\
&\quad -\frac{YD}{P\big(\widetilde{Y}_l < t_l\big)}\frac{1}{\sqrt{1+\beta_l}}\int_{-\infty}^{t_l}\phi\Big(\sqrt{1+\beta_l}s_l - \tilde{y}_l/\sqrt{1+\beta_l}\Big)P\big(\widetilde{Y}_l < \tilde{y}_l\big)\mathrm{d}\tilde{y}_l.
\end{aligned}$$

Using integration by parts once more and dividing by the yield, we arrive at

$$\begin{aligned}
&P\big(X_l > s_l\big|\widetilde{Y}_r < t_r \forall r\big)\\
&\leq 1 - \frac{1}{P\big(\widetilde{Y}_l < t_l\big)}\int_{-\infty}^{t_l}\Phi\Big(\sqrt{1+\beta_l}s_l - \tilde{y}_l/\sqrt{1+\beta_l}\Big) f_{\widetilde{Y}_l}(\tilde{y}_l)\mathrm{d}\tilde{y}_l\\
&= 1 - P\big(X_l < s_l\big|\widetilde{Y}_l < t_l\big) = P\big(X_l > s_l\big|\widetilde{Y}_l < t_l\big). \qquad \square
\end{aligned}$$

Theorem 4.1 together with (4.2) implies that if the linear combinations are all positively correlated, then

$$CR \leq \sum_{l=1}^{k} P\big(X_l > s_l\big|\widetilde{Y}_r < t_r \forall r\big) \leq \sum_{l=1}^{k} P\big(X_l > s_l\big|\widetilde{Y}_l < t_l\big). \tag{4.7}$$

That means that we have found an upper bound for CR, which is the sum of individual consumer risks $CR_l = P\big(X_l > s_l \big| \widetilde{Y}_l < t_l\big)$. However, as also negative correlations between the true values can occur, the linear combinations are not necessarily positively correlated. This implies that $P\big(X_l > s_l \big| \widetilde{Y}_l < t_l\big)$ may be a lower bound for $P\big(X_l > s_l \big| \widetilde{Y}_r < t_r \forall r\big)$ instead of an upper bound if linear combinations are negatively correlated. This is for example the case if $k = 2$ and $\rho(\widetilde{Y}_1, \widetilde{Y}_2) < 0$, which can be shown in the same way as in which we proved theorem 4.1. Note however, that negative correlations will not occur very often and the negative correlations that do occur will not be very large. Therefore, if $\sum_l P\big(X_l > s_l \big| \widetilde{Y}_l < t_l\big)$ is not an upper bound for $\sum_l P\big(X_l > s_l \big| \widetilde{Y}_r < t_r \forall r\big)$, then it still will be approximately equal to $\sum_l P\big(X_l > s_l \big| \widetilde{Y}_r < t_r \forall r\big)$ (which in its turn is approximately equal to CR).

We can now determine a such that the "upper bound" $\sum_l P\big(X_l > s_l \big| \widetilde{Y}_l < t_l\big)$, denoted by CRU, is equal to γ. In case the linear combinations are positively correlated, this results in a consumer risk that falls below the prescribed bound γ. Although it may be possible to numerically calculate a such that CRU is equal to γ, we prefer to find an approximate solution for a. The reason for doing so is that in case parameters are unknown, correction of a is needed to control the effects on the consumer risk of having to estimate the parameters. Unlike the exact numerical results, the approximation of a is of a simple structure and is explicit in terms of the parameters, which makes it possible to evaluate such correction terms.

Define

$$\bar{X}_l = \frac{X_l - \mu_{X_l}}{\sigma_{X_l}},\ \bar{s}_l = \frac{s_l - \mu_{X_l}}{\sigma_{X_l}},\ \bar{Z}_l = -\frac{Z_l}{\sigma_{Z_l}}, \tag{4.8}$$

then for the individual consumer risk $CR_l = P\big(X_l > s_l \big| \widetilde{Y}_l < t_l\big)$ we have

$$\begin{aligned} CR_l &= \frac{P(X_l > s_l, \widetilde{Y}_l < t_l)}{P(\widetilde{Y}_l < t_l)} = \frac{P(\bar{X}_l > \bar{s}_l, \bar{X}_l - \sigma_l \bar{Z}_l < \bar{s}_l - a\sigma_l)}{P(\bar{X}_l - \sigma_l \bar{Z}_l < \bar{s}_l - a\sigma_l)} \\ &= \frac{\int_a^\infty \left[\Phi[\bar{s}_l + \sigma_l(\bar{z}_l - a)] - \Phi(\bar{s}_l)\right] \phi(\bar{z}_l) \mathrm{d}\bar{z}_l}{\Phi[\rho_l(\bar{s}_l - a\sigma_l)]}. \end{aligned} \tag{4.9}$$

As explained in section 3.2, σ_l will be small and we can approximate CR_l using expansion in powers of σ_l. This results in

$$\begin{aligned} &CRU(a) := \sum_{l=1}^{k} CR_l = \\ &\sum_{l=1}^{k} \sigma_l \frac{\phi(\bar{s}_l)}{\Phi(\bar{s}_l)} \left[g_1(a) - \tfrac{1}{2}\sigma_l \bar{s}_l g_2(a) + \frac{\Phi(\bar{s}_l) - \Phi[\rho_l(\bar{s}_l - a\sigma_l)]}{\Phi(\bar{s}_l)} g_1(a) \right] + O(\textstyle\sum_l \sigma_l^3), \end{aligned} \tag{4.10}$$

where

$$g_k(a) = \int_a^\infty (e - a)^k \phi(e) \mathrm{d}e.$$

Let $a_{\mathrm{e}}^{\mathrm{U}}$ be the value of a that satisfies $CRU(a_{\mathrm{e}}^{\mathrm{U}}) = \gamma$. Using the approximation of $CRU(a)$ in (4.10), we determine a first and second order approximation of $a_{\mathrm{e}}^{\mathrm{U}}$ denoted by a_1^{U} and a_2^{U}, respectively, that satisfy $CRU(a_1^{\mathrm{U}}) = \gamma + O(\sum_l \sigma_l^2)$ and $CRU(a_2^{\mathrm{U}}) = \gamma + O(\sum_l \sigma_l^3)$.

Define

$$A_l = \sigma_l \frac{\phi(\bar{s}_l)}{\Phi(\bar{s}_l)},\ C_l = -\tfrac{1}{2}\sigma_l \bar{s}_l \ \text{ and } \ D_l(a) = \frac{\Phi(\bar{s}_l) - \Phi[\rho_l(\bar{s}_l - a\sigma_l)]}{\Phi(\bar{s}_l)}, \tag{4.11}$$

then

$$CRU(a) = \sum_{l=1}^{k} A_l\{g_1(a) + C_l g_2(a) + D_l(a) g_1(a)\} + O(\textstyle\sum_l \sigma_l^3). \tag{4.12}$$

The first order approximation a_1^{U} is found by setting the leading term in (4.12), $\sum_l A_l g_1(a)$, equal to γ. This results in

$$a_1^{\mathrm{U}} = g_1^{-1}\Big(\gamma \Big/ \sum_{l=1}^{k} A_l\Big). \tag{4.13}$$

The second order approximation is obtained by letting $a_2^{\mathrm{U}} = a_1^{\mathrm{U}} + \delta$, expanding $CRU(a_2^{\mathrm{U}})$ around $CRU(a_1^{\mathrm{U}})$ using (4.12), and then determining δ such that $CRU(a_2^{\mathrm{U}}) = \gamma + O(\sum_l \sigma_l^3)$. This results in

$$\begin{aligned} a_2^{\mathrm{U}} = a_1^{\mathrm{U}} + \sum_{l=1}^{k} \Big\{ &\left[(a_1^{\mathrm{U}})^2 + 1 - a_1^{\mathrm{U}} k(a_1^{\mathrm{U}})\right] A_l C_l \\ &+ \left[k(a_1^{\mathrm{U}}) - a_1^{\mathrm{U}}\right] A_l D_l(a_1^{\mathrm{U}}) \Big\} \Big/ \sum_{l=1}^{k} A_l, \end{aligned} \tag{4.14}$$

where

$$k(a) = \phi(a)/[1 - \Phi(a)].$$

To see how accurate this second order approximation of $a_{\mathrm{e}}^{\mathrm{U}}$ is, we have calculated a variety of numerical examples, where two or three characteristics have to satisfy a specification limit, with γ varying from 1 up to 100 ppm, the relative size of the measurement errors $(\sigma_{U_l}/\sigma_{X_l})$ varying from 0.1 up to 0.3, the standardized specification limits $(\bar{s}_l)$ varying from 1.5 up to 2.5 and the correlations between the true values of the characteristics $(\rho_{12} = \rho(X_1, X_2))$ varying from -0.5 up to 0.99. We did not consider the examples with small specification limits in combination with large measurement errors, as for these examples the yield becomes unacceptably low, under 50%.

From these examples we can conclude that a_2^{U} is a very accurate approximation of $a_{\mathrm{e}}^{\mathrm{U}}$, as $CRU(a_2^{\mathrm{U}})$ is very close to γ. The relative error $(CRU(a_2^{\mathrm{U}}) - \gamma)/\gamma$ is not larger than 0.05 in the examples we considered, and often even much smaller, see Arts (1998).

Besides $CRU(a_2^{\mathrm{U}})$, we also evaluated $CR(a_2^{\mathrm{U}})$, the actual consumer risk using the test limit $t_{2l}^{\mathrm{U}} = \alpha_l + \beta_l s_l - a_2^{\mathrm{U}}\sigma_{Z_l}$. It turns out that the test limits can be very conservative in case large positive correlations occur. If $k = 2$ and $\rho_{12} = 0.99$, the largest correlation between the true values we considered, then the relative error $(CR(a_2^{\mathrm{U}}) - \gamma)/\gamma$ varies from -0.25 down to -0.39. Also for $\rho_{12} = 0.7$, the relative error is quite large, varying from -0.15 down to -0.25. If $k = 3$, the relative error is about -0.50 if all correlations are very large (≥ 0.95).

As we already mentioned, if negative correlations occur, then CRU is no longer an upper bound, and using the test limits such that CRU is approximately equal to γ, may result in a consumer risk larger than γ. However, the relative error in the consumer risk does not become very large, as large negative correlations do not occur in practice. For the numerical examples with $k = 2$, we see that the relative error $(CR(a_2^{\mathrm{U}}) - \gamma)/\gamma$ varies from 0.01 up to 0.28 if $\rho_{12} = -0.5$. If $k = 3$ and some of the correlations are negative, then relative errors are found of up to 0.25.

In general we can conclude that the test limits $t_{2l}^{\mathrm{U}} = \alpha_l + \beta_l s_l - a_2^{\mathrm{U}}\sigma_{Z_l}$, $l = 1, \ldots, k$, result in a consumer risk quite close to γ if the correlations are small, or if some of the correlations are negative. However, if all correlations are positive and some of the correlations are large, then the test limits are very conservative and the yield can be improved. In the next section we present a modification of a_2^{U}, which results in a consumer risk that is quite close to γ.

4.2 Improvement of the test limits

The improved test limit for the l^{th} linear combination is equal to $t_{2l} = \alpha_l + \beta_l s_l - a_2\sigma_{Z_l}$, where

$$a_2 = a_2^{\mathrm{U}} + [k(a_1^{\mathrm{U}}) - a_1^{\mathrm{U}}]\sum_{l=1}^{k} A_l B_l \Big/ \sum_{l=1}^{k} A_l, \tag{4.15}$$

with a_2^{U} from (4.14) and

$$B_l = \frac{P\big(\bar{X}_r < \bar{s}_r \forall r \neq l \big| \bar{X}_l = \bar{s}_l\big)}{P\big(\bar{X}_r < \bar{s}_r \forall r \neq l \big| \bar{X}_l < \bar{s}_l\big)} - 1. \tag{4.16}$$

For technical details on how this test limit is derived, we refer to Arts (1998), and in the Appendix we briefly clarify the modification of a_2^{U}. Here we explain the effect on the consumer risk of using a_2 instead of a_2^{U}.

Note that if the characteristics are positively correlated, $B_l \leq 0$, so that $a_2 \leq a_2^{\mathrm{U}}$ and the test limit $t_{2l} = \alpha_l + \beta_l s_l - a_2\sigma_{Z_l}$ is less strict than the test limit $t_{2l}^{\mathrm{U}} = \alpha_l + \beta_l s_l - a_2^{\mathrm{U}}\sigma_{Z_l}$. So using the test limits t_{2l}, $l = 1, \ldots, k$, instead of t_{2l}^{U}, $l = 1, \ldots, k$, results in a higher yield if the characteristics are positively correlated. Of course this test limit is only acceptable if the consumer risk is not much larger than γ. Note that if the correlations are small, the consumer risk

using a_2^{U} is close to γ, and the increase in the consumer risk replacing a_2^{U} by a_2 should be small in that case. As B_l will be close to zero if the correlations are small, indeed a_2 will be close to a_2^{U} and $CR(a_2)$ will just as $CR(a_2^{\mathrm{U}})$ be close to γ. On the other hand, if large positive correlations occur, B_l will be close to -1 and $CR(a_2)$ will be much larger than $CR(a_2^{\mathrm{U}})$. Note however that in that case $CR(a_2^{\mathrm{U}})$ was much smaller than γ, and the consumer risk can increase quite a bit, before it will exceed γ. To examine whether the consumer risk using a_2 instead of a_2^{U} exceeds γ, and if it does, how much larger the consumer risk is, we evaluate a number of numerical examples, the results of which are presented in tables 4.1 ($k = 2$) and 4.2 ($k = 3$). We evaluate $\pi = P\big(\exists l : X_l > s_l\big)$, a_2, the realized consumer risk $CR(a_2)$ and the realized yield $YD(a_2)$.

Table 4.1. The value of π, a_2, the corresponding consumer risk $CR(a_2)$ and the yield $YD(a_2)$, with $\gamma = 20$ ppm and $k = 2$ for different values of ρ_{12}, $\bar{s}_1$, $\bar{s}_2$ and $\sigma_{U_1}/\sigma_{X_1} = \sigma_{U_2}/\sigma_{X_2}$. The measurement errors are independent.

ρ_{12}	π	a_2	$CR(a_2)$	$YD(a_2)$	π	a_2	$CR(a_2)$	$YD(a_2)$
	$\bar{s}_1 = 1.5$, $\bar{s}_2 = 1.5$				$\bar{s}_1 = 2.0$, $\bar{s}_2 = 2.0$			
	$\sigma_{U_1}/\sigma_{X_1} = \sigma_{U_2}/\sigma_{X_2} = 0.1$				$\sigma_{U_1}/\sigma_{X_1} = \sigma_{U_2}/\sigma_{X_2} = 0.3$			
−0.50	0.13	2.84	20.95	77.50	0.05	2.86	22.79	73.30
−0.30	0.13	2.83	20.56	77.81	0.05	2.86	21.83	73.38
−0.10	0.13	2.82	20.19	78.36	0.05	2.86	20.75	74.03
0.00	0.13	2.82	20.03	78.72	0.04	2.85	20.24	74.53
0.10	0.13	2.81	19.87	79.15	0.04	2.85	19.78	75.14
0.30	0.12	2.79	19.58	80.17	0.04	2.84	19.01	76.68
0.50	0.12	2.77	19.28	81.47	0.04	2.82	18.44	78.74
0.70	0.11	2.75	18.91	83.18	0.04	2.79	18.10	81.61
0.90	0.09	2.71	18.38	85.85	0.03	2.73	18.50	86.53
0.95	0.08	2.69	18.30	87.03	0.03	2.69	19.50	88.73
0.99	0.07	2.64	19.34	89.01	0.03	2.63	22.40	91.31
	$\bar{s}_1 = 1.5$, $\bar{s}_2 = 2.0$				$\bar{s}_1 = 2.0$, $\bar{s}_2 = 2.5$			
	$\sigma_{U_1}/\sigma_{X_1} = \sigma_{U_2}/\sigma_{X_2} = 0.1$				$\sigma_{U_1}/\sigma_{X_1} = \sigma_{U_2}/\sigma_{X_2} = 0.3$			
−0.50	0.09	2.72	20.59	84.68	0.03	2.72	21.56	82.23
−0.30	0.09	2.72	20.39	84.76	0.03	2.72	21.17	82.09
−0.10	0.09	2.71	20.14	84.98	0.03	2.72	20.56	82.29
0.00	0.09	2.71	20.02	85.15	0.03	2.72	20.23	82.52
0.10	0.09	2.70	19.91	85.35	0.03	2.72	19.91	82.82
0.30	0.08	2.69	19.70	85.88	0.03	2.71	19.36	83.66
0.50	0.08	2.67	19.53	86.61	0.03	2.69	19.00	84.87
0.70	0.08	2.65	19.36	87.56	0.03	2.67	18.96	86.61
0.90	0.07	2.62	19.27	88.86	0.02	2.62	19.59	89.40
0.95	0.07	2.61	19.24	89.25	0.02	2.59	19.62	90.48
0.99	0.07	2.57	19.11	89.83	0.02	2.55	19.54	91.85

In both tables 4.1 and 4.2, we see that in almost every situation with positive correlations, the consumer risk is smaller than γ. So, the test limit $t_{2l} = \alpha_l + \beta_l s_l -$

Table 4.2. The value of π, a_2, the corresponding consumer risk $CR(a_2)$ and the yield $YD(a_2)$, with $\gamma = 20$ ppm and $k = 3$, for different values of ρ_{12}, ρ_{13} and ρ_{23}, $\bar{s}_1$, $\bar{s}_2$ and $\bar{s}_3$ and $\sigma_{U_1}/\sigma_{X_1} = \sigma_{U_2}/\sigma_{X_2} = \sigma_{U_3}/\sigma_{X_3}$. The measurement errors are independent.

$(\rho_{12}, \rho_{13}, \rho_{23})$	π	a_2	$CR(a_2)$	$YD(a_2)$	π	a_2	$CR(a_2)$	$YD(a_2)$
	$\bar{s}_1 = \bar{s}_2 = 1.5$, $\bar{s}_3 = 2.0$ $\sigma_{U_l}/\sigma_{X_l} = 0.1$				$\bar{s}_1 = \bar{s}_2 = \bar{s}_3 = 2.0$ $\sigma_{U_l}/\sigma_{X_l} = 0.3$			
(−0.50,−0.50,0.95)	0.13	2.83	21.00	77.66	0.05	2.88	22.64	74.52
(−0.50,−0.50,0.70)	0.14	2.86	20.92	75.91	0.06	2.94	22.02	67.02
(−0.50,−0.50,0.50)	0.15	2.87	20.96	74.92	0.06	2.96	22.45	64.14
(−0.50,−0.50,0.30)	0.15	2.88	21.04	74.17	0.07	2.97	23.08	62.18
(−0.30,−0.30,0.95)	0.13	2.83	20.55	77.97	0.05	2.88	21.71	74.66
(−0.30,−0.30,0.70)	0.14	2.85	20.45	76.26	0.06	2.94	20.78	67.39
(−0.30,−0.30,0.30)	0.15	2.88	20.55	74.57	0.07	2.97	21.57	62.50
(−0.30,−0.30,0.10)	0.15	2.89	20.67	74.02	0.07	2.98	22.27	60.99
(0.00, 0.00,0.00)	0.15	2.87	20.03	75.08	0.07	2.98	20.27	62.63
(0.30, 0.30,0.30)	0.14	2.83	19.26	77.63	0.06	2.95	17.99	67.92
(0.30, 0.30,0.70)	0.13	2.82	19.11	78.92	0.06	2.92	17.61	71.77
(0.30, 0.30,0.95)	0.12	2.80	19.15	80.28	0.05	2.86	18.76	77.67
(0.50, 0.50,0.70)	0.12	2.80	18.65	80.36	0.05	2.90	16.94	74.21
(0.50, 0.50,0.95)	0.12	2.78	18.65	81.56	0.05	2.85	17.96	79.52
(0.70, 0.70,0.70)	0.11	2.77	18.09	82.28	0.05	2.87	16.48	77.56
(0.90, 0.90,0.90)	0.09	2.73	17.18	85.73	0.04	2.79	16.58	85.64
(0.95, 0.95,0.95)	0.08	2.71	16.94	87.06	0.03	2.74	17.72	88.90
(0.95, 0.95,0.99)	0.08	2.69	17.00	87.32	0.03	2.72	18.60	89.87
	$\bar{s}_1 = 1.5$, $\bar{s}_2 = \bar{s}_3 = 2.0$ $\sigma_{U_l}/\sigma_{X_l} = 0.1$				$\bar{s}_1 = \bar{s}_2 = 2.0$, $\bar{s}_3 = 2.5$ $\sigma_{U_l}/\sigma_{X_l} = 0.3$			
(−0.50,−0.50,0.95)	0.10	2.74	20.66	83.70	0.05	2.83	22.33	76.37
(−0.50,−0.50,0.70)	0.10	2.77	20.64	81.91	0.05	2.87	22.26	72.17
(−0.50,−0.50,0.50)	0.11	2.78	20.71	81.20	0.05	2.88	22.43	70.39
(−0.50,−0.50,0.30)	0.11	2.79	20.80	80.74	0.05	2.89	22.80	69.25
(−0.30,−0.30,0.95)	0.10	2.74	20.40	83.81	0.05	2.83	21.55	76.45
(−0.30,−0.30,0.70)	0.10	2.77	20.34	82.07	0.05	2.87	21.25	72.32
(−0.30,−0.30,0.30)	0.11	2.79	20.47	80.91	0.05	2.89	21.65	69.28
(−0.30,−0.30,0.10)	0.11	2.79	20.57	80.61	0.05	2.90	22.09	68.43
(0.00, 0.00,0.00)	0.11	2.78	20.02	81.27	0.05	2.90	20.26	69.72
(0.30, 0.30,0.30)	0.10	2.75	19.36	82.95	0.05	2.87	18.33	73.59
(0.30, 0.30,0.70)	0.10	2.73	19.25	83.78	0.05	2.85	18.31	75.88
(0.30, 0.30,0.95)	0.09	2.71	19.33	85.12	0.04	2.82	18.90	79.15
(0.50, 0.50,0.70)	0.09	2.72	18.88	84.83	0.04	2.83	17.67	78.03
(0.50, 0.50,0.95)	0.09	2.70	18.93	85.95	0.04	2.80	18.25	80.87
(0.70, 0.70,0.70)	0.08	2.69	18.44	86.26	0.04	2.80	17.24	81.10
(0.90, 0.90,0.90)	0.07	2.65	17.86	88.57	0.03	2.73	17.61	87.33
(0.95, 0.95,0.95)	0.07	2.63	17.69	89.22	0.03	2.68	18.36	89.84
(0.95, 0.95,0.99)	0.07	2.63	18.00	89.23	0.03	2.67	18.13	90.16

$a_2\sigma_{Z_l}$ is more strict than necessary in most situations, but it is less conservative than the test $t_{2l}^{\mathrm{U}} = \alpha_l + \beta_l s_l - a_2^{\mathrm{U}}\sigma_{Z_l}$. The test limits result in a consumer risk quite close to γ if $k = 2$. If $k = 3$, the consumer risk can be a bit small for ρ_{lr} equal to 0.7 and 0.9, especially if the specification limits are all equal. If some of the correlations are negative, then the consumer risk is quite close to γ, for both $k = 2$ and $k = 3$.

Of course their are many more combinations of specification limits, measurement errors and values of γ possible, than the ones we presented in tables 4.1 and 4.2. First of all, in most situations the relative error in the consumer risk, $(CR(a_2) - \gamma)/\gamma$ is getting smaller if γ increases. If $k = 2$, the relative error for the same examples as in table 4.1, varies from -0.11 up to 0.22 if γ equals 1 ppm, from -0.10 up to 0.14 if γ equals 20 ppm, and from -0.08 up to 0.04 if γ equals 100 ppm. We see that especially the positive relative error, which occurs if the characteristics are negatively correlated, becomes smaller if γ increases. If $k = 3$, then the relative error in the consumer risk varies from -0.20 up to 0.25 if γ is equal to 1 ppm, from -0.23 up to 0.15 if γ equals 20 ppm and from -0.16 up to 0.10 if γ equals 100 ppm.

Secondly, we consider examples with the same value of γ and the same measurement errors and correlations as in tables 4.1 and 4.2, but other specification limits. If the specification limits are larger than the specification limits from tables 4.1 and 4.2, the relative error in the consumer risk is smaller. Note that smaller specification limits are not of practical interest, as then the percentage of nonconforming items will be too large.

We conclude with examples in which the relative size of the measurement errors differ from the ones in tables 4.1 and 4.2. For larger measurement errors, we observe a larger consumer risk. That implies that if the consumer risk is larger than γ, the relative error will become larger if the measurement error becomes larger. For example, if $k = 2$ and $\bar{s}_1 = \bar{s}_2 = 1.5$, then the consumer risk goes up to 22.83 if $\sigma_{U_l}/\sigma_{X_l}$ is equal to 0.2 for $l = 1, 2$. If the correlations are positive, then the consumer risk is smaller than γ, hence in that case smaller measurement errors will result in larger relative errors. However, the relative error in the consumer risk is still not very large. For $k = 2$, with the specification limits both equal to 1.5, the consumer risk goes down to 17.28 if $\sigma_{U_l}/\sigma_{X_l}$, $l = 1, 2$ is equal to 0.01.

As the test limits determined in this section result in a consumer risk quite close to γ, these are the test limits we propose and which we will apply to an example from semiconductor industry in the next section.

5 An example from semiconductor industry

In this section we will illustrate by means of an example from Philips Semiconductors Nijmegen, how the yield can be improved when inspection takes place by inspecting each characteristic by means of a linear combination instead of the

measurement of the characteristic. As we mentioned in section 4, if parameters are unknown, the test limits should be modified to correct for the effects on the consumer risk of estimating the parameters. As the modifications needed will hardly influence the difference between the yield if linear combinations are used and the yield if the measurements of the characteristics are used, here we will neglect these modifications. However, if in practice one wants to ensure that on the long run the average consumer risk is (approximately) equal to γ, then these modifications are necessary. The values of the parameters we will mention are actually estimates of the parameters.

The example concerns a product of which two characteristics are measured with a large measurement error. The relative sizes of the measurement errors are $\sigma_{U_1}/\sigma_{X_1} = 0.347$ and $\sigma_{U_2}/\sigma_{X_2} = 0.371$. The standardized specification limits for these two characteristics are $\bar{s}_1 = 1.85$ and $\bar{s}_2 = 2.40$. Besides these two characteristics there are another 8 characteristics that are inspected (characteristic 3 up to 10) and the measurements of these characteristics are somewhat correlated with the measurements of the first two characteristics. Characteristics 3 up to 6 are mainly correlated with the first characteristic (correlations between the true values of 0.35 up to 0.45 with characteristic 1 and 0.07 up to 0.18 with characteristic 2) and characteristics 7 up to 10 are mainly correlated with the second characteristic (correlations between the true values of 0.03 up to 0.16 with characteristic 1 and 0.36 up to 0.60 with characteristic 2). Except for characteristic 6, the specification limits of characteristics 3 up to 10 are very large. The standardized specification limits ($\bar{s}_l$) are larger than 8. As the measurement errors in measuring these characteristics are not extremely large ($\sigma_{U_l}/\sigma_{X_l} < 0.18$), the probability that at least one of these characteristics is nonconforming is negligible. For characteristic 6 we have a bit smaller but still quite large standardized specification limit of $\bar{s}_6 = 3.04$, and a very small measurement error: $\sigma_{U_6}/\sigma_{X_6} = 0.01$. Consequently, the consumer risk will be mainly determined by the errors made in accepting the first two characteristics. The percentage of items that conforms to the specification limits, before inspection takes place, is equal to 95.9%.

First of all we take a look at the consumer risk and yield when the characteristics are inspected by means of their own measurements only. The test limits for the measurements of the characteristics are determined similarly as in section 4. If γ equals 20 ppm, then $a = 2.87$ and the standardized test limits for the measurements $\widetilde{X}_1$, $\widetilde{X}_2$ and $\widetilde{X}_6$ are $\bar{t}_1 = (t_1 - \mu_{X_1})/\sigma_{\widetilde{X}_1} = 0.80$, $\bar{t}_2 = (t_2 - \mu_{X_2})/\sigma_{\widetilde{X}_2} = 1.25$ and $\bar{t}_6 = (t_6 - \mu_{X_6})/\sigma_{\widetilde{X}_6} = 3.01$, which results in a yield equal to 69.35%. The actual consumer risk is equal to 18.49 ppm, so the test limits are a bit conservative. If we take γ equal to 100 ppm, then $a = 2.34$ and the test limits are equal to $\bar{t}_1 = 0.98$, $\bar{t}_2 = 1.44$ and $\bar{t}_6 = 3.01$, and the yield equals 75.53%. In that case the actual consumer risk is equal to 91.75, so again the test limits are a bit conservative.

Note that if we use the measurements of the characteristics themselves for

inspection, the relative error in predicting the true value of the l^{th} characteristic is given by $\sigma_{U_l}/\sigma_{X_l}$. If we use linear combinations, the relative error in predicting the true value of the l^{th} characteristic is given by $\sigma_l = \sigma_{Z_l}/(\beta_l \sigma_{X_l})$, which is smaller than $\sigma_{U_l}/\sigma_{X_l}$ (cf. section 3.2). For characteristic 6, the relative error in predicting the true value by its measurement is already small ($\sigma_{U_6}/\sigma_{X_6} = 0.0104$) and the improvement using a linear combination of the measurements of all 10 characteristics is very small ($\sigma_6 = 0.0102$). For the first two characteristics the improvements using linear combinations instead of the measurement of the characteristic are a bit larger, but still small. We have $\sigma_1 = 0.329$ and $\sigma_2 = 0.334$, while $\sigma_{U_1}/\sigma_{X_1} = 0.347$ and $\sigma_{U_2}/\sigma_{X_2} = 0.371$. Although the improvement in the prediction error seems to be very small, it still results in a gain in yield of a few percents. The coefficients (accurate up to 2 decimals) in the standardized linear combinations $\widetilde{Y}_l' = (\widetilde{Y}_l - \mu_{\widetilde{Y}_l})/\sigma_{\widetilde{Y}_l}$, of the 10 standardized measurements $\widetilde{X}_l' = (\widetilde{X}_l - \mu_{\widetilde{X}_l})/\sigma_{\widetilde{X}_l}$, are as follows:

	$\widetilde{X}_1'$	$\widetilde{X}_2'$	$\widetilde{X}_3'$	$\widetilde{X}_4'$	$\widetilde{X}_5'$	$\widetilde{X}_6'$	$\widetilde{X}_7'$	$\widetilde{X}_8'$	$\widetilde{X}_9'$	$\widetilde{X}_{10}'$
$\widetilde{Y}_1'$	0.95	−0.01	−0.19	−0.01	0.18	−0.00	−0.08	0.13	−0.18	−0.09
$\widetilde{Y}_2'$	0.00	0.93	0.05	0.04	−0.15	0.03	−0.66	0.48	0.06	−0.02
$\widetilde{Y}_6'$	0.00	0.00	0.00	−0.01	0.00	1.00	−0.00	0.01	−0.00	0.00

From the linear combinations it is not immediately clear which correlated characteristics have the largest correlations with the characteristic of interest. This is caused by the fact that the characteristics correlated with the characteristic of interest, are not independent. However, we do see that the measurement of the characteristic itself gets the highest weight in the linear combination. Especially in the linear combination corresponding to the 6^{th} characteristic, the coefficient of the measurement of the characteristic itself is much larger than the other coefficients, which is caused by the fact that this characteristic has a very small measurement error. (Note that 0.00 and −0.00 correspond to coefficients of which the absolute value is less than 0.005.)

If γ equals 20 ppm, then $a = 2.85$ and the standardized test limits for the linear combinations $\widetilde{Y}_1'$, $\widetilde{Y}_2'$ and $\widetilde{Y}_6'$ are $\bar{t}_1 = (t_1 - \mu_{\widetilde{Y}_1})/\sigma_{\widetilde{Y}_1} = 0.86$, $\bar{t}_2 = (t_2 - \mu_{\widetilde{Y}_2})/\sigma_{\widetilde{Y}_2} = 1.37$ and $\bar{t}_6 = (t_6 - \mu_{\widetilde{Y}_6})/\sigma_{\widetilde{Y}_6} = 3.01$ and the yield equals 74.44%, which is 5.09% higher than when the measurements themselves are used for inspection. The actual consumer risk is very close to γ, namely 19.77 ppm. If γ equals 100 ppm, then $a = 2.32$ and $\bar{t}_1 = 1.03$, $\bar{t}_2 = 1.54$ and $\bar{t}_6 = 3.01$, and the yield equals 79.99%, which is 4.46% larger than when the measurements themselves are used for inspection. The consumer risk is equal to 98.66 ppm.

It is clear that using linear combinations for the inspection of an item, results in a much larger yield than when inspection of the characteristics takes place by means of their own measurements only. Although the consumer risk if linear combinations are used is larger than when the measurements of the characteristics

are used for inspection, even when they are equal, the gain in yield would still be over 4%.

6 Test limits if parameters are unknown

Together, sections 3 and 4 resulted in a test region

$$T_2 = \left\{ \tilde{\mathbf{x}} \in \mathbb{R}^k \middle| \mathbf{b}_l^{\mathrm{T}} \Sigma_l^{-1} \tilde{\mathbf{x}} < \alpha_l + \beta_l s_l - a_2 \sigma_{Z_l}, \forall\, l = 1, \ldots, k \right\}, \tag{6.1}$$

which gives a consumer risk close to γ if parameters are known. However, in practice it is more likely that the parameters are unknown. To estimate the parameters we assume that we have a sample of n items, of which we have two measurements. So we have observations $\widetilde{\mathbf{X}}_{ij}$ for $i = 1, \ldots, n, j = 1, 2$. The fact that on each item we have two measurements, makes it possible to estimate the covariance matrix of the measurement error $\mathbf{U}$. Of course it is possible to have more than two repeated measurements on each item. Note, however, that the measurement error is small compared to the total variation in the measurement. Therefore, it is more useful to measure the items only twice, and to spend the available budget on two repeated measurements of some additional items instead, which results in better estimators of the parameters other than $\sigma^2_{U_0}$.

Using the two repeated measurements of n items, the estimators of the parameters are given by

$$\begin{aligned}
&\widehat{\Sigma}_{\mathbf{UU}} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{2} (\widetilde{\mathbf{X}}_{ij} - \widetilde{\mathbf{X}}_{i\bullet})(\widetilde{\mathbf{X}}_{ij} - \widetilde{\mathbf{X}}_{i\bullet})^{\mathrm{T}}, \\
&\widehat{\mu}_{\mathbf{X}} = \frac{1}{n} \sum_{i=1}^{n} \widetilde{\mathbf{X}}_{i\bullet}, \;\; \widehat{\Sigma}_{\mathbf{XX}} = \frac{1}{n-1} \sum_{i=1}^{n} (\widetilde{\mathbf{X}}_{i\bullet} - \widehat{\mu}_{\mathbf{X}})(\widetilde{\mathbf{X}}_{i\bullet} - \widehat{\mu}_{\mathbf{X}})^{\mathrm{T}} - \tfrac{1}{2}\widehat{\Sigma}_{\mathbf{UU}}, \\
&\widehat{\mathbf{b}}_l = \widehat{\Sigma}_{X_l \mathbf{X}} / \widehat{\sigma}^2_{X_l}, \;\; \widehat{\mathbf{a}}_l = \widehat{\mu}_{\mathbf{X}} - \widehat{\mathbf{b}}_l \widehat{\mu}_{X_l}, \;\; \widehat{\Sigma}_l = \widehat{\Sigma}_{\mathbf{XX}} + \widehat{\Sigma}_{\mathbf{UU}} - \widehat{\mathbf{b}}_l \widehat{\mathbf{b}}_l^{\mathrm{T}} \widehat{\sigma}^2_{X_l} \\
&\widehat{\beta}_l = \widehat{\mathbf{b}}_l^{\mathrm{T}} \widehat{\Sigma}_l^{-1} \widehat{\mathbf{b}}_l, \;\; \widehat{\alpha}_l = \widehat{\mathbf{b}}_l^{\mathrm{T}} \widehat{\Sigma}_l^{-1} \widehat{\mathbf{a}}_l, \;\; \widehat{\sigma}^2_{Z_l} = \widehat{\mathbf{b}}_l^{\mathrm{T}} \widehat{\Sigma}_l^{-1} \widehat{\mathbf{b}}_l,
\end{aligned} \tag{6.2}$$

where $\widetilde{\mathbf{X}}_{i\bullet} = (\widetilde{\mathbf{X}}_{i1} + \widetilde{\mathbf{X}}_{i2})/2$. Using these estimators, we can estimate the test region T_2 from (6.1) by

$$\widehat{T}_2 = \left\{ \tilde{\mathbf{x}} \in \mathbb{R}^k \middle| \widehat{\mathbf{b}}_l^{\mathrm{T}} \widehat{\Sigma}_l^{-1} \tilde{\mathbf{x}} < \widehat{\alpha}_l + \widehat{\beta}_l - \widehat{a}_2 \widehat{\sigma}_{Z_l} \forall l = 1, \ldots, k \right\}, \tag{6.3}$$

where $\widehat{a}_2$ can be found by plugging in the estimators from (6.2) into (3.5), (3.9), (4.8), (4.11), (4.13), (4.14), (4.16) and (4.15). Unfortunately, using this estimated version of the test region results in a consumer risk of which the expected value is much larger than γ. This phenomenon also occurs in the inspection of one characteristic, using either the measurement of the characteristic itself (see Albers, Kallenberg and Otten (1994a)) or the measurements of one or more correlated characteristics (see Albers, Arts and Kallenberg (1998a) and Arts (1998)). To

ensure that on the long run the average consumer risk does not exceed γ by much, modification of the test limits $\widehat{t}_l = \widehat{\alpha}_l + \widehat{\beta}_l s_l - \widehat{a}_2 \widehat{\sigma}_{Z_l}$ is needed. The modified test limits are determined in such a way that the expected consumer risk if parameters are estimated by (6.2), is approximately equal to the consumer risk if parameters are known, using the test region T_2 from (6.1). In determining these modified test limits, we can benefit from using the approximate though explicit test limits as derived in section 4.1, instead of the exact but numerical result. For technical details we refer to Arts (1998), here we just give the result. The modified test region equals

$$\widehat{T}_{\mathrm{u}} = \Big\{ \tilde{\mathbf{x}} \in \mathbb{R}^k \Big| \widehat{\mathbf{b}}_l^{\mathrm{T}} \widehat{\Sigma}_l^{-1} \tilde{\mathbf{x}} < \widehat{\alpha}_l + \widehat{\beta}_l - (\widehat{a}_2 + \widehat{d}_l) \widehat{\sigma}_{Z_l} \forall l = 1, \ldots, k \Big\},$$

where the correction term d_l in the test limit for the l^{th} linear combination, is equal to

$$\begin{aligned} d_l = & \frac{1}{4n} \left[\kappa_{1l}^2 - \kappa_{0l}^2 + \tfrac{1}{2}\kappa_{0l}^4 + \tfrac{1}{2}\kappa_{1l}^4 \right] k(a_1)[1 + 2a_1 k(a_1) - a_1^2] \\ & + \frac{1}{4n} \left[\bar{s}_l^4 + 4\bar{s}_l^2 + 1 + (3 + \bar{s}_l^2) \frac{\bar{s}_l \phi(\bar{s}_l)}{\Phi(\bar{s}_l)} \right] [k(a_1) - a_1] \\ & + \frac{1}{n} \Big\{ (\tfrac{1}{2}\bar{s}_l^2 + k + 1) \left[\tfrac{1}{2}\kappa_{0l}^2 - \tfrac{1}{2}\kappa_{1l}^2 \right] + \tfrac{1}{2}[\mathrm{tr}(\Sigma_l^{-1}\Sigma_{\mathbf{UU}}) - 1] \\ & \qquad + 1 - \kappa_{1l}^2 + \tfrac{1}{2}[\kappa_{1l}^2 \mathrm{tr}(\Sigma_l^{-1}\Sigma_{\mathbf{UU}}) + \frac{1}{\beta_l} \Sigma_{U_l^{\mathrm{L}}\mathbf{U}}^{\mathrm{T}} \Sigma_l^{-1} \Sigma_{U_l^{\mathrm{L}}\mathbf{U}} - 2\kappa_{1l}^4] \Big\} k(a_1), \end{aligned}$$

where

$$\begin{aligned} & \sigma_{U_l^{\mathrm{L}}}^2 = \mathbf{b}_l^{\mathrm{T}} \Sigma_l^{-1} \Sigma_{\mathbf{UU}} \Sigma_l^{-1} \mathbf{b}_l, \ \Sigma_{U_l^{\mathrm{L}}\mathbf{U}} = \mathbf{b}_l^{\mathrm{T}} \Sigma_l^{-1} \Sigma_{\mathbf{UU}}, \\ & \kappa_{0l} = \frac{\beta_l \sigma_{U_l}}{\sigma_{Z_l}}, \ \kappa_{1l} = \frac{\sigma_{U_l^{\mathrm{L}}}}{\sigma_{Z_l}}, \end{aligned}$$

and $\mathrm{tr}(\Sigma_l^{-1}\Sigma_{\mathbf{UU}})$ is the trace of $\Sigma_l^{-1}\Sigma_{\mathbf{UU}}$, i.e. the sum of the diagonal elements of $\Sigma_l^{-1}\Sigma_{\mathbf{UU}}$.

In Arts (1998) a simulation study is performed, for the same examples as we considered in tables 4.1 and 4.2, with a sample size equal to $n = 150$ in case two characteristics have to be inspected and a sample size equal to $n = 200$ in case three characteristics have to be inspected. From these simulations, we can conclude that the modification of the test limits works very well: the average consumer risk using the estimated test region $\widehat{T}_{\mathrm{u}}$ from (6.3), is approximately equal to the consumer risk using the test region T_2 if parameters are known. That means that just as the consumer risk using T_2 if parameters are known, the expected consumer risk using $\widehat{T}_{\mathrm{u}}$ if parameters are estimated by (6.2), is close to γ.

Appendix: Clarification of modified test limit

The conservatism of the test limit based on a_2^{U} is mainly due to neglecting the events $\widetilde{Y}_r < t_r$, $\forall r \neq l$ in $P\big(X_l > s_l \big| \widetilde{Y}_r < t_r \forall r\big)$. Instead of neglecting these events beforehand, we approximate $P\big(X_l > s_l \big| \widetilde{Y}_r < t_r \forall r\big) = P\big(X_l > s_l, \widetilde{Y}_l < t_l \big| \widetilde{Y}_r < t_r \forall r \neq l\big) \big/ P\big(\widetilde{Y}_l < t_l \big| \widetilde{Y}_r < t_r \forall r \neq l\big)$ by expanding it in powers of σ_l up to second order terms (similar to the approximation of CR_l from (4.9) that occurs in (4.10)) and neglect the events $\widetilde{Y}_r < t_r \forall r \neq l$ only in terms of second order, i.e. of order σ_l^2. This results in

$$\sum_{l=1}^{k} \Big\{ \sigma_l \frac{\int_a^\infty f_{\bar{X}_l | \widetilde{Y}_r < t_r \forall r \neq l}(\bar{s}_l)(\bar{z}_l - a) f_{\bar{Z}_l | \bar{X}_l = \bar{s}_l, \widetilde{Y}_r < t_r \forall r \neq l}(\bar{z}_l) \mathrm{d}\bar{z}_l}{P\big(\bar{X}_l < \bar{s}_l \big| \widetilde{Y}_r < t_r \forall r \neq l\big)} + A_l [C_l g_2(a) + D_l(a) g_1(a)] \Big\}.$$

In addition, we neglect the events $\widetilde{Y}_r < t_r \forall r \neq l$ in the conditional density of $\bar{Z}_l$, leading to

$$\begin{aligned} &\sum_{l=1}^{k} \Big\{ \sigma_l g_1(a) \frac{f_{\bar{X}_l | \widetilde{Y}_r < t_r \forall r \neq l}(\bar{s}_l)}{P\big(\bar{X}_l < \bar{s}_l \big| \widetilde{Y}_r < t_r \forall r \neq l\big)} + A_l [C_l g_2(a) + D_l(a) g_1(a)] \Big\} = \\ &\sum_{l=1}^{k} \Big\{ A_l [g_1(a) + B_l g_1(a) + C_l g_2(a) + D_l(a) g_1(a)] \\ &\qquad + \sigma_l g_1(a) \Big[\frac{f_{\bar{X}_l | \widetilde{Y}_r < t_r \forall r \neq l}(\bar{s}_l)}{P\big(\bar{X}_l < \bar{s}_l \big| \widetilde{Y}_r < t_r \forall r \neq l\big)} - \frac{f_{\bar{X}_l | X_r < s_r \forall r \neq l}(\bar{s}_l)}{P\big(\bar{X}_l < \bar{s}_l \big| X_r < s_r \forall r \neq l\big)} \Big] \Big\}, \end{aligned} \tag{A.1}$$

with B_l from (4.16). Note that if we, just as before, neglect the conditional events in terms of second order, then the term on the last line of (A.1) vanishes and we arrive at

$$CRA(a) := \sum_{l=1}^{k} A_l [g_1(a) + B_l g_1(a) + C_l g_2(a) + D_l(a) g_1(a)]. \tag{A.2}$$

With CRA of the same structure as (the approximation of) CRU in (4.12), the result for a_2 can be found similarly as a_2^{U}.

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