



MIXMAX CHARTS

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Abstract

For attribute data with (very) small failure rates, control charts based on subsequent groups of r failure times, for some $r \geq 1$, have been shown to be attractive. This especially holds for charts which stop once the maximum (*MAX*) of such a group is sufficiently small, as this choice allows a nonparametric adaptation already for Phase I samples of ordinary size. The choice of r is dictated by the suspected rate of change in failure rate once the process goes out-of-control: for large (small) changes, r should be small (large). Typically, the actual rate of change will be unknown and hence some flexibility w.r.t. the choice of r seems advisable. In the present paper, this goal is achieved by mixing a *MAX*-chart for a large r with one for which r is relatively small.

1. Introduction and Motivation

High-quality processes are increasingly common in industrial settings, as

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production standards have steadily been improving over the last decades. Moreover, in the application area of health care monitoring, they, in fact, are the norm. Here failures should be rare by nature, because these correspond to thoroughly unpleasant events (e.g., delayed emergency vehicles, surgical errors, recurrence of cancer or birth defects). For both the areas, control charts are recognized as important tools to improve and maintain quality (see e.g., Sonesson and Bock [8], Thor et al. [9] and Shaha [7] for some review papers). In view of the above, the (average) failure probability p during the in-control (*IC*) phase is typically very small, which strongly suggests to base the charts on the waiting times from one failure till the next. After each group of size r ($r \geq 1$) of such waiting times, a signal is given if their total value is too small. (Typically, this one-sided version is of primary interest, as it is aimed at detecting increases of p during the out-of-control (*OoC*) phase.) What actually constitutes ‘too small’ can be made precise by specifying a false alarm rate (*FAR*) which is the expected frequency of signals during *IC*.

In Albers [1], the basic homogeneous case is described, where p is the same for all items or patients. Then the waiting times are geometrically distributed and the total of r of these thus has a negative binomial distribution. Among others, a rule of thumb is mentioned to find the best r for given underlying parameters (see the paper [1] for details and additional references). Even in this simple setup, a major complication remains: p typically is unknown and requires estimation on the basis of a Phase I sample. Common practice is to just plug in the resulting \hat{p} and to act as if nothing has happened. However, as the required *FAR* by its nature is tiny, the relative estimation error involved is definitely not negligible for sample sizes used in practice. Hence Albers [1] also discussed how to analyze the magnitude of such errors and how to subsequently correct for them.

Quite often, however, the estimation issue is not solved by dealing with a single parameter. In health care applications, patients may exhibit considerable variation and thus the homogeneity assumption fails. The same then holds for the distributional assumption: the waiting times will no longer be geometric. To continue with the negative binomial approach hence will

produce not only an estimation error, but also a model error as well. The latter will not become negligible, even if huge Phase I samples are available. An adequate way to deal with this further complication, of course, is to try a nonparametric approach. However, a straightforward adaptation in this direction typically will not work satisfactorily either. The problem is that the estimation error, which was already non-negligible in the parametric case, tends to become huge in the nonparametric one. To see this, note that the Phase I sample size used in practice (e.g., 100) will typically be (much) smaller than $1/FAR$ (e.g., 1000) and hence estimating the quantile corresponding to this FAR will be a problem. Consequently, in this way, one problem (a non-vanishing model error) would be removed by creating a new one (explosion of estimation error).

The way to overcome this problem is to no longer consider the total of the r waiting times for each group, but to use their maximum instead. The advantage is easily illustrated: e.g., for $r = 3$, this maximum falls below the 0.1-quantile with probability $(0.1)^3 = 0.001$. This is nicely small, while estimating a 0.1-quantile is quite feasible based on a sample of usual moderate size, and thus the estimation error is brought back to similar proportions as in the parametric case. However, before adopting this solution, another aspect has to be checked. If homogeneity happens to be true after all, then it is easily verified that the negative binomial chart is the optimal choice when using r waiting times. Hence using their maximum, rather than their total, will cause some loss of detection power under these ideal circumstances. But if this loss is small, then it can be viewed as an insurance premium paid for the protection against non-vanishing model errors if homogeneity does fail. In Albers [2], it is shown that this is indeed the situation, and hence the robust alternative approach based on the *MAX*-chart makes sense. The estimation aspects of this chart are subsequently checked and the resulting empirical chart is both easy to understand and to apply.

In principle, the problem has now been solved in a quite satisfactory manner. Nevertheless, an interesting aspect still deserves further attention. In the above, it was mentioned that guidelines are available to find the best

value of r for given underlying parameters. To be specific, assume that the intended average run length (ARL) during IC should be equal to $1/\alpha$ for some prescribed small $\alpha > 0$ and moreover that p is supposed to change during OoC into θp , for some given $\theta > 1$. Then the rule of thumb for the value of r which approximately minimizes ARL for given α and θ (see Albers [1] for details) suggests using

$$r^{\text{opt}} = \frac{1}{\alpha(2.6\theta + 2) + 0.01(4\theta - 3)} \quad (1.1)$$

for values of α in (0.001, 0.01) and θ in (3/2, 4). This leads to (cf. Table 3 in Albers [1]) quite a variety of values for r^{opt} , including rather large ones, up to 30. Subsequently, it is argued there that using too large r feels awkward in practice and that hence a truncated version like $\min(5, r^{\text{opt}})$ might be preferable. One reason is that already a substantial part of the attainable improvement over $r = 1$ will have been realized at this upper value $r = 5$. But a maybe even more important reason for this cautious step is the fact that, while α may be given, the actual θ , in fact, is not known. The value θ_D of θ we use in (1.1) is merely the one against which we design the chart to have optimal detection power. If the actual θ happens to be really large, then a chart with a rather small r , like $r = 3$, may already be the actual winner, and then we would be most unhappy if we have selected a value like $r = 15$ (or even larger). On the other hand, such a value of r is quite attractive in terms of detection power when the actual θ is indeed small.

A solution for this type of dilemma is to combine a chart for large(er) θ with one intended for small(er) θ . Indeed, a similar program was carried out successfully by Albers and Kallenberg [4] for the continuous case of controlling a process mean. Here hybrid approaches such as ‘fast initial response’ $CUSUM$ and Shewhart- $CUSUM$ (see e.g., Ryan [6]) in itself were already available, but these are relatively complicated and apparently not very popular, in practice, and hence a new type of approach was considered to be useful. Arguing along the same lines in the present context of high

quality attribute data, then suggests to start by combining the individual (*IND*) chart for $r = 1$ with a *MAX*-chart for some larger value of r . The resulting *INDMAX*-chart now signals if either a single waiting time falls below a really small lower limit k , or if a group of r such waiting times all fall below a moderately small lower limit n . Here we keep working with fixed groups of size r , which keeps matters very simple indeed.

Note that in Albers and Kallenberg [4], a slightly more complicated cumulative approach is used, which would translate here into signaling as soon as r consecutive waiting times all fall below n . In the context of controlling a process mean, such a cumulative version provides some improvement over working with fixed groups (see Albers and Kallenberg [5]) and hence there it is worthwhile to add this aspect. However, for attribute data, this is not really the case, as has been demonstrated somewhat surprisingly in Albers [3]: *MAX* and *CUMAX* are very similar in performance. Consequently, the desire for simplicity suggests carrying on with *INDMAX*, rather than with *INDCUMAX*.

An additional reason to strive for simplicity is the fact that yet another step has to be taken. As we observed above, a straightforward nonparametric approach runs into trouble due to very large stochastic errors, and hence the step towards *MAX* was advocated. However, if we next mix this chart with *IND*, then we are back at the original problem of having to estimate a (too) small quantile. In fact, it will even be smaller than α , as *IND* has to share the possibility of causing a false alarm with *MAX*. The solution for this final complication (cf. Albers and Kallenberg [4]) is to apply the grouping step twice. First, the waiting times are considered t at a time, where t is relatively small (say 3 to 5). The maxima of these blocks are subsequently used as input for the *INDMAX*-chart described above. Hence stopping can occur after each t waiting times, with an additional option after each rt ones. Taking r , e.g., also 3 to 5, we get choices like $(t, rt) = (3, 9), (4, 16)$ or even $(5, 25)$. Indeed this provides the required flexibility in comparison to a *MAX*-chart with a fixed r like 6, 10, or even 15. A major change in p (i.e., a large θ) can now still be picked up after already 3-5 steps, while good detection power

against a small change in p also exists, due to the presence of larger r in the range 9-25, as well.

Incidentally, the description given here in terms of maxima used as input for *INDMAX*, has been adopted because it will turn out to be convenient for understanding and analyzing the behavior of the chart. However, do note that, in fact, we are using nothing but a mixture of two *MAX*-charts: one with group size t and another with group size rt . Hence we shall call the resulting chart *MIXMAX* (or *MIXMAX*(t, rt) in full) with $INDMAX(r) = MIXMAX(1, r)$ as a boundary case. Another comment in passing is that the pairs (t, rt) used in the present application are typically larger than those in the continuous case of controlling a process mean. There combinations like $(t, rt) = (2, 6)$ or $(3, 9)$ are used. The reason for this difference is that the mean shifts considered are usually quite large when translated into terms of the θ 's used in the present case of attribute data. We come back to this point in Section 2.

After the motivating outline presented in this introduction, we shall study in Section 2 the behavior of our new chart in a more systematic manner. As explained above, our starting point will always be the basic homogeneous case, i.e., where the underlying distributions are simply geometric. The empirical nonparametric version of the *MIXMAX*-chart is subsequently discussed in Section 3. Note that it also offers an attractive alternative to existing methods of *CUSUM* type, which may be slightly more efficient under optimal conditions. However, as these conditions are rarely available, such superiority is rather dubious. Here as well, deviations from model assumptions and estimation effects will have serious effects and neglecting the resulting model and stochastic errors requires a seemingly unfounded optimism. As *CUSUM* procedures have a more complicated structure, analysis of the effects in question (so far) seems out of reach. Consequently, from a robustness point of view, the proposed *MIXMAX* approach is indeed a serious competitor. For convenience, the conclusions reached, as well as a summary of the resulting procedure, are presented in Section 4.

2. The Homogeneous Case

As explained in the introduction, before considering the nonparametric version of the proposed chart, we first analyze the homogeneous case. Hence we use D_1, D_2, \dots , a sequence of independent identically distributed random variables (r.v.'s), with failure probability $P(D_1 = 1) = 1 - P(D_1 = 0) = p$ during *IC*. During *OoC*, this p becomes θp , for some $\theta > 1$, and the process should be stopped as quickly as possible. This sequence of D_i 's gives rise to a new sequence of geometric r.v.'s $X_i, i = 1, 2, \dots$, defined as the times from the $(i - 1)$ th up to and including the i th failure. Hence

$$P(X_i = k) = p(1 - p)^{k-1}, \quad (2.1)$$

with $k = 1, 2, \dots$. For some $r \geq 1$, the *MAX*-chart from Albers [2] now gives a signal if $\max(X_1, \dots, X_r)$ is too small; otherwise, the next group of size r is considered, and so on. To allow a fair comparison among different values of r , the boundary value n should satisfy

$$FAR = P(\max(X_1, \dots, X_r) \leq n) = r\alpha, \quad (2.2)$$

for some small $\alpha > 0$. Then the *ARL* during *IC* will indeed have the same value $r/(r\alpha) = 1/\alpha$ for all r . As $P(X_1 \leq n) = 1 - (1 - p)^n$, it readily follows from (2.2) that

$$n = \frac{\log(1 - \{r\alpha\}^{1/r})}{\log(1 - p)}. \quad (2.3)$$

(Either use standard interpolation in (2.3) or let n be the largest integer such that $FAR \leq r\alpha$; in practice, the differences involved will be negligible.)

During *OoC*, the alarm rate becomes $\{1 - (1 - \theta p)^n\}^r$. To compare the detection power for various r , we note that in view of (2.3) this result subsequently shows that

$$ARL = ARL_{r, \theta} = \frac{r}{\{1 - (1 - \{r\alpha\}^{1/r})^{g(\theta, p)}\}^r}, \quad (2.4)$$

where $g(\theta, p) = \log(1 - \theta p)/\log(1 - p) \approx \theta\{1 + (\theta - 1)p/2\}$. It follows that for the small p we are interested in the expression from (2.4) equals $r/\{1 - (1 - \{r\alpha\}^{1/r})^\theta\}^r$ to high precision. Hence the dependence of ARL on the actual p is negligible, which explains the notation $ARL_{r,\theta}$ in (2.4). Clearly, this $ARL_{r,\theta}$ as a function of θ decreases from $1/\alpha$ at $\theta = 1$ to a lower limit r as θ becomes very large. In particular, for the *IND*-chart, where $r = 1$, we get $ARL_{1,\theta} \approx 1/\{1 - (1 - \alpha)^\theta\} \approx 1/(\theta\alpha)$, showing that this simple geometric chart indeed performs rather poorly, unless θ is quite large. Hence the suggestion to use larger r , the more so if the excess of θ over 1 is supposed to be smaller. In fact, the rule of thumb already mentioned in (1.1) for the negative binomial charts is also applicable for the *MAX*-charts (cf. Albers [2], among others Lemma 3.1).

Using the above, we are now able to characterize the new proposals *INDMAX* and *MIXMAX*. We shall be considering two lower limits: a really small one k and a moderately small one n . If a waiting time falls at or below k , then we call it an '*L*', if it falls above k and at or below n , then we call it an '*M*', and otherwise, we call it an '*H*'. Let us denote the corresponding probabilities by α_L , α_M and $1 - \alpha_L - \alpha_M$, respectively, i.e.,

$$\alpha_L = P(X_i \leq k), \quad \alpha_L + \alpha_M = P(X_i \leq n). \quad (2.5)$$

In view of (2.1), it follows from (2.5) that $k = \log(1 - \alpha_L)/\log(1 - p)$ and likewise $n = \log(1 - \alpha_L - \alpha_M)/\log(1 - p)$. The *INDMAX*(r)-chart now stops at each step if it produces an '*L*' and in addition after each r steps if these are all '*M*'. Moreover, let '*Y* is $G(\theta)$ ' mean that an r.v. Y has a geometric distribution with parameter θ (e.g., the X_i from (2.1) are $G(p)$) and moreover, let '*Z* is $G_r(\theta)$ ' mean that Z has an r -truncated geometric distribution, in the sense that $P(Z = j) = P(Y = j | Y \leq r)$, $j = 1, \dots, r$, where Y is $G(\theta)$. Then we have the following characterization.

Theorem 2.1. *The run length of the INDMAX(r)-chart is distributed as*

$$RL = rV + W(B - r), \quad (2.6)$$

where V , W and B are independent r.v.'s such that V is $G(\tau)$ with

$$\tau = 1 - (1 - \alpha_L)^r + \alpha_M^r, \quad (2.7)$$

while B is $G_r(\alpha_L)$ and $P(W = 0) = 1 - P(W = 1) = \alpha_M^r / \tau$. Moreover,

$$ARL = \frac{1 - (1 - \alpha_L)^r}{\alpha_L \tau}. \quad (2.8)$$

Proof. Consider the process after each r steps. Then a success means that either an 'L' has occurred, which happens with probability $1 - (1 - \alpha_L)^r$, or a straight sequence of r 'M', which has probability α_M^r . Hence the number V of attempts required is $G(\tau)$, with τ as in (2.7). Consequently, to first order approximation RL in (2.6) simply equals rV , but note that an exact result requires a slight correction step. The point is that all failed sequences indeed have length r , but the last and successful sequence may be shorter. This happens if the occurrence of an 'L' is the reason for stopping, which has probability $1 - \alpha_M^r / \tau$, and thus leads to the factor W in the second term in (2.6). In the actual correction $(B - r)$ the r.v. B stands for the waiting time till an 'L', given that it will occur within r steps, i.e., indeed B is $G_r(\alpha_L)$. Next, it is a straightforward exercise to obtain that $EB = r - \{r/[1 - (1 - \alpha_L)^r] - 1/\alpha_L\}$. Consequently, $EW(EB - r) = \{1 - (1 - \alpha_L)^r\} / (\alpha_L \tau) - r/\tau$, from which (2.8) readily follows. \square

Remark 2.1. To illuminate the result from (2.8), the following comments may be useful:

(i) As α_L is small, we have that $1 - (1 - \alpha_L)^r \approx r\alpha_L\{1 - (r - 1)\alpha_L/2\}$ and thus that $ARL = (r/\tau)\{1 - (r - 1)\alpha_L/2\}$ which is indeed only marginally smaller than $rEV = r/\tau$.

(ii) In a bit more detail, since $P(B = j) = \alpha_L(1 - \alpha_L)^{j-1} / \{1 - (1 - \alpha_L)^r\} \approx 1/r$ for $j = 1, \dots, r$ and α_L small, $EB \approx (r + 1)/2$ and thus the correction $(EB - r) \approx -(r - 1)/2$. Together with $EW = 1 - \alpha_M^r/\tau \approx r\alpha_L/\tau$ this again produces that $ARL/(r/\tau) \approx 1 - (r - 1)\alpha_L/2$.

(iii) Also, observe that $1/ARL = \alpha_L + \alpha_M^r\alpha_L/\{1 - (1 - \alpha_L)^r\} \approx \alpha_L + (\alpha_M^r/r)\{1 + (r - 1)\alpha_L/2\}$. Now, $\alpha_L = 1/ARL_{IND}$ and $\alpha_M^r/r = 1/ARL_{MAX(r)}$, and thus to first order $1/ARL = 1/ARL_{IND} + 1/ARL_{MAX(r)}$, a result similar to, e.g., Lemma 2.1 of Albers and Kallenberg [4]. The fact that this nice type of result does not hold exactly here, but only to first approximation, is due to the fact that the two patterns involved ('L' and 'M ... M') in the present context of fixed groups slightly interfere. \square

It is not difficult to derive further properties of RL , see e.g., Corollary 2.2 from Albers and Kallenberg [4]. However, to avoid repetition, we shall refrain from pursuing this here. We do observe that from (2.8) (see also Remark 2.1(iii) above) it is immediate that $ARL = 1/\alpha$ will follow if α_L and α_M from (2.5) are chosen such that

$$\alpha_L + \frac{\alpha_M^r \alpha_L}{1 - (1 - \alpha_L)^r} = \alpha. \quad (2.9)$$

Suppose that for some γ with $0 \leq \gamma \leq 1$ we let $\alpha_L = \gamma\alpha$. Then (2.9) implies that $\alpha_M^r = (1 - \gamma)\{1 - (1 - \gamma\alpha)^r\}/\gamma \approx (1 - \gamma)r\alpha\{1 - \gamma(r - 1)\alpha/2\}$. In fact, the latter value is a lower bound and thus using $\alpha_M \approx ((1 - \gamma)r\alpha\{1 - \gamma(r - 1)\alpha/2\})^{1/r}$ will produce a total at most α in (2.9) and hence an ARL which is almost equal to and also at least $1/\alpha$. However, usually even the simple first order approximation $\alpha_M \approx \{(1 - \gamma)r\alpha\}^{1/r}$ will be adequate. To see this, observe that the realization in (2.9) for this choice of α_M equals $\gamma\alpha + (1 - \gamma)r\alpha\{\gamma\alpha/1 - (1 - \gamma\alpha)^r\}$, which exceeds α by at

most $(1 - \gamma)\alpha\{1/(1 - (r - 1)\gamma\alpha/2) - 1\}$. Hence the resulting relative error is $\leq \gamma(1 - \gamma)(r - 1)\alpha/\{2 - (r - 1)\gamma\alpha\}$ which is very small indeed. A final comment is that for $\gamma = 0$ we are clearly back at the $MAX(r)$ -chart with $\alpha_M = (r\alpha)^{1/r}$ (cf. (2.2) and (2.3)) while for $\gamma = 1$ we get the IND -chart. For application, in practice, the obvious choice will be $\gamma = 1/2$.

Given the setup for $INDMAX$, it now becomes quite easy to make the final step towards $MIXMAX$. Here, as described in the introduction, we no longer use individual X_i as input, but instead maxima of relatively small groups of such waiting times. To be precise, we consider $Y_1 = \max(X_1, \dots, X_t)$, $Y_2 = \max(X_{t+1}, \dots, X_{2t})$, Only two modifications are required to accommodate this change. In the first place, as (2.5) should now hold for the Y_i rather than the X_i , the fact that $P(Y_j \leq j) = P^t(X_j \leq j)$ thus implies that now

$$\alpha_L^{1/t} = P(X_i \leq k), \quad (\alpha_L + \alpha_M)^{1/t} = P(X_i \leq n), \quad (2.10)$$

and therefore that $k = \log(1 - \alpha_L^{1/t})/\log(1 - p)$ and

$$n = \log\{1 - (\alpha_L + \alpha_M)^{1/t}\}/\log(1 - p).$$

The second change entails that (2.6) and (2.8) are replaced by $RL = t\{rV + W(B - r)\}$ and $ARL = t\{1 - (1 - \alpha_L)^r\}/(\alpha_L\tau)$, respectively. Consequently, to once more arrive at the desired result $ARL = 1/\alpha$ in view of (2.9) requires $\alpha_L + \alpha_M^r\alpha_L/\{1 - (1 - \alpha_L)^r\} = t\alpha$. Hence

$$\begin{aligned} \alpha_L = \gamma t\alpha, \quad \alpha_M &= \left\{ (1 - \gamma) \frac{1 - (1 - \gamma t\alpha)^r}{\gamma} \right\}^{1/r} \\ &\approx \left\{ (1 - \gamma) r t\alpha \left(\frac{1 - \gamma(r - 1)t\alpha}{2} \right) \right\}^{1/r} \end{aligned} \quad (2.11)$$

will do the trick. Indeed, for $\gamma = 0$, the chart boils down to merely using $n = \log(1 - \{rt\alpha\}^{1/(rt)})/\log(1 - p)$ while for $\gamma = 1$ we merely use $k = \log(1 - \{t\alpha\}^{1/t})/\log(1 - p)$. For values in between (with typically $\gamma = 1/2$ as our default), we get a mixture of $MAX(rt)$ and $MAX(t)$.

In the above, we have considered the *IC* situation. Next, we move on to the *OoC* case, where p has changed into θp for some $\theta > 1$. We have:

Theorem 2.2. *During OoC, the average run length of the MIXMAX-chart equals*

$$ARL_{\theta} = \frac{t\{1 - (1 - \alpha_{L,\theta})^r\}}{(\alpha_{L,\theta}\tau_{\theta})}, \quad (2.12)$$

where now $\alpha_{L,\theta} = \{1 - (1 - \alpha_L^{1/t})^{g(\theta,p)}\}^t$,

$$\alpha_{M,\theta} = \{1 - (1 - (\alpha_L + \alpha_M)^{1/t})^{g(\theta,p)}\}^t - \alpha_{L,\theta}$$

and $\tau_{\theta} = 1 - (1 - \alpha_{L,\theta})^r + \alpha_{M,\theta}^r$, in which $g(\theta, p) = \log(1 - \theta p)/\log(1 - p)$. If α_L and α_M are chosen according to (2.11), then we have in particular that $ARL_1 = 1/\alpha$.

Proof. For simplicity, first consider the *INDMAX* case. During *OoC*, the structure described in (2.6)-(2.8) continues to hold. Only the probabilities from (2.5) should be replaced by $\alpha_{L,\theta} = P_{\theta}(X_i \leq k)$ and $\alpha_{L,\theta} + \alpha_{M,\theta} = P_{\theta}(X_i \leq n)$. Consequently, (2.8) can be translated directly into (2.12). To characterize these new probabilities in terms of the old ones, we proceed as follows. In analogy to (2.3) and (2.4), we obtain that $P_{\theta}(X_i \leq k) = 1 - (1 - \theta p)^k = 1 - (1 - \alpha_L)^{g(\theta,p)}$, where again $g(\theta, p) = \log(1 - \theta p)/\log(1 - p)$. Likewise, $P_{\theta}(X_i \leq n) = 1 - (1 - \theta p)^n = 1 - (1 - \alpha_L - \alpha_M)^{g(\theta,p)}$. The step towards *MIXMAX* subsequently follows by taking (2.10) rather than (2.5) as our starting point. Hence, e.g., $\alpha_{L,\theta}^{1/t} = P_{\theta}(X_i \leq k)$ with $k = \log(1 -$

$\alpha_L^{1/t})/\log(1-p)$, and thus $\alpha_{L,\theta} = \{1 - (1 - \alpha_L^{1/t})^{g(\theta,p)}\}^t$, as proposed. As concerns ARL_1 , in the boundary case $\theta = 1$ we simply have $g(1, p) = 1$ and therefore $\alpha_{L,1} = \alpha_L$, $\alpha_{M,1} = \alpha_M$. Choosing these as in (2.11) indeed produces $ARL = 1/\alpha$. \square

Now we are in a position to compare the performance during *OoC* of the *MIXMAX*-chart to that of the *MAX*-chart. In fact, two such comparisons seem appropriate. In the first place, of course, that of *MIXMAX*(t, rt) to *MAX*(t). Here the issue is what advantage is achieved by adding a kind of ‘second opinion’ after each rt steps to the standard stopping option after each t steps. But also interesting is the comparison between *MIXMAX*(t, rt) and *MAX*(q), where $q = [t(r+1)/2]$, with $[z]$ denoting from now on the largest integer $\leq z$. This latter chart hence represents the fixed choice for the (possibly truncated) average of t and rt . As concerns the values of t and r to be used, we have the following observations. Of course, these values can be chosen freely, but some guidance can also be derived from the rule of thumb (1.1). Suppose that instead of a single design value θ_D we use an interval (θ_L, θ_U) . Then for t we propose $t = [r_U^{\text{opt}}]$, where r_U^{opt} is the result from (1.1) for θ_U and the given α . Likewise, we obtain r_L^{opt} using θ_L and subsequently choose $r = [r_L^{\text{opt}}/t]$.

In this way, we, e.g., obtain for $(\theta_L, \theta_U) = (3/2, 5)$ and $\alpha = 0.001, 0.005$ and 0.01 , the triplets $(t, r, q) = (5, 5, 15), (4, 4, 10)$ and $(3, 3, 6)$, respectively. In Table 2.1, we use these configurations to illustrate the behavior of the three competing charts. From this table, we observe that the *MAX*(t)-chart indeed performs nicely if θ becomes large: the *ARL* then tends to its lower limit t . However, on the other hand, if θ happens to be small, then its *ARL* can still be unpleasantly large. This effect can be remedied by averaging t with the then more appropriate choice rt , as is done by the *MAX*(q)-chart. Indeed, for such smaller values of θ , the improvement is

substantial, up to well above a factor 2. But once more, on the opposite side, the picture is less nice: the lower limit for large θ now is q rather than t , which again can mean a factor well above 2, but now in the wrong direction. In between the rows for these two *MAX*-charts, we have tabulated the results for the corresponding *MIXMAX*-chart. This illustrates clearly that this chart indeed performs as we designed it to do: for small θ , it behaves like the then suitable *MAX*(q)-chart, while for large θ , it follows the behavior of the *MAX*(t)-chart. Hence it combines the best of both the worlds, often meaning a dramatic improvement over the worst of its two competitors, and never losing more than marginally compared to the best of the two.

Table 2.1. Comparison of the *ARL*'s of *MAX*(t) (upper value), *MIXMAX*(t, r) (middle value) and *MAX*(q) (lower value) for various θ

| $\alpha \backslash \theta$ | 5/4 | 3/2 | 2 | 3 | 4 | 6 | 9 | 12 | 16 |
|-----------------------------------|------|------|------|------|------|------|------|------|------|
| 0.001 $(t, r, q) = (5, 5, 15)$ | 418 | 214 | 80.8 | 25.6 | 13.6 | 7.48 | 5.57 | 5.15 | 5.03 |
| | 256 | 103 | 39.4 | 20.6 | 15.1 | 9.04 | 6.10 | 5.34 | 5.08 |
| | 253 | 103 | 37.7 | 18.7 | 15.8 | 15.0 | 15.0 | 15.0 | 15.0 |
| 0.005 $(t, r, q) = (4, 4, 10)$ | 102 | 60.4 | 28.7 | 12.2 | 7.70 | 5.09 | 4.23 | 4.05 | 4.00 |
| | 77.3 | 41.1 | 20.5 | 12.0 | 9.09 | 6.05 | 4.56 | 4.17 | 4.03 |
| | 77.0 | 41.0 | 20.0 | 11.9 | 10.5 | 10.0 | 10.0 | 10.0 | 10.0 |
| 0.01 $(t, r, q) = (3, 3, 6)$ | 58.2 | 38.3 | 20.7 | 9.84 | 6.45 | 4.20 | 3.33 | 3.10 | 3.02 |
| | 47.7 | 28.2 | 14.7 | 8.43 | 6.65 | 4.98 | 3.78 | 3.33 | 3.10 |
| | 47.9 | 28.5 | 14.8 | 8.28 | 6.75 | 6.10 | 6.00 | 6.00 | 6.00 |

Note that in Table 2.1 we allowed θ to increase to values well above the design interval $(3/2, 5)$. One reason is that such cases nicely illustrate the risk of being stuck with a large value of q in *MAX*(q): for these larger θ , soon hardly more than, e.g., 5 observations would suffice, and a lower bound of e.g., 15 then clearly remains an obstacle. But, in fact, there is more. Our

choice for the range of values of θ to be considered has been inspired by the desire to tune into the application at hand. The failure probability p typically corresponds to some (very) unpleasant event, like death due to surgical error. Rates of increasing θ which already compel us to detect such going *OoC* will therefore be not really large and an upper bound like $\theta_U = 5$ seems amply sufficient. Nevertheless, protection against sudden dramatic increases remains quite desirable as well.

Moreover, other types of applications might warrant the use of much larger values of θ . By way of illustration, we once more draw a parallel to the continuous case of monitoring a normal mean. Without loss of generality, let X be $N(0, 1)$ during *IC*. Then the usual 3σ -chart in its one-sided version has an $ARL = 1/p$, with $p = P(X > 3) = 1/740$. During *OoC*, a shift d will occur, which typically is supposed to vary from $d = 1$ ('small/moderate') to large $d = 3$ ('large'). This means that then p is replaced by $p_1 = P(X > 3 - d)$, which varies from 0.023 to $1/2$ over this range. As this, p_1 obviously plays the role of θp from the attribute case, it follows that the corresponding θ varies from a small/moderate value 17 to a large value 370. Indeed, the interval (17, 370) is quite different from the interval $(3/2, 5)$ used above (also, cf. a remark to this effect already in Albers [1] after (3.2)). For an application of this type, choices like $(t, r) = (2, 3), (2, 5)$ or $(3, 3)$ (cf. Albers and Kallenberg [4]) are suitable, leading to similar results as in Table 2.1 above when letting θ increase to, e.g., 100 rather than just to 16. The *MIXMAX*-chart again stays close to (and often is) the winner over the whole range.

3. The Nonparametric Chart

After demonstrating in Section 2 that the *MIXMAX*-chart indeed lives up to expectations for the basic homogeneous case, the next step consists of dropping the rather artificial assumption of a known underlying distribution. In this way, we arrive at our main proposal: an empirical nonparametric

version of the *MIXMAX*-chart, which provides the robust alternative to existing methods, as discussed at the end of Section 1. To avoid duplication, we try to be rather brief here. The emphasis will be on the ideas and the actual implementation. For further details and properties, we will often refer to our earlier papers dealing with nonparametric proposals for this area (Albers and Kallenberg [5] and Albers [2]).

Hence, from now on, we no longer rely on the geometric assumption (2.1), but instead have an unknown underlying distribution function (df) F for the X_i . A Phase I sample X_1, \dots, X_m should thus be available, from which we obtain $F_m(x) = m^{-1} \# \{X_i \leq x\}$, the corresponding empirical df, and F_m^{-1} , the quantile function, i.e., $F_m^{-1}(t) = \inf \{x \mid F_m(x) \geq t\}$. Note that $F_m^{-1}(t)$ equals $X_{(i)}$ for $(i-1)/m < t \leq i/m$, with $X_{(1)} < \dots < X_{(m)}$ the order statistics for the sample. Consequently, a q -quantile $l = F^{-1}(q)$ can be estimated by

$$\hat{l} = F_m^{-1}(q) = X_{(s)}, \quad (3.1)$$

where $s = \{mq\}$, with $\{z\}$ denoting the smallest integer $\geq z$. For example, when applying these steps to the *MAX*-chart, (2.2) turns into $FAR = F^r(n) = r\alpha$ and thus (2.3) into $n = F^{-1}((r\alpha)^{1/r})$. From (3.1), the estimated version $\hat{n} = X_{(s)}$ then follows, with $s = \{m(r\alpha)^{1/r}\}$. Hence for the empirical nonparametric version of the *MAX*-chart, we just check after each group of r waiting times Y_1, \dots, Y_r whether $\max(Y_1, \dots, Y_r) \leq X_{(s)}$. If so, then we give an alarm; if not, then we consider the next batch of size r .

Note that increasing r indeed helps: for $r = 1$, typically $s = \{m\alpha\}$ will equal 1, while for $r > 1$ the use of extreme order statistics (and thus the occurrence of relatively huge stochastic errors) is effectively avoided. To be more specific, conditional on X_1, \dots, X_m , we obtain during *IC* that

$ARL = r/F^r(X^{(s)}) \stackrel{D}{=} r/(U_{(s)})^r$, where ‘ $\stackrel{D}{=}$ ’ means ‘distributed as’ and $U_{(1)} < \dots < U_{(m)}$ are the order statistics for a sample of size m from the uniform distribution on $(0, 1)$. Indeed, for common choices like $(m, \alpha) = (100, 0.001)$, we will typically find for $r = 1$ that $s = \{m\alpha\} = 1$, and thus that $ARL = 1/U_{(1)}$, which is clearly ill-behaved. However, for $r > 1$, this problem disappears. Take, e.g., $r = 3$, then $s = \{14.4\} = 15$, which is not at all extreme anymore. (If desired, then see Albers [2] for further details.)

Next, we proceed along these lines to the *MIXMAX*-chart. After the Phase I sample X_1, \dots, X_m , we now consider as input for a *MIXMAX*(t, rt)-chart the two sequences

$$Y_1 = \max(X_{m+1}, \dots, X_{m+t}), \quad Y_2 = \max(X_{m+t+1}, \dots, X_{m+2t}), \dots$$

and

$$Z_1 = \max(Y_1, \dots, Y_r), \quad Z_2 = \max(Y_{r+1}, \dots, Y_{2r}), \dots$$

A signal results if a Y_i is at most some estimated small lower bound \hat{k} , or if a Z_i is at most some estimated moderate lower bound \hat{n} . We have the following result.

Theorem 3.1. *The nonparametric version of the MIXMAX-chart follows by choosing*

$$\hat{k} = X_{(s)} \quad \text{and} \quad \hat{n} = X_{(v)}, \tag{3.2}$$

where $s = \{m(\alpha_L)^{1/t}\}$ and $v = \{m(\alpha_L + \alpha_M)^{1/t}\}$, with α_L and α_M once more as in (2.11). Moreover, conditional on X_1, \dots, X_m , its ARL during IC

satisfies $ARL \stackrel{D}{=} 1/W$, with

$$W = \frac{U_{(s)}^t + \frac{\{U_{(v)}^t - U_{(s)}^t\}U_{(s)}^t}{1 - (1 - U_{(s)}^t)^r}}{t}. \tag{3.3}$$

Proof. In the geometric case (2.10) implied that $k = F^{-1}(\alpha_L^{1/t}) = \log(1 - \alpha_L^{1/t})/\log(1 - p)$. In the present context, this has to be replaced by $\hat{k} = F_m^{-1}(\alpha_L^{1/t})$, from which (3.2) immediately follows (cf. (3.1)). For $\hat{n} = F_m^{-1}((\alpha_L + \alpha_M)^{1/t})$, the argument is completely similar. As concerns the *ARL* of the chart, note that in the geometric case (cf. Remark 2.1(iii)), we have $t/ARL = \alpha_L + \alpha_M^r \alpha_L / \{1 - (1 - \alpha_L)^r\}$. Since α_L now transforms into $F^t(\hat{k}) = F^t(X_{(s)}) \stackrel{D}{=} U_{(s)}^t$, and likewise α_M becomes $F^t(\hat{n}) - F^t(\hat{k}) \stackrel{D}{=} U_{(v)}^t - U_{(s)}^t$, the result in (3.3) follows. \square

Observe that (3.3) shows that the chart is indeed truly nonparametric. Moreover, its application is easy to understand and simple to perform, as the following example illustrates.

Example 3.1. Let $m = 100$, $\alpha = 0.001$, $\gamma = 1/2$ and $t = r = 5$ (cf. Table 2.1). Then (2.11) and (3.1) together produce the values $s = \{30.2\} = 31$ and $v = \{84.0\} = 84$. Hence we stop after a group of 5 X_i that are all at most $X_{(31)}$, or after one of 25 $X_{(i)}$ which are all at most $X_{(80)}$. Some additional remarks are:

(i) A minor refinement can be achieved by using interpolation, e.g., implying here that $X_{(31)}$ is replaced by $0.8X_{(30)} + 0.2X_{(31)}$.

(ii) Using the first order approximation $\alpha_M \approx \{(1 - \gamma)rt\alpha\}^{1/r}$ in (2.11) leads to the simple $\tilde{v} = \{m(\gamma t\alpha + ((1 - \gamma)rt\alpha)^{1/r})^{1/t}\}$. As observed after (2.9), the error involved is tiny. Indeed, here we find $\tilde{v} = \{84.0\} = 84$ as well (in more decimals, the underlying outcomes are 84.006 and 84.023, respectively).

(iii) By way of comparison, note that for $\gamma = 1$ we get the *MAX(5)*-chart with $k = 34.7$, while $\gamma = 0$ gives the *MAX(25)*-chart with $n = 86.3$.

Moreover, $q = \lceil t(r + 1)/2 \rceil$ (cf. e.g., Table 2.1) equals 15 here and the $MAX(15)$ -chart uses 75.6 as a lower bound. (Indeed, $(0.347)^5/5 = (0.863)^{25}/25 = (0.756)^{15}/15 = 0.001$).

Hence, with the above, we have all that is needed for the implementation of the chart. However, its actual behavior still deserves further study, since a performance characteristic like the ARL is now no longer fixed at a given value $1/\alpha$, but instead stochastic, viewed conditionally on the Phase I sample

X_1, \dots, X_m . Of course, it is immediate that, e.g., $U_{(s)}^t \xrightarrow{P} \alpha_L$, and thus in

view of (2.11) that the ARL given through (3.3) will satisfy $ARL \xrightarrow{P} 1/\alpha$. However, this mere fact is not really sufficient. Do remember the discussion from the introduction: typically estimation effects lead to non-negligible stochastic errors. Subsequently, we noted that deviations from (rather optimistic) routine assumptions produce model errors as well, which additional drawback can be removed by using a nonparametric approach. This step was seen to require some additional adaptations, in order to avoid that the removal of such model errors produced huge stochastic ones in return. Indeed, we succeeded in bringing these stochastic errors back to the same proportions as those from the parametric competitors at the starting point: not huge, but also not negligible.

One way to control this stochastic behavior is to monitor left exceedance probabilities like $P(ARL < 1/\{\alpha(1 + \varepsilon)\})$ for some small $\varepsilon > 0$, like $\varepsilon = 0.25$. Ideally, the occurrence of relative errors on the left side in the ARL larger than a fraction ε should be rare, e.g.,

$$P_{Exc} = P\left(ARL < \frac{1}{\alpha(1 + \varepsilon)}\right) \leq \beta \tag{3.4}$$

should hold for some likewise small $\beta > 0$, e.g., $\beta = 0.2$. Interesting questions are what values of m are required to achieve equality in (3.4) for

given ε and β and, if for given m the upper bound β is violated, then what type of - hopefully small - correction can be applied to the bounds \hat{k} and \hat{n} from (3.2) in order to ensure compliance with (3.4) after all. Let Φ denote the standard normal distribution function. Then we have

Theorem 3.2. *For $m \rightarrow \infty$, the exceedance probability from (3.4) satisfies*

$$P_{Exc} \rightarrow 1 - \Phi\left(\frac{m^{1/2}\varepsilon\alpha}{\sigma(\alpha_L, \alpha_M)}\right), \quad (3.5)$$

where

$$\begin{aligned} \sigma^2(x, y) &= (x + y^r)2\{(x + y)^{-1/t} - 1\} \\ &\quad + x^2(1 - y^{r-1})^2\{x^{-1/t} - (x + y)^{-1/t}\}. \end{aligned}$$

Proof. Rather precise and detailed proofs for similar results can be found in Albers and Kallenberg [4] and Albers and Kallenberg [5]. To avoid repetition, we shall be brief here. Using (mostly) standard results on means, (co)variances and asymptotic normality of uniform order statistics, together with a one-step Taylor expansion, it can be shown that W from (3.3) is asymptotically normal with mean α and variance $m^{-1}\sigma^2(\alpha_L, \alpha_M)$. As (3.4) implies that $P_{Exc} = P(W > \alpha(1 + \varepsilon))$, the desired result then readily follows. \square

Remark 3.1. Some comments may again be helpful to illuminate this result.

(i) Using (2.11), the variance $m^{-1}\sigma^2(\alpha_L, \alpha_M)$ can be expressed in terms of α , t , r and γ . Moreover, since $(\alpha_L + \alpha_M^r) = \{1 + \gamma(r - 1)\}t\alpha$ to first order and $\alpha_L = \gamma t\alpha$, it follows that $\sigma(\alpha_L, \alpha_M)$ contains a factor $t\alpha$, which further simplifies the result in (3.5).

(ii) For $\gamma = 1$, we have $\sigma^2(x, 0) = x^2 \{x^{-1/t} - 1\}$ with $x = t\alpha$, while for $\gamma = 0$ we have $\sigma^2(0, y) = y^{2r} \{y^{-1/t} - 1\}$ with $y = (rt\alpha)^{1/r}$. Indeed, these are the expressions for the $MAX(r)$ - and the $MAX(rt)$ -charts, respectively.

If the value of P_{Exc} obtained from (3.5) for given m exceeds the prescribed β , then one option is to increase this sample size to a value which will be satisfactory in this respect. In fact, let u_β be such that $\beta = 1 - \Phi(u_\beta)$. Then equality to first approximation will clearly be achieved in (3.4) by letting

$$m = \left(u_\beta \frac{\sigma(\alpha_L, \alpha_M)}{\alpha} \right)^2. \quad (3.6)$$

If increasing m is no option, then the alternative for given sample size is to slightly lower the value of α which is used. This adaptation will somewhat increase the ARL , and thus lower the corresponding left exceedance probability. In this way, compliance to (3.4) can be achieved as well. We have:

Theorem 3.3. *For some small $\delta > 0$, replace α in (2.11) by $\alpha(1 - \delta)$. Use the resulting α_L^* and α_M^* to define new variables s^* , v^* , \hat{k}^* and \hat{n}^* according to (3.2). The modified chart will achieve approximate equality in (3.4) if δ is selected as*

$$\delta = \frac{m^{-1/2} u_\beta \sigma(\alpha_L, \alpha_M)}{\alpha} - \varepsilon. \quad (3.7)$$

Proof. The replacement of α by $\alpha(1 - \delta)$ will clearly transform W from (3.3) into a W^* which is asymptotically normal with mean $\alpha(1 - \delta)$ and variance $m^{-1} \sigma^2(\alpha_L^*, \alpha_M^*)$. Consequently, the corresponding P_{Exc}^* then will equal to $P(W^* > \alpha(1 + \varepsilon)) \approx 1 - \Phi(m^{1/2}(\varepsilon + \delta)\alpha / \sigma(\alpha_L^*, \alpha_M^*))$. This will

agree with $1 - \Phi(u_\beta)$ for $\delta = m^{-1/2} u_\beta \sigma(\alpha_L^*, \alpha_M^*) / \alpha - \varepsilon$. As this δ is typically small, a further approximation step leads to the result in (3.7). \square

Applying this modified *MIXMAX*-chart is only marginally less simple than using the original proposal. With δ from (3.7), we, e.g., replace s from (3.2) by

$$\begin{aligned} s^* &= \{m(\alpha(1 - \delta))^{1/t}\} \approx s(1 - \delta/t) \\ &= s(1 + \varepsilon/t) - m^{-1/2} u_\beta s \sigma(\alpha_L, \alpha_M) / (t\alpha). \end{aligned}$$

In view of Remark 3.1(i) above, this can still be simplified somewhat further. For example, in the boundary case $\gamma = 1$ (cf. Remark 3.1(ii)), it follows that $\sigma(\alpha_L, \alpha_M) / (t\alpha) = \sigma(t\alpha, 0) / (t\alpha) = \{(t\alpha)^{-1/t} - 1\}^{1/2} \approx \{m/s - 1\}^{1/2}$ and hence that $s^* \approx s(1 + \varepsilon/t) - u_\beta \{s(1 - s/m)\}^{1/2}$ (cf. (4.8) in Albers [2]). To conclude this section, we update our example:

Example 3.1 (Continued). For the configuration used here, together with $\varepsilon = 0.25$, we obtain in (3.5) from Theorem 3.2 an approximate value 0.37 for P_{Exc} . Incidentally, this outcome is very stable in γ : if we move from $\gamma = 1/2$ to the extreme case $\gamma = 1$, then it changes into 0.36. Either way, such a value will often be considered too large. To achieve, e.g., a boundary value $\beta = 0.2$, the modification in Theorem 3.3 through δ from (3.7) then leads for our default $\gamma = 1/2$ to lowering $s = \{30.2\}$ to $s^* = \{27.5\}$ and $v = \{84.0\}$ to $v^* = \{82.4\}$. In comparison, for $\gamma = 1$ we have that 34.7 becomes 32.0, while for $\gamma = 0$ 86.3 becomes 83.3. \square

4. Conclusions and Summary

From Albers [2], we know that the *MAX*(r)-chart is attractive in many

aspects. It produces an alarm once all waiting times Y_i in a group of size r are at or below a boundary value $X_{(s)}$, which is a suitably chosen order statistic from a Phase I sample. The chart has good detection power and, as it is nonparametric, it has no model error. Moreover, for $r > 1$ its stochastic error is comparable to that of parametric competitors, which usually suffer from a non-vanishing model error. Guidance on how to optimally choose r is given in the simple rule of thumb (1.1). However, a problem remains that this requires specification of the supposed rate of change θ . In practice, this rate is not really known and misspecification can produce unsatisfactory behavior. If θ is quite large, then quick stopping would have been feasible, but using a large r will prevent that. If θ is small, then using a small r leads to low sensitivity and stopping can take really very long.

In the present paper, this problem is addressed by introducing in Section 2 the $MIXMAX(t, rt)$ -chart, which is a suitable *MIX*tture of a $MAX(t)$ - and a $MAX(rt)$ -chart, where both t and r are small to moderate (and thus rt is (quite) large). In the basic homogeneous case, it is subsequently demonstrated that this combined procedure indeed performs as intended: it closely follows (or even beats) the best of the two underlying MAX -charts (i.e., the one with rt for small θ and the one with t for large θ) and (very) substantially improves on the worst of the two (i.e., the other way around: t for small θ and rt for large θ). See once more Table 2.1 for illustration. Consequently, it makes sense to develop in Section 3 the empirical nonparametric version of the $MIXMAX$ -chart, which does not rely on the often rather dubious assumption of a known underlying distribution. The impact of the estimation step is analyzed and minor corrections are derived to control relevant exceedance probabilities.

For convenience, we summarize the application of the $MIXMAX(t, rt)$ -chart:

1. Select a desired in-control $ARL = 1/\alpha$.
2. Choose an interval (θ_L, θ_U) for the rate of change θ that should be protected against.
3. Apply rule of thumb (1.1) to obtain the best t and r .
4. Choose a value for the mixing parameter γ (the default is $\gamma = 1/2$).
5. Compute $\alpha_L = \gamma t \alpha$ and $\alpha_M = \{(1 - \gamma)\{1 - (1 - \gamma t \alpha)^r\}/\gamma\}^{1/r}$ (cf. (2.11)).
6. Select m (e.g., $m = 100$) and collect a Phase I sample of waiting times X_1, \dots, X_m .
7. Compute the smallest integer $s \geq m(\alpha_L)^{1/t}$ and $v \geq m(\alpha_L + \alpha_M)^{1/t}$ (or use interpolation).
8. Find the order statistics $\hat{k} = X_{(s)}$ and $\hat{n} = X_{(v)}$ from X_1, \dots, X_m (cf. (3.2)).
9. Start monitoring:
 - a. check after each t waiting times X_i whether all of these are $\leq X_{(s)}$;
 - b. check after each rt steps whether all X_i involved are $\leq X_{(v)}$.
10. If desired, then select small ε and β such that $P(A\hat{R}L < 1/\{\alpha(1 + \varepsilon)\}) = \beta$ should hold.
11. Compute $\delta = m^{-1/2} u_\beta \sigma(\alpha_L, \alpha_M)/\alpha - \varepsilon$ (cf. (3.7)).
12. Replace α by $\alpha(1 - \delta)$ and reapply steps 5 to 9.

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