Longest Cycles in Tough Graphs

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Abstract: In this article, we establish bounds for the length of a longest cycle C in a 2-connected graph G in terms of the minimum degree δ and the toughness t. It is shown that C is a Hamiltonian cycle or $|C| \geq (t+1)\delta + t$. J Graph Theory 31: 107–127, 1999

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1. INTRODUCTION

In this article, we establish bounds for the length $c(G)$ of a longest cycle in a graph G in terms of the minimum degree $\delta(G) = \delta$ and the toughness $t(G) = t = \min(\frac{|S|}{\omega(G-S)} : S \subset V(G),\ \omega(G-S) > 1)$, where $\omega(G-S)$ denotes the number of components of $G-S$.

Theorem 1. *If G is a 2-connected graph of toughness t and minimum degree δ on n vertices, then*

$$c(G) \geq \min((t+1)\delta + t, n).$$

A result of this type has been conjectured by v. d. Heuvel [4]. He asked whether there is a constant $A(t)$ such that $c(G) \geq (t+1)\delta - A(t)$ for every non-Hamiltonian graph G of toughness $t \geq 1$. Theorem 1 can be considered as a common generalization of Theorem 2 and Theorem 3 for $t \geq 1$.

Theorem 2 (Dirac [2]). *Let G be a 2-connected graph of minimum degree δ on n vertices. Then*

$$c(G) \geq \min(2\delta, n).$$

It is well-known that $c(G) \geq 2\delta + 2$ for a non-Hamiltonian graph G of toughness $t \geq 1$.

Theorem 3 (Bauer et al. [1]). *Let G be a graph of toughness t and minimum degree δ on $n \geq 3$ vertices. If $n < (t+1)\delta + t + 1$, then G is Hamiltonian.*

2. NOTATION AND PRELIMINARIES

The following definitions refer to a given graph G. If K, H are subsets of $V(G)$ or subgraphs of G, let $N_K(H)$ denote the set of all vertices in K that are adjacent to some vertex in H. For $\{u_1, \ldots, u_m\} \subseteq V(G)$ we use $N_K(u_1, \ldots, u_m)$ instead of $N_K(\{u_1, \ldots, u_m\})$. A path P, which has its end vertices on a given cycle C in G and is openly disjoint with C, is called a C-arc. For subgraphs H of G we also use the notation $G - H$ short for $G - V(H)$, and $|H|$ short for $|V(H)|$.

To illustrate our approach, we start with the following lemma from Bauer et al. [1] about maximal cycles in G, that is, cycles C such that there is no cycle C' in G with $V(C) \subset V(C')$.

Lemma 1. *Let C be a maximal cycle in a graph G of toughness t and let H be a component of $G - C$. Then $|C| \geq (t+1)|N_C(H)| + t$.*

Proof. We fix a cyclic orientation on C and label $N_C(H) = \{x_1, \ldots, x_m\}$ accordingly. As C is maximal, we derive that, for each $x_i \in N_C(H)$, the successor x_i^+ on C is not in $N_C(H)$, and no pair of successors x_i^+, x_j^+ $(i \neq j)$ is linked by a C-arc. Therefore, the set $N_C^+(H) = \{x_1^+, \ldots, x_m^+\}$ is independent in G, and so $G - S$ has at least $m+1$ components, where $S = V(C) - N_C^+(H)$. Hence, indeed

$$t(m+1) \leq |S| = |C| - m$$

□

Notice that Lemma 1 already yields our main result if $|N_C(H)| \geq \delta$. In the sequel, we generalize the underlying idea of the proof with the help of the following definitions.

For vertices a and b in a graph H, let $D_H(a, b)$ denote the length of a longest (a, b)-path in H. If H is not separable, we set $D(H) = \min(D_H(a, b) : a, b \in V(H), a \neq b)$. For $|H| = 1$, set $D(H) = 0$. In terms of this definition, we provide the following standard lemma, which will be frequently used in the sequel. Given a cycle C with a fixed cyclic orientation and vertices $x, y \in V(C)$, we use $C[x, y], C[x, y), Cx, y]$, and $C(x, y)$ for the corresponding subpaths of C.

Lemma 2. *Let C be a cycle in a graph G such that $h = c(G) - |C|$, and let H be a component of $G - C$. Further, let x_1, x_2 be distinct vertices on C and let $v_1 \in N_H(x_1), v_2 \in N_H(x_2)$. Let P be a longest (v_1, v_2)-path in H, and let $Q = Q[z_1, z_2]$ be a C-arc from $C(x_1, x_2]$ to $C(x_2, x_1]$ such that $Q \cap P = \emptyset$. Then*

(i) $|C(x_1, x_2)| \geq D_H(v_1, v_2) + 1 - h$
and
(ii) $|C(x_1, z_1)| + |C(x_2, z_2)| \geq D_H(v_1, v_2) + 1 + (|Q| - 2) - h.$

Proof. Since P and $C[x_2, x_1]$ define another cycle C', we obtain

$$|C| + h = c(G) \geq |C'| \geq |C| - |C(x_1, x_2)| + D_H(v_1, v_2) + 1,$$

hence, (i). Also, $C[z_1, x_2] \cup C[z_2, x_1] \cup Q$ and P define a cycle C'' such that

$$|C| + h = c(G) \geq |C''| \geq |C| - (|C(x_1, z_1)| + |C(x_2, z_2)|) + D_H(v_1, v_2) + 1 + (|Q| - 2).$$

Hence, (ii). □

Lemma 3. *Let a, b be distinct vertices and P a longest (a, b)-path in a 2- connected graph H, and let K be a component of $H - P$. There exist distinct vertices $v_1, v_2 \in V(H)$ such that*

(i) $|\{v_1, v_2\} \cap K| = min(2, |K|)$, *and*
(ii) $|P| \geq d_H(v_i) + 1$*; moreover,* $|P| \geq 2d_H(v_i) - 1$ *if H is 3-connected* $(i = 1, 2)$.

Proof. We proceed by induction on $|H|$ and label $N_P(K) = \{x_1, \ldots .x_m\}$ in the order from a to b. Notice that $x_i^+ \notin N_P(K)$ for $1 \leq i < m$, since P is a longest (a, b)-path.

Case 1. $|K| \geq 2$ and K has no cut vertex.
Let J denote the set of integers j $(1 \leq j < m)$, such that $|N_K(x_j, x_{j+1})| \geq 2$. For $j \in J$, we obtain that x_j and x_{j+1} have distinct neighbors $a', b' \in V(K)$, and since P is a longest (a, b)-path in H, it follows that $|P(x_j, x_{j+1})| \geq D_K(a', b') + 1 \geq D(K) + 1$. Therefore:

$$|P| \geq 2m - 1 + |J|D(K). \tag{1}$$

If $D(K) = |K| - 1$, we take any distinct vertices $v_1, v_2 \in V(K)$ to obtain $d_K(v_i) \leq D(K)$ for $i = 1, 2$. If $D(K) < |K| - 1$, we make use of the induction hypothesis and determine distinct vertices $v_1, v_2 \in V(K)$ such that $D(K) \geq d_K(v_i)$ $(i = 1, 2)$. If $|J| \geq 2$, we derive $|P| \geq 2d_H(v_i) - 1$ $(i = 1, 2)$ from (1). On the other hand, $|J| \geq 1$, since H is 2-connected. Hence, $|P| \geq m + 1 + D(K) \geq d_H(v_i) + 1$ $(i = 1, 2)$ again by (1). It remains to consider the subcase when H is 3-connected and $|J| = 1$. In this event, necessarily $|K| = 2$ and $N_P(v_1) \cap N_P(v_2) = \emptyset$. Consequently, $|P| \geq 2d_H(v_i) - 1$ $(i = 1, 2)$. This settles Case 1.

Case 2. K has a cut vertex.
Consider an endblock B of K, and let c denote the unique cut vertex of K in $V(B)$. Now let J denote the set of integers j $(1 \leq j < m)$, such that $N_P(B - c) \cap \{x_j, x_{j+1}\} \neq \emptyset$ and $|N_K(x_j, x_{j+1})| \geq 2$. Again $|P(x_j, x_{j+1})| \geq D(B) + 1$, since P is a longest (a, b) path in H. Therefore:

$$|P| \geq 2m - 1 + |J|D(B). \tag{2}$$

Note that $|J| \geq 1$. If H is 3-connected. Moreover, $|J| \geq 2$ unless $|B| = 2$ and $d_P(v_1) < m$. Using the same arguments as in Case 1, we determine a vertex v in $B - c$, such that $D(B) \geq d_B(v)$. Consequently, $D(B) \geq d_K(v)$. The last inequality and (2) yield $|P| \geq d_H(v) - 1$; moreover, $|P| \geq 2d_H(v) - 1$, if H is 3-connected. This settles Case 2, since K has at least two endblocks.

Case 3. $|K| = 1$.
Clearly $|P| \geq 2d_H(v) - 1$ for the vertex v of K. In view of the preceding cases, we are done if $H - P$ has another component. Finally, let $|H| = |P| + 1$. As P is a longest path, no two vertices x_j^+, x_k^+ $(1 \leq j, k \leq m)$ are adjacent, and therefore, $|P| \geq d_H(x_i^+) + m - 1$ $(1 \leq i < m)$. Thus, we are done unless H is 3-connected. If H is 3-connected, then $m \geq 3$ and a standard argument yields $|P| \geq d_H(x_j^+) + d_H(x_k^+) - 1$ $(1 \leq j < k < m)$. This settles Case 3. □

3. SEPARABLE COMPONENTS

In this section, let C be a longest cycle in the 2-connected graph G. We will first show an improvement of the main result in the case when some component off C has a cut vertex.

Theorem 4. *Let C be a longest cycle in a 2-connected graph G. Each separable component H of $G - V(C)$ contains a vertex $v \in V(H)$ such that $|C| \geq (t + 1)d(v) + 3t$.*

We determine endblocks B_1 and B_2 of H and denote for $h = 1, 2$ by c_h the unique cut vertex of H in $V(B_h)$. Further, let $D_1 = D(B_1), D_2 = D(B_2)$ and $D = D_1 + D_2 + D_H(c_1, c_2)$. Using Lemma 3, we determine for $h = 1, 2$ vertices v_h in $B_h - c_h$ such that $d_H(v_h) \leq D_h$. Consequently, $d(v_h) \leq d_C(v_h) + D_h$. Let $d = min(d(v_1), d(v_2))$. Next we define for $h = 1, 2$

$$\begin{aligned} Z_h &= N_C(B_h - c_h) \\ X_h &= \{x \in Z_h : |N_H(x, x')| \geq 2 \text{ for all } x' \in N_C(H) - \{x\}\} \\ Y_h &= \{y \in V(B_h - c_h) : N_C(y) - X_h \neq \emptyset\}. \end{aligned}$$

Now we fix a cyclic orientation on C and label $N_C(H) = \{x_1, \ldots, x_m\}$ accordingly. Note that $m \geq 2$, since G is 2-connected. Clearly, $m \geq 2t$; moreover, $m \geq 3t - 1$, since $G - (N_C(H) \cup \{c_1\})$ has at least 3 components. Next we provide some lower bounds for $|C|$ in technical terms. Let us abbreviate $Z = Z_1 \cap Z_2$ and $l_0 = |Z|$.

Lemma 4. *Let* $|C| = (t+1)m + 2t - 1 + r$. *Then*

(i) $r \geq l_0 \lfloor \frac{D_1+D_2}{2} \rfloor$
(ii) $r \geq l_0 \lfloor \frac{D}{2} \rfloor$, *if* $N_C(H) = Z_1 \cup Z_2$
(iii) $r \geq (l_0+1) \lfloor \frac{D}{2} \rfloor$ *if* $N_C(H) = Z_1 \cup Z_2$ *and* $Z_1 \cup Z_2 \neq Z$.

Proof. Set $D' = max(D_1, D_2)$. For $x_i \in Z$ let u_i denote the vertex on C such that $|C(x_i, u_i)| = \lfloor \frac{D_1+D_2+2}{2} \rfloor$.

For $x_i \in N_{C-Z}(H)$, let $u_i = x_i^{++}$. For $x_i \in Z$, it follows from Lemma 2 (i) that $|C(x_i, x_{i+1})| \geq D' + 1$. Therefore, $C(x_i, u_i) \cap N_C(H) = \emptyset$ and by Lemma 2 (ii) no C-arc joins distinct segments of the form $C(x_i, u_i)$.

Let $S = V(\bigcup_{i=1}^{m} C[u_i, x_{i+1}])$. Then $G - (S \cup \{c_1\})$ has at least $m+2$ components. Consequently, $t(m+2) \leq |S| + 1$ and, equivalently, (i).

Now let $N_C(H) = Z_1 \cup Z_2$. If there is a vertex $x_q \in (Z_1 \cup Z_2) - Z$, let $Z' = Z \cup \{x_q\}$; otherwise, let $Z' = Z$. For $x_i \in Z'$, let w_i denote the vertex on C such that $|C(x_i, w_i)| = \lfloor \frac{D+2}{2} \rfloor$; otherwise, let $w_i = x_i^{++}$. Because C is a longest cycle, it follows again by Lemma 2(ii) that no C-arc joins distinct segments of the form $C(x_i, w_i)$. Setting $S = V(\bigcup_{i=1}^{m} C[w_i, x_{i+1}])$, we infer that $G - (S \cup \{c_1\})$ has at least $m+2$ components. Hence, $t(m+2) \leq |S| + 1$ and, equivalently, (ii) and (iii). □

Proof of Theorem 4. We divide the proof into several cases.

Case 1. $Z_1 \not\subseteq Z_2$ and $Z_2 \not\subseteq Z_1$.

We determine for each $y \in Y_1 \cup Y_2$ a vertex $\hat{y} = x_i \in N_C(H)$ such that $N_H(x_i) = \{y\}$ and $N_H(x_{i+1}) \neq \{y\}$. For $h = 1, 2$, let $\hat{X}_h = X_h \cup \{\hat{y} : y \in Y_h\}$ and $l_h = |\hat{X}_h| - |Z_1 \cap Z_2| - 1$. Considering the segments with respect to $Z_1 \cup Z_2$, we apply Lemma 2(i) to obtain that

$$\begin{aligned} |C| &\geq 2|Z_1 \cup Z_2| + (l_0+2)D + l_1 D_1 + l_2 D_2 \\ &= 2|Z_1| + 2|Z_2| + 2D + l_0(D-2) + l_1 D_1 + l_2 D_2 \end{aligned}$$

and, therefore,

$$|C| \geq 2d(v_1) + 2d(v_2) + l_0(D-2) + l_1 D_1 + l_2 D_2. \qquad (3)$$

This settles Theorem 4, if $t \leq \frac{3}{2}$, since $d \geq 2t$ and then $|C| \geq 4d = \frac{5}{2}d + \frac{3}{2}d \geq (t+1)d + 3t$. Let $t > \frac{3}{2}$ for the rest of Case 1.

Next we settle the subcase when $D_1 = 1$ or $D_2 = 1$, say $D_1 = 1$. In view of Lemma 4(i), we may assume $m = d_C(v_1) + 1$ so that $N_C(H) = N_C(v_1) \cup \{z\}$, where $z \in Z_2 - Z_1$. Then $Y_2 = \emptyset$ and $|Z_2| = |\hat{X}_2| \geq 2t - 1 > 2$. Hence, $|Z_2| \geq t + 1$, and we obtain $|Z| = |Z_2| - 1 \geq t$. Thus, the claim follows by Lemma 4 (iii).

Now let $D_1 \geq 2$ and $D_2 \geq 2$. For $h = 1, 2$ label $X = \hat{X}_1 \cup \hat{X}_2 \cup N_C(v_1) = \{z_1, \dots, z_p\}$ according to the given orientation on C. If $Y_h = V(B_h - c_h)$ for $h = 1$ or $h = 2$, say $Y_1 = V(B_1 - c_1)$, then

$$m \geq d_C(v_1) + 2(|B_1| - 2) + 1 \geq d_C(v_1) + (|B_1| - 1) + 1 \geq d(v_1) + 1,$$

and the result follows from Lemma 4 (i).

It remains to consider the subcase when $Y_h \neq V(B_h - c_h)$ for $h = 1, 2$. Then $X_h \cup Y_h \cup \{c_h\}$ is a cut set of G and, consequently, $l_0 + l_h = |\hat{X}_h| - 1 \geq \kappa - 2$ for $h = 1, 2$, where κ is the connectivity of G. It follows from (3) that

$$|C| \geq 4d + 2(l_0 + l_1 + l_2) \geq 4d + 2\kappa - 4.$$

If $\frac{3}{2} < t \leq 2$, then $\kappa \geq 4$ and $|C| \geq 4d + 4 \geq 3d + 8$. If $2 < t \leq \frac{5}{2}$, then $\kappa \geq 5$ and $|C| \geq 4d + 6 \geq \frac{7}{2}d + \frac{17}{2}$.

For the rest of Case 1, we assume that $t > \frac{5}{2}$ and $D_1 \leq D_2$. For $z_i \in Z$, let u_i denote the vertex on C such that $|C(z_i, u_i)| = \lfloor \frac{D_1 + D_2 + 2}{2} \rfloor$. If $z_i \in N_C(v_1) - X_1$, set $u_i = z_i^{++}$. For the remaining $z_i \in \hat{X}_h$, let $|C(z_i, u_i)| = \lfloor \frac{D_h + 2}{2} \rfloor$ for $h = 1, 2$.

Because C is a longest cycle, it follows that $C(z_i, u_i) \cap N_C(H) = \emptyset$. By Lemma 2(ii), no C-arc joins distinct segments of the form $C(z_i, u_i)$.

Setting $S = V(\bigcup_{i=1}^{p} C[u_i, z_{i+1}])$, we infer that $G - (S \cup \{c_1\})$ has at least $p+2$ components. Hence, $t(p + 2) \leq |S| + 1$ and, equivalently,

$$|C| \geq (t + 1)p + 2t - 1 + r', \tag{4}$$

where $r' = \sum_{z_i \in X} |C(z_i^+, u_i)|$. By construction,

$$r' \geq l_0 \lfloor \frac{D_1 + D_2}{2} \rfloor + l_1 \lfloor \frac{D_1}{2} \rfloor + (l_2 + 1) \lfloor \frac{D_2}{2} \rfloor \geq l_0 D_1 + (2\kappa - 3 - 2l_0) \frac{D_1 - 1}{2},$$

hence,

$$|C| \geq (t + 1)p + 2t - 1 + (2\kappa - 3) \frac{D_1 - 1}{2} + l_0. \tag{5}$$

Notice that $p \geq d_C(v_1) + |\hat{X}_2 - Z| \geq d_C(v_1) + 1$ and $2\kappa - 3 \geq \kappa + 3 \geq 2t + 3$. If $p \geq d_C(v_1) + 2$, then (5) implies $|C| \geq (t + 1)d(v_1) + 3t$. If $p = d_C(v_1) + 1$, then $|\hat{X}_2 - Z| = 1$. Hence, $l_0 = |\hat{X}_2| - 1 \geq \kappa - 2 \geq t + 1$, and again (5) yields the result. Thus, Case 1 is settled.

For the rest of the proof, we assume that $Z_1 \subseteq Z_2$ or $Z_2 \subseteq Z_1$, say $Z_2 \subseteq Z_1$. Then $\emptyset \neq Z = Z_2$ and $|Z| \geq 2t - 1$, since $Z_2 \cup \{c_2\}$ is a cut set.

Case 2. $m \geq |Z| + 2$.

If $|Z| = 1$, then $t \leq 1$ and Lemma 2(i) yields $|C| \geq 2m + 2D_2 \geq 2D_2 + 6 \geq 2d(v_2) + 4$. If $|Z| = 2$, then $t \leq \frac{3}{2}$. If $x_i, x_{i+1} \in Z$ for some i, then Lemma 2(i) yields $|C| \geq 2m + D + 2D_2 \geq 3D_2 + 9$. If there is no such i, then $|C| \geq 2m + 4D_2 \geq 3D_2 + 9$. Anyway, $|C| \geq 3D_2 + 9 \geq \frac{5}{2}(D_2 + 2) + \frac{9}{2}$.

Now let $|Z| \geq 3$. Consequently, $4 \geq 2t$ and $|Z| \geq t + 1$. If $D_1 \geq D_2 - 2$, then $l_0 \lfloor \frac{D_1 + D_2}{2} \rfloor \geq (t + 1)(D_2 - 1)$ and Lemma 4(i) yields our result, since $m \geq$

$d_C(v_2)+2$. If $D_1 \le D_2-2$, then $l_0\lfloor\frac{D_1+D_2}{2}\rfloor \ge (t+1)(D_1+1)$ and Lemma 4(i) yields our result.

Case 3. $m > |Z_1 \cup Z_2|$.
In view of Case 2, we may assume that $m = |Z|+1$, so that $Z_1 = Z_2 = Z$. If $|Z| \ge 3$, then $l_0\lfloor\frac{D_1+D_2}{2}\rfloor \ge (t+1)D_h$ for $h=1$ or $h=2$, while $m \ge d_C(v_h)+1$ for $h=1$ and $h=2$. The desired estimate follows from Lemma 4(i). Now let $|Z| \le 2$ for the rest of this case.

Case 3.1. $c_1 \ne c_2$.
We can pick $x_j \in N_C(H)-(Z_1\cup Z_2)$ and we may assume $N_H(x_j) \ne \{c_2\}$. Note that $3t-2 \le |Z|$, since $G-(Z\cup\{c_1,c_2\})$ has at least 3 components. If $|Z|=1$, then $t \le 1$ and $|C| \ge 2(D_2+3) \ge 2d(v_2)+4$. If $|Z|=2$, then $t \le \frac{4}{3}$ and $|C| \ge 2(D_2+3)+D+2 \ge 3D_2+10 \ge 3d(v_2)+4$. Thus, Case 3.1 is settled.

Case 3.2. $c_1 = c_2$.
In this subcase, $|Z| \ge 3t-1$, because $G-(Z\cup\{c_2\})$ has at least 3 components. If $|Z|=1$, then $t \le \frac{2}{3}$ and $|C| \ge 2D_2+4 \ge 2d(v_2)+2 \ge 2d+3t$. If $|Z|=2$, then $t \le 1$ and $|C| \ge D+2+2(D_2+2) \ge 2D_2+8 \ge 2d(v_2)+4$. This settles Case 3.

For the remainder of the proof, we assume that $m = |Z_1\cup Z_2|$ and $m \le |Z|+1$. Now Lemma 4(ii) and (iii) yield

$$|C| \ge (t+1)m + 2t - 1 + m\lfloor\frac{D}{2}\rfloor. \tag{6}$$

Case 4. $D \ge 4$.
For $h=1$ or $h=2$ we have $D \ge 2D_h$ and $D - D_h \ge 2$.

If $m \ge \frac{3}{2}(t+1)$, the claim follows by (6), since $m\lfloor\frac{D}{2}\rfloor = \frac{m}{3}\lfloor\frac{D}{2}\rfloor + \frac{2m}{3}D_h \ge (t+1)(1+D_h)$ for $h=1$ or $h=2$.

Now let $m < \frac{3}{2}(t+1)$. Then $6t-2 \le 2m < 3t+3$. Consequently, $3t<5$, which in turn implies $m \le 3$. If $m=2$, then $t \le 1$ and $|C| \ge 2(D+2) \ge 2D_h+8 \ge 2(D_h+2)+4$. If $m=3$, then $t \le \frac{4}{3}$ and $|C| \ge 3(D+2) \ge 3D_h+12 \ge \frac{7}{3}(D_h+3)+5$. This settles Case 4.

Case 5. $D \le 3$.
In this last case, we have $d \le m+1$, because $D_1 = 1$ or $D_2 = 1$.

Let M denote the set of all x_i^+ such that no C-arc joins x_i^+ to an element of $\{x_1^-,\ldots,x_m^-\}$. Symmetrically let $\hat{M}$ denote the set of all x_{i+1}^- such that no C-arc joins x_{i+1}^- to an element of $\{x_1^+,\ldots,x_m^+\}$.

Case 5.1. $|M| \ge 2$ or $|\hat{M}| \ge 2$.
In view of the symmetry, we deal only with the case that $|\hat{M}| \ge 2$. Let $S_0 = V(\bigcup C(x_i^+, x_{i+1}]) - \hat{M}$. By construction, $G-(S_0\cup\{c_1\})$ has at least $m+|\hat{M}|+2$ components. Therefore, $t(m+|\hat{M}|+2) \le |C|-(m+|\hat{M}|)+1$ and

$$|C| \ge (t+1)m + 2t - 1 + |\hat{M}|(t+1) \ge (t+1)(m+1)+3t.$$

Case 5.2. $|M| \le 1$ and $|\hat{M}| \le 1$.
For $h = 0,1,\ldots$, let X'_h denote the set of all $x_i \in N_C(H)$ such that $|C(x_i,x_{i+1})| = D+1+h$, and let $m_h = |X'_h|$. For $x_i \in N_C(H)$, let $u_i = x_i^{+++}$ and $S =$

$V(\bigcup C[u_i, x_{i+1}))$. By the assumptions before Case 4, we have $x_j \in N_C(B_1 - c_1)$ and $x_k \in N_C(B_2 - c_2)$, or vice versa for any distinct $x_j, x_k \in N_C(H)$. In particular, $|C(x_j, x_{j+1})| \geq D + 2$ for all $x_j \in N_C(H)$. If there exists a C-arc between u_j and u_k, Lemma 2(ii) with the inverse direction on C yields $|C(x_j, x_{j+1}) \cup C(x_k, x_{k+1})| \geq 6 + (D + 1)$.

Claim 1. If $D = 3$, then $|C| \geq (t + 1)m + 2t - 1 + m + m_0$.

Proof of Claim 1. Let $S_1 = S - \{u_i : x_i \in X_0'\}$. If there exists a C-arc $Q = Q[z_j, z_k]$ in $G - S_1$ between distinct segments $C(x_j, u_j]$ and $C(x_k, u_k]$, then Lemma 2(ii) yields $(z_j, z_k) = (u_j, u_k)$ and, by construction, $x_j, x_k \in X_0'$. This clearly is a contradiction, since $|C(x_j, x_{j+1}) \cup C(x_k, x_{k+1})| \geq 2D + 4$, as observed above. Therefore, $G - (S_1 \cup \{c_1\})$ has at least $m + 2$ components. Hence, $t(m + 2) \leq |S_1| + 1$ and, equivalently, Claim 1.

Claim 2. If $D = 2$, then $|C| \geq (t + 1)m + 2t - 1 + m + m_0 + m_1 - 1$.

Proof of Claim 2. Let $S_2 = S - \{u_i : x_i \in X_0' \cup X_1'\}$. If no C-arc in $G - S_2$ joins distinct segments $C(x_i, u_i]$, then $G - (S_2 \cup \{c_1\})$ has at least $m + 2$ components. Then $t(m+2) \leq |S_2| + 1$ and, equivalently, $|C| \geq (t+1)m + 2t - 1 + m + m_0 + m_1$.

Now consider a C-arc $Q = Q[z_j, z_k]$ in $G - S_2$, which links distinct segments $C(x_j, u_j]$ and $C(x_k, u_k]$. If $(z_j, z_k) = (u_j, u_k)$, then, by construction, $x_j, x_k \in X_0' \cup X_1'$, which contradicts $|C(x_j, x_{j+1})| + |C(x_k, x_{k+1})| \geq 2D + 5$. Hence, in fact, $Q = Q[u_j, u_k^-]$ or $Q = Q[u_j^-, u_k]$ by Lemma 2(ii), say $Q = Q[u_j, u_k^-]$. Then $x_j \in X_0' \cup X_1'$. Consequently, $|C(u_j, x_{j+1})| \leq 1$. It is not difficult to see that $x_k^+ \in M$, since otherwise we could construct a longer cycle. Therefore, $M = \{x_k^+\}$ by the hypothesis of Case 5.2. This shows that each such Q must have the end vertex x_k^{++}. Therefore, $G - (S_2 \cup \{c_1, x_k^{++}\})$ has at least $m + 2$ components. Hence, indeed $t(m + 2) \leq |S_2| + 2$ and, equivalently, Claim 2.

Using (6) and Claims 1 and 2, we can quickly settle Case 5.2. If $m \geq 2t + 2$, we are done by (6). Now let $m < 2t + 2$. Using $3t - 1 \leq m$, we deduce that $t < 3$ and $m < 8$.

If $D = 3$ and $m_0 \geq 2$, our result follows by Claim 1, since $m \geq 2t$. If $D = 3$ and $m_0 \leq 1$, then $|C| \geq 6m - 1$. In this subcase, the result follows, since $m + 1 \geq 3t$ and

$$6m - 1 \geq \frac{m + 4}{3}(m + 1) + m + 3$$

for $2 \leq m \leq 7$.

If $D = 2$ and $m_0 + m_1 \geq 3$, we are done by Claim 2. If $D = 2$ and $m_0 + m_1 \leq 2$, then $|C| \geq 6m - 4$. For $3 \leq m \leq 7$, the result follows since

$$6m - 4 \geq \frac{m + 4}{3}(m + 1) + m + 1.$$

If $m = 2 = D$ and $|C| \geq 10$, then $t \leq 1$ and $|C| \geq 2(m + 1) + 4$. Finally, let $m = 2 = D$ and $|C| \leq 9$. As shown above, no C-arc joins the segments $C(x_1, x_2)$ and $C(x_2, x_1)$. Therefore, $G - \{x_1, x_2\}$ has at least 3 components; hence, $t \leq \frac{2}{3}$. Now $|C| \geq 2(D + 2) \geq 2(D_2 + 3) \geq 2d(v_2) + 2$. The proof of Theorem 4 is complete. $\square$

4. NONSEPARABLE COMPONENTS

For the proof of the main result, we still have to investigate nonseparable components of longest cycles. For an induced subgraph H of G, we set

$$\begin{aligned} X(H) = &\ \{x \in N_{G-H}(H) : |N_H(x, x')| \geq 2 \text{ for each } x' \in N_{G-H}(H) - \{x\}\} \\ &\text{and} \\ Y(H) = &\ \{y \in V(H) : N_{G-H-X(H)}(y) \neq \emptyset\}. \end{aligned}$$

Notice that, in fact, $|N_{G-H-X(H)}(y)| \geq 2$ for all $y \in Y(H)$. If $V(H) = Y(H)$, set $\mu_H = \max_{v \in V(H)} d(v)$. Otherwise, let $\mu_H = |X(H) \cup Y(H)|$.

If $N_{G-H}(H) \neq V(G-H)$ and $V(H) \neq Y(H)$, then $X(H) \cup Y(H)$ is a cut set of G. Anyway, $\mu_H \geq \kappa_G \geq 2t$, where κ_G denotes the connectivity of G.

Lemma 5. *Let C be a maximal cycle in a graph G, and let H be a nonseparable component of $G - V(C)$. If $Y(H) = V(H)$, then $|C| \geq (t+1)(\mu_H + |H| - 1) + t$.*

Proof. Let $X = X(H)$ and $Y = Y(H)$. By the definition of X and Y, we obtain for each $v \in V(H)$ that

$$|N_C(H)| \geq |X| + d_{C-X}(v) + 2|Y| - 2 \geq d(v) + |Y| - 1.$$

Therefore, $|N_C(H)| \geq \mu_H + |H| - 1$ and the result follows from Lemma 1. □

Lemma 6. *Let C be a longest cycle in a 2-connected graph G, and let H be a component of $G - V(C)$ such that $|H| \leq 2$. Then $|C| \geq (t+1)\delta + t$.*

Proof. If $|H| = 1$, we obtain our result from Lemma 1. Now let $V(H) = \{v, w\}$. We fix a cyclic orientation on C and label $N_C(H) = \{x_1, \ldots, x_m\}$ accordingly. By Lemma 1, we have $|C| \geq (t+1)m + t$ and, thus, we are done, if $d_C(v) < m$ or $d_C(w) < m$. For the rest of this proof, let $N_C(v) = N_C(w)$. Then $|C(x_i, x_{i+1})| \geq 2$ for $1 \leq i \leq m$. It suffices to show that

$$|C| \geq (t+1)(m+1) + t. \tag{7}$$

We consider the cut set $S = V(C) - \{x_1^+, \ldots, x_m^+\}$. Clearly, there is no C-arc $Q = Q[x_j^+, x_k^+]$ or $Q = Q[x_j^{++}, x_k^+]$ for $1 \leq j < k \leq m$.

If there is no C-arc $Q = Q[x_j^{++}, x_k^{++}]$, then $G - (S - \{x_1^{++}, \ldots, x_m^{++}\})$ has at least $m+1$ components. In this event, $t(m+1) \leq |C| - 2m$. This yields (7), since $m = \frac{m}{2} + \frac{m}{2} \geq 1 + t$.

Now let us consider a C-arc $Q = Q[x_j^{++}, x_k^{++}]$. This Q gives rise to a cycle C' with vertex set $V(Q) \cup \{v, w\} \cup V(C) - \{x_j^+, x_k^+\}$. Hence, $|Q| = 2$ and C' is a longest cycle.

Let H_j denote the component of $G - C[x_j^{++}, x_j]$ that contains x_j^+. As H_j is a component of $G - C'$, we are done by Lemma 1 if $|H_j| = 1$. Hence, we may assume $|H_j| \geq 2$ and, therefore, $N_{G-C}(x_j^+) \neq \emptyset$.

Abbreviate $w_j = x_j^{+++}$ and let H_j' denote the component of $G - C[w_j, x_j]$ that contains H_j. Since C is a longest cycle, we derive that x_j^+ is a cut vertex of H_j'.

If $w_j \in N_C(H)$ or there exists a C-arc $Q' = Q'[x_i^+, w_j]$ for some $x_i \in N_C(H) - \{x_j\}$, then there exists a cycle C'' with vertex set $V(C) \cup \{v, w\} - \{x_j^+, x_j^{++}\}$, and H_j' is a component of $G - C''$. Since C'' must be a longest cycle, we are done by Theorem 4.

In the remaining case, when $|C(x_j, x_{j+1})| \geq 3$ and there is no C-arc $Q' = Q'[x_i^+, w_j]$ with $x_i \in N_C(H) - \{x_j\}$, we consider the component H_j'' of $G - C(w_j, x_j]$ that contains H_j'. It follows that x_j^+ is a cut vertex also of H_j''. For otherwise there exists a C-arc $Q' = Q'[w_j, x_j^+]$. But Q' and Q give rise to a cycle C'' with vertex set $V(Q') \cup \{v, w\} \cup V(C) - \{x_k^+\}$, which would be longer than C. Therefore, in fact, $G - (S \cup \{x_j^+\} - \{x_j^{++}, w_j\})$ has at least $m + 2$ components. Thus, $t(m+2) \leq (|S| - 2) + 1 = |C| - m - 1$. Equivalently, (7). This completes the proof of Lemma 6. □

The next lemma indicates that a 2-connected graph H either has many vertices v such that $d_H(v) \leq D(H)$ or contains some vertex v such that $d_H(v) \leq \frac{D(H)+2}{2}$. It is used to settle the cases addressed in Lemma 8.

Lemma 7. *Let H be a a 2-connected graph and let $Y' = \{v \in V(H) : D(H) \geq d_H(v)\}$. If $d_H(v) \geq 2|Y'| - 1$ for each $v \in V(H)$, then $|Y'| \geq 10$.*

Proof. Determine $a, b \in V(H)$ such that $D = D(H) = D_H(a, b)$, and let P be a longest (a, b)- path in H. Let r denote the number of components of $H - P$. Then $r > 0$, since otherwise $|Y'| = |H| = D + 1 \geq d_H(v) + 1 \geq 2|Y'|$, which is absurd.

Claim 1. $|Y' - V(P)| \geq 2r$.

Proof of Claim 1. Choose a component L_0 of $H - P$ and label $N_{P[a,b)}(L_0) = \{x_1, \ldots, x_m\}$ in order from a to b. If x_i^+ has a neighbor outside P, pick a component L_i of $H - P$ such that $x_i^+ \in N_P(L_i)$. Because P is longest, we have $L_j \neq L_k$ for distinct x_i^+, x_k^+. Applying Lemma 3, we obtain

$$|Y'| \geq m + \min(|L_0|, 2) \geq 2.$$

If $V(L_0) = \{w_0\}$, then $w_0 \in Y'$ and $|Y'| \geq m + 1 \geq d_H(w_0) \geq 2|Y'| - 1$, contrary to $|Y'| \geq 2$. Hence, in fact, each component of $G - P$ has at least two vertices and Lemma 3 yields Claim 1.

Claim 2. $|Y' \cap V(P)| \geq \frac{D - r(r-1)}{2}$.

Proof of Claim 2. Let $L_1, \dots, L_r$ be the components of $H - P$. We "color" the edge vw of P by the pair (i, j), if $v \in N_P(L_i)$ and $w \in N_P(L_j)$. Since P is a longest (a, b)-path, we have $i \neq j$, and each color occurs at most once on P. Therefore, at least $D - r(r-1)$ edges are uncolored. Each of those edges has at least one end vertex in $V(P) - N_P(H - P)$ and, consequently, in Y'. Hence, Claim 2.

For $v \in Y'$, we have $D \geq d_H(v) \geq 2|Y'| - 1$. We infer by the above claims that

$$2|Y'| \geq 4r + D - r(r-1) \geq 4r + 2|Y'| - 1 - r(r-1).$$

Hence, $4r \leq r(r-1) + 1$ and, therefore, $r \geq 5$. Claim 1 yields $|Y'| \geq 10$. □

Lemma 8. *Let C be a maximal cycle in a 2-connected graph G such that $c(G) - |C| \leq 2$, and let H be a nonseparable component of $G - V(C)$. Further, let $Y' = \{v \in V(H) : D(H) \geq d_H(v)\}$. If $Y' \subseteq Y(H)$ or $|C| \geq (t+1)(\mu_H + D(H)) + t$, then $|C| \geq (t+1)d(v) + t$ for some $v \in V(H)$.*

Proof. We abbreviate $Y = Y(H)$, $X = X(H)$, and $D = D(H)$. In view of Lemma 5, we may assume that $V(H) \neq Y$, so that $X \cup Y$ is a cut set. For $v \in Y' - Y$, we have $d_H(v) \leq D$ and $d_C(v) \leq |X|$. Consequently,

$$|C| \geq (t+1)(\mu_H + D(H)) + t \geq (t+1)d(v) + t.$$

Therefore, it remains to consider the case when $Y' \subseteq Y$.

If $d_H(v) \leq 2|Y| - 2$ for some $v \in V(H)$, then $|N_C(H)| \geq d_C(v) + 2|Y| - 2 \geq d_C(v) + d_H(v) \geq d(v)$, and the result follows from Lemma 1.

Now assume that $d_H(v) \geq 2|Y| - 1$ for all $v \in V(H)$. Then $|H| \geq 3$, since otherwise for any $v \in V(H)$ we would have $|Y| \geq |Y'| = |H| \geq d_H(v) + 1 \geq 2|Y|$, which is absurd. Therefore, H is 2-connected, and we can apply Lemma 7 to obtain $|Y'| \geq 10$.

Fix a cyclic orientation on C. For each $x \in N_C(H)$, let $x^\star$ denote the first vertex on $C(x, x]$ in $N_C(H)$. For each $y \in Y(H)$, we determine a vertex $\hat{y} \in N_{C-X}(y)$ such that $|N_H(\hat{y}, \hat{y}^\star)| \geq 2$. Let $\hat{Y} = \{\hat{y} : y \in Y\}$, and abbreviate $\hat{X} = X \cup \hat{Y}$. By construction, we obtain that $|N_H(x, x')| \geq 2$ for any distinct $x, x' \in \hat{X}$.

Picking some $y_0 \in Y'$, we label $X \cup (\hat{Y} - \{\hat{y}_0\}) \cup N_{C-X}(y_0) = \{x_1, \dots, x_m\}$ according to the fixed orientation. For $x_i \in X \cup \hat{Y} - \{\hat{y}_0\}$ let u_i, denote the vertex on C such that $|C(x_i, u_i)| = \lfloor \frac{D+2-h}{2} \rfloor$, where $h = c(G) - |C|$. For the remaining x_i, set $u_i = x_i^{++}$. By Lemma 2, no C-arc joins distinct segments of the form $C(x_i, u_i)$. Therefore, $S = V(\cup_{i=1}^m C[u_i, x_{i+1}])$ is a cut set of G, and $G - S$ has at least $|X \cup Y| - 1 + |N_{C-X}(y_0)| + 1$ components. Hence, $t(m+1) \leq |S| = |C| - m - (|X \cup Y| - 1)\lfloor \frac{D-h}{2} \rfloor$ and, equivalently,

$$|C| \geq (t+1)m + t + (|X \cup Y)| - 1)\lfloor \frac{D-h}{2} \rfloor. \tag{8}$$

If $|X \cup Y| \geq 2t+3$, then (8) yields

$$|C| \geq (t+1)d_C(y_0)+t+(t+1)(D-1-h+|Y|-1) \geq (t+1)(d(y_0)+6)+t.$$

In the remaining case, when $|X \cup Y| < 2t+3$, let us first assume that $D < 2d_H(v)-2$ for all $v \in V(H)$. Then the graph H has a cut set $T = \{a, b\}$ by Lemma 3. Let L be a component of $H - T$ such that $|V(L) \cap Y|$ is minimal. If $V(L) \subseteq Y$, we pick $v \in V(L)$ and obtain $2|Y|-1 \leq d_H(v) < |Y|-1+|T| = |Y|+1$, contrary to $|Y| \geq |Y'| \geq 10$. If $V(L) - Y \neq \emptyset$, then

$$2t \leq |X|+|T|+|Y \cap V(L)| \leq |X|+2+\frac{|Y|}{2} < 2t+5-\frac{|Y|}{2} \leq 2t,$$

which is absurd.

Hence, in fact, $D \geq 2d_H(v)-2$ for some $v \in V(H)$. As $v \in Y'$, we can apply (8). From $|X \cup Y| < 2t+3$, we derive $t > \frac{7}{2}$ and, in particular, $|X \cup Y|-1 \geq 2t-1 > t+2$. Thus, we obtain

$$|C| \geq (t+1)d_C(v)+t+(t+1)(|Y|-1+d_H(v)-\frac{h+3}{2}) > (t+1)(d(v)+6)+t.$$

□

Lemma 9. *Let C be a longest cycle in a 2-connected graph G, and let H be a 2-connected component of $G - C$ such that $|C| < (t+1)(\mu_H + D(H)) + t$. Then $D < 2t$ and*

$$|C| \geq \mu_H(D(H)+2) + min(D(H)+2, t+1). \tag{9}$$

Proof. We fix a cyclic orientation on C and abbreviate $X = X(H), Y = Y(H)$, and $D = D(H)$. Note that $D \geq 2$, since H is 2-connected.

If $D \geq 2t$ and (9), we obtain a contradiction to the hypothesis of Lemma 9, since

$$\begin{aligned}\mu_H(D+2) &= \frac{\mu_H+2}{2}D + D\frac{\mu_H-2}{2} + 2\mu_H \\ &\geq (t+1)D + t(\mu_H-2) + 2\mu_H \\ &\geq (t+1)(D+\mu_H).\end{aligned}$$

Therefore, it suffices to show (9).

For $x \in N_C(H)$, let $x^\star$ denote the first vertex on $C(x,x]$ such that $x^\star \in N_C(H)$. Let $\hat{X}$ denote the set of all $x \in N_C(H)$ such that $|N_H(x, x^\star)| \geq 2$ and label $\hat{X} = \{x_1, \ldots, x_m\}$, according to the given orientation. Then $|C(x_i, x_i^\star)| \geq D+1$ by Lemma 2(i) and, therefore,

$$|C| = m(D+2) + R_1, \quad \text{where } R_1 \geq 2|N_{C-\hat{X}}(H)|. \tag{10}$$

If $m > \mu_H$, then (10) immediately yields (9). For the rest of the proof, let $m = \mu_H$. Then, for each $y \in Y$, there exists a unique vertex $\hat{y} \in N_{C-X}(y) \cap \hat{X}$ and, therefore, $|N_H(x_j, x_k)| \geq 2$ for any distinct $x_j, x_k \in \hat{X}$.

Consider a C-arc $Q = Q[z_j, z_k]$ between distinct segments $C(x_j, x_{j+1})$ and $C(x_k, x_{k+1})$ such that $Q \cap V(H) = \emptyset$. By Lemma 2(ii), we have

$$|C(x_j, z_j)| + |C(x_k, z_k)| \geq D + 1. \tag{11}$$

Since $|N_H(x_{j+1}, x_{k+1})| \geq 2$, we are allowed to use the same argument with the orientation of C reversed. Therefore, $|C(z_j, x_{j+1})| + |C(z_k, x_{k+1})| \geq D + 1$ and, by (11),

$$|C(x_j, x_{j+1})| + |C(x_k, x_{k+1})| \geq 2D + 4. \tag{12}$$

If $R_1 \leq 1$, then clearly $N_C(H) = \hat{X}$ and, by (12), there is no C-arc Q between distinct segments of the form $C(x_i, x_{i+1})$. In this case, $t \leq \frac{m}{m+1}$ and (10) yields a contradiction, since

$$m(D+2) \geq 2m+2D = \frac{2m+1}{m+1}(m+D) + \frac{1}{m+1}(m+D) > (t+1)(\mu_H + D) + 1.$$

For the rest of the proof, let us assume $2 \leq R_1 < t + 1$. For $h = 0, 1$, set

$$X_h = \{x_j \in \hat{X} : |C(x_j, x_{j+1})| = D + 1 + h\}, \text{ and let } X_2 = \hat{X} - (X_0 \cup X_1).$$

For $x_i \in X_0$, set $w_i = x_{i+1}$. If $X_1 \neq \emptyset$ pick $x_s \in X_1$, set $\epsilon = 1$ and $w_s = x_{s+1}^{-}$. If $X_1 = \emptyset$, set $\epsilon = 0$. Further, let $w_i = x_i^{++}$ for $x_i \in \hat{X} - (X_0 \cup \{x_s\})$.

It readily follows from (11) and (12) that there is no C-arc Q between distinct segments of the form $C(x_i, w_i)$. Therefore, $G - V(\cup_{i=1}^{m} C[w_i, x_{i+1}])$ has at least $m + 1$ components and, consequently, $t(m + 1) \leq |C| - |V(\cup_{i=1}^{m} C(x_i, w_i))|$. Hence,

$$|C| \geq (t + 1)m + t + (|X_0| + \epsilon)D. \tag{13}$$

As $|X_1| + 2|X_2| \leq R_1 < t + 1 \leq \frac{m+2}{2}$. we have $|X_1| + 2|X_2| \leq \frac{m+1}{2}$. By (13) and the hypothesis of Lemma 9 also $|X_0| + \epsilon < t + 1$, and, hence, $|X_0| + \epsilon \leq \frac{m+1}{2}$. Now $|X_0| + |X_1| + 2|X_2| + \epsilon \leq m + 1$ and $|X_2| + \epsilon \leq 1$. On the other hand, $|X_2| + \epsilon \geq 1$, since $R_1 > 0$. Consequently, $|X_2| + \epsilon = 1$ and

$$|X_0| + \epsilon = |X_1| + 2|X_2| = \frac{m+1}{2}. \tag{14}$$

If $X_2 = \emptyset$ then $Y = \emptyset$, and $G - \hat{X} = G - X$ has at least $|X_0| + 2$ components. In this event,

$$1 < t \leq \frac{m}{|X_0| + 2} = \frac{2|X_0| + 1}{|X_0| + 2} < 2,$$

but $|X_0| + \epsilon = |X_0| + 1 < t + 1$ and, therefore, $|X_0| = 1$, a contradiction.

Hence, in fact, $|X_2| = 1$ and $\epsilon = 0$. Consequently, $X_1 = \emptyset$ and, by (14), $m = 3 \geq 2t$. Since $D \geq 2$, we obtain from (10)

$$|C| = \frac{5}{2}(D + 3) - \frac{5}{2} + \frac{1}{2}(D + 2) + R_1 \geq (t + 1)(m + D) + \frac{3}{2}.$$

This contradiction completes the proof of Lemma 9. □

For the next two lemmas, we need some technical notations. Let C be a cycle in a graph G, and let H_1, H_2 be nonseparable components of $G - C$. We say that (C, H_1) is of type (h, A), if $|C| = c(G) - h$ and $D(H_1) \le A$. If $H_1 \neq H_2$, we say that (C, H_1, H_2) is of type (h, A, B), if $|C| \ge c(G) - h$ and $D(H_1) \le A$, and $D(H_2) \le B$. We say that C is of type (h, A) resp. of type (h, A, B), if there exist pairs or triples of that type.

Now consider a component H of $G - C$. Let $x, x', z, z' \in V(C)$ be distinct vertices on C such that $z \in C(x, x')$ and $z' \in C(x', x)$. Further, let $Q = Q[z, z']$ be a C-arc such that $Q \cap V(H) = \emptyset$, and let $P = P[x, x']$ be a longest C-arc with all its inner vertices in H. Note that $P \cap Q = \emptyset$. We call the cycle $C' = C - ((C(x, z) \cup C(x', z')) \cup Q \cup P$ a Q-reduction.

The next two lemmas refer to a longest cycle C in the 2-connected graph G and a 2-connected component H of $G - C$. Their proofs have the same technical approach, and thus are merged.

Lemma 10. *Let G allow no cycles of type* $(0, \frac{D(H)+1}{2})$*, then*

$$|C| > (t + 1)(\mu_H + D(H)) - t. \tag{15}$$

If, moreover, G allows no cycles of type $(1, \frac{D(H)}{2}, \frac{D(H)+1}{2})$*, then*

$$|C| > (t + 1)(\mu_H + D(H)). \tag{16}$$

Lemma 11. *Let G allow neither cycles of type* $(0, \frac{D(H)+1}{2})$ *nor cycles of type* $(1, \frac{D(H)}{2}, \frac{D(H)+1}{2})$*. Then*

$$|C| \ge (t + 1)(\mu_H + D(H)) + t \tag{17}$$

or $D(H)$ is odd, $D(H) \ge 5$ and there exists a triple (C_1, H_1, H_2) of type $(2, \frac{D(H)+1}{2}, \frac{D(H)+1}{2})$ such that H_1 and H_2 are 3-connected.

Proof of Lemma 10 and Lemma 11. We abbreviate $D = D(H)$ and fix a cyclic orientation of C. If $Y(H) = V(H)$ or $D \ge 2t$, then (17) follows from Lemma 5 and Lemma 9. Assuming $Y(H) \neq V(H)$ and $D < 2t$, we obtain that $\mu_H \ge 2t > D \ge 2$.

For each $x \in N_C(H)$, let $x^\star$ denote the first vertex on $C(x, x]$ such that $x^\star \in N_C(H)$. For $w \in C(x, x^\star]$, let $H(w)$ denote the component of $G - C[w, x]$ that includes $C[x^+, w^-]$. For each $y \in Y(H)$, we determine a vertex $\hat{y} \in N_{C-X(H)}(y)$ such that $|N_H(\hat{y}, \hat{y}^\star)| \ge 2$. Clearly, $\hat{y_1} \neq \hat{y_2}$ for distinct $y_1, y_2 \in Y(H)$. Set $\hat{X} = X(H) \cup \{\hat{y} : y \in Y(H)\}$ and label $\hat{X} = \{x_1, \ldots, x_m\}$ according to the given direction. Then $m = \mu_H$ and $|N_H(x_j, x_k)| \ge 2$ for any distinct $x_j, x_k \in \hat{X}$.

For $i = 1, \dots, m$, let w_i denote the vertex on C such that $|C(x_i, w_i)| = \lfloor \frac{D+2}{2} \rfloor$. Note that $|C(x_i, x_i^\star)| \geq D+1$ by Lemma 2(i) and, therefore, $C(x_i, w_i] \subseteq C(x_i, x_i^\star)$ for any x_i.

Now set $X_1 = \{x_i \in \hat{X} : H(w_i) \text{ is separable}\}$, and let X_2 denote the set of all vertices $x_i \in \hat{X} - X_1$ such that $H(w_i^+)$ has a cut set of at most two vertices. First, we use the cut set $S = V(\cup_{i=1}^m C(w_i, x_{i+1}]) \cup \{w_i : x_i \in X_1\}$.

Case 1. Some C-arc Q in $G - S$ joins distinct segments of the form $C(x_i, w_i]$. Let $Q = Q[z_j, z_k]$, where z_i is on $C(x_i, w_i]$ $(i = j, k)$. As already noted, $C(x_i, w_i] \subseteq C(x_i, x_i^\star)$ for all x_i. Therefore, Q gives rise to a Q-reduction C' such that

$$|C| - |C'| \leq |C(x_j, z_j)| + |C(x_k, z_k)| - (D + 1).$$

In particular, $z_j = w_j$ or $z_k = w_k$, say $z_j = w_j$. Hence, $H(w_j)$ and $H(z_k)$ are distinct components of $G - C'$. By construction, $x_j \in \hat{X} - X_1$ and, therefore, $H(w_j)$ is not separable. Furthermore, $D(H(w_j)) \leq |C(x_j, w_j)| - 1$, since C is a longest cycle. Consequently, $|C'| < |C|$, since otherwise $(C', H(w_j))$ would be of type $(0, \frac{D}{2})$.

From $|C'| < |C|$, we deduce that D must be even and also $z_k = w_k$. Hence, in fact, $(C', H(w_j), H(w_k))$ is of type $(1, \frac{D}{2}, \frac{D}{2})$. Therefore, it remains to prove (15).

We also have shown that $G - V(\cup_{i=1}^m C[w_i, x_{i+1}])$ has the distinct components $H, H(w_1), \dots, H(w_m)$. Thus, $t(m+1) \leq |C| - m\frac{D+2}{2}$ and, equivalently, $|C| \geq (t+1)m + t + m\frac{D}{2}$. This inequality yields (15), since $m\frac{D}{2} \geq tD$ and $D < 2t$. This settles Case 1.

For the rest of this proof, we assume that there exists no C-arc as in Case 1, and derive that $G - S$ has the distinct components $H, H(w_i)$ $(x_i \in X_1)$ and $H(w_i^+)$ $(x_i \in \hat{X} - X_1)$. Therefore,

$$t(m + 1 + |X_1 \cup X_2|) \leq |S| + |X_1| + 2|X_2| = |C| - \lfloor m\frac{D+4}{2} \rfloor + 2|X_1 \cup X_2|$$

and, equivalently,

$$|C| \geq (t+1)m + t + m\lfloor \frac{D+2}{2} \rfloor + (t-2)|X_1 \cup X_2|. \tag{18}$$

Case 2. D is even.
If $t \geq 2$, then (18) yields (17), since $m\frac{D}{2} \geq tD$ and $m \geq 2t > D$. If $\frac{3}{2} < t < 2$, again (18) implies (17), since, in this event, $D = 2$ and $2m + (t-2)|X_1 \cup X_2| > 2m - \frac{m}{2} \geq 6 \geq 2(t+1)$. If $t \leq \frac{3}{2}$, Lemma 9 yields (17), since $m \geq 2t > D = 2$ and $4m + 3 \geq \frac{5}{2}(m+3)$. Hence, Case 2 is settled.

Case 3. D is odd and $t \leq \frac{7}{3}$.
Then $D = 3$ and $m \geq 2t > 3$. If $t \leq 2$, then

$$m(D+2) + 3 = 5m + 3 = 3(m+3) + 2m - 6 \geq (t+1)(m+D) + 2.$$

If $2 < t \leq \frac{7}{3}$, then $m \geq 5$ and

$$m(D+2)+4 = 5m+4 \geq \frac{10}{3}(m+3)+\frac{7}{3} \geq (t+1)(m+D)+t.$$

In either subcase, Lemma 9 yields (17).

Case 4. $|X_1 \cup X_2| \geq 4$.

In view of the preceding cases, we may assume that D is odd and $t > \frac{7}{3}$. If $D \geq 5$, then $m \geq 6$ and $\frac{m}{2} + 4(t-2) > 3 + 2D - 8 \geq D$. If $D = 3$, then $\frac{m}{2} + 4(t-2) \geq 2 + \frac{4}{3} > D$. In either subcase, (18) yields (17).

For the rest of this proof, we assume that D is odd, $t > \frac{7}{3}$, and $|X_1 \cup X_2| \leq 3$. Then $m \geq 5 > |X_1 \cup X_2|$, and (18) implies (16). Then the proof of Lemma 10 is complete, and we assume, in addition, that there exist no cycles of type $(1, \frac{D-1}{2}, \frac{D+1}{2})$.

Using the latter assumption, we pick $x_p \in \hat{X} - (X_1 \cup X_2)$ and consider the cut set $S_1 = S - \{w_p^+\}$. Note that w_p^+ is on $C(x_p, x_p^\star)$, since $D \geq 3$. By the same arguments as above, we obtain that $G - S_1$ has at least $m+1$ components and deduce

$$|C| \geq (t+1)m + t + m\frac{D+1}{2} + 1 + (t-2)|X_1 \cup X_2|. \tag{19}$$

If $D = 3$, then $m\frac{D+1}{2} + 1 = 2m + 1 = \frac{3}{2}m + \frac{m}{2} + 1 > 3t + 3 = (t+1)D$ and, therefore, (19) yields (17). For the rest of this proof, let also $D \geq 5$, so that $t > \frac{5}{2}$ and $S_2 = S - \{w_i^+ : x_i \in \hat{X} - (X_1 \cup X_2)\}$.

Case 5. No C-arc in $G - S_2$ joins distinct segments of the form $C(x_i, w_i^+]$.

We derive that

$$t(m + 1 + |X_1 \cup X_2|) \leq |S_2| + |X_1| + 2|X_2|$$

and, equivalently,

$$|C| \geq (t+1)m + t + m\frac{D+3}{2} + (t-3)|X_1 \cup X_2|. \tag{20}$$

Now $\frac{3}{2}m + (t-3)|X_1 \cup X_2| \geq m > D$, hence (17).

Case 6. Some C-arc Q in $G - S_2$ joins the distinct segments $C(x_j, w_j^+]$ and $C(x_k, w_k^+]$.

Let C' be a Q-reduction. If $x_i \in \hat{X} - (X_1 \cup X_2)$, then $H(w_i^+)$ is 3-connected and $D(H(w_i^+)) \leq |C(x_i, w_i)| = \frac{D+1}{2}$.

Q cannot have an end vertex in $C(x_j, w_j)$, since otherwise $|C'| = |C|$ and $Q = Q[w_j^-, w_k^+]$. In this event $x_k \notin X_1 \cup X_2$ and, therefore, $(C', H(w_k^+))$ would be of type $(0, \frac{D+1}{2})$. Similarly, a contradiction is obtained if Q was of the form $Q[w_j, w_k]$. If $Q = Q[w_j, w_k^+]$, then $x_j \notin X_1$ and $x_k \notin X_1 \cup X_2$. In this event, $(C', H(w_j), H(w_k^+))$ is of type $(1, \frac{D-1}{2}, \frac{D+1}{2})$, contrary to the hypothesis of Lemma 11.

We have shown that necessarily $Q = Q[w_j^+, w_k^+]$ and, therefore, $x_j, x_k \in \hat{X} - (X_1 \cup X_2)$. Then $H(w_j^+)$ and $H(w_k^+)$ are distinct 3-connected components of

$G - C'$. Since $|C| - |C'| \leq 2$, we obtain indeed that $(C', H(w_j^+), H(w_k^+))$ is of type $(2, \frac{D+1}{2}, \frac{D+1}{2})$, and the proof of Lemma 11 is complete. $\square$

Proof of Theorem 1. We choose a longest cycle C_0 and a nonseparable component H_0 of $G - C_0$ such that $D_0 = D(H_0)$ is minimal. If $D_0 \leq 1$, we are done by Lemma 6. So, let us assume $D_0 \geq 2$. Then H_0 is 2-connected, and there is no cycle of type $(0, \frac{D_0+1}{2})$, since $\frac{D_0+1}{2} < D_0$. If $V(H_0) = Y(H_0)$ or $c(G) \geq (t+1)(\mu_{H_0} + D_0) + t$, we obtain our result by Lemma 5 and Lemma 8. For the rest of the proof, let us assume that $V(H_0) \neq Y(H_0)$ and $c(G) < (t+1)(\mu_{H_0} + D_0) + t$. Then $2 \leq D_0 < 2t$ by Lemma 9. Recall that $\mu_{H_0} \geq 2t$. We distinguish two disjoint cases, namely

(a) there exists a triple of type $(1, \frac{D_0+1}{2}, \frac{D_0+1}{2})$
(b) there exists no such triple.

If (a), we determine a triple (C, H_1, H_2) of type $(1, \frac{D_0+1}{2}, \frac{D_0+1}{2})$ with minimum $D(H_1)$. Then $|C| = c(G) - 1$ and $D(H_1) \leq \frac{D_0+1}{2} < D_0$. Furthermore, by Lemma 10,

$$c(G) > (t+1)(\mu_{H_0} + D_0 - 1). \tag{21}$$

If (b), we apply Lemma 11 to obtain a triple (C, H_1, H_2) of type $(2, \frac{D_0+1}{2}, \frac{D_0+1}{2})$ such that $D(H_1) \leq D(H_2)$ and H_1 and H_2 are 3-connected. In particular, $D(H_i) \geq 3$ for $i = 1, 2$. We also know from Lemma 11 and Lemma 10 that D_0 is odd, $D_0 \geq 5$, and

$$c(G) > (t+1)(\mu_{H_0} + D_0). \tag{22}$$

Furthermore, $|C| = c(G) - 2$, by construction. Let $D_1 = D(H_1)$.
Claim 1. C is a maximal cycle.

Proof of Claim 1. Suppose, to the contrary, that there exists a cycle C' in G such that $V(C) \subseteq V(C')$ and $|C'| > |C|$. If C' is a longest cycle, then neither H_1 nor H_2 can be a component of $G - C'$. because there is no cycle of type $(0, \frac{D_0+1}{2})$. Hence, $V(C') = V(C) \cup \{v_1, v_2\}$, where $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$, and we are in case (b). Therefore, $H_1 - v_1$ and $H_2 - v_2$ are 2-connected components of $G - C'$. Since $D(H_1 - v_1) \leq D_1 \leq \frac{D_0+1}{2}$, we obtain that $(C', H_1 - v_1)$ is of type $(0, \frac{D_0+1}{2})$, which is a contradiction. If $c(G) > |C'| = |C| + 1$, we are again in case (b). We may assume that C' contains a vertex v_1 of H_1, and obtain that $(C', H_1 - v_1, H_2)$ is of type $(1, \frac{D_0+1}{2}, \frac{D_0+1}{2})$. But then we would be in case (a), which is a contradiction. Hence, Claim 1.

Now fix a cyclic orientation on C, and let $X = X(H_1)$ and $Y = Y(H_1)$. For each $x \in N_C(H_1)$, let $x^\star$ denote the first vertex on $C(x, x]$ such that $x^\star \in N_C(H_1)$. If $w \in C(x, x^\star]$, let $H(w)$ denote the component of $G - C[w, x]$, which includes $C[x^+, w^-]$. For each $y \in Y$, we determine a vertex $\hat{y} \in N_{C-X}(y)$ such that $|N_{H_1}(\hat{y}, \hat{y}^\star)| \geq 2$. Let $\hat{Y} = \{\hat{y} : y \in Y\}$, and label $\hat{X} = X \cup \hat{Y} = \{x_1, \ldots, x_m\}$ according to the fixed orientation. By construction, we have $|N_{H_1}(x, x')| \geq 2$ for each pair of distinct vertices in $\hat{X}$.

If $Y = V(H_1)$, we obtain our result by Claim 1, Lemma 5, and Lemma 8. For the rest of this proof, let $Y \neq V(H_1)$, which yields $D_1 \geq 1$ and $m \geq 2t$. In several subcases, we will show that

$$c(G) \geq (t+1)(m+D_1)+t. \tag{23}$$

Note that (23) yields the desired result, since either Lemma 8 applies, or there exists a vertex $v \in V(H_1) - Y$ such that $d_{H_1}(v) \leq D_1$ and $d_C(v) \leq |X| \leq m$.

Case 1. $D_1 = 1$.
Then we are in case (a), therefore $c(G) = |C| + 1$ and $m \geq 2t > D_0 \geq 2$. By Claim 1, there is no C-arc $Q = Q[x_j^+, x_k^+]$ for distinct $x_j, x_k \in \hat{X}$. Let $S = V(C) - N_C^+(H_1)$.

If $N_C(H_2) \cap N_C^+(H_1) = \emptyset$, then $G - S$ has at least $m+2$ components. Hence, $t(m+2) \leq |C| - m$. Consequently, (23).

Now let $N_C(H_2) \cap N_C^+(H_1) \neq \emptyset$. It follows by Claim 1 that $N_C(H_2) \cap N_C^+(H_1) = \{x_s^+\}$. Then $|C(x_i, x_i^\star)| \geq 2$ for all $x_i \neq x_s$, since otherwise we could find a longest cycle C' such that H_2 is a component of $G - C'$, which contradicts $D(H_2) < D_0$. If there is some vertex $x_q \neq x_s$ such that $d_{G-C}(x_q^+) \geq 1$, then $H(x_q^{+++})$ must be separable by Claim 1. A C-arc $Q = Q[x_j^+, x_q^{++}]$ would give rise to a Q-reduction C' such that H_2 is a component of $G - C'$ and $|C'| > |C|$. Then (C', H_2) would be of type $(0, \frac{D_0+1}{2})$, which is a contradiction. Hence, in fact, $G - (S - \{x_q^{++}\})$ has at least $m+1$ components, among them the separable one $H(x_q^{+++})$. Therefore, $t(m+2) \leq |C| - m$ and (23) follows as above.

If there is no such vertex x_q, let $S_1 = S - \{x_i^{++} : x_i \neq x_s\}$. A C-arc $Q = Q[z_j, z_k]$ in GS_1 gives rise to a Q-reduction of type $(0,1)$ or $(1,0,0)$, which is a contradiction. Hence, in fact, $G - S_1$ has at least $m+1$ components and, therefore, $t(m+1) \leq |C| - (2m-1)$. Equivalently, $|C| \geq (t+1)m + t + m - 1$. Hence, we obtain (23), because $m = \frac{m}{2} + \frac{m}{2} \geq t+1$.

Case 2. $D_1 = 2$.
Again we are in case (a) and, therefore, $c(G) = |C| + 1$. If there is no cycle of type $(1, \frac{D_0}{2}, \frac{D_0+1}{2})$, then $D_1 = \frac{D_0+1}{2}$ by the inequality constraints, that is $D_0 = 3$. But $D_0 \geq 5$ by Lemma 11. Hence, in fact, $D_1 \leq \frac{D_0}{2}$ and, therefore, $m \geq 2t > D_0 \geq 4$. If $m = 5$, then $D_0 = 4$ and $2 < t \leq \frac{5}{2}$. As $\mu_{H_0}(D_0+2)+4 = \frac{7}{2}(\mu_{H_0}+4) - 10 + \frac{5}{2}\mu_{H_0}$, Lemma 9 and $\mu_{H_0} \geq 2t > 4$ would yield $c(G) \geq (t+1)(\mu_{H_0} + D_0) + t$, contrary to the assumption at the outset of the proof of Theorem 1. Hence, $m \geq 6$. Let $S_0 = V(C) - V(\bigcup_{i=1}^m C[x_i^+, x_i^{++}])$ and consider a C-arc $Q = Q[z_j, z_k]$ in $C - S_0$. By Lemma 2, we have $Q = Q[x_j^{++}, x_k^{++}]$ and $|C(x_i, x_i^\star)| \geq 2$ for all $x_i \in \hat{X}$. Now Q gives rise to a Q-reduction C', which is a longest cycle. Therefore, H_2 cannot be a component of $G - C'$, say $x_k^+ \in N_C(H_2)$. Furthermore, $d_{G-C}(x_j^+) \neq 0$, since otherwise $H(x_j^{++})$ would be a trivial component of $G - C'$. Therefore, $H(x_j^{+++})$ is separable.

If such a C-arc exists, we set $S = S_0 \cup \{x_k^{++}\}$. Since all C-arcs that link vertices of $C - S_0$ must have the end vertex x_k^{++}, we obtain that $G - S$ has at least $m+1$ components including the separable component $H(x_j^{+++})$. Therefore,

$t(m+2) \le |S|+1 = |C|-2m+2$ and, equivalently, $|C| \ge (t+1)m+t+m+t-2$. Hence, (23), since $m+t-2 \ge \frac{m}{2}+2t-2 \ge 2t+1 = (t+1)D_1-1$.

If no such C-arc exists, then $G-S_0$ has at least $m+1$ components. Let p denote the number of separable ones. Then $t(m+1+p) \le |C|-2m+p$ and, equivalently, $|C| \ge (t+1)m+m+t+(t-1)p$. If $p>0$, then we obtain (23). If $p=0$, then $N_C(H_2) \cap \{x_i^+, x_i^{++}\} = \emptyset$ for all $x_i \in \hat{X}$. Hence, $t(m+2) \le |C|-2m$ and again we get (23).

Case 3. $D_1 \ge 3$.

For each $x_i \in \hat{X}$, let $u_i \in V(C)$ be such that $|C(x_i, u_i)| = \lfloor \frac{D_1+1}{2} \rfloor$. We call $x_s \in \hat{X}$ a special vertex, if $N_C(H_2) \cap C(x_s, u_s) \neq \emptyset$.

If $C(x_s, u) \cap N_C(H_2) \neq \emptyset$ and $2 \le |C(x_s, u)| \le \frac{D_1+1}{2}$, then $H(u)$ is separable. For otherwise there exist distinct vertices x, x' on $C(x_s, u)$ such that $|N_{H_2}(x, x')| \ge 2$. Applying Lemma 2(i), we would obtain $|C(x_s, u)| \ge 2+D(H_2)-1 \ge D_1+1$, an obvious contradiction. Let further

$$\begin{aligned} X_1 &= \{x_i \in \hat{X} : H(u_i) \text{ is separable}\} \text{ and} \\ X_2 &= \{x_i \in \hat{X} - X_1 : H(u_i^+) \text{ is separable}\}. \end{aligned}$$

If x_s is a special vertex, then clearly $x_s \in X_1$. Let $S_0 = V(\bigcup_{i=1}^{m} C(u_i, x_{i+1}]) \cup \{u_i : x_i \in X_1\}$. In order to get upper bounds for t, we use different variants of S_0 as cut sets. In the subcase when $c(G) = |C|+2$, D_1 is odd and $N_C(H_2) \cap C(x_s, u_s^-) \neq \emptyset$, we set $S = S_0 \cup \{u_s^-\}$; otherwise $S = S_0$.

Claim 2. If $C(x_i, x_i^\star) \subseteq C(x_i, u_i)$, then $D_1 = 3$ and x_i is a special vertex. Furthermore, $x_i^\star = u_i$ and $x_i \in X_1$.

Proof Claim 2. Let P^i be a longest $(x_i, x_i^\star)$-path with inner vertices in H_1. As $|P^i|-2 \ge D_1+1$, we can construct a cycle C^i such that

$$2 \ge |C^i| - |C| \ge D_1+1-|C(x_i, x_i^\star)| \ge D_1+1-\lfloor \frac{D_1+1}{2} \rfloor \ge 2.$$

Therefore, $|C^i|-|C| = 2$ and we are in case (b) with $D_1 = 3$ and $|C(x_i, x_i^\star)| = 2 = |C(x_i, u_i)|$. Hence, C^i is a longest cycle and H_2 is no component of $G-C^i$. Therefore, $N_C(H_2) \cap C(x_i, u_i) \neq \emptyset$, and the proof of Claim 2 is complete.

Claim 3. Let $Q = Q[z_j, z_k]$ be a C-arc, which links distinct segments $C(x_j, u_j]$ and $C(x_k, u_k]$ in $G-S$. Then $z_j = u_j$, $z_k = u_k$, $c(G) = |C|+2$, and D_1 is odd.

Proof of Claim 3. First observe that z_i is on $C(x_i, x_i^\star)$ for $i = j, k$ by Claim 2. Therefore, Q gives rise to a Q-reduction C' such that

$$2 \ge |C'| - |C| \ge D_1+1-(|C(x_j, z_j)|+|C(x_k, z_k)|).$$

If $z_j \neq u_j$ and $z_k \neq u_k$, then $|C'| = |C|+2$ and, necessarily, $z_i = u_i^-$ $(i = j, k)$. Furthermore, D_1 is odd. Since there is no cycle of type $(0, \frac{D_0+1}{2})$, we obtain that H_2 cannot be a component of $G-C'$. In this event, we may assume that

$C(x_k, u_k^-) \cap N_C(H_2) \neq \emptyset$ and we obtain that x_k is a special vertex. But then $u_k^- \in S$, by construction, contrary to $u_k^- = z_k$ and $z_k \notin S$.

Hence, in fact, $u_j = z_j$ or $u_k = z_k$, say $u_j = z_j$. Consequently, $x_j \notin X_1$. As no C-arc in $G - S$ links $C(x_j, u_j)$ and $C(x_k, u_k)$, we obtain that $H(u_j)$ is a nonseparable component of GC'. Since $D(H(u_j)) \leq \frac{D_0+1}{2}$ and there is no cycle of type $(0, \frac{D_0+1}{2})$, it follows that C' is no longest cycle. Consequently, $z_k \in \{u_k^-, u_k\}$.

If $z_k = u_k^-$, we obtain that D_1 is odd and $|C'| = |C|+1$. Hence, we are in case (b) and, therefore, H_2 is no component of $G - C'$, because otherwise $(C', H(u_j), H_2)$ would be of type $(1, \frac{D_0+1}{2}, \frac{D_0+1}{2})$. Therefore, x_k is a special vertex. In this subcase again, $u_k^- \in S$, by construction, contrary to $u_k^- = z_k$ and $z_k \notin S$. If $z_k = u_k$ and D_1 is even, we obtain again $|C'| = |C| + 1$ and we are in case (b). As above, H_2 is no component of $G - C'$ and x_k is a special vertex. Again $u_k \notin S$, contrary to $x_k \in X_1$.

We have shown that $z_i = u_i$ for $i = j, k$ and D_1 is odd. By construction, $H(u_j)$ and $H(u_k)$ are nonseparable components of $G - C'$. Therefore, $|C| = c(G) - 2$ by the minimality constraints, consequently Claim 3.

As an immediate consequence of Claim 3, there exists at most one special vertex.

Case 3.1. $c(G) = |C| + 1$ or D_1 is even.

In this subcase. Claim 3 implies that H_1, $H(u_i)$ $(x_i \in X_1)$ and $H(u_i^+)$ $(x_i \in \hat{X} - X_1)$ are distinct components of $G - S$, including $p = |X_1 \cup X_2|$ separable ones. Note that either $p > 0$ or H_2 is a component of $G - S$. We derive

$$t(m + 1 + p + \epsilon) \leq |S| + p = |C| - (m\lfloor\frac{D_1 + 3}{2}\rfloor - |X_1|) + p, \qquad (24)$$

where $\epsilon = 1$, if $p = 0$, and $\epsilon = 0$ otherwise. Therefore, (24) implies

$$|C| \geq (t + 1)m + t + m\frac{D_1}{2} + t - 2. \qquad (25)$$

If $m \geq 2t + 1$, then $m\frac{D_1}{2} + t - 2 \geq tD_1 + \frac{D_1}{2} + t - 2 \geq (t + 1)D_1 - 1$, and we obtain (23) from (25). If $m < 2t + 1$, then $\mu_{H_0} \geq 2t > m - 1$ and, therefore, $\mu_{H_0} \geq m$. As $D_0 \geq 2D_1 - 1 \geq D_1 + 2$, an application of (21) yields $c(G) \geq (t+1)(\mu_{H_0} + D_0 - 1) \geq (t+1)(m + D_1 + 1)$. Hence, again we get (23).

Case 3.2. $c(G) = |C| + 2$ and D_1 is odd.

Let us first note that it suffices to show that

$$|C| \geq (t + 1)m + t + m\frac{D_1 - 1}{2} + t - 1. \qquad (26)$$

Indeed, if $m \geq 2t+2$, then $m\frac{D_1-1}{2} + t - 1 \geq (t+1)D_1 - 2$ and (23) follows. If $m < 2t+2$, then $\mu_{H_0} \geq m-1$. Since we are in case (b), we can make use of (22) to obtain $c(G) \geq (t+1)(\mu_{H_0}+D_0) \geq (t+1)(m-1+2D_1-1) \geq (t+1)(m+D_1+1)$, and again we get (23).

Now let $S_1 = V(\bigcup_{i=1}^{m} C[u_i, x_{i+1}]) \cup \{u_s^-\}$. If $\hat{X} = X_1$, we derive from Claim 3 that $G - S_1$ has at least $m + 1$ distinct components, including at least $m - 1$

separable ones. Then

$$t(m+1+m-1) \leq |S_1| + m - 1 = |C| - (m\frac{D_1+1}{2} - 1) + m - 1$$

and, equivalently, $|C| \geq (t+1)m + t + m\frac{D_1-1}{2} + (t-1)(m-1) - 1$. Since $m \geq 2t > D_0 \geq 5$, we get $(t-1)(m-1) - 1 \geq (t-1) + 4(t-1) - 1$ and (26) follows.

If $\hat{X} \neq X_1$, pick $x_p \in \hat{X} - X_1$ and set $S_2 = S_1 - \{u_p\}$. It follows by Claim 3 that $G - S_2$ has at least $m+1$ distinct components. If H_2 is a component of $G - S_2$ or if one of the components is separable, we obtain $t(m+2) \leq |S_2| + 1 \leq |C| - m\frac{D_1+1}{2} + 1$ and again we get (26).

It remains to consider the subcase when $X_1 = \{x_s\}$, $N_C(H_2) \cap C(x_s, u_s^-) \neq \emptyset$ and $H(u_s^-)$ is not separable. Then $|C(x_s, u_s^-)| = 1$, that is, $D_1 = 3$. Let $S_3 = V(\bigcup_{i=1}^m C[u_i, x_{i+1}])$.

If there is no C-arc between distinct segments of the form $C(x_i, u_i)$, then $G - S_3$ has at least $m+1$ components, including the separable one $H(u_s)$. Then $t(m+2) \leq |S_3| + 1$ and we derive (26).

If there is such a C-arc Q, it must have end vertices u_s^- and u_k^- for some $x_k \in \hat{X} - \{x_s\}$. This gives rise to a Q-reduction C'. But C' is a longest cycle and $H(u_k^-)$ is a component of $G - C'$. Since $H(u_k)$ is not separable, necessarily $H(u_k^-) = \{x_k^+\}$, and we obtain a contradiction. □

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