A dynamic market microstructure model with market orders and random order book depth

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Abstract

This paper studies a dynamic market microstructure model, in which a strategic market maker competes with an informed trader. We include the presence of noise traders and limit order traders in our setup. Our model is a $N-$period model. We give necessary and sufficient conditions for an equilibrium to exist and provide conditions for it to be unique. Moreover, both the informed trader and the market maker try to maximize their profits. The resulting recursive equations lead to various economic interpretations. We investigate the interplay of different information sets. Finally we consider the competitive situation for the market maker. Our framework is general enough to obtain several well-known models as a particular case, among them the models by Kyle (1985) as well as Bondarenko and Sung (2003).

1 Introduction

We consider a multiperiod market microstructure model. Our model has four different types of agents: limit order traders, noise traders, one insider and one market maker where the market maker can choose how much liquidity he provides to the market. The first main feature is that different agents possess different information, in particular, the influence of price and quantities is considered. The trading behaviour of market makers and informed traders is of utmost importance in financial markets and therefore, it is of interest in market microstructure. The market maker usually is the one who provides liquidity to the market. However, many real markets work as hybrid markets, where liquidity is supplied both by the market maker and limit order traders. A rule on the NYSE and Amex states that the specialists have to consider the highest bid and the lowest ask prices in the limit order book, when they determine the price. This has also been a topic of empirical research. In this research, see e.g. Madhavan and Sofianos (1998) [MS98], it has been reported, that liquidity provision significantly depends on the stock which is traded. Some of the stocks basically trade in quote-driven markets, where most of the liquidity is provided by the specialist, where for other stocks, it might be just the other way around. Another study by Harris and Hasbrouck (1996) [HH96] shows that more than half of the orders submitted through SuperDOT are limit orders. Therefore, one might want to include both limit order traders and specialists in a microstructure model. Models which consider only one of both ways to provide liquidity to the market cannot model the interaction between those two types. One of the most obvious result in our paper in this direction is of course, that we do have an equilibrium under certain conditions once the order book depth is not zero. The interested reader can compare this to the case of strategic market

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makers when there are no limit order traders present. Results in this area have usually shown the existence of an equilibrium, however there was the additional assumption that there are more than two market makers. With a nontrivial order book, the limit order traders naturally compete with the market maker for liquidity. This has the nice effect that there can be an equilibrium even if there is only one market maker. Of course, this shows up in reality, since there are several exchanges which are organized in a way that there is a monopolist market maker for certain stocks. This specialist can choose whether and how much to use from his own inventory, since the limit order traders come up for the remaining liquidity which needs to be provided.

Furthermore, the insider's trading strategies as well as the specialist's pricing rule are derived endogenously. This extension allows us to thoroughly investigate the influence of time-varying information and the learning behaviour of different agents. All agents are risk-neutral and both the market maker and the informed trader try to maximize their expected profits, whereas the limit order trader's strategy satisfies a no expected profit condition. This allows us to explicitly model the information advantage of the market maker.

Our model is in the spirit of Kyle (1985) [Kyl85] and Bondarenko and Sung (2003) [BS03]. Bondarenko and Sung (2003) [BS03] work with a one-period model which we will extend to a multiperiod model. Other papers in this area are Back (1992) [Bac92], Bhattacharya and Spiegel (1991) [BS91], Chung and Charoenwong (1998) [CC98], Dutta and Madhavan (1997) [DM97], Bondarenko (2001) [Bon01], Chakravarty and Holden (1995) [CH95], Chung, Van Ness and Van Ness (1999) [CVNVN99], Dennert 1993 [Den93] and Glosten 1989 [Glo89]. In particular we mention here the model by Bondarenko 2001 [Bon01] . He also works with a multiperiod model, however, he does not include limit order traders and his setup is different: There, the market makers post price schedules in the beginning, before observing the order flow. Therefore he cannot investigate one of our main goals, namely the effects of being able to observe the order flow before posting a price. Furthermore he concludes that there is no equilibrium, when there are less than three market makers, whereas we will show that there exists a (unique) equilibrium even if there is only one market maker. Our general setup also allows us to obtain Kyle's model [Kyl85] in a very straightforward and easy way.

There has been extensive research on trading behaviour of market makers which are perfectly competitive. This leads to a zero-profit condition which greatly simplifies the analysis of those models. However, this contradicts empirical literature quite often, see e.g. Christie and Schultz (1994) [CS94], Christie et al. (1994) [CHS94] or Hasbrouck and Sofianos (1993) [HS93] and Sofianos (1995) [Sof95]. Those findings are supported by both static and dynamic research which investigate the effects of strategic market makers. However, most of those models have been of static nature, and, to the best of our knowledge, there has not been any investigation as far as a dynamic model is concerned, which included both a nontrivial order book and a strategic market maker.

We consider a $N-$period model. In every period the noise traders and the insider submit their orders to the market maker. The market maker then chooses the price after having observed the limit order book and the combined order flow from the insider and the noise traders, such that the market is cleared. A precise definition and conditions on the distribution of the order book depth which are necessary and sufficient for a market equilibrium to exist, will be given later. After the last trading round, the true value of the asset will become known and everyone realizes his gains or losses on the positions which he still has. We note here the particular role which information plays in this model. The insider knows the true value of the asset, whereas the market maker observes the order book depth and the combined order flow from the insider and the noise traders. This shows that we deal with two kinds of information: One is the information about the asset price, the other is the information on the order flow. It is the interplay between these two aspects which we will investigate and show that both the insider and the market maker will gain from their information whereas the noise traders will lose money in this model. We will show that there exists a market equilibrium and investigate the mutual interdependence of the various parameters. We will also give conditions under which the market equilibrium is unique.

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The remainder of this paper is organized as follows. In the next section we describe the basic model. In Section 3 we give the main theorems. The following section is supposed to describe the four market participants in more detail. Section 4 describes the particular case, when the order book depth is predictable. Next we compare our setup to the case of a competitive market maker, arising from a Bertrand auction e.g. We finish with a short section on comparative statistics. In the appendix we shall give proofs of the more complicated theorems.

2 The basic model

Throughout the paper, we shall fix a probability space $(\Omega, \mathcal{F}, P)$. The setup is as follows: The distribution of the asset value ν at time T is a normal distribution with mean ν_0 and variance σ_0^2. This distribution is known to all participants in the financial market. We consider a $N-$period model from time 0 to T, with the shares being traded at times $0 < t_1 < \cdots < t_N < T$. The length of the n-th period is denoted by Δt_n. The market maker sets the price p_n at time Δt_n.

We model the information through $\sigma-$algebras. Information which is known to the limit order traders just before time t_n is described by the $\sigma-$algebra $\mathcal{F}_{n-1}$ generated by the prices $p_1, \ldots, p_{n-1}$, i.e.

$$\mathcal{F}_{n-1} = \sigma\left(p_1, \ldots, p_{n-1}\right)$$

The limit order traders base their decision at time t_n on the information $\mathcal{F}_{n-1}$. The distribution of α_n is known to everyone, however only the market makers know the realization of α_n.

The order quantities are denoted as follows: The number of shares ordered by the insider are x_n, the noise traders submit z_n, the market maker y_n and the limit order traders $\alpha_n\left(\nu_{n-1}^L - p_n\right)$, where ν_{n-1}^L denotes the limit order traders' expected price of the true asset based on the information they have at time t_n. We assume that $P\left(\alpha_n > 0\right) = 1$. Of course, this setup implies that the order book has a uniform depth of α_n and limit orders are placed for all prices. The assumption of a linear limit order supply schedules has been used in a number of important market microstructure models see e.g. Kyle (1989) [Kyl89] and Black (1995) [Bla95] and Glosten (1994) [Glo89]. Just before the new prices p_n are published, the insider knows all previous prices, his own order flow and the true value of the asset ν.

$$\mathcal{I}_{n-1} = \sigma\left(p_1, \ldots, p_{n-1}, x_1, \ldots, x_{n-1}, x_n, \nu\right)$$

The insider bases his decision on the information $\mathcal{I}_{n-1}$.

After the market maker receives the combined order flow from the insider and the noise trader (He can only observe the combined order flow.), he knows all previous prices, the combined order flow from the insider and the noise traders and his own order flow.

$$\mathcal{M}_n = \sigma\left(\alpha_1, \ldots, \alpha_n, p_1, \ldots, p_{n-1}, p_n, x_1 + z_1, \ldots, x_n + z_n, y_1, \ldots, y_n\right)$$

The market maker bases his decision on the information $\mathcal{M}_n$.

The market participants update their information on the true value of the asset in each trading round. Each of the market participants has a different information set. Therefore we have to define the following updated price of the asset at time t_n as follows: The price as seen by the market maker:

$$\nu_n = E\left(\nu \left| \mathcal{M}_n\right.\right)$$

The price as seen by the limit order traders:

$$\nu_n^L = E\left(\nu \left| \mathcal{F}_n\right.\right)$$

Of course, the insider knows the true value:

$$\nu = E\left(\nu \left| \mathcal{I}_n\right.\right)$$

Furthermore we define the updated variance of the true price ν as seen by the limit order traders at time t_n as

$$\begin{aligned} \sigma_n^2 &= Var\left(\nu \left| \mathcal{M}_n \right.\right) \\ &= E\left(\left(\nu - E\left(\nu \left| \mathcal{M}_n \right.\right)\right)^2 \left| \mathcal{M}_n \right.\right) \\ &= E\left(\left(\nu - \nu_n\right)^2 \left| \mathcal{M}_n \right.\right) \end{aligned}$$

We denote the profits of the insider in trading rounds n through N as π_n^I and the profits of the market maker as π_n^M :

$$\begin{aligned} \pi_n^I &= \sum_{k=n}^{N} \left(\nu - p_k\right) x_k \\ \pi_n^M &= \sum_{k=n}^{N} \left(\nu - p_k\right) y_k = \sum_{k=n}^{N} \left(p_k - \nu\right) \left(x_k + z_k + \alpha \left(\nu_{k-1}^L - p_n\right)\right) \end{aligned}$$

The market maker has to ensure that the market clearing condition is satisfied:

$$x_n + y_n + z_n + \alpha_n \left(\nu_{n-1}^L - p_n\right) = 0 \qquad n = 1, 2, \ldots, N \tag{1}$$

Of course, it is natural to assume that the limit order traders base their decision on their estimate of the true asset price ν_{n-1}^L as compared to the price p_{n-1}. In order to solve for an equilibrium, we have to make several assumptions: First, as customary in the literature on market microstructure, we assume that the quantity x_n traded by the insider at time t_n is linear in the difference between the actual asset price ν and the expected value of p_n, given the information the insider has at time t_n, when he makes his decision. We also assume that the price chosen by the market maker is linear in the total quantity submitted by the insider and the noise trader:

$$x_n = \beta_n \left(\nu - E\left(p_n \left| \mathcal{I}_{n-1} \right.\right)\right) \Delta t_n \tag{2}$$

$$p_n = p_{n-1} + \lambda_n \left(x_n + z_n\right) \tag{3}$$

where $P\left(\beta_n > 0\right) = 1$.

We assume that z_n is normally distributed with mean zero and instantaneous variance $\sigma_z^2 \Delta t_n$, independent of ν. The distribution of ν is public knowledge (we need this e.g. later for modelling the limit order traders decision on α_n. This assumption allows us to calculate $E\left(\nu \left| \mathcal{M}_n \right.\right)$). We define trading strategies as $N-$tuples:

Definition 1 *A trading strategy for the insider is defined as a function* $X_N : \{t_1, t_2, \ldots, t_N\} \rightarrow \mathbb{R}$ *with* $X_N = \left(x_1, \ldots, x_N\right)$.

Definition 2 *A trading strategy for the market maker is defined as a function* $Y_N : \{t_1, t_2, \ldots, t_N\} \rightarrow \mathbb{R}$ *with* $Y_N = \left(y_1, \ldots, y_N\right)$.

Definition 3 *Each trader is represented by its trading strategy.*

The vector of order book depths is defined as $\alpha = \left(\alpha_1, \ldots, \alpha_N\right)$.

Definition 4 (Market Equilibrium) *A security market equilibrium is a triplet* (X, Y, α) *such that the following conditions are satisfied:*

1. *The market maker maximizes his expected profits:*

$$\left(y_{n+1}, \ldots, y_N\right) \in \arg\max E\left(\pi_{n+1}^M \left| \mathcal{M}_{n+1} \right.\right), n = 0 \ldots N-1$$

2. *The informed trader maximizes his expected profits:*

$$(x_{n+1}, \dots, x_N) \in \arg\max E\left(\pi^I_{n+1} \left| \mathcal{I}_n \right.\right), n = 0 \dots N-1$$

3. *The order book depth α_n satisfies a no-expected profit condition:*

$$E\left((\nu - p_n)\, \alpha_n \left(\nu^L_{n-1} - p_n\right) \left| \mathcal{F}_n \right.\right) = 0, \qquad n = 1 \dots N$$

4. *The market clearing condition is satisfied:*

$$x_n + y_n + z_n + \alpha_n \left(\nu^L_{n-1} - p_n\right) = 0 \qquad n = 1, \dots, N$$

We only consider linear market equilibria and later we will show under what conditions a linear market equilibrium exists. Solutions to the informed trader's and the market maker's optimization problem are called optimal strategies for the informed trader and the market maker respectively.

Definition 5 (Linear market equilibrium) *A linear market equilibrium is defined as a market equilibrium where x_n and p_n satisfy in addition*

$$\begin{aligned} x_n &= \beta_n \left(\nu - E\left(p_n \left| \mathcal{I}_{n-1} \right.\right)\right) \Delta t_n \\ p_n &= p_{n-1} + \lambda_n \left(x_n + z_n\right) \end{aligned}$$

for some λ_n which is a function of α_n, such that λ_n is $\sigma\left(p_1, \dots, p_{n-1}\right)$ −measurable.

3 General calculations and main results

The market maker has an informational advantage as compared to the limit order traders, since he observes the combined order flow from the noise traders and the insider. For a model where the market can observe the noise trader order flow separately, see Rochet and Vila (1994) [RV94]. However the market maker also has an informational disadvantage as compared to the insider, since he does not know the true value of the asset. It is this interplay between those two aspects which is one of our main points of investigation. We shall first state our main theorem on the existence of an equilibrium.

Theorem 6 *There is an equilibrium in this financial market which, for $n = 1, \dots, N$ is given by*

1. *The insider's strategy at time t_n is*

$$x_n = \beta_n \Delta t_n \left(\nu - E\left(p_n \left| \mathcal{I}_{n-1} \right.\right)\right)$$

2. *The market maker sets the price p_n at time t_n as*

$$p_n = p_{n-1} + \lambda_n \left(\alpha_n\right) \left(x_n + z_n\right)$$

3. *The depth of the market is distributed such that the conditional expected profit of the limit order traders satisfies a zero profit condition*

$$E\left((\nu - p_n)\, \alpha_n \left(\nu^L_{n-1} - p_n\right) \left| \mathcal{F}_{n-1} \right.\right) = 0$$

4. *The market is cleared, i.e.*

$$0 = x_n + y_n + z_n + \alpha_n \left(\nu^L_{n-1} - p_n\right)$$

The maximal conditional expected profit of the insider is given by

$$\max_{(x_{n+1},\ldots,x_N)} E\left(\pi_{n+1}^I \left| \mathcal{I}_n \right.\right) = a_n \left(\nu - p_n\right)^2 + b_n$$

whereas the maximal conditional expected profit of the market maker is given by

$$\max_{(\lambda_{n+1},\ldots,\lambda_N)} E\left(\pi_{n+1}^M \left| \mathcal{M}_{n+1} \right.\right) = c_{n+1}\left(x_{n+1}+z_{n+1}\right)^2 + d_{n+1}\left(x_{n+1}+z_{n+1}\right) + e_{n+1}$$

The coefficients $a_n, b_n, c_n, d_n, e_n, \beta_n$ are $\mathcal{F}_{n-1}-$measurable and together with α_n, λ_n satisfy the following recursive system of equations for $n = 1, \ldots, N$:

$$a_{n-1} = a_n + \gamma_n \left(1 - 2a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right) - \gamma_n^2 \left(E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) - a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)\right)$$

$$b_{n-1} = a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) \sigma_z^2 \Delta t_n + b_n$$

with the boundary conditions

$$a_N = b_N = 0$$

and

$$\gamma_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)}.$$

The parameter β_n chosen by the informed trader satisfies

$$\frac{2\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)} \left(E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) - a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)\right) = 1 - 2a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)$$

with the second-order conditions given by

$$E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) > a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) \tag{4}$$

The market maker chooses the optimal λ_n given by

$$\begin{aligned} & 2\lambda_n\left(\alpha_n\right)\left(x_n + z_n\right)\left(\alpha_n - 2c_{n+1}\gamma_{n+1}^2\right) \\ = \; & x_n + z_n + \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + \nu_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1} - \nu_n\right) - d_{n+1}\gamma_{n+1} \end{aligned}$$

where

$$\nu_n = \nu_{n-1} + \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}} \left(x_n + z_n - \gamma_n\left(\nu_{n-1} - p_{n-1}\right)\right)$$

and

$$\sigma_n^2 = Var\left(\nu \left|\mathcal{M}_n\right.\right) = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}$$

The second-order conditions read as

$$c_{n+1}\gamma_{n+1}^2 - \alpha_n < 0$$

and

$$\alpha_n > 0.$$

The distribution of α_n and λ_n satisfies

$$\begin{aligned} & E\left(\alpha_n \left|\mathcal{F}_{n-1}\right.\right)\left(\nu_{n-1}^L - p_{n-1}\right)^2 \\ & + E\left(\alpha_n \lambda_n\left(\alpha_n\right) \left|\mathcal{F}_{n-1}\right.\right)\gamma_n\left(-E\left(\nu^2 \left|\mathcal{F}_{n-1}\right.\right) - \left(\nu_{n-1}^L\right)^2 + 4\nu_{n-1}^L p_{n-1} - 2p_{n-1}^2\right) \\ & + E\left(\alpha_n \lambda_n^2\left(\alpha_n\right) \left|\mathcal{F}_{n-1}\right.\right)\left(\gamma_n^2 E\left(\nu^2 \left|\mathcal{F}_{n-1}\right.\right) + \gamma_n^2 p_{n-1}^2 - 2\gamma_n^2 \nu_{n-1}^L p_{n-1} + \sigma_z^2 \Delta t_n\right) \\ = \; & 0 \end{aligned} \tag{5}$$

The exact recursive forms of c_n, d_n and e_n are given in the appendix. They satisfy the boundary condition

$$c_{N+1} = d_{N+1} = e_{N+1} = 0$$

Our second theorem concerns the solution of the informed trader's maximization problem.

Theorem 7 *Assume that* $E(\lambda_n |\mathcal{I}_{n-1}) \neq a_n E(\lambda_n^2 |\mathcal{I}_{n-1})$ *for all* $n = 1, \ldots N$. *Then the informed trader has a unique optimal strategy, whereas the informed trader has several optimal strategies in periods where* $E(\lambda_n |\mathcal{I}_{n-1}) = a_n E(\lambda_n^2 |\mathcal{I}_{n-1})$.

Next, we look at the market maker's optimization problem.

Theorem 8 *Assume that* $\alpha_n - c_{n+1}\gamma_{n+1}^2 \neq 0$ *for all* $n = 1, \ldots N$. *Then the market maker has a unique optimal strategy, whereas he has several optimal strategies in periods where* $\alpha_n - c_{n+1}\gamma_{n+1}^2 = 0$.

Uniqueness of the equilibrium is provided in the next theorem.

Theorem 9 *The market equilibrium is unique if for all* $n = 1, \ldots N$, $\alpha_n - c_{n+1}\gamma_{n+1}^2 \neq 0$, $E(\lambda_n |\mathcal{I}_{n-1}) \neq a_n E(\lambda_n^2 |\mathcal{I}_{n-1})$ *and if the distribution of* α_n *is uniquely characterized by Conditions* (4) *and* (5).

Now we look at the different market participants separately and we start with a look at the informed trader.

3.1 The informed trader

Recall that ν is $\mathcal{I}_0-$measurable. We start with writing the quantity x_n as a linear function of ν and p_{n-1}.

Lemma 10 *Assume that* ν *is not* $\mathcal{M}_n-$*measurable for every* n. *The quantity* x_n *ordered by the informed trader is linear in the difference between* ν *and* p_{n-1}. *More precisely,*

$$\begin{aligned} x_n &= \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n |\mathcal{I}_{n-1})} (\nu - p_{n-1}) \\ &= : \gamma_n (\nu - p_{n-1}) \end{aligned} \tag{6}$$

Proof. We use equations (2) and (3).

$$\begin{aligned} x_n &= \beta_n \Delta t_n (\nu - p_{n-1} - x_n E(\lambda_n |\mathcal{I}_{n-1})) \\ &\Longrightarrow x_n (1 + \beta_n \Delta t_n E(\lambda_n |\mathcal{I}_{n-1})) = \beta_n \Delta t_n (\nu - p_{n-1}) \\ &\Longrightarrow x_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n |\mathcal{I}_{n-1})} (\nu - p_{n-1}) \end{aligned}$$

provided

$$1 + \beta_n \Delta t_n E(\lambda_n |\mathcal{I}_{n-1}) \neq 0$$

The condition

$$1 + \beta_n \Delta t_n E(\lambda_n |\mathcal{I}_{n-1}) = 0$$

implies that $\beta_n \Delta t_n (\nu - p_{n-1}) = 0$. This holds if and only if $\beta_n = 0$ or $\nu = p_{n-1}$. However, these are the cases which we excluded by assumption. ■

Assume that β_n is $\mathcal{I}_{n-1}$ measurable. The informed trader knows the realization of ν and chooses the quantity x_n. To solve his optimization problem, we make the following inductive hypothesis:

$$\max_{(x_{n+1}, \ldots, x_N)} E(\pi_{n+1}^I |\mathcal{I}_n) = a_n (\nu - p_n)^2 + b_n$$

for some $\mathcal{I}_{n-1}$ measurable functions a_n and b_n with the boundary conditions

$$a_N = b_N = 0$$

This backward iteration procedure approach already appears in Kyle 1985.

Remark 11 *The equation*

$$\max_{(x_{n+1},\dots,x_N)} E\left(\pi^I_{n+1}\left|\mathcal{I}_n\right.\right) = a_n\left(\nu - p_n\right)^2 + b_n$$

leads to our first interpretation. If the market maker sets a price p_n which is far away from the true asset value ν, the conditional expected profit of the insider is larger than if p_n were close to ν. Of course, this shows how the lack of information on the side of the market maker leads to additional profits on the side of the insider. (Note that we will show later that indeed, $a_n \geqslant 0$ and $b_n \geqslant 0$.)

The informed trader's optimization problem: Choose x_n such as to maximize the expected profit based on the information just before time t_n :

$$\max_{x_n} E\left(\left(\nu - p_n\right)x_n + \pi^I_{n+1}\left|\mathcal{I}_{n-1}\right.\right)$$

Our main result reads as follows.

Theorem 12 *Assume that $E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) \neq a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)$ for all n. Then necessary conditions for an equilibrium are*

$$\frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right)} = \frac{1 - 2a_n E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right)}{2E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) - 2a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)}$$

and

$$E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) > a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)$$

Proof. see Appendix ■

In the next theorem we look at the case where $E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) = a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)$ for some n.

Theorem 13 *Assume that $E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) = a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)$ for some n. Then $a_n E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) = \frac{1}{2}$ is a necessary condition for having an equilibrium. However in this case the strategy of the insider is not unique anymore. If we have $a_n E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) \neq \frac{1}{2}$ then there is no equilibrium.*

Proof. see Appendix ■

Remark 14 *This in contrast to the results in Kyle* 1985 *[Kyl85] where we always have a unique equilibrium. If however, as in Kyle* 1985 *[Kyl85], λ_n is constant for all n, we get a unique strategy for the insider, which can be seen from the following lemma.*

Lemma 15 *Assume that $\lambda_n \in \mathcal{I}_{n-1}$ for $n = 1, \dots, N$. The condition $E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) = a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)$ implies that $a_n E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) = \frac{1}{2}$ cannot be true. Therefore we get a unique strategy for the insider. In particular, $\lambda_n \in \mathcal{I}_{n-1}$ if λ_n is constant as in Kyle [Kyl85].*

Proof. Assume that $E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) = a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)$ for some n and therefore $\lambda_n = a_n \lambda_n^2$. Together with $a_n \lambda_n = \frac{1}{2}$ this implies $a_n \lambda_n = 1$ which leads to a contradiction. ■

We now look at the maximal conditional expected profit

$$\max_{(x_{n+1},\dots,x_N)} E\left(\pi^I_{n+1}\left|\mathcal{I}_n\right.\right) = a_n\left(\nu - p_n\right)^2 + b_n$$

and the coefficients a_n and b_n.

Lemma 16 *Let $n \in \mathbb{N}$ and assume that $E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) \neq a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)$. Then the coefficients a_n and b_n are given by the following recursive formula*

$$b_{n-1} := a_n E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right)\sigma_z^2 \Delta t_n + b_n$$

and

$$a_{n-1} := \gamma_n - \gamma_n^2 E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right) + a_n + a_n \gamma_n^2 E\left(\lambda_n^2\left|\mathcal{I}_{n-1}\right.\right) - 2a_n \gamma_n E\left(\lambda_n\left|\mathcal{I}_{n-1}\right.\right)$$

with the boundary conditions $a_N = b_N = 0$. If $E(\lambda_n | \mathcal{I}_{n-1}) = a_n E(\lambda_n^2 | \mathcal{I}_{n-1})$ for some $n \in \mathbb{N}$, then necessarily for an equilibrium $a_n E(\lambda_n | \mathcal{I}_{n-1}) = \frac{1}{2}$ and the coefficients a_n and b_n are given by

$$a_{n-1} := a_n$$

and

$$b_{n-1} := E(\lambda_n | \mathcal{I}_{n-1}) \sigma_z^2 \Delta t_n + b_n$$

Proof. See Appendix ■

3.2 The market maker

Recall that α_n is $\mathcal{M}_n-$measurable. Assume that λ_n is independent of $\mathcal{I}_n$. (Clearly, this could be justified by assuming that α_n is independent of $\mathcal{I}_n$, which is one of our basic features of the model, and the assumption that λ_n is e.g. a deterministic function of α_n. We also assume that ν is independent of α_n.

The following lemma will be used in the sequel.

Lemma 17 *We have $x_n + z_n \neq 0$ $P-a.s.$*

Proof. Assume that $P(x_n + z_n = 0) > 0$. Then on this set

$$\beta_n \Delta t_n (\nu - E(p_n | \mathcal{I}_{n-1})) = -z_n$$

However by the independence of the distributions of ν and z_n, this cannot hold true with positive probability. ■

In the next subsection we want to calculate the conditional expected value of the asset based on the market maker's information. He does not have as much information on ν as the informed trader, however the market maker observes the combined order flow from the liquidity traders and the informed trader. Therefore he does have at least partial information on ν. To make this statement more precise is the purpose of the next subsection.

3.2.1 $E(\nu | \mathcal{M}_n)$

As a preliminary step, we calculate

$$E(\nu | \mathcal{M}_n)$$

Theorem 18 *The expected value of the asset conditional on the market maker's information is given by*

$$\nu_n = E(\nu | \mathcal{M}_n) = \nu_{n-1} + \gamma_n \frac{\sigma_n^2}{\sigma_z^2 \Delta t_{n-1}} (x_n + z_n) - \frac{\sigma_n^2}{\sigma_z^2 \Delta t_{n-1}} \gamma_n^2 (\nu_{n-1} - p_{n-1})$$

The conditional variance reads as

$$\sigma_n^2 = Var(\nu | \mathcal{M}_n) = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}$$

Proof. see appendix ■

Based on those formulas, we are ready to solve the maximization problem of the marker maker.

3.2.2 Maximization problem of the market maker

As before we proceed by backward induction. We make the following inductive hypotheses:

$$\max_{(\lambda_{n+1},\ldots,\lambda_N)} E\left(\pi^M_{n+1} \left| \mathcal{M}_{n+1}\right.\right) = c_{n+1}\left(x_{n+1}+z_{n+1}\right)^2 + d_{n+1}\left(x_{n+1}+z_{n+1}\right) + e_{n+1}$$

with the boundary conditions

$$c_{N+1} = d_{N+1} = e_{N+1} = 0$$

and the assumption that c_n, d_n and e_n are $\mathcal{M}_{n-1}-$measurable for all n. The market maker solves

$$\max_{\lambda_n} E\left(\left(p_n-\nu\right)\left(x_n+z_n+\alpha_n\left(\nu^L_{n-1}-p_n\right)\right)+c_{n+1}\left(x_{n+1}+z_{n+1}\right)^2+d_{n+1}\left(x_{n+1}+z_{n+1}\right)+e_{n+1}\left|\mathcal{M}_n\right.\right) \tag{7}$$

The results are summarized in the following two theorems.

Theorem 19 *A solution to the market maker's maximization problem exists and the maximizing λ_n is given by*

$$\begin{aligned} & \lambda_n\left(\alpha_n\right)\left(2\left(x_n+z_n\right)\alpha_n - 2c_{n+1}\gamma^2_{n+1}\left(x_n+z_n\right)\right) \\ = \;& x_n+z_n+\alpha_n\left(\nu^L_{n-1}-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma^2_{n+1}p_{n-1}-d_{n+1}\gamma_{n+1}-2c_{n+1}\gamma^2_{n+1}\nu_n \end{aligned}$$

Uniqueness of the solution is given by the following theorem.

Theorem 20 *Assume that $\alpha_n - c_{n+1}\gamma^2_{n+1} \neq 0$. Then a unique solution to the market maker's maximization problem exists and the maximizing λ_n is given by*

$$\lambda_n\left(\alpha_n\right) = \frac{x_n+z_n+\alpha_n\left(\nu^L_{n-1}-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma^2_{n+1}p_{n-1}-d_{n+1}\gamma_{n+1}-2c_{n+1}\gamma^2_{n+1}\nu_n}{2\left(x_n+z_n\right)\alpha_n-2c_{n+1}\gamma^2_{n+1}\left(x_n+z_n\right)}$$

The second-order conditions are given by

$$c_{n+1}\gamma^2_{n+1} < \alpha_n$$

and

$$\alpha_n > 0$$

Proof. see Appendix ■

We also give a condition, when there is no equilibrium.

Lemma 21 *Assume that $\alpha_n - c_{n+1}\gamma^2_{n+1} = 0$ for some n. If in addition,*

$$\left(x_n+z_n\right)\left(x_n+z_n+\alpha_n\nu^L_{n-1}-\alpha_n\nu_n-d_{n+1}\gamma_{n+1}\right) \neq 0$$

then there is no equilibrium. If

$$\left(x_n+z_n\right)\left(x_n+z_n+\alpha_n\nu^L_{n-1}-\alpha_n\nu_n-d_{n+1}\gamma_{n+1}\right) = 0$$

there will be multiple optimal strategies for the market maker.

In the appendix we provide precise recursive formulas for the coefficients c_n, d_n and e_n. Based on those formulas, one can also show the sufficiency of the above conditions.

3.3 Noise traders

They submit orders z_n where z_n is normally distributed with expectation 0 and instantaneous variance σ_z^2.

$$\begin{aligned} x_n + y_n + z_n + \alpha_n \left(\nu_{n-1}^L - p_n\right) &= 0 \\ \iff & \quad z_n = -x_n - y_n - \alpha_n \left(\nu_{n-1}^L - p_n\right) \end{aligned}$$

Their profit is given by

$$\begin{aligned} & \nu \sum_{n=1}^{N} z_n - \sum_{n=1}^{N} z_n p_n = \nu \sum_{n=1}^{N} z_n - \sum_{n=1}^{N} z_n \left(\nu_{n-1} + \lambda_n \left(x_n + z_n\right)\right) \\ = & \sum_{n=1}^{N} z_n \left(\nu - \nu_{n-1}\right) - \sum_{n=1}^{N} z_n \lambda_n \left(x_n + z_n\right) = \sum_{n=1}^{N} z_n \left(\nu - \nu_{n-1}\right) - \sum_{n=1}^{N} z_n \left(p_n - \nu_{n-1}\right) \end{aligned}$$

They do participate, no matter how much they lose. A reason for this seemingly paradox behaviour could be liquidity reasons for example, see Kyle (1989) [Kyl89] among others.

3.4 Limit order traders

Here we derive a necessary condition on the distribution of α_n and $\lambda_n(\alpha_n)$ such that the expected profit of the limit order traders satisfies a zero profit condition. A note on ν_{n-1}^L : This denotes the opinion of the limit order traders on the true value of the asset ν, at time t_n, just when the limit order traders decide on the price. This means that the limit order traders buy $\alpha_n \left(\nu_{n-1}^L - p_n\right)$ shares at time t_n and ν_{n-1}^L is given by

$$\nu_{n-1}^L = E\left(\nu \left| \mathcal{F}_{n-1} \right.\right)$$

where $\mathcal{F}_{n-1}$ denotes the information of the limit order trader after trading round $n-1$ but before he knows about the price p_n at time t_n.

The limit order traders create the limit order book with depth α_n. At time t_n they buy $\alpha_n \left(\nu_{n-1}^L - p_n\right)$ shares if $\nu_{n-1}^L - p_n$ is positive and sell them if this expression is negative. This means the limit order traders observe the market and update their beliefs on the expected value of ν. In order to find a condition on α_n, in other words, to find out whether and to what extent limit order traders are willing to participate, we assume that their demand $\alpha_n \left(\nu_{n-1}^L - p_n\right)$ satisfies a zero expected profit condition: For each $n = 1, \ldots, N$ we assume that

$$E\left(\left(\nu - p_n\right) \alpha_n \left(\nu_{n-1}^L - p_n\right) \left| \mathcal{F}_{n-1} \right.\right) = 0$$

Our result reads as follows.

Theorem 22 *Assume that* $E\left(\left(\nu - p_n\right) \alpha_n \left(\nu_{n-1}^L - p_n\right) \left| \mathcal{F}_{n-1} \right.\right) = 0$. *Then* α_n *and* λ_n *have to satisfy*

$$\begin{aligned} & E\left(\alpha_n \left| \mathcal{F}_{n-1} \right.\right) \left(\left(\nu_{n-1}^L\right)^2 - 2p_{n-1}\nu_{n-1}^L + p_{n-1}^2\right) \\ & + E\left(\alpha_n \lambda_n\left(\alpha_n\right) \left| \mathcal{F}_{n-1} \right.\right) \gamma_n \left(-E\left(\nu^2 \left| \mathcal{F}_{n-1} \right.\right) - \left(\nu_{n-1}^L\right)^2 + 4\nu_{n-1}^L p_{n-1} - 2p_{n-1}^2\right) \\ & + E\left(\alpha_n \lambda_n^2\left(\alpha_n\right) \left| \mathcal{F}_{n-1} \right.\right) \left(\gamma_n^2 E\left(\nu^2 \left| \mathcal{F}_{n-1} \right.\right) + \gamma_n^2 p_{n-1}^2 - 2\gamma_n^2 \nu_{n-1}^L p_{n-1} + \sigma_z^2 \Delta t_n\right) \\ = & \ 0 \end{aligned}$$

Proof. see Appendix ■

4 Predictable order book depth

Here we consider the particular case where the order book depth α_n is $\mathcal{F}_{n-1}-$measurable. Furthermore we assume that λ_n is a function of α_n which is $\mathcal{F}_{n-1}-$measurable. Our first lemma shows that $E\left(p_n \left|\mathcal{I}_{n-1}\right.\right)$ is $\mathcal{M}_n-$measurable.

Lemma 23 *Assume that α_n is independent of $\mathcal{I}_{n-1}$ and λ_n is a deterministic function of α_n. Then $E\left(p_n \left|\mathcal{I}_{n-1}\right.\right)$ is $\mathcal{M}_n$-measurable.*

Proof.

$$\begin{aligned}
& E\left(E\left(p_n \left|\mathcal{I}_{n-1}\right.\right)\left|\mathcal{M}_n\right.\right) \\
= \ & E\left(E\left(p_{n-1}+x_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\left|\mathcal{I}_{n-1}\right.\right)\left|\mathcal{M}_n\right.\right) \\
= \ & p_{n-1}+E\left(x_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\left|\mathcal{M}_n\right.\right) \\
= \ & E\left(p_{n-1}+x_n E\left(\lambda_n\right)+z_n E\left(\lambda_n\right)\left|\mathcal{M}_n\right.\right)-z_n E\left(\lambda_n\right) \\
= \ & E\left(p_{n-1}+\left(x_n+z_n\right) E\left(\lambda_n\right)\left|\mathcal{M}_n\right.\right)-z_n E\left(\lambda_n\right) \\
= \ & E\left(p_{n-1}+\left(x_n+z_n\right) E\left(\lambda_n\right)\left|\mathcal{M}_n\right.\right)-z_n E\left(\lambda_n\right) \\
= \ & p_{n-1}+\left(x_n+z_n\right) E\left(\lambda_n\right)=E\left(p_n \left|\mathcal{I}_{n-1}\right.\right)
\end{aligned}$$

Next we show that the information of the market maker is exactly one period ahead of the information which the limit order traders have. ■

Lemma 24 *Assume that $\alpha_n \in \mathcal{F}_{n-1}$ and λ_n is a deterministic function of α_n. We claim that*

$$\mathcal{M}_n=\mathcal{F}_n \tag{8}$$

In other words, if the order book depth is nonrandom, then the noise and limit order traders will have the same information as the market maker, only one period later.

Proof. Recall that

$$\mathcal{M}_n=\sigma\left(\alpha_1, \ldots, \alpha_n, p_1, \ldots, p_n, x_1+z_1, \ldots, x_n+z_n, y_1, \ldots, y_n\right)$$

and

$$\mathcal{F}_n=\sigma\left(p_1, \ldots, p_n\right)$$

$$\begin{aligned}
x_n &= \beta_n \Delta t_n\left(\nu-E\left(p_n \left|\mathcal{I}_{n-1}\right.\right)\right) \\
p_n &= p_{n-1}+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right) \\
0 &= x_n+y_n+z_n+\alpha_n\left(\nu_{n-1}-p_n\right)
\end{aligned}$$

We use equations (2) and (3) as well as the market clearing condition (1). We proceed by induction: $\mathcal{M}_1=\mathcal{F}_1$: It is clear that $\mathcal{F}_1 \subset \mathcal{M}_1$, so we need to show that $\mathcal{M}_1 \subset \mathcal{F}_1$. By assumption, p_1 is $\mathcal{F}_1-$measurable. This, together with the assumption of F_0-measurable α_1 and $\mathcal{F}_1-$measurable p_0, implies that x_1+z_1 is $\mathcal{F}_1-$measurable. Therefore the market clearing condition implies that y_1 is also $\mathcal{F}_1-$measurable. Now assume that

$$\mathcal{M}_{n-1}=\mathcal{F}_{n-1}$$

To show $\mathcal{M}_n \subset \mathcal{F}_n$ we proceed as follows: By assumption, p_n and p_{n-1} are $\mathcal{F}_n-$measurable. Then x_n+z_n is $\mathcal{F}_n-$measurable and as before this implies that y_n is $\mathcal{F}_n-$measurable. Here we also used that ν_{n-1} is $\mathcal{F}_{n-1}-$measurable. This leads us to conclude that $\mathcal{M}_n \subset \mathcal{F}_n$. ■

After those preliminary remarks we look at the equations we got in earlier sections and consider the particular case where α_n and λ_n are $\mathcal{F}_{n-1}-$measurable. The previous lemma implies that $\nu_n^L=\nu_n$.

Theorem 25 *We have the following necessary conditions for an equilibrium:*

1. *The market clearing condition is given by*

$$x_n + y_n + z_n + \alpha_n (\nu_{n-1} - p_n) = 0$$

2. *The parameters* γ_n *and* β_n *are given by*

$$\begin{aligned} \gamma_n &= \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n \lambda_n}. \\ \beta_n \Delta t_n &= \frac{1}{\lambda_n} - 2a_n \end{aligned}$$

3. *The second-order conditions are*

$$1 > a_n \lambda_n,$$
$$c_{n+1} \gamma_{n+1}^2 < \alpha_n$$

and

$$\alpha_n > 0.$$

4. *The parameters* a_n *and* b_n *are recursively given by*

$$b_{n-1} = a_n \lambda_n^2 \sigma_z^2 \Delta t_n + b_n$$

and

$$a_{n-1} = a_n + \frac{1}{2} \gamma_n (1 - 2a_n \lambda_n)$$

with the boundary conditions

$$a_N = b_N = 0.$$

5. *The parameter* λ_n *is given by*

$$\lambda_n (\alpha_n) = \frac{x_n + z_n + \alpha_n \left(\nu_{n-1}^L - 2p_{n-1} + \nu_n \right) + 2c_{n+1} \gamma_{n+1}^2 (p_{n-1} - \nu_n) - d_{n+1} \gamma_{n+1}}{2 (x_n + z_n) \alpha_n - 2c_{n+1} \gamma_{n+1}^2 (x_n + z_n)}.$$

6. *The conditional expected value of* ν *is given by*

$$\nu_n = \nu_{n-1} + \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}} (x_n + z_n - \gamma_n (\nu_{n-1} - p_{n-1}))$$

and the conditional variance is

$$\sigma_n^2 = Var (\nu \,| \mathcal{M}_n) = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}.$$

7. *The equations for* c_n, d_n *and* e_n *are given in the Appendix.*

8. *The condition on* λ_n *is as follows*

$$\begin{aligned} &\nu_{n-1}^2 - 2p_{n-1} \nu_{n-1} + p_{n-1}^2 + \\ &\lambda_n (\alpha_n) \gamma_n \left(-E \left(\nu^2 \,| \mathcal{F}_{n-1} \right) - \nu_{n-1}^2 + 4\nu_{n-1} p_{n-1} - 2p_{n-1}^2 \right) + \\ &\lambda_n^2 (\alpha_n) \left(\gamma_n^2 E \left(\nu^2 \,| \mathcal{F}_{n-1} \right) + \gamma_n^2 p_{n-1}^2 - 2\gamma_n^2 \nu_{n-1} p_{n-1} + \sigma_z^2 \Delta t_n \right) = 0. \end{aligned}$$

Proof. The only equations which remain to be shown are the equations for a_n and β_n.

$$\begin{aligned} a_{n-1} &= a_n + \gamma_n (1 - 2a_n \lambda_n) - \gamma_n^2 \left(\lambda_n - a_n \lambda_n^2 \right) \\ &= a_n + \frac{1 - 2a_n \lambda_n}{2\lambda_n - 2a_n \lambda_n^2} (1 - 2a_n \lambda_n) - \frac{(1 - 2a_n \lambda_n)^2}{4 \left(\lambda_n - a_n \lambda_n^2 \right)^2} \left(\lambda_n - a_n \lambda_n^2 \right) \\ &= a_n + \frac{(1 - 2a_n \lambda_n)^2}{2 \left(\lambda_n - a_n \lambda_n^2 \right)} - \frac{(1 - 2a_n \lambda_n)^2}{4 \left(\lambda_n - a_n \lambda_n^2 \right)} = a_n + \frac{(1 - 2a_n \lambda_n)^2}{4 \left(\lambda_n - a_n \lambda_n^2 \right)} \\ &= a_n + \frac{1}{2} \gamma_n (1 - 2a_n \lambda_n) = a_n (1 - \gamma_n \lambda_n) + \frac{1}{2} \gamma_n \end{aligned}$$

For β_n we obtain

$$\begin{aligned}
\frac{\beta_n \Delta t_n}{1+\beta_n \Delta t_n \lambda_n} &= \frac{1-2a_n\lambda_n}{2\lambda_n - 2a_n\lambda_n^2} \\
\iff & \beta_n \Delta t_n \left(2\lambda_n - 2a_n\lambda_n^2\right) = \left(1+\beta_n \Delta t_n \lambda_n\right)\left(1-2a_n\lambda_n\right) \\
\iff & \beta_n \Delta t_n \left(2\lambda_n - 2a_n\lambda_n^2\right) = 1-2a_n\lambda_n + \beta_n \Delta t_n \lambda_n - \beta_n \Delta t_n \lambda_n 2a_n\lambda_n \\
\iff & \beta_n \Delta t_n \lambda_n = 1-2a_n\lambda_n \\
\iff & \beta_n \Delta t_n = \frac{1}{\lambda_n} - 2a_n
\end{aligned}$$

■

If we look at period $N-1$, we obtain several interesting results.

Lemma 26 *a_{N-1} and b_{N-1} are given by*

$$a_{N-1} = \frac{1}{4\lambda_N}$$

and

$$b_{N-1} = 0$$

This clearly has an economic interpretation: Assume that λ_N is a strictly increasing function of α_N. Then the larger the order book depth, the larger is the slope of the pricing rule, i.e. the larger λ_N, the lower is the informed trader's profit. In other words, if there are many limit order traders in the market, the profit of the informed trader is low.

Proof. We use that $a_N = b_N = 0$. Then

$$b_{N-1} = 0$$

and

$$a_{N-1} = \gamma_N - \gamma_N^2 \lambda_N \tag{9}$$

from the equations for a_{N-1} and b_{N-1} respectively. From the equation for β_N we get

$$\begin{aligned}
\frac{\beta_N \Delta t_N}{1+\beta_N \Delta t_N \lambda_N} &= \frac{1}{2\lambda_N} \\
\iff & 2\lambda_N \beta_N \Delta t_N = 1 + \beta_N \Delta t_N \lambda_N \\
\iff & \lambda_N \beta_N \Delta t_N = 1
\end{aligned}$$

Therefore

$$\gamma_N = \frac{1}{2\lambda_N}$$

Plugging this into Equation (9) yields

$$\begin{aligned}
a_{N-1} &= \frac{1}{2\lambda_N} - \frac{1}{4\lambda_N^2}\lambda_N \\
&= \frac{1}{4\lambda_N}
\end{aligned}$$

■

Lemma 27 *The second-order conditions are fulfilled in the last period, provided $\alpha_N > 0$ and $\lambda_N > 0$.*

Proof. The second-order conditions are

$$E\left(\lambda_N\left|\mathcal{I}_{N-1}\right.\right) > a_N E\left(\lambda_N^2\left|\mathcal{I}_{N-1}\right.\right)$$

$$c_{N+1}\gamma_{N+1}^2 < \alpha_N$$

and

$$\alpha_N > 0$$

which are equivalent to

$$\lambda_N > 0$$

and

$$\alpha_N > 0$$

since $c_{N+1} = a_N = 0$. ■

Lemma 28 *The condition $\lambda_N > 0$ is satisfied in equilibrium provided*

$$-2\left(\nu_{N-1}^L\right)^2 + 2p_{N-1}\nu_{N-1}^L - p_{N-1}^2 + E\left(\nu^2\left|\mathcal{F}_{N-1}\right.\right) > 0.$$

Proof. From the condition on the limit order book and the equation

$$\gamma_N = \frac{1}{2\lambda_N}$$

we get

$$\begin{aligned}
&\left(\nu_{N-1}^L\right)^2 - 2p_{N-1}\nu_{N-1}^L + p_{N-1}^2 \\
&+\frac{1}{2}\left(-E\left(\nu^2\left|\mathcal{F}_{N-1}\right.\right) - \left(\nu_{N-1}^L\right)^2 + 4\nu_{N-1}^L p_{N-1} - 2p_{N-1}^2\right) \\
&+\frac{1}{4}E\left(\nu^2\left|\mathcal{F}_{N-1}\right.\right) + \frac{1}{4}p_{N-1}^2 - \frac{1}{2}\nu_{N-1}^L p_{N-1} + \lambda_N^2\left(\alpha_N\right)\sigma_z^2\Delta t_N = 0
\end{aligned}$$

This holds if and only if

$$2\left(\nu_{N-1}^L\right)^2 - 2p_{N-1}\nu_{N-1}^L + p_{N-1}^2 - E\left(\nu^2\left|\mathcal{F}_{N-1}\right.\right) + 4\lambda_N^2\left(\alpha_N\right)\sigma_z^2\Delta t_N = 0$$

This gives

$$4\lambda_N^2\left(\alpha_N\right)\sigma_z^2\Delta t_N = -2\left(\nu_{N-1}^L\right)^2 + 2p_{N-1}\nu_{N-1}^L - p_{N-1}^2 + E\left(\nu^2\left|\mathcal{F}_{N-1}\right.\right)$$

If we now assume that

$$\begin{aligned}
& 2p_{N-1}\nu_{N-1}^L - 2\left(\nu_{N-1}^L\right)^2 - p_{N-1}^2 + E\left(\nu^2\left|\mathcal{F}_{N-1}\right.\right) > 0 \\
\iff\quad & -2\left(\nu_{N-1}^L\right)^2 + 2p_{N-1}\nu_{N-1}^L - p_{N-1}^2 + E\left(\nu^2\left|\mathcal{F}_{N-1}\right.\right) > 0
\end{aligned}$$

there is exactly one positive solution for λ_N. ■

Lemma 29 *The condition*

$$-2\left(\nu_{N-1}^L\right)^2 + 2p_{N-1}\nu_{N-1}^L - p_{N-1}^2 + E\left(\nu^2\left|\mathcal{F}_{N-1}\right.\right) > 0$$

holds e.g. if the conditional variance of ν is large compared to the conditional mean.

Proof. We look at the condition

$$-2\left(\nu_{N-1}^L\right)^2 + 2p_{N-1}\nu_{N-1}^L - p_{N-1}^2 + E\left(\nu^2\left|\mathcal{F}_{N-1}\right.\right) > 0$$

as a quadratic term in p_{N-1}^2. Then this condition is satisfied if and only if

$$\begin{aligned}
& \left(\nu_{N-1}^L\right)^2 - 4\left(-1\right)\left(-2\left(\nu_{N-1}^L\right)^2 + E\left(\nu^2\left|\mathcal{F}_{N-1}\right.\right)\right) > 0 \\
\iff\quad & -3\left(\nu_{N-1}^L\right)^2 + 4Var\left(\nu^2\left|\mathcal{F}_{N-1}\right.\right) > 0 \\
\iff\quad & Var\left(\nu^2\left|\mathcal{F}_{N-1}\right.\right) > \frac{3}{4}\left(\nu_{N-1}^L\right)^2
\end{aligned}$$

■

5 Competitive market maker

5.1 Competitive market maker with order book

Here we modify our previous setup slightly. Quite often in the literature on market microstructure we do not have a maximizing market maker, but instead the maximizing behaviour is replaced by the condition

$$E(\nu \,|\mathcal{M}_n) = p_n$$

which actually is a zero expected profit condition for the market maker, which we shall show later. For a justification of this in a game-theoretic context see e.g. Kyle (1985) [Kyl85]. We call this the competitive case.

Lemma 30 *We have that* $\nu_n^L = p_n$, $n = 1, \ldots, N$.

Proof. The proof uses the assumption

$$E(\nu \,|\mathcal{M}_n) = p_n$$

and the fact that $\mathcal{F}_n \subset \mathcal{M}_n$. Then

$$\begin{aligned} \nu_n^L &= E(\nu \,|\mathcal{F}_n) = E((\nu \,|\mathcal{M}_n) \,|\mathcal{F}_n) \\ &= E(p_n \,|\mathcal{F}_n) = p_n \end{aligned}$$

■

The next lemma shows that indeed the condition $E(\nu \,|\mathcal{M}_n) = p_n$ is a zero-profit condition for the market maker.

Lemma 31 *The market maker has a zero expected conditional profit.*

Proof. Different from before, we now make the inductive hypotheses that

$$\max_{(\lambda_{n+1},\ldots,\lambda_N)} E\left(\pi_{n+1}^M \,|\mathcal{M}_{n+1}\right) = e_{n+1}$$

with $e_{n+1} \in \mathcal{M}_{n+1}$ and the boundary condition $e_{N+1} = 0$. From before we get for the conditional expected profit of the market maker:

$$\begin{aligned} &\max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi_n^M \,|\mathcal{M}_n\right) = \\ &e_{n+1} + (p_{n-1} - \nu_n)\left(x_n + z_n + \alpha_n \nu_{n-1}^L - \alpha_n p_{n-1}\right) + \lambda_n(\alpha_n)(x_n + z_n)^2 \\ &+\lambda_n(\alpha_n)(x_n + z_n)\alpha_n\left(\nu_{n-1}^L - p_{n-1}\right) \\ &-(p_{n-1} - \nu_n)\alpha_n\lambda_n(\alpha_n)(x_n + z_n) - \lambda_n^2(\alpha_n)(x_n + z_n)^2\alpha_n \end{aligned}$$

Setting $\nu_n = p_n$ and $\nu_{n-1}^L = p_{n-1}$ gives

$$\begin{aligned} &\max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi_n^M \,|\mathcal{M}_n\right) \\ = \;& e_{n+1} + (p_{n-1} - p_n)(x_n + z_n + \alpha_n p_{n-1} - \alpha_n p_{n-1}) + \lambda_n(\alpha_n)(x_n + z_n)^2 \\ &-(p_{n-1} - p_n)\alpha_n\lambda_n(\alpha_n)(x_n + z_n) - \lambda_n^2(\alpha_n)(x_n + z_n)^2\alpha_n \\ = \;& e_{n+1} - \lambda_n(\alpha_n)(x_n + z_n)^2 + \lambda_n(\alpha_n)(x_n + z_n)^2 \\ &+\alpha_n\lambda_n^2(\alpha_n)(x_n + z_n)^2 - \alpha_n\lambda_n^2(\alpha_n)(x_n + z_n)^2 \\ = \;& e_{n+1} \end{aligned}$$

Of course, the boundary condition $e_{N+1} = 0$ now implies that $e_n = 0$ for all $n = 1, \ldots, N+1$.
■

Theorem 32 *The equilibrium in the competitive case is given by the following recursive system of equations*

$$\begin{aligned} x_n &= \beta_n \Delta t_n \left(\nu - E\left(p_n \left| \mathcal{I}_{n-1} \right.\right)\right) \\ p_n &= p_{n-1} + \lambda_n\left(\alpha_n\right)\left(x_n + z_n\right) \\ 0 &= \left(x_n + z_n\right)\left(1 - \alpha_n \lambda_n\left(\alpha_n\right)\right) + y_n \end{aligned}$$

with

$$E\left(\nu \left| \mathcal{M}_n \right.\right) = p_n$$

$$\sigma_n^2 = Var\left(\nu \left| \mathcal{M}_n \right.\right) = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}$$

$$\lambda_n\left(\alpha_n\right) = \frac{\gamma_n \sigma_n^2}{\sigma_z^2 \Delta t_{n-1}}$$

$$\begin{aligned} &\left(p_{n-1}^2 - E\left(\nu^2 \left| \mathcal{F}_{n-1} \right.\right)\right) \gamma_n \left(E\left(\alpha_n \lambda_n\left(\alpha_n\right) \left| \mathcal{F}_{n-1} \right.\right) + E\left(\alpha_n \lambda_n^2\left(\alpha_n\right) \left| \mathcal{F}_{n-1} \right.\right) \gamma_n\right) + \\ &E\left(\alpha_n \lambda_n^2\left(\alpha_n\right) \left| \mathcal{F}_{n-1} \right.\right) \sigma_z^2 \Delta t_n = 0 \end{aligned}$$

$$\beta_n \Delta t_n \left(2E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) - 2a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right)\right) = 1 + \beta_n \Delta t_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right)\left(1 - 2a_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right)\right)$$

and the second-order conditions

$$E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) > a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right)$$

where

$$\gamma_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right)}$$

$$b_{n-1} := a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right) \sigma_z^2 \Delta t_n + b_n$$

$$a_{n-1} := a_n + \gamma_n \left(1 - 2a_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right)\right) - \gamma_n^2 \left(E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) - a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right)\right)$$

and the boundary conditions

$$a_N = b_N = 0$$

Proof. We use that

$$p_n = p_{n-1} + \lambda_n\left(\alpha_n\right)\left(x_n + z_n\right)$$

Before we have shown that

$$\nu_n = \nu_{n-1} + \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}\left(x_n + z_n\right)$$

and

$$\sigma_n^2 = Var\left(\nu \left| \mathcal{M}_n \right.\right) = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}$$

This immediately implies

$$\begin{aligned} \lambda_n\left(\alpha_n\right) &= \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}} \\ &= \frac{\gamma_n \sigma_n^2}{\sigma_z^2 \Delta t_{n-1}} \end{aligned}$$

The market clearing condition now reads as

$$0 = \left(x_n + z_n\right)\left(1 - \alpha_n \lambda_n\left(\alpha_n\right)\right) + y_n$$

Rewriting the condition on the order book depth coming from the no expected profit condition for the limit order traders yields

$$\gamma_n E\left(\alpha_n \lambda_n\left(\alpha_n\right) \left| \mathcal{F}_{n-1}\right.\right)\left(p_{n-1}^2 - E\left(\nu^2 \left| \mathcal{F}_{n-1}\right.\right)\right) \\ - \gamma_n^2 E\left(\alpha_n \lambda_n^2\left(\alpha_n\right) \left| \mathcal{F}_{n-1}\right.\right)\left(p_{n-1}^2 - E\left(\nu^2 \left| \mathcal{F}_{n-1}\right.\right)\right) + E\left(\alpha_n \lambda_n^2\left(\alpha_n\right) \left| \mathcal{F}_{n-1}\right.\right) \sigma_z^2 \Delta t_n = 0$$

■

Lemma 33 *In the competitive case, the prices p_n follow a martingale with respect to the filtration $(\mathcal{F}_n)$ and with respect to the filtration $(\mathcal{M}_n)$.*

Proof. We look at

$$\begin{aligned} E\left(p_n \left| \mathcal{F}_{n-1}\right.\right) &= E\left(\left(\nu \left| \mathcal{M}_n\right.\right) \left| \mathcal{F}_{n-1}\right.\right) \\ &= E\left(\nu \left| \mathcal{F}_{n-1}\right.\right) = p_{n-1} \end{aligned}$$

and

$$\begin{aligned} E\left(p_n \left| \mathcal{M}_{n-1}\right.\right) &= E\left(\left(\nu \left| \mathcal{M}_n\right.\right) \left| \mathcal{M}_{n-1}\right.\right) \\ &= E\left(\nu \left| \mathcal{M}_{n-1}\right.\right) = p_{n-1} \end{aligned}$$

■

5.2 Competitive case with no order book

If we look at the competitive case with no order book, i.e. $\alpha_n = 0$ for $n = 1, \ldots, N$, we obtain a particularly nice result. In the sequel we show that this case can easily be treated with our setup.

Theorem 34 *The equilibrium in the competitive case is given by the following recursive system of equations*

$$\begin{aligned} x_n &= \beta_n \Delta t_n \left(\nu - E\left(p_n \left| \mathcal{I}_{n-1}\right.\right)\right) \\ p_n &= p_{n-1} + \lambda_n\left(\alpha_n\right)\left(x_n + z_n\right) \\ 0 &= x_n + z_n + y_n \end{aligned}$$

with

$$E\left(\nu \left| \mathcal{M}_n\right.\right) = p_n$$

$$\sigma_n^2 = Var\left(\nu \left| \mathcal{M}_n\right.\right) = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}$$

$$\lambda_n = \frac{\gamma_n \sigma_n^2}{\sigma_z^2 \Delta t_{n-1}} = \frac{1}{\beta_n \Delta t_n}$$

and the second-order conditions

$$1 > a_n \lambda_n$$

where

$$\gamma_n = \frac{\beta_n \Delta t_n}{2}$$

$$b_{n-1} := a_n \lambda_n^2 \sigma_z^2 \Delta t_n + b_n$$

$$a_{n-1} = \frac{1}{4}\left(\frac{1}{\lambda_n} + a_n\right)$$

and the boundary conditions

$$a_N = b_N = 0$$

Proof. We use that

$$p_n = p_{n-1} + \lambda_n (x_n + z_n)$$

Before we have shown that

$$\nu_n = \nu_{n-1} + \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}} (x_n + z_n)$$

and

$$\sigma_n^2 = Var(\nu \,|\mathcal{M}_n) = \frac{\sigma_{n-1}^2 \sigma_z^2 \Delta t_{n-1}}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}$$

This immediately implies that

$$\begin{aligned} \lambda_n &= \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}} \\ &= \frac{\gamma_n \sigma_n^2}{\sigma_z^2 \Delta t_{n-1}} \end{aligned}$$

The market clearing condition now reads as

$$0 = x_n + z_n + y_n$$

$$\gamma_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n \lambda_n} = \frac{\beta_n \Delta t_n}{2}$$

$$\begin{aligned} \beta_n \Delta t_n \left(2\lambda_n - 2a_n \lambda_n^2\right) &= 1 + \beta_n \Delta t_n \lambda_n (1 - 2a_n \lambda_n) \\ &\iff 2\beta_n \Delta t_n \lambda_n = 1 + \beta_n \Delta t_n \lambda_n \\ &\iff \beta_n \Delta t_n \lambda_n = 1 \\ &\iff \lambda_n = \frac{1}{\beta_n \Delta t_n} \end{aligned}$$

$$b_{n-1} := a_n \lambda_n^2 \sigma_z^2 \Delta t_n + b_n$$

$$\begin{aligned} a_{n-1} &:= a_n + \gamma_n (1 - 2a_n \lambda_n) - \gamma_n^2 \left(\lambda_n - a_n \lambda_n^2\right) \\ &= a_n + \frac{\beta_n \Delta t_n}{2} \left(1 - 2a_n \frac{1}{\beta_n \Delta t_n}\right) - \frac{\beta_n^2 \Delta t_n^2}{4} \left(\frac{1}{\beta_n \Delta t_n} - a_n \frac{1}{\beta_n^2 \Delta t_n^2}\right) \\ &= a_n + \frac{\beta_n \Delta t_n}{2} - a_n - \frac{\beta_n \Delta t_n}{4} + \frac{a_n}{4} = \frac{1}{4} (\beta_n \Delta t_n + a_n) = \frac{1}{4} \left(\frac{1}{\lambda_n} + a_n\right) \end{aligned}$$

■

The careful reader observes that these results are consistent with the results in Kyle (1985) [Kyl85].

6 Two particular cases

6.1 No order book

Here we consider the particular case, when there is no order book at some time t_n. Based on our previous results, there should be no equilibrium. As a necessary condition for an equilibrium to exist, we derived that $\alpha_n > 0$. Therefore, no equilibrium can exist with $\alpha_n = 0$. Here we want to show intuitively how the market maker can make infinite profits in this case and how the other market participants can deviate from their strategy.

Theorem 35 *If $\alpha_n = 0$ for some $n = 1, \ldots, N$, then the market maker can make infinite profits with probability one.*

Proof. Assume that, for some $n = 1, \ldots, N$, $\alpha_n = 0$. Then the market maker observes the total order flow $x_n + z_n$. If $x_n + z_n > 0$, he chooses λ_n as large as possible. His profit is then

$$-\left(x_n + z_n\right)\left(\nu - p_n\right) \rightarrow \infty \text{ as } \lambda_n \rightarrow \infty$$

Of course, given a strategy at time t_n for the market maker, represented by λ_n, the market maker will always deviate by choosing a larger λ_n. Therefore, no equilibrium can exist. Similarly if $x_n + z_n < 0$, no equilibrium exist. The case $x_n + z_n = 0$ has probability zero and will therefore not be considered by us. ■

Again, this shows that for the existence of our equilibrium it was crucial to have a nontrivial order book. This result is consistent with earlier results in the literature. In a static model, Dennert [Den93],has already shown a nonexistence result. Even the existence of two market makers is not sufficient to guarantee an equilibrium, see e.g. Bondarenko [Bon01].

6.2 The one-period model

If we choose $N = 1$ we are able to obtain several previous results, known in the literature, in an easy and straightforward way. Again our setup is easy and general enough to allow for particular cases in an extremely convenient way. In particular, we obtain the model by Bondarenko and Sung [BS03]: We first observe that $\nu_0 = p_0 = \nu_0^L$. The interested reader can easily do the calculations by using the following formulas

a)

$$\theta := \frac{\gamma^2 \sigma_0^2 + \sigma_z^2}{\gamma \sigma_0^2} = \frac{\gamma^2 + \rho^2}{\gamma}$$

where $\rho^2 = \frac{\sigma_z^2}{\sigma_0^2}$,

b)

$$s := \left(E\left(\alpha\right) E\left(\frac{1}{\alpha}\right)\right)^{\frac{1}{4}},$$

c)

$$\frac{\rho}{s} = \gamma,$$

d)

$$E\left(\alpha\right) = \rho\left(s + s^3\right),$$

and

e)

$$\lambda\left(\alpha\right) = \frac{1}{2\alpha} + \frac{1}{2\rho} \frac{s}{s^2 + 1}.$$

Furthermore, we also obtain a formula for the expected profit of the market maker, which is not in their paper and which is given by

$$\max_{\lambda} E\left(\pi^M \left|\mathcal{M}_1\right.\right) = c\left(x + z\right)^2$$

where

$$c = \frac{\left(\theta - \alpha\right)^2}{4\alpha\theta^2} \geqslant 0$$

Intuitively, the further θ and α are apart, the larger is the expected profit of the market maker.

7 Comparative statistics

We end this paper with a look at a few very interesting lemmas and return to the setup of the basic model as in Section 2. We want to look at the behaviour of certain parameters: First we consider $\lambda_n(\alpha_n)$.

Lemma 36 *Assume that $\alpha_n - c_{n+1}\gamma_{n+1}^2 > 0$. If the market order quantity is positive then the slope of the pricing rule increases if the expectation about the asset value as seen by the market maker increases. The pricing rule becomes less steep if ν_n increases provided the total market order quantity is negative. In short, we can write*

$$\begin{aligned} \frac{\partial \lambda_n}{\partial \nu_n} &> 0 \qquad \text{if} \qquad x_n + z_n > 0 \\ \frac{\partial \lambda_n}{\partial \nu_n} &< 0 \qquad \text{if} \qquad x_n + z_n < 0 \end{aligned}$$

Economically this is meaningful, since the market maker has to adjust the new price p_n to the new conditions.

Proof. The proof is straightforward:

$$\begin{aligned} \frac{\partial \lambda_n}{\partial \nu_n} &= \frac{\partial}{\partial \nu_n} \frac{\alpha_n \nu_n - 2c_{n+1}\gamma_{n+1}^2 \nu_n}{2(x_n + z_n)\alpha_n - 2c_{n+1}\gamma_{n+1}^2 (x_n + z_n)} \\ &= \frac{1}{2(x_n + z_n)\left(\alpha_n - 2c_{n+1}\gamma_{n+1}^2\right)} \cdot \left(\alpha_n - 2c_{n+1}\gamma_{n+1}^2\right) \\ &= \frac{1}{2(x_n + z_n)} \end{aligned}$$

■

Lemma 37 *We get*

$$\frac{\partial x_n}{\partial E(\lambda_n | \mathcal{I}_{n-1})} > 0 \qquad \text{if} \qquad \nu < p_{n-1}$$

and

$$\frac{\partial x_n}{\partial E(\lambda_n | \mathcal{I}_{n-1})} < 0 \qquad \text{if} \qquad \nu > p_{n-1}$$

Economically, this makes sense. If the previous price p_{n-1} is greater than the true price ν and the expected conditional value $E(\lambda_n | \mathcal{I}_{n-1})$ increases, the order quantity x_n increases.

Proof. This immediately follows from

$$x_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n | \mathcal{I}_{n-1})} (\nu - p_{n-1})$$

■

Lemma 38 *Clearly, the market maker learns about the true asset value ν from the combined order flow $x_n + z_n$. If this order flow increases, then ν_n also increases, i.e.*

$$\frac{\partial \nu_n}{\partial (x_n + z_n)} > 0$$

Proof. Clearly, looking at the formula for ν_n and taking the partial derivative with respect to $x_n + z_n$ yields

$$\nu_n = \nu_{n-1} + \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}} \left(x_n + z_n - \gamma_n (\nu_{n-1} - p_{n-1})\right)$$

and

$$\frac{\partial \nu_n}{\partial (x_n + z_n)} = \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}$$

■

Lemma 39 *Assume that* λ_n *is independent of* $\mathcal{I}_{n-1}$. *Then*

$$-(\nu - p_{n-1}) \leqslant E(\nu - p_n | \mathcal{I}_{n-1}) \leqslant (\nu - p_{n-1})$$

More precisely,

$$-(\nu - p_{n-1})(1 - \gamma_n E(\lambda_n(\alpha_n))) \leqslant E(\nu - p_n | \mathcal{I}_{n-1}) \leqslant (1 - \gamma_n E(\lambda_n(\alpha_n)))(\nu - p_{n-1})$$

The larger $\gamma_n \lambda_n(\alpha_n)$ *is, the closer will be the price* p_n *to* ν. *In other words, if the insider trades very aggressively on his information, i.e. if* β_n *is very large, the price will approach the true value very fast.*

Proof. We calculate

$$\begin{aligned}
& E(\nu - p_n | \mathcal{I}_{n-1}) \\
= \; & E(\nu - p_{n-1} - \lambda_n(\alpha_n)(x_n + z_n) | \mathcal{I}_{n-1}) \\
= \; & \nu - p_{n-1} - E(\lambda_n(\alpha_n) x_n | \mathcal{I}_{n-1}) \\
= \; & \nu - p_{n-1} - E(\lambda_n(\alpha_n) \gamma_n (\nu - p_{n-1}) | \mathcal{I}_{n-1}) \\
= \; & \nu - p_{n-1} - \gamma_n (\nu - p_{n-1}) E(\lambda_n(\alpha_n)) \\
= \; & (\nu - p_{n-1})(1 - \gamma_n E(\lambda_n(\alpha_n)))
\end{aligned}$$

and look at

$$\begin{aligned}
& 1 - \gamma_n \lambda_n(\alpha_n) \\
= \; & 1 - \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n(\alpha_n))} E(\lambda_n(\alpha_n)).
\end{aligned}$$

Clearly

$$0 \leqslant 1 - \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n(\alpha_n))} E(\lambda_n(\alpha_n)) \leqslant 1$$

■

Here we want to show that $a_n \geqslant 0$ and $b_n \geqslant 0$, which we already used in a previous remark.

Lemma 40 *Assume that* $E(\lambda_n | \mathcal{I}_{n-1}) > a_n E(\lambda_n^2 | \mathcal{I}_{n-1})$ *for all* $n = 1, \ldots, N$ *and* $\beta_n \geqslant 0$. *Then the coefficients* a_n *and* b_n *are nonnegative.*

Proof. We proceed by backward induction: $\alpha_{N+1} = 0 = b_{N+1}$. Now assume that $a_n, b_n \geqslant 0$. We use the recursive formulas for a_n and b_n.

$$b_{n-1} = a_n E(\lambda_n^2 | \mathcal{I}_{n-1}) \sigma_z^2 \Delta t_n + b_n$$

and

$$a_{n-1} = \gamma_n - \gamma_n^2 E(\lambda_n | \mathcal{I}_{n-1}) + a_n + a_n \gamma_n^2 E(\lambda_n^2 | \mathcal{I}_{n-1}) - 2 a_n \gamma_n E(\lambda_n | \mathcal{I}_{n-1})$$

Clearly, this shows already, that $b_{n-1} \geqslant 0$. Furthermore γ_n is given by

$$\gamma_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n | \mathcal{I}_{n-1})}$$

From the second-order conditions

$$E(\lambda_n | \mathcal{I}_{n-1}) > a_n E(\lambda_n^2 | \mathcal{I}_{n-1})$$

and the equation

$$\frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n | \mathcal{I}_{n-1})} = \frac{1 - 2 a_n E(\lambda_n | \mathcal{I}_{n-1})}{2E(\lambda_n | \mathcal{I}_{n-1}) - 2 a_n E(\lambda_n^2 | \mathcal{I}_{n-1})} = \gamma_n$$

we obtain that

$$\begin{aligned} 1 - 2a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) &\geqslant 0, \\ \gamma_n &\leqslant 1 \end{aligned}$$

and

$$1 - 2a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) \leqslant 2E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) - 2a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)$$

We can use these results to estimate

$$\begin{aligned} a_{n-1} &= \gamma_n - \gamma_n^2 E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) + a_n + a_n \gamma_n^2 E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) - 2a_n \gamma_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) \\ &= \gamma_n \left(1 - 2a_n \gamma_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) + \gamma_n \left(a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) - E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right)\right) + a_n \\ &= \gamma_n \left(1 - 2a_n \gamma_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) + \frac{\left(1 - 2a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right)\left(a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right) - E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right)}{2E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) - 2a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)}\right) + a_n \\ &= \gamma_n \left(1 - 2a_n \gamma_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) - \frac{1}{2}\left(1 - 2a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right)\right) + a_n \\ &= \gamma_n \left(\frac{1}{2} + a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\left(1 - 2\gamma_n\right)\right) + a_n \\ &\geqslant \gamma_n \left(\frac{1}{2} - a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right) + a_n \geqslant a_n \end{aligned}$$

as desired. ■

Here we look at the behaviour of λ_n if α_n changes. Based on the formula $p_n = p_{n-1} + \lambda_n(\alpha_n)(x_n + z_n)$ one would expect that λ_n increases if α_n increases, provided that the limit order traders and the market traders actually order a positive amount. This is what we want to show next.

Lemma 41

$$\frac{\partial \lambda_n}{\partial \alpha_n} \begin{cases} > 0 & \text{if } \lambda_n < \frac{\nu_{n-1}^L - p_{n-1}}{x_n + z_n} \\ < 0 & \text{if } \lambda_n > \frac{\nu_{n-1}^L - p_{n-1}}{x_n + z_n} \end{cases}$$

Proof. We use the market clearing condition

$$x_n + y_n + z_n + \alpha_n \left(\nu_{n-1}^L - p_{n-1} - \lambda_n(\alpha_n)(x_n + z_n)\right) = 0 \tag{10}$$

to find the partial derivative of λ_n with respect to α_n. Solving 10 for $\lambda_n(\alpha_n)$ yields

$$\lambda_n(\alpha_n) = \frac{x_n + y_n + z_n + \alpha_n \nu_{n-1}^L - \alpha_n p_{n-1}}{\alpha_n (x_n + z_n)}$$

Taking the derivative:

$$\frac{\partial \lambda_n}{\partial \alpha_n} = \frac{\alpha_n (x_n + z_n)\left(\nu_{n-1}^L - p_{n-1}\right) - \left(x_n + y_n + z_n + \alpha_n \nu_{n-1}^L - \alpha_n p_{n-1}\right)(x_n + z_n)}{\alpha_n^2 (x_n + z_n)^2}$$

$$= -\frac{(x_n + y_n + z_n)}{\alpha_n^2 (x_n + z_n)} = \frac{\alpha_n \left(\nu_{n-1}^L - p_n\right)}{\alpha_n^2 (x_n + z_n)} = \frac{\nu_{n-1}^L - p_n}{\alpha_n (x_n + z_n)}$$

First we make the assumption that $x_n + z_n > 0$, i.e. the cumulative market order is positive. Then

$$\begin{aligned} \frac{\partial \lambda_n}{\partial \alpha_n} &> 0 \quad \text{if} \quad \nu_{n-1}^L - p_n > 0 \\ \frac{\partial \lambda_n}{\partial \alpha_n} &< 0 \quad \text{if} \quad \nu_{n-1}^L - p_n < 0 \end{aligned}$$

Clearly, if the market depth increases and both the market and the limit orders are positive, the market maker sets a steeper pricing rule. For the case $x_n + z_n < 0$ we get similar results. This yields

$$\frac{\partial \lambda_n}{\partial \alpha_n} \begin{cases} > 0 & \text{if } \left(\nu_{n-1}^L - p_n > 0 \text{ and } x_n + z_n > 0\right) \text{ or } \left(\nu_{n-1}^L - p_n < 0 \text{ and } x_n + z_n < 0\right) \\ < 0 & \text{if } \left(\nu_{n-1}^L - p_n > 0 \text{ and } x_n + z_n < 0\right) \text{ or } \left(\nu_{n-1}^L - p_n < 0 \text{ and } x_n + z_n > 0\right) \end{cases}$$

We can express this differently and get

$$\frac{\partial \lambda_n}{\partial \alpha_n} \begin{cases} > 0 & \text{if } \lambda_n < \frac{\nu_{n-1}^L - p_{n-1}}{x_n + z_n} \\ < 0 & \text{if } \lambda_n > \frac{\nu_{n-1}^L - p_{n-1}}{x_n + z_n} \end{cases}$$

■

Next we want to look at the signed participation rate $R_n(\alpha_n, x_n + z_n)$ defined as

$$R_n(\alpha_n, x_n + z_n) = \frac{y_n}{x_n + z_n}$$

Lemma 42

$$\begin{aligned} R_n(\alpha_n, x_n + z_n) &> 0 \\ &\iff \alpha_n > \frac{x_n + z_n}{p_n - \nu_{n-1}^L} \text{ and } (x_n + z_n)\left(\nu_{n-1}^L - p_n\right) < 0 \\ \text{or } \alpha_n &< \frac{x_n + z_n}{p_n - \nu_{n-1}^L} \textit{ and } (x_n + z_n)\left(\nu_{n-1}^L - p_n\right) > 0 \end{aligned}$$

Proof. We estimate

$$\begin{aligned} R_n(\alpha_n, x_n + z_n) &> 0 \\ &\iff \frac{\alpha_n\left(p_n - \nu_{n-1}^L\right)}{x_n + z_n} > 1 \end{aligned}$$

First assume that $x_n + z_n > 0$ and $p_n - \nu_{n-1}^L > 0$. Then

$$\begin{aligned} R_n(\alpha_n, x_n + z_n) &> 0 \iff \frac{\alpha_n\left(p_n - \nu_{n-1}^L\right)}{x_n + z_n} > 1 \\ &\iff \alpha_n > \frac{x_n + z_n}{p_n - \nu_{n-1}^L} \end{aligned}$$

Therefore, if α_n is large enough, the market maker and the market traders trade in the same direction. Obviously, if α_n is large, the limit order traders provide the necessary liquidity and therefore, both market traders and the market maker can take liquidity out of the market. With similar calculations, we get

$$\begin{aligned} R_n(\alpha_n, x_n + z_n) &> 0 \\ &\iff \alpha_n > \frac{x_n + z_n}{p_n - \nu_{n-1}^L} \text{ and } (x_n + z_n)\left(\nu_{n-1}^L - p_n\right) < 0 \\ \text{or } \alpha_n &< \frac{x_n + z_n}{p_n - \nu_{n-1}^L} \text{ and } (x_n + z_n)\left(\nu_{n-1}^L - p_n\right) > 0 \end{aligned}$$

■

Remark 43 *This means that the market maker and the market traders trade in the same direction provided α_n is large and both market and limit traders trade in different directions. Furthermore the market maker and the market traders trade also in the same direction provided α_n is small and market traders and limit traders trade in different directions.*

These short observations finish the main part of this paper. In the appendix we provide the remaining proofs.

Appendix

Before starting with the main calculations, we state and prove a preliminary lemma.

Lemma 44 *ν is not $\mathcal{M}_n-$measurable for all n.*

Proof. Recall that

$$\mathcal{M}_n = \sigma\left(\alpha_1, \ldots, \alpha_n, p_1, \ldots, p_{n-1}, p_n, x_1 + z_1, \ldots, x_n + z_n, y_1, \ldots, y_n\right)$$

and that z_n is independent of both ν and $\alpha_k, 1 \leqslant k \leqslant n$. From this we immediately see that $\nu \notin \mathcal{M}_n$. ■

Remark 45 *Clearly, the previous result is necessary in order for us to be able to set up the model the way we wanted to: The insider should really be the only one who knows the exact realization of ν.*

From now on, we shall use that for $n = 1, \ldots, N$, ν is not $\mathcal{M}_n-$measurable.

A The informed trader

Here we want to prove Theorems 12, 13 as well as Lemma 16. We start with a preliminary lemma.

Lemma 46 *From the assumption*

$$\max_{(x_{n+1},\ldots,x_N)} E\left(\pi_{n+1}^I \left| \mathcal{I}_n\right.\right) = a_n \left(\nu - p_n\right)^2 + b_n$$

we get

$$\max_{(x_{n+1},\ldots,x_N)} E\left(\pi_{n+1}^I \left| \mathcal{I}_{n-1}\right.\right) = \max_{(x_{n+1},\ldots,x_N)} E\left(E\left(\pi_{n+1}^I \left| \mathcal{I}_n\right.\right) \left| \mathcal{I}_{n-1}\right.\right) = \max_{(x_{n+1},\ldots,x_N)} E\left(a_n\left(\nu - p_n\right)^2 + b_n \left| \mathcal{I}_{n-1}\right.\right)$$

The informed trader solves

$$\begin{aligned} & \max_{(x_n,\ldots,x_N)} E\left(\left(\nu - p_n\right) x_n + \pi_{n+1}^I \left| \mathcal{I}_{n-1}\right.\right) \\ = & \max_{x_n} E\left(\left(\nu - p_n\right) x_n + a_n\left(\nu - p_n\right)^2 + b_n \left| \mathcal{I}_{n-1}\right.\right) \end{aligned}$$

We shall assume that a_n and b_n are $\mathcal{I}_{n-1}-$measurable.

$$\begin{aligned} & \max_{(x_n,\ldots,x_N)} E\left(\left(\nu - p_n\right) x_n + \pi_{n+1}^I \left| \mathcal{I}_{n-1}\right.\right) \\ = & \max_{x_n} E\left(\left(\nu - p_{n-1} - \lambda_n\left(x_n + z_n\right)\right) x_n + a_n\left(\nu - p_{n-1} - \lambda_n\left(x_n + z_n\right)\right)^2 + b_n \left| \mathcal{I}_{n-1}\right.\right) \\ = & \max_{x_n} E\left(\left(\nu - p_{n-1} - \lambda_n\left(x_n + z_n\right)\right) x_n \left| \mathcal{I}_{n-1}\right.\right) \\ & + E\left(a_n\left(\left(\nu - p_{n-1}\right)^2 + \lambda_n^2\left(x_n + z_n\right)^2 - 2\left(\nu - p_{n-1}\right)\lambda_n\left(x_n + z_n\right)\right) + b_n \left| \mathcal{I}_{n-1}\right.\right) \\ = & \max_{x_n}\left(\nu - p_{n-1} - E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) x_n\right) x_n \\ & + E\left(a_n\left(\nu - p_{n-1}\right)^2 + a_n\lambda_n^2\left(x_n^2 + z_n^2 + 2x_nz_n\right) - 2a_n\left(\nu - p_{n-1}\right)\lambda_n\left(x_n + z_n\right) + b_n \left| \mathcal{I}_{n-1}\right.\right) \\ = & \max_{x_n}\left(\nu - p_{n-1} - E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) x_n\right) x_n + a_n\left(\nu - p_{n-1}\right)^2 + a_n x_n^2 E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right) \\ & + a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right)\sigma_z^2 \Delta t_n - 2a_n\left(\nu - p_{n-1}\right) x_n E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) + b_n \end{aligned}$$

Note that we implicitly used that

$$E\left(z_n \left| \mathcal{I}_{n-1}\right.\right) = 0$$

and

$$
\begin{aligned}
E\left(z_n^2 \left| \mathcal{I}_{n-1}\right.\right) &= E\left(z_n^2\right) \\
&= \sigma_z^2 \Delta t_n
\end{aligned}
$$

since z_n is independent of $\mathcal{I}_{n-1}$. In a first step we assume that $E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) \neq a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right)$. Looking at the first order conditions and differentiating the following expression with respect to x_n

$$
\begin{aligned}
&\left(\nu - p_{n-1} - E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) x_n\right) x_n + a_n \left(\nu - p_{n-1}\right)^2 \\
&+ a_n x_n^2 E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right) + a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right) \sigma_z^2 \Delta t_n - 2a_n \left(\nu - p_{n-1}\right) x_n E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) + b_n
\end{aligned}
$$

yields

$$
\begin{aligned}
& \nu - p_{n-1} - 2E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) x_n + 2a_n x_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right) - 2a_n \left(\nu - p_{n-1}\right) E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) = 0 \\
\iff \quad & x_n \left(-2E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) + 2a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right)\right) = -\nu + p_{n-1} + 2a_n \left(\nu - p_{n-1}\right) E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) \\
\iff \quad & x_n = \frac{-\nu + p_{n-1} + 2a_n \left(\nu - p_{n-1}\right) E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right)}{-2E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) + 2a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right)} \\
\iff \quad & x_n = \frac{-1 + 2a_n E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right)}{-2E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) + 2a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right)} \left(v - p_{n-1}\right)
\end{aligned}
$$

Therefore, using Lemma 10 we get

$$
\frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right)} = \frac{-1 + 2a_n E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right)}{-2E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) + 2a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right)}
$$

The second-order conditions yield

$$
\begin{aligned}
-2E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) + 2a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right) &< 0 \\
&\iff E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) - a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right) > 0
\end{aligned}
$$

In the next theorem we look at the case where $E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) = a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right)$ for some n, i.e. we prove Theorem 13.

Proof of Theorem 13

Consider some n for which

$$
E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) = a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right)
$$

then the first order conditions are zero if and only if

$$
\begin{aligned}
\nu - p_{n-1} - 2a_n \left(\nu - p_{n-1}\right) E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) &= 0 \\
&\iff \left(\nu - p_{n-1}\right)\left(1 - 2a_n E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right)\right) = 0 \\
&\iff \nu = p_{n-1} \text{ or } a_n E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) = \frac{1}{2}
\end{aligned}
$$

The first condition implies $\nu \in \mathcal{M}_{n-1}$. This means that ν is known to the market maker after round $n-1$. We discussed this case before and excluded it. From the second condition

$$
a_n E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) = \frac{1}{2}
$$

we get with

$$
E\left(\lambda_n \left| \mathcal{I}_{n-1}\right.\right) = a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right)
$$

that

$$
\frac{1}{2} = a_n^2 E\left(\lambda_n^2 \left| \mathcal{I}_{n-1}\right.\right)
$$

Here the first order conditions are always zero. Therefore

$$\begin{aligned} & \max_{x_n} \left(\nu - p_{n-1} - E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) x_n\right) x_n + a_n \left(\nu - p_{n-1}\right)^2 + a_n x_n^2 E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right) \\ & + a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right) \sigma_z^2 \Delta t_n - 2a_n \left(\nu - p_{n-1}\right) x_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) + b_n \\ = \; & a_n \left(\nu - p_{n-1}\right)^2 + E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) \sigma_z^2 \Delta t_n + b_n \end{aligned}$$

From this we set

$$a_{n-1} := a_n$$

and

$$b_{n-1} := E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) \sigma_z^2 \Delta t_n + b_n$$

Now, in period t_n we therefore have several choices of x_n. The conditional expected profit of the insider will be the same, no matter what x_n he chooses. Therefore the strategy of the insider is not unique anymore. We want to add a small remark: The condition $a_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) = \frac{1}{2}$ cannot be satisfied in the last trading period since $a_N = 0$.

For the case $E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) \neq a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right)$ for all $n = 1, \ldots, N$ we have a closer look at the maximal expected conditional profit. This allows us to compute the recursive formulas for a_n and b_n.

$$\begin{aligned} & \max_{(x_n, \ldots, x_N)} E\left(\left(\nu - p_n\right) x_n + \pi_{n+1}^I \left| \mathcal{I}_{n-1} \right.\right) \\ = \; & \max_{x_n} \left(\nu - p_{n-1} - E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) x_n\right) x_n + a_n \left(\nu - p_{n-1}\right)^2 \\ & + a_n x_n^2 E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right) + a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right) \sigma_z^2 \Delta t_n - 2a_n \left(\nu - p_{n-1}\right) x_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) + b_n \\ = \; & \left(\nu - p_{n-1} - E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) \gamma_n \left(\nu - p_{n-1}\right)\right) \gamma_n \left(\nu - p_{n-1}\right) \\ & + a_n \left(\nu - p_{n-1}\right)^2 + a_n \left(\gamma_n \left(\nu - p_{n-1}\right)\right)^2 E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right) \\ & + a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right) \sigma_z^2 \Delta t_n - 2a_n \left(\nu - p_{n-1}\right)^2 \gamma_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) + b_n \\ = \; & \left(\nu - p_{n-1}\right)^2 \left(\gamma_n - \gamma_n^2 E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) + a_n + a_n \gamma_n^2 E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right) - 2a_n \gamma_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right)\right) \\ & + a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right) \sigma_z^2 \Delta t_n + b_n \end{aligned}$$

Recall that

$$\gamma_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right)}$$

From this we set

$$b_{n-1} := a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right) \sigma_z^2 \Delta t_n + b_n$$

and

$$\begin{aligned} a_{n-1} & := \gamma_n - \gamma_n^2 E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) + a_n + a_n \gamma_n^2 E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right) - 2a_n \gamma_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) \\ & = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right)} - \left(\frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right)}\right)^2 E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) \\ & + a_n + a_n \left(\frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right)}\right)^2 E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right) \\ & - 2a_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right)} \\ & = a_n + \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right)} \left(1 - 2a_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right)\right) \\ & - \left(\frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right)}\right)^2 \left(E\left(\lambda_n \left| \mathcal{I}_{n-1} \right.\right) - a_n E\left(\lambda_n^2 \left| \mathcal{I}_{n-1} \right.\right)\right) \end{aligned}$$

Clearly, this implies that a_n and b_n are indeed $\mathcal{I}_{n-1}-$measurable.

We consider two more cases. First

$$(\nu - p_{n-1})\left(1 - 2a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right) < 0$$

Then the insider chooses the quantity minus infinity, so there is no equilibrium. Second

$$(\nu - p_{n-1})\left(1 - 2a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)\right) > 0$$

Then the insider chooses a quantity plus infinity and he gains an infinite amount of money. Then y_n would be minus infinity and the market maker would make an infinite loss. Of course the market maker can deviate from this strategy by picking a different λ_n. Therefore he will do so and this case cannot occur in equilibrium. Summing up, we see that λ_n will not be chosen such that $E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) = a_n E\left(\lambda_n^2 \left|\mathcal{I}_{n-1}\right.\right)$, except if in addition $a_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right) = \frac{1}{2}$.

B The market maker

Assume that α_n, β_n and λ_n are $\mathcal{M}_n-$measurable. Furthermore assume that λ_n is independent of $\mathcal{I}_n$. (Clearly, this could be justified by assuming that α_n is independent of $\mathcal{I}_n$, which is one of our basic features of the model, and the assumption that λ_n is e.g. a deterministic function of α_n) We also assume that ν is independent of α_n. First we start with proving the formulas for ν_n and σ_n^2 as in Theorem 18. We look at

$$x_n + z_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E\left(\lambda_n \left|\mathcal{I}_{n-1}\right.\right)}(\nu - p_{n-1}) + z_n$$

Assume that x_n is $\mathcal{M}_n-$measurable and λ_n is independent of $\mathcal{I}_{n-1}$. Then

$$x_n + z_n = \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n)}(\nu - p_{n-1}) + z_n$$

We assume that β_n is $\mathcal{M}_{n-1}-$measurable and independent of ν. For the forthcoming calculations, also note that ν is independent of α_n. We start with the conditional expected value of ν.

$$\begin{aligned} E\left(\nu \left|\mathcal{M}_n\right.\right) &= E\left(v \left|\mathcal{M}_{n-1}\right.\right) + \frac{Cov\left(\nu, x_n + z_n \left|\mathcal{M}_{n-1}\right.\right)}{Var\left(x_n + z_n \left|\mathcal{M}_{n-1}\right.\right)}\left(x_n + z_n - E\left(x_n + z_n \left|\mathcal{M}_{n-1}\right.\right)\right) \\ &= \nu_{n-1} + \frac{Cov\left(\nu, x_n + z_n \left|\mathcal{M}_{n-1}\right.\right)}{Var\left(x_n + z_n \left|\mathcal{M}_{n-1}\right.\right)}\left(x_n + z_n - E\left(x_n \left|\mathcal{M}_{n-1}\right.\right)\right) \end{aligned}$$

First we calculate

$$\begin{aligned} & Cov\left(\nu, x_n + z_n \left|\mathcal{M}_{n-1}\right.\right) = Cov\left(\nu, \frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n)}(\nu - p_{n-1}) + z_n \left|\mathcal{M}_{n-1}\right.\right) \\ = \; & Cov\left(\nu, \gamma_n (\nu - p_{n-1}) + z_n \left|\mathcal{M}_{n-1}\right.\right) = Cov\left(\nu, \gamma_n (\nu - p_{n-1}) \left|\mathcal{M}_{n-1}\right.\right) = \gamma_n \sigma_{n-1}^2 \end{aligned}$$

and

$$\begin{aligned} & Var\left(x_n + z_n \left|\mathcal{M}_{n-1}\right.\right) = Var\left(\frac{\beta_n \Delta t_n}{1 + \beta_n \Delta t_n E(\lambda_n)}(\nu - p_{n-1}) + z_n \left|\mathcal{M}_{n-1}\right.\right) \\ = \; & Var\left(\gamma_n (\nu - p_{n-1}) + z_n \left|\mathcal{M}_{n-1}\right.\right) = Var\left(\gamma_n (\nu - p_{n-1}) + z_n \left|\mathcal{M}_{n-1}\right.\right) = \gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1} \end{aligned}$$

We assume that $\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1} \neq 0$. Using this, we can calculate

$$\begin{aligned} E\left(\nu \left|\mathcal{M}_n\right.\right) &= \nu_{n-1} + \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}\left(x_n + z_n - \gamma_n (\nu_{n-1} - p_{n-1})\right) \\ &= \nu_{n-1} + \frac{\gamma_n \sigma_{n-1}^2}{\gamma_n^2 \sigma_{n-1}^2 + \sigma_z^2 \Delta t_{n-1}}\left(x_n + z_n - \gamma_n (\nu_{n-1} - p_{n-1})\right) \end{aligned}$$

as well as

$$\begin{aligned}
\sigma_n^2 &= Var\left(\nu\left|\mathcal{M}_n\right.\right) = Var\left(\nu\left|\mathcal{M}_{n-1}\right.\right) - \frac{\left(Cov\left(\nu, x_n + z_n\right)\left|\mathcal{M}_{n-1}\right.\right)^2}{Var\left(x_n + z_n\left|\mathcal{M}_{n-1}\right.\right)} \\
&= \sigma_{n-1}^2 - \frac{\gamma_n^2\sigma_{n-1}^4}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}} = \sigma_{n-1}^2 - \frac{\gamma_n^2\sigma_{n-1}^4}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}} \\
&= \frac{\gamma_n^2\sigma_{n-1}^4 + \sigma_{n-1}^2\sigma_z^2\Delta t_{n-1} - \gamma_n^2\sigma_{n-1}^4}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}} = \frac{\sigma_{n-1}^2\sigma_z^2\Delta t_{n-1}}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}
\end{aligned}$$

(Note that λ_n is independent of ν) From this

$$\gamma_n \frac{\sigma_n^2}{\sigma_z^2\Delta t_{n-1}} = \frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}$$

and

$$\frac{\sigma_n^2}{\sigma_z^2\Delta t_{n-1}} = \frac{\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}$$

Then

$$\begin{aligned}
\nu_n &= \nu_{n-1} + \gamma_n \frac{\sigma_n^2}{\sigma_z^2\Delta t_{n-1}}\left(x_n + z_n - \gamma_n\left(\nu_{n-1} - p_{n-1}\right)\right) \\
&= \nu_{n-1} + \gamma_n \frac{\sigma_n^2}{\sigma_z^2\Delta t_{n-1}}\left(x_n + z_n\right) - \frac{\sigma_n^2}{\sigma_z^2\Delta t_{n-1}}\gamma_n^2\left(\nu_{n-1} - p_{n-1}\right)
\end{aligned}$$

Furthermore

$$E\left(\nu^2\left|\mathcal{M}_n\right.\right) = \frac{\sigma_{n-1}^2\sigma_z^2\Delta t_{n-1}}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}} + \nu_n^2$$

Next we give proofs for Theorems 19 and 20.

Proofs of Theorems 19 and 20

We make the following inductive hypotheses:

$$\max_{(\lambda_{n+1},\ldots,\lambda_N)} E\left(\pi_{n+1}^M\left|\mathcal{M}_{n+1}\right.\right) = c_{n+1}\left(x_{n+1} + z_{n+1}\right)^2 + d_{n+1}\left(x_{n+1} + z_{n+1}\right) + e_{n+1}$$

with the boundary conditions

$$c_{N+1} = d_{N+1} = e_{N+1} = 0$$

and the assumption that c_{n+1}, d_{n+1} and e_{n+1} are $\mathcal{M}_n-$measurable. The market maker solves

$$\max_{\lambda_n} E\left(\left(p_n - \nu\right)\left(x_n + z_n + \alpha_n\left(\nu_{n-1}^L - p_n\right)\right) + c_{n+1}\left(x_{n+1} + z_{n+1}\right)^2 + d_{n+1}\left(x_{n+1} + z_{n+1}\right) + e_{n+1}\left|\mathcal{M}_n\right.\right) \quad (11)$$

First assume that $\alpha_n - c_{n+1}\gamma_{n+1}^2 \neq 0$. Use that λ_n is independent of $\mathcal{I}_n$ and therefore

$$x_{n+1} = \frac{\beta_{n+1}\Delta t_{n+1}}{1 + \beta_{n+1}\Delta t_{n+1}E\left(\lambda_{n+1}\right)}\left(\nu - p_n\right)$$

We proceed in two steps: First we calculate

$$
\begin{aligned}
& E\left(c_{n+1}\left(x_{n+1}+z_{n+1}\right)^2+d_{n+1}\left(x_{n+1}+z_{n+1}\right)+e_{n+1}\left|\mathcal{M}_n\right.\right) \\
= & E\left(c_{n+1}x_{n+1}^2+d_{n+1}x_{n+1}\left|\mathcal{M}_n\right.\right)+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1} \\
= & E\left(c_{n+1}\left(\frac{\beta_{n+1}\Delta t_{n+1}}{1+\beta_{n+1}\Delta t_{n+1}E\left(\lambda_{n+1}\right)}\left(\nu-p_n\right)\right)^2+d_{n+1}\frac{\beta_{n+1}\Delta t_{n+1}}{1+\beta_{n+1}\Delta t_{n+1}E\left(\lambda_{n+1}\right)}\left(\nu-p_n\right)\left|\mathcal{M}_n\right.\right) \\
& +c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1} \\
= & E\left(c_{n+1}\frac{\beta_{n+1}^2\Delta^2 t_{n+1}}{\left(1+\beta_{n+1}\Delta t_{n+1}E\left(\lambda_{n+1}\right)\right)^2}\left(\nu-p_n\right)^2+d_{n+1}\frac{\beta_{n+1}\Delta t_{n+1}}{1+\beta_{n+1}\Delta t_{n+1}E\left(\lambda_{n+1}\right)}\left(\nu-p_n\right)\left|\mathcal{M}_n\right.\right) \\
& +c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1} \\
= & E\left(c_{n+1}\gamma_{n+1}^2\left(\nu-p_n\right)^2+d_{n+1}\gamma_{n+1}\left(\nu-p_n\right)\left|\mathcal{M}_n\right.\right)+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1} \\
= & E\left(c_{n+1}\gamma_{n+1}^2\left(\nu-p_n\right)^2+d_{n+1}\gamma_{n+1}\left(\nu-p_n\right)\left|\mathcal{M}_n\right.\right)+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1} \\
= & c_{n+1}\gamma_{n+1}^2E\left(\left(\nu-p_n\right)^2\left|\mathcal{M}_n\right.\right)+d_{n+1}\gamma_{n+1}E\left(\left(\nu-p_n\right)\left|\mathcal{M}_n\right.\right)+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}
\end{aligned}
$$

Continuing we get

$$
\begin{aligned}
& E\left(c_{n+1}\left(x_{n+1}+z_{n+1}\right)^2+d_{n+1}\left(x_{n+1}+z_{n+1}\right)+e_{n+1}\left|\mathcal{M}_n\right.\right) \\
= & c_{n+1}\gamma_{n+1}^2E\left(\left(\nu-p_n\right)^2\left|\mathcal{M}_n\right.\right)+d_{n+1}\gamma_{n+1}E\left(\left(\nu-p_n\right)\left|\mathcal{M}_n\right.\right)+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1} \\
= & c_{n+1}\gamma_{n+1}^2E\left(\nu^2\left|\mathcal{M}_n\right.\right)-c_{n+1}\gamma_{n+1}^2E\left(2\nu p_n\left|\mathcal{M}_n\right.\right)+c_{n+1}\gamma_{n+1}^2E\left(p_n^2\left|\mathcal{M}_n\right.\right)+ \\
& d_{n+1}\gamma_{n+1}E\left(\nu\left|\mathcal{M}_n\right.\right)-d_{n+1}\gamma_{n+1}E\left(p_n\left|\mathcal{M}_n\right.\right)+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1} \\
= & c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2+\nu_n^2\right)-2p_nc_{n+1}\gamma_{n+1}^2\nu_n+c_{n+1}\gamma_{n+1}^2p_n^2+ \\
& d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}p_n+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}
\end{aligned}
$$

Plugging in the equation for p_n

$$
p_n=p_{n-1}+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)
$$

yields

$$
\begin{aligned}
& E\left(c_{n+1}\left(x_{n+1}+z_{n+1}\right)^2+d_{n+1}\left(x_{n+1}+z_{n+1}\right)+e_{n+1}\left|\mathcal{M}_n\right.\right) \\
= & c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2+\nu_n^2\right)-2\left(p_{n-1}+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\right)c_{n+1}\gamma_{n+1}^2\nu_n \\
& +c_{n+1}\gamma_{n+1}^2\left(p_{n-1}+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\right)^2+ \\
& d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}\left(p_{n-1}+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\right)+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}
\end{aligned}
\tag{12}
$$

Differentiating with respect to λ_n leads to:

$$
\begin{aligned}
& \frac{\partial}{\partial\lambda_n}E\left(c_{n+1}\left(x_{n+1}+z_{n+1}\right)^2+d_{n+1}\left(x_{n+1}+z_{n+1}\right)+e_{n+1}\left|\mathcal{M}_n\right.\right) \\
= & -2\left(x_n+z_n\right)c_{n+1}\gamma_{n+1}^2\nu_n+c_{n+1}\gamma_{n+1}^2 2\left(p_{n-1}+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\right)\left(x_n+z_n\right) \\
& -d_{n+1}\gamma_{n+1}\left(x_n+z_n\right)
\end{aligned}
$$

To continue, we assume that α_{n+1} is independent of $\mathcal{M}_n$. In a second step we calculate

$$
\begin{aligned}
& E\left(\left(p_{n-1}+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)-\nu\right)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-p_{n-1}-\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\right)\right)\left|\mathcal{M}_n\right.\right) \\
= & \left(p_{n-1}+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)-\nu_n\right)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_np_{n-1}-\alpha_n\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\right) \\
= & \left(p_{n-1}-\nu_n\right)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_np_{n-1}\right)+\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)^2 \\
& +\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right)\alpha_n\left(\nu_{n-1}^L-p_{n-1}\right)-\left(p_{n-1}-\nu_n\right)\alpha_n\lambda_n\left(\alpha_n\right)\left(x_n+z_n\right) \\
& -\lambda_n^2\left(\alpha_n\right)\left(x_n+z_n\right)^2\alpha_n
\end{aligned}
$$

(Clearly, $\nu_{n-1}^L = E(\nu \,|\mathcal{F}_{n-1})$ is $\mathcal{M}_n-$measurable.) Differentiating this last expression

$$\begin{aligned}
&(p_{n-1}-\nu_n)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+\lambda_n(\alpha_n)(x_n+z_n)^2\\
&+\lambda_n(\alpha_n)(x_n+z_n)\alpha_n\left(\nu_{n-1}^L-p_{n-1}\right)\\
&-(p_{n-1}-\nu_n)\alpha_n\lambda_n(\alpha_n)(x_n+z_n)-\lambda_n^2(\alpha_n)(x_n+z_n)^2\alpha_n
\end{aligned}$$

with respect to λ_n yields

$$\begin{aligned}
&(x_n+z_n)^2+(x_n+z_n)\alpha_n\left(\nu_{n-1}^L-p_{n-1}\right)\\
&-(p_{n-1}-\nu_n)\alpha_n(x_n+z_n)-2\lambda_n(\alpha_n)(x_n+z_n)^2\alpha_n
\end{aligned} \tag{13}$$

We have to ensure that this term has a finite maximum, therefore we take the second derivative, which yields

$$-2(x_n+z_n)^2\alpha_n$$

This has be to negative, which is equivalent to the condition $\alpha_n > 0$ (with probability one). We add the term (13) to the expression (12) and set the derivative of the sum to zero.

$$\begin{aligned}
&(x_n+z_n)^2+(x_n+z_n)\alpha_n\left(\nu_{n-1}^L-p_{n-1}\right)+-(p_{n-1}-\nu_n)\alpha_n(x_n+z_n)\\
&-2\lambda_n(\alpha_n)(x_n+z_n)^2\alpha_n-2(x_n+z_n)c_{n+1}\gamma_{n+1}^2\nu_n+\\
&2c_{n+1}\gamma_{n+1}^2(p_{n-1}+\lambda_n(\alpha_n)(x_n+z_n))(x_n+z_n)-d_{n+1}\gamma_{n+1}(x_n+z_n)=0
\end{aligned}$$

Solving this for λ_n yields

$$\lambda_n(\alpha_n)=\frac{x_n+z_n+\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma_{n+1}^2(p_{n-1}-\nu_n)-d_{n+1}\gamma_{n+1}}{2(x_n+z_n)\alpha_n-2c_{n+1}\gamma_{n+1}^2(x_n+z_n)} \tag{14}$$

Multiplying by x_n+z_n and $(x_n+z_n)^2$ respectively, we obtain

$$\lambda_n(\alpha_n)(x_n+z_n)=\frac{x_n+z_n+\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma_{n+1}^2(p_{n-1}-\nu_n)-d_{n+1}\gamma_{n+1}}{2\alpha_n-2c_{n+1}\gamma_{n+1}^2}$$

and

$$\lambda_n^2(\alpha_n)(x_n+z_n)^2=\frac{\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma_{n+1}^2(p_{n-1}-\nu_n)-d_{n+1}\gamma_{n+1}\right)^2}{\left(2\alpha_n-2c_{n+1}\gamma_{n+1}^2\right)^2}$$

The maximal conditional expected profit is then

$$\begin{aligned}
&(p_{n-1}-\nu_n)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+\lambda_n(\alpha_n)(x_n+z_n)^2\\
&+\lambda_n(\alpha_n)(x_n+z_n)\alpha_n\left(\nu_{n-1}^L-p_{n-1}\right)\\
&-(p_{n-1}-\nu_n)\alpha_n\lambda_n(\alpha_n)(x_n+z_n)-\lambda_n^2(\alpha_n)(x_n+z_n)^2\alpha_n\\
&+c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2+\nu_n^2\right)-2(p_{n-1}+\lambda_n(\alpha_n)(x_n+z_n))c_{n+1}\gamma_{n+1}^2\nu_n\\
&+c_{n+1}\gamma_{n+1}^2(p_{n-1}+\lambda_n(\alpha_n)(x_n+z_n))^2+\\
&+d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}(p_{n-1}+\lambda_n(\alpha_n)(x_n+z_n))+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}
\end{aligned}$$

Factorizing λ_n and λ_n^2: The maximal profit reads as:

$$\begin{aligned}
&\lambda_n(\alpha_n)(x_n+z_n)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma_{n+1}^2(p_{n-1}-\nu_n)-d_{n+1}\gamma_{n+1}\right)\\
&+\lambda_n^2(\alpha_n)(x_n+z_n)^2\left(-\alpha_n+c_{n+1}\gamma_{n+1}^2\right)+(p_{n-1}-\nu_n)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+\\
&+c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2+\nu_n^2\right)-2p_{n-1}c_{n+1}\gamma_{n+1}^2\nu_n+c_{n+1}\gamma_{n+1}^2p_{n-1}^2+\\
&+d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}p_{n-1}+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}
\end{aligned}$$

Further simplifying and plugging in $\lambda_n(\alpha_n)$ from Equation (14) :

$$\begin{aligned}
&\max_{(\lambda_n,\dots,\lambda_N)} E\left(\pi_n^M \,|\mathcal{M}_n\right) = \\
&\frac{x_n + z_n + \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + \nu_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1} - \nu_n\right) - d_{n+1}\gamma_{n+1}}{2\alpha_n - 2c_{n+1}\gamma_{n+1}^2} \\
&\times\left(x_n + z_n + \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + \nu_n\right) - d_{n+1}\gamma_{n+1} + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1} - \nu_n\right)\right) \\
&+\frac{\left(x_n + z_n + \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + \nu_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1} - \nu_n\right) - d_{n+1}\gamma_{n+1}\right)^2}{\left(2\alpha_n - 2c_{n+1}\gamma_{n+1}^2\right)^2}\left(c_{n+1}\gamma_{n+1}^2 - \alpha_n\right) \\
&+\left(p_{n-1} - \nu_n\right)\left(x_n + z_n + \alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right) + c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2 + \nu_n^2\right) - 2p_{n-1}c_{n+1}\gamma_{n+1}^2\nu_n \\
&+c_{n+1}\gamma_{n+1}^2 p_{n-1}^2 + d_{n+1}\gamma_{n+1}\nu_n - d_{n+1}\gamma_{n+1}p_{n-1} + c_{n+1}\sigma_z^2\Delta t_{n+1} + e_{n+1} \\
=\; &\frac{\left(x_n + z_n + \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + \nu_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1} - \nu_n\right) - d_{n+1}\gamma_{n+1}\right)^2}{4\alpha_n - 4c_{n+1}\gamma_{n+1}^2} \\
&+\left(p_{n-1} - \nu_n\right)\left(x_n + z_n + \alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right) + c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2 + \nu_n^2\right) - 2p_{n-1}c_{n+1}\gamma_{n+1}^2\nu_n \\
&+c_{n+1}\gamma_{n+1}^2 p_{n-1}^2 + d_{n+1}\gamma_{n+1}\nu_n - d_{n+1}\gamma_{n+1}p_{n-1} + c_{n+1}\sigma_z^2\Delta t_{n+1} + e_{n+1}
\end{aligned}$$

Now plugging in the expression for ν_n,

$$\nu_n = \nu_{n-1} + \frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2 + \sigma_z^2\Delta t_{n-1}}\left(x_n + z_n - \gamma_n\left(\nu_{n-1} - p_{n-1}\right)\right)$$

and looking at the terms with $x_n + z_n$, one can see that we can write

$$\max_{(\lambda_n,\dots,\lambda_N)} E\left(\pi_n^M \,|\mathcal{M}_n\right) = c_n\left(x_n + z_n\right)^2 + d_n\left(x_n + z_n\right) + e_n$$

for some $\mathcal{M}_n-$measurable functions c_n, d_n and e_n. From this calculation one also gets that indeed $c_{n+1}, d_{n+1}, e_{n+1}$ are $\mathcal{M}_n-$measurable. Now we look at the second-order conditions. We differentiate the following expression again with respect to λ_n :

$$\begin{aligned}
&\left(x_n + z_n\right)^2 + \left(x_n + z_n\right)\alpha_n\left(\nu_{n-1}^L - p_{n-1}\right) - \left(p_{n-1} - \nu_n\right)\alpha_n\left(x_n + z_n\right) - 2\lambda_n\left(\alpha_n\right)\left(x_n + z_n\right)^2\alpha_n \\
&-2\left(x_n + z_n\right)c_{n+1}\gamma_{n+1}^2\nu_n + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1} + \lambda_n\left(\alpha_n\right)\left(x_n + z_n\right)\right)\left(x_n + z_n\right) - d_{n+1}\gamma_{n+1}\left(x_n + z_n\right)
\end{aligned}$$

which yields

$$\begin{aligned}
-2\left(x_n + z_n\right)^2\alpha_n + 2c_{n+1}\gamma_{n+1}^2\left(x_n + z_n\right)^2 \quad &< \quad 0 \\
\iff \quad &c_{n+1}\gamma_{n+1}^2 < \alpha_n
\end{aligned}$$

This finishes the proof of Theorems 19 and 20.

In the following we want to prove Lemma 21.

Proof of Lemma 21

We have a look at the maximal expected conditional profit.

$$\begin{aligned}
& \max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi_n^M \left| \mathcal{M}_n\right.\right) \\
= \;& \lambda_n(\alpha_n)(x_n+z_n)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma_{n+1}^2(p_{n-1}-\nu_n)-d_{n+1}\gamma_{n+1}\right) \\
& +\lambda_n^2(\alpha_n)(x_n+z_n)^2\left(-\alpha_n+c_{n+1}\gamma_{n+1}^2\right)+(p_{n-1}-\nu_n)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+ \\
& +c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2+\nu_n^2\right)-2p_{n-1}c_{n+1}\gamma_{n+1}^2\nu_n+c_{n+1}\gamma_{n+1}^2p_{n-1}^2+ \\
& +d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}p_{n-1}+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1} \\
= \;& \lambda_n(\alpha_n)(x_n+z_n)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_n\right)+2\alpha_n(p_{n-1}-\nu_n)-d_{n+1}\gamma_{n+1}\right) \\
& +(p_{n-1}-\nu_n)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+\alpha_n\left(\sigma_n^2+\nu_n^2\right)-2p_{n-1}\alpha_n\nu_n \\
& +\alpha_n p_{n-1}^2+d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}p_{n-1}+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1} \\
= \;& \lambda_n(\alpha_n)(x_n+z_n)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L+\nu_n\right)-2\alpha_n\nu_n-d_{n+1}\gamma_{n+1}\right) \\
& +(p_{n-1}-\nu_n)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+\alpha_n\left(\sigma_n^2+\nu_n^2\right)-2p_{n-1}\alpha_n\nu_n \\
& +\alpha_n p_{n-1}^2+d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}p_{n-1}+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}
\end{aligned}$$

We consider three cases. First

$$(x_n+z_n)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L+\nu_n\right)-2\alpha_n\nu_n-d_{n+1}\gamma_{n+1}\right)<0$$

Then the market maker would choose λ_n to be minus infinity, and there will be no equilibrium. (Note that the informed trader could deviate from his strategy by picking a different x_n which gives him a higher profit.) Second

$$(x_n+z_n)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L+\nu_n\right)-2\alpha_n\nu_n-d_{n+1}\gamma_{n+1}\right)>0$$

Then the market maker would choose λ_n to be plus infinity, and there will be no equilibrium. Third

$$(x_n+z_n)\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L+\nu_n\right)-2\alpha_n\nu_n-d_{n+1}\gamma_{n+1}\right)=0$$

Then the conditional expected profit of the market maker will be

$$\begin{aligned}
& (p_{n-1}-\nu_n)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+\alpha_n\left(\sigma_n^2+\nu_n^2\right)-2p_{n-1}\alpha_n\nu_n \\
& +\alpha_n p_{n-1}^2+d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}p_{n-1}+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}
\end{aligned}$$

independent of λ_n. Therefore the market maker has several strategies, he can pick whatever λ_n he desires.

C Calculation of c_n, d_n and e_n

The purpose of this section is to make a previous remark more precise. In the previous subsection we omitted the explicit calculation of the parameters c_n, d_n and e_n. Now we shall give explicit expressions for them. The calculations are quite lengthy and they are just stated here for the convenience of the reader. However they are necessary since they will provide us with the necessary result that c_{n+1}, d_{n+1} and e_{n+1} are $\mathcal{M}_n-$measurable functions.

$$\begin{aligned}
&\max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi_n^M \left|\mathcal{M}_n\right.\right) = \\
&\frac{x_n+z_n+\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-\nu_n\right)-d_{n+1}\gamma_{n+1}}{2\alpha_n-2c_{n+1}\gamma_{n+1}^2} \\
&\times\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_n\right)-d_{n+1}\gamma_{n+1}+2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-\nu_n\right)\right) \\
&+\frac{\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-\nu_n\right)-d_{n+1}\gamma_{n+1}\right)^2}{\left(2\alpha_n-2c_{n+1}\gamma_{n+1}^2\right)^2}\left(c_{n+1}\gamma_{n+1}^2-\alpha_n\right) \\
&+\left(p_{n-1}-\nu_n\right)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2+\nu_n^2\right)-2p_{n-1}c_{n+1}\gamma_{n+1}^2\nu_n \\
&+c_{n+1}\gamma_{n+1}^2p_{n-1}^2+d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}p_{n-1}+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1} \\
=\;&\frac{\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-\nu_n\right)-d_{n+1}\gamma_{n+1}\right)^2}{2\left(\alpha_n-c_{n+1}\gamma_{n+1}^2\right)} \\
&-\frac{\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-\nu_n\right)-d_{n+1}\gamma_{n+1}\right)^2}{4\left(\alpha_n-c_{n+1}\gamma_{n+1}^2\right)} \\
&+\left(p_{n-1}-\nu_n\right)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2+\nu_n^2\right)-2p_{n-1}c_{n+1}\gamma_{n+1}^2\nu_n \\
&+c_{n+1}\gamma_{n+1}^2p_{n-1}^2+d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}p_{n-1}+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1} \\
=\;&\frac{\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_n\right)+2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-\nu_n\right)-d_{n+1}\gamma_{n+1}\right)^2}{4\left(\alpha_n-c_{n+1}\gamma_{n+1}^2\right)} \\
&+\left(p_{n-1}-\nu_n\right)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2+\nu_n^2\right)-2p_{n-1}c_{n+1}\gamma_{n+1}^2\nu_n \\
&+c_{n+1}\gamma_{n+1}^2p_{n-1}^2+d_{n+1}\gamma_{n+1}\nu_n-d_{n+1}\gamma_{n+1}p_{n-1}+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}
\end{aligned}$$

Now plugging in the expression for ν_n,

$$\begin{aligned}
\nu_n &= \nu_{n-1}+\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(x_n+z_n-\gamma_n\left(\nu_{n-1}-p_{n-1}\right)\right) \\
&= \nu_{n-1}+g_n\left(x_n+z_n\right)-g_n\left(\nu_{n-1}-p_{n-1}\right) \\
&= h_n+g_n\left(x_n+z_n\right)
\end{aligned}$$

where

$$\begin{aligned}
g_n &:= \frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}} \\
h_n &:= \nu_{n-1}-g_n\left(\nu_{n-1}-p_{n-1}\right)
\end{aligned}$$

one gets

$$\begin{aligned}
&\max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi_n^M \left|\mathcal{M}_n\right.\right) = \\
&\frac{\left(x_n+z_n+\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+h_n+g_n\left(x_n+z_n\right)\right)+2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-\left(h_n+g_n\left(x_n+z_n\right)\right)\right)-d_{n+1}\gamma_{n+1}\right)^2}{4\left(\alpha_n-c_{n+1}\gamma_{n+1}^2\right)} \\
&+\left(p_{n-1}-\left(h_n+g_n\left(x_n+z_n\right)\right)\right)\left(x_n+z_n+\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+c_{n+1}\gamma_{n+1}^2\left(\sigma_n^2+\left(h_n+g_n\left(x_n+z_n\right)\right)^2\right) \\
&-2p_{n-1}c_{n+1}\gamma_{n+1}^2\left(h_n+g_n\left(x_n+z_n\right)\right)+c_{n+1}\gamma_{n+1}^2p_{n-1}^2+d_{n+1}\gamma_{n+1}\left(h_n+g_n\left(x_n+z_n\right)\right) \\
&-d_{n+1}\gamma_{n+1}p_{n-1}+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}
\end{aligned}$$

Simplifying leads to

$$\max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi_n^M \left|\mathcal{M}_n\right.\right) =$$

$$\frac{\left(x_n + z_n + \alpha_n g_n\left(x_n+z_n\right) - 2c_{n+1}\gamma_{n+1}^2 g_n\left(x_n+z_n\right) + \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + h_n\right)\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)}$$
$$+\frac{\left(2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-h_n\right) - d_{n+1}\gamma_{n+1}\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)}$$
$$+\frac{x_n + z_n + \alpha_n g_n\left(x_n+z_n\right) - 2c_{n+1}\gamma_{n+1}^2 g_n\left(x_n+z_n\right) + \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + h_n\right)}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)}$$
$$\times\frac{2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-h_n\right) - d_{n+1}\gamma_{n+1}}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)}$$
$$+\left(p_{n-1}-h_n\right)\left(x_n+z_n\right) + \left(p_{n-1}-h_n\right)\left(\alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right)$$
$$-g_n\left(x_n+z_n\right)\left(x_n+z_n\right) - g_n\left(x_n+z_n\right)\left(\alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right)$$
$$+c_{n+1}\gamma_{n+1}^2\sigma_n^2 + c_{n+1}\gamma_{n+1}^2 h_n^2 + 2c_{n+1}\gamma_{n+1}^2 h_n g_n\left(x_n+z_n\right)$$
$$+c_{n+1}\gamma_{n+1}^2 g_n^2\left(x_n+z_n\right)^2 - 2p_{n-1}c_{n+1}\gamma_{n+1}^2 h_n - 2p_{n-1}c_{n+1}\gamma_{n+1}^2 g_n\left(x_n+z_n\right)$$
$$+c_{n+1}\gamma_{n+1}^2 p_{n-1}^2 + d_{n+1}\gamma_{n+1}h_n + d_{n+1}\gamma_{n+1}g_n\left(x_n+z_n\right) - d_{n+1}\gamma_{n+1}p_{n-1} + c_{n+1}\sigma_z^2\Delta t_{n+1} + e_{n+1}$$

Factorizing $x_n + z_n$

$$\max_{(\lambda_n,\ldots,\lambda_N)} E\left(\pi_n^M \left|\mathcal{M}_n\right.\right) =$$

$$\frac{\left(\left(x_n+z_n\right)\left(1+\alpha_n g_n - 2c_{n+1}\gamma_{n+1}^2 g_n\right) + \alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + h_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-h_n\right) - d_{n+1}\gamma_{n+1}\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)}$$
$$+\left(x_n+z_n\right)\left(\left(p_{n-1}-h_n\right) - g_n\left(\alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right) + 2c_{n+1}\gamma_{n+1}^2 h_n g_n - 2p_{n-1}c_{n+1}\gamma_{n+1}^2 g_n + d_{n+1}\gamma_{n+1}g_n\right)$$
$$+\left(x_n+z_n\right)^2\left(-g_n + c_{n+1}\gamma_{n+1}^2 g_n^2\right) + \left(p_{n-1}-h_n\right)\left(\alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right) + c_{n+1}\gamma_{n+1}^2\sigma_n^2 + c_{n+1}\gamma_{n+1}^2 h_n^2$$
$$-2p_{n-1}c_{n+1}\gamma_{n+1}^2 h_n + c_{n+1}\gamma_{n+1}^2 p_{n-1}^2 + d_{n+1}\gamma_{n+1}h_n - d_{n+1}\gamma_{n+1}p_{n-1} + c_{n+1}\sigma_z^2\Delta t_{n+1} + e_{n+1}$$
$$= \quad \left(x_n+z_n\right)^2\frac{\left(1+\alpha_n g_n - 2c_{n+1}\gamma_{n+1}^2 g_n\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)}$$
$$+\left(x_n+z_n\right)\frac{2\left(1+\alpha_n g_n - 2c_{n+1}\gamma_{n+1}^2 g_n\right)\left(\alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + h_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-h_n\right) - d_{n+1}\gamma_{n+1}\right)}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)}$$
$$+\frac{\left(\alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + h_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-h_n\right) - d_{n+1}\gamma_{n+1}\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)}$$
$$+\left(x_n+z_n\right)\left(p_{n-1} - h_n - g_n\left(\alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right) + 2c_{n+1}\gamma_{n+1}^2 h_n g_n - 2p_{n-1}c_{n+1}\gamma_{n+1}^2 g_n + d_{n+1}\gamma_{n+1}g_n\right)$$
$$+\left(x_n+z_n\right)^2\left(-g_n + c_{n+1}\gamma_{n+1}^2 g_n^2\right) + \left(p_{n-1}-h_n\right)\left(\alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right) + c_{n+1}\gamma_{n+1}^2\sigma_n^2 + c_{n+1}\gamma_{n+1}^2 h_n^2$$
$$-2p_{n-1}c_{n+1}\gamma_{n+1}^2 h_n + c_{n+1}\gamma_{n+1}^2 p_{n-1}^2 + d_{n+1}\gamma_{n+1}h_n - d_{n+1}\gamma_{n+1}p_{n-1} + c_{n+1}\sigma_z^2\Delta t_{n+1} + e_{n+1}$$

We set

$$c_n := \frac{\left(1+\alpha_n g_n - 2c_{n+1}\gamma_{n+1}^2 g_n\right)^2}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)} - g_n + c_{n+1}\gamma_{n+1}^2 g_n^2$$

$$d_n \quad : \quad = \frac{2\left(1+\alpha_n g_n - 2c_{n+1}\gamma_{n+1}^2 g_n\right)\left(\alpha_n\left(\nu_{n-1}^L - 2p_{n-1} + h_n\right) + 2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-h_n\right) - d_{n+1}\gamma_{n+1}\right)}{4\left(\alpha_n - c_{n+1}\gamma_{n+1}^2\right)}$$
$$+p_{n-1} - h_n - g_n\left(\alpha_n\nu_{n-1}^L - \alpha_n p_{n-1}\right) + 2c_{n+1}\gamma_{n+1}^2 h_n g_n - 2p_{n-1}c_{n+1}\gamma_{n+1}^2 g_n + d_{n+1}\gamma_{n+1}g_n$$

$$
\begin{aligned}
e_n \quad : \quad =& \frac{\left(\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+h_n\right)+2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-h_n\right)-d_{n+1}\gamma_{n+1}\right)^2}{4\left(\alpha_n-c_{n+1}\gamma_{n+1}^2\right)} \\
&+\left(p_{n-1}-h_n\right)\left(\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right)+c_{n+1}\gamma_{n+1}^2\sigma_n^2+c_{n+1}\gamma_{n+1}^2 h_n^2 \\
&-2p_{n-1}c_{n+1}\gamma_{n+1}^2 h_n+c_{n+1}\gamma_{n+1}^2 p_{n-1}^2+d_{n+1}\gamma_{n+1}h_n-d_{n+1}\gamma_{n+1}p_{n-1}+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}
\end{aligned}
$$

Resubstituting h_n and g_n yields

$$
\begin{aligned}
c_n \quad =& \quad \frac{\left(1+\alpha_n\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}-2c_{n+1}\gamma_{n+1}^2\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\right)^2}{4\left(\alpha_n-c_{n+1}\gamma_{n+1}^2\right)}-\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}} \\
&+c_{n+1}\gamma_{n+1}^2\left(\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
d_n \quad =& \quad \frac{2\left(1+\alpha_n\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}-2c_{n+1}\gamma_{n+1}^2\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\right)}{4\left(\alpha_n-c_{n+1}\gamma_{n+1}^2\right)} \\
&\times\left(\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_{n-1}-\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right)\right. \\
&\left.+2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-\nu_{n-1}+\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right)-d_{n+1}\gamma_{n+1}\right) \\
&+p_{n-1}-\nu_{n-1}+\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)-\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right) \\
&+2c_{n+1}\gamma_{n+1}^2\left(\nu_{n-1}-\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right)\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}} \\
&-2p_{n-1}c_{n+1}\gamma_{n+1}^2\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}+d_{n+1}\gamma_{n+1}\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}
\end{aligned}
$$

$$
\begin{aligned}
e_n \quad =& \\
&\frac{\alpha_n^2\left(\nu_{n-1}^L-2p_{n-1}+\nu_{n-1}-\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right)^2}{4\left(\alpha_n-c_{n+1}\gamma_{n+1}^2\right)}+ \\
&\frac{\left(2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-\left(\nu_{n-1}-\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right)\right)-d_{n+1}\gamma_{n+1}\right)^2}{4\left(\alpha_n-c_{n+1}\gamma_{n+1}^2\right)}+ \\
&\frac{2\alpha_n\left(\nu_{n-1}^L-2p_{n-1}+\nu_{n-1}-\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right)}{4\left(\alpha_n-c_{n+1}\gamma_{n+1}^2\right)} \\
&\times\left(2c_{n+1}\gamma_{n+1}^2\left(p_{n-1}-\left(\nu_{n-1}-\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right)\right)-d_{n+1}\gamma_{n+1}\right) \\
&+\left(p_{n-1}-\left(\nu_{n-1}-\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right)\right)\left(\alpha_n\nu_{n-1}^L-\alpha_n p_{n-1}\right) \\
&+c_{n+1}\gamma_{n+1}^2\sigma_n^2+c_{n+1}\gamma_{n+1}^2\left(\nu_{n-1}-\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right)^2 \\
&-2p_{n-1}c_{n+1}\gamma_{n+1}^2\left(\nu_{n-1}-\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right) \\
&+c_{n+1}\gamma_{n+1}^2 p_{n-1}^2+d_{n+1}\gamma_{n+1}\left(\nu_{n-1}-\frac{\gamma_n\sigma_{n-1}^2}{\gamma_n^2\sigma_{n-1}^2+\sigma_z^2\Delta t_{n-1}}\left(\nu_{n-1}-p_{n-1}\right)\right) \\
&-d_{n+1}\gamma_{n+1}p_{n-1}+c_{n+1}\sigma_z^2\Delta t_{n+1}+e_{n+1}
\end{aligned}
$$

This shows that indeed, $c_{n+1}, d_{n+1}, e_{n+1}$ are $\mathcal{M}_n$−measurable. Obviously, here we used that $\gamma_{n+1} \in \mathcal{M}_n$.

D Limit order traders

For each $n = 1, \ldots, N$ we assume that

$$E\left(\left(\nu - p_n\right)\alpha_n\left(\nu_{n-1}^L - p_n\right)|\mathcal{F}_{n-1}\right) = 0$$

From this we get:

$$\begin{aligned}
& E\left(\left(\nu - p_n\right)\alpha_n\left(\nu_{n-1}^L - p_n\right)|\mathcal{F}_{n-1}\right) = 0 \\
\iff\quad & E\left(\nu\alpha_n\nu_{n-1}^L - \nu\alpha_n p_n - p_n\alpha_n\nu_{n-1}^L + \alpha_n p_n^2\,|\mathcal{F}_{n-1}\right) = 0 \\
\iff\quad & \nu_{n-1}^L E\left(\nu\alpha_n\,|\mathcal{F}_{n-1}\right) - p_{n-1}E\left(\nu\alpha_n\,|\mathcal{F}_{n-1}\right) - E\left(\nu\alpha_n\lambda_n\left(\alpha_n\right)\left(x_n + z_n\right)|\mathcal{F}_{n-1}\right) - \\
& E\left(\left(p_{n-1} + \lambda_n\left(\alpha_n\right)\left(x_n + z_n\right)\right)\alpha_n\nu_{n-1}^L\,|\mathcal{F}_{n-1}\right) + E\left(\alpha_n\left(p_{n-1} + \lambda_n\left(\alpha_n\right)\left(x_n + z_n\right)\right)^2|\mathcal{F}_{n-1}\right) = 0 \\
\iff\quad & \left(\nu_{n-1}^L\right)^2 E\left(\alpha_n\,|\mathcal{F}_{n-1}\right) - 2p_{n-1}\nu_{n-1}^L E\left(\alpha_n\,|\mathcal{F}_{n-1}\right) - E\left(\nu\alpha_n\lambda_n\left(\alpha_n\right)x_n\,|\mathcal{F}_{n-1}\right) - \\
& E\left(\lambda_n\left(\alpha_n\right)x_n\alpha_n\nu_{n-1}^L\,|\mathcal{F}_{n-1}\right) + E\left(\alpha_n p_{n-1}^2\,|\mathcal{F}_{n-1}\right) + E\left(\alpha_n\lambda_n^2\left(\alpha_n\right)\left(x_n + z_n\right)^2|\mathcal{F}_{n-1}\right) + \\
& E\left(2\alpha_n p_{n-1}\lambda_n\left(\alpha_n\right)\left(x_n + z_n\right)|\mathcal{F}_{n-1}\right) = 0
\end{aligned}$$

Further simplifying this leads to

$$\begin{aligned}
& E\left(\left(\nu - p_n\right)\alpha_n\left(\nu_{n-1}^L - p_n\right)|\mathcal{F}_{n-1}\right) = 0 \\
\iff\quad & \left(\nu_{n-1}^L\right)^2 E\left(\alpha_n\,|\mathcal{F}_{n-1}\right) - 2p_{n-1}\nu_{n-1}^L E\left(\alpha_n\,|\mathcal{F}_{n-1}\right) - E\left(\nu\alpha_n\lambda_n\left(\alpha_n\right)\gamma_n\left(\nu - p_{n-1}\right)|\mathcal{F}_{n-1}\right) \\
& -E\left(\lambda_n\left(\alpha_n\right)\gamma_n\left(\nu - p_{n-1}\right)\alpha_n\nu_{n-1}^L\,|\mathcal{F}_{n-1}\right) + p_{n-1}^2 E\left(\alpha_n\,|\mathcal{F}_{n-1}\right) \\
& +E\left(\alpha_n\lambda_n^2\left(\alpha_n\right)\gamma_n^2\left(\nu - p_{n-1}\right)^2|\mathcal{F}_{n-1}\right) + \sigma_z^2\Delta t_n E\left(\alpha_n\lambda_n^2\left(\alpha_n\right)|\mathcal{F}_{n-1}\right) \\
& +E\left(2\alpha_n p_{n-1}\lambda_n\left(\alpha_n\right)\gamma_n\left(\nu - p_{n-1}\right)|\mathcal{F}_{n-1}\right) = 0
\end{aligned}$$

Factorizing the terms with α_n and λ_n :

$$\begin{aligned}
& E\left(\left(\nu - p_n\right)\alpha_n\left(\nu_{n-1}^L - p_n\right)|\mathcal{F}_{n-1}\right) = 0 \\
\iff\quad & \left(\nu_{n-1}^L\right)^2 E\left(\alpha_n\,|\mathcal{F}_{n-1}\right) - 2p_{n-1}\nu_{n-1}^L E\left(\alpha_n\,|\mathcal{F}_{n-1}\right) \\
& -\gamma_n E\left(\nu^2\,|\mathcal{F}_{n-1}\right)E\left(\alpha_n\lambda_n\left(\alpha_n\right)|\mathcal{F}_{n-1}\right) + \nu_{n-1}^L\gamma_n p_{n-1}E\left(\alpha_n\lambda_n\left(\alpha_n\right)|\mathcal{F}_{n-1}\right) \\
& -\gamma_n\left(\nu_{n-1}^L\right)^2 E\left(\lambda_n\left(\alpha_n\right)\alpha_n\,|\mathcal{F}_{n-1}\right) + \gamma_n p_{n-1}\nu_{n-1}^L E\left(\lambda_n\left(\alpha_n\right)\alpha_n\,|\mathcal{F}_{n-1}\right) \\
& +p_{n-1}^2 E\left(\alpha_n\,|\mathcal{F}_{n-1}\right) + \gamma_n^2 E\left(\nu^2\,|\mathcal{F}_{n-1}\right)E\left(\alpha_n\lambda_n^2\left(\alpha_n\right)|\mathcal{F}_{n-1}\right) \\
& +\gamma_n^2 p_{n-1}^2 E\left(\alpha_n\lambda_n^2\left(\alpha_n\right)|\mathcal{F}_{n-1}\right) - 2\gamma_n^2\nu_{n-1}^L p_{n-1}E\left(\alpha_n\lambda_n^2\left(\alpha_n\right)|\mathcal{F}_{n-1}\right) \\
& +\sigma_z^2\Delta t_n E\left(\alpha_n\lambda_n^2\left(\alpha_n\right)|\mathcal{F}_{n-1}\right) + 2p_{n-1}\gamma_n\nu_{n-1}^L E\left(\alpha_n\lambda_n\left(\alpha_n\right)|\mathcal{F}_{n-1}\right) \\
& -2p_{n-1}^2\gamma_n E\left(\alpha_n\lambda_n\left(\alpha_n\right)|\mathcal{F}_{n-1}\right) = 0 \\
\iff\quad & E\left(\alpha_n\,|\mathcal{F}_{n-1}\right)\left(\left(\nu_{n-1}^L\right)^2 - 2p_{n-1}\nu_{n-1}^L + p_{n-1}^2\right) \\
& +E\left(\alpha_n\lambda_n\left(\alpha_n\right)|\mathcal{F}_{n-1}\right)\gamma_n\left(-E\left(\nu^2\,|\mathcal{F}_{n-1}\right) - \left(\nu_{n-1}^L\right)^2 + 4\nu_{n-1}^L p_{n-1} - 2p_{n-1}^2\right) + \\
& E\left(\alpha_n\lambda_n^2\left(\alpha_n\right)|\mathcal{F}_{n-1}\right)\left(\gamma_n^2 E\left(\nu^2\,|\mathcal{F}_{n-1}\right) + \gamma_n^2 p_{n-1}^2 - 2\gamma_n^2\nu_{n-1}^L p_{n-1} + \sigma_z^2\Delta t_n\right) = 0
\end{aligned}$$

For $\alpha_n \in \mathcal{F}_{n-1}$ and λ_n a deterministic function of α_n, we get:

$$
\begin{aligned}
& E\left(\left(\nu-p_n\right)\alpha_n\left(\nu_{n-1}^L-p_n\right)|\mathcal{F}_{n-1}\right)=0 \\
\iff\quad & \alpha_n\left(\left(\nu_{n-1}^L\right)^2-2p_{n-1}\nu_{n-1}^L+p_{n-1}^2\right) \\
& +\alpha_n\lambda_n\left(\alpha_n\right)\left(-E\left(\nu^2|\mathcal{F}_{n-1}\right)-\left(\nu_{n-1}^L\right)^2+4\nu_{n-1}^L p_{n-1}-2p_{n-1}^2\right) \\
& +\alpha_n\lambda_n^2\left(\alpha_n\right)\left(\gamma_n^2 E\left(\nu^2|\mathcal{F}_{n-1}\right)+\gamma_n^2 p_{n-1}^2-2\gamma_n^2\nu_{n-1}^L p_{n-1}+\sigma_z^2\Delta t_n\right)=0 \\
\iff\quad & \left(\nu_{n-1}^L\right)^2-2p_{n-1}\nu_{n-1}^L+p_{n-1}^2+ \\
& \lambda_n\left(\alpha_n\right)\left(-E\left(\nu^2|\mathcal{F}_{n-1}\right)-\left(\nu_{n-1}^L\right)^2+4\nu_{n-1}^L p_{n-1}-2p_{n-1}^2\right)+ \\
& \lambda_n^2\left(\alpha_n\right)\left(\gamma_n^2 E\left(\nu^2|\mathcal{F}_{n-1}\right)+\gamma_n^2 p_{n-1}^2-2\gamma_n^2\nu_{n-1}^L p_{n-1}+\sigma_z^2\Delta t_n\right)=0
\end{aligned}
$$

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