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Analysis of a mixed discontinuous Galerkin method for the time-harmonic Maxwell equations with minimal smoothness requirements

Kaifang Liu*
Department of Applied Mathematics, University of Twente, PO Box 217, 7500 AE Enschede, The Netherlands
*Corresponding author: k.liu@utwente.nl

Dietmar Gallistl
Institut für Mathematik, Universität Jena, 07743 Jena, Germany

and

Matthias Schlottbom and J.J.W. van der Vegt
Department of Applied Mathematics, University of Twente, PO Box 217, 7500 AE Enschede, The Netherlands

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An error analysis of a mixed discontinuous Galerkin (DG) method with lifting operators as numerical fluxes for the time-harmonic Maxwell equations with minimal smoothness requirements is presented. The key difficulty in the error analysis for the DG method is that due to the low regularity the tangential trace of the exact solution is not well defined on the faces of the computational mesh. This difficulty is addressed by adopting the face-to-cell lifting introduced by Ern & Guermond (2021, Quasi-optimal nonconforming approximation of elliptic PDEs with contrasted coefficients and H^{1+r}, $r > 0$, regularity. *Found. Comput. Math.*, 1–36). To obtain optimal local interpolation estimates, we introduce Scott–Zhang-type interpolations that are well defined for $H(\text{curl})$ and $H(\text{div})$ functions with minimal regularity requirements. As a by-product of penalizing the lifting of the tangential jumps, an explicit and easily computable stabilization parameter is given.

Keywords: discontinuous Galerkin method; time-harmonic Maxwell equations; minimal regularity; lifting operator; quasi-interpolation.

1. Introduction

We consider the analysis of a mixed discontinuous Galerkin (DG) approximation for the time-harmonic Maxwell equations with low regularity solutions: find $\boldsymbol{u}, p$ such that

$$\nabla \times (\mu^{-1}\nabla \times \boldsymbol{u}) - k^2\varepsilon\boldsymbol{u} - \varepsilon\nabla p = \boldsymbol{j} \quad \text{in } \Omega, \tag{1.1a}$$

$$\nabla \cdot (\varepsilon\boldsymbol{u}) = 0 \quad \text{in } \Omega, \tag{1.1b}$$

$$\boldsymbol{n} \times \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{1.1c}$$

$$p = 0 \quad \text{on } \Gamma. \tag{1.1d}$$

Here, $\boldsymbol{u}$ represents the electrical field, p is the Lagrange multiplier used to enforce the divergence constraint (1.1b), k is the wave number and $\boldsymbol{j} \in [L^2(\Omega)]^3$ is the source term. The piecewise constant matrix-valued functions μ and ε are the magnetic permeability and electrical permittivity of the media, respectively. We assume that $\Omega \subset \mathbb{R}^3$ is a simply connected Lipschitz domain with connected boundary Γ and $\boldsymbol{n}$ is the external unit normal vector.

Several numerical methods for the approximation of the time-harmonic Maxwell equations have been investigated, such as finite-difference time-domain (FDTD) methods (see, e.g., Taflove & Hagness, 2005; Gedney, 2011), conforming finite element methods (FEMs) (see, e.g., Monk, 2003; Ern & Guermond, 2004; Brenner & Scott, 2008) and DG methods (see, e.g., Arnold *et al.*, 2002; Perugia & Schötzau, 2003; Houston *et al.*, 2005a; Di Pietro & Ern, 2012). Standard FDTD methods suffer from serious accuracy loss near curved boundaries and singularities; see, e.g., Nicolaides (2004) for a modified scheme for complex geometries and further references. Furthermore, the corresponding error analysis for low regularity problems is challenging; let us refer to Jovanović & Süli (2014) for the analysis of finite difference schemes for certain linear elliptic, parabolic and hyperbolic equations with minimal regularity assumptions on the solution. While curl-conforming FEMs have a proper mathematical foundation, it is difficult to construct curl-conforming approximations in the context of *hp*-adaptation (see, e.g., Demkowicz, 2003; Monk, 2003). In comparison, DG methods are well suited for complex geometries, *hp*-adaptation and parallel computing.

There are several papers devoted to solving the time-harmonic Maxwell equations using DG methods. In Perugia *et al.* (2002), an interior-penalty DG method was proposed for the indefinite time-harmonic Maxwell equations with smooth coefficients. The method of Perugia *et al.* (2002), however, involves many terms and parameters, which makes the practical implementation difficult. Houston *et al.* (2004) introduced a mixed DG method for (1.1) with $k = 0$, which gives a significantly simplified DG formulation with fewer terms and allows piecewise constant coefficients μ and ε. Also, by adding an auxiliary variable to transform the DG discretization into a standard mixed formulation, i.e., a saddle-point problem without penalty, the error analysis was greatly simplified. Subsequently, the formulation was further simplified by removing the standard penalization term of the normal jump at mesh interfaces (Houston *et al.*, 2005c). This simplification allowed the use of the discrete Helmholtz decomposition for the analysis of the DG method.

In the mentioned DG methods (Perugia *et al.*, 2002; Houston *et al.*, 2004, 2005a), the *a priori* error estimates require relatively high regularity of the exact solution of the time-harmonic Maxwell equations. However, strong smoothness assumptions are not realistic in general, since the solution of the Maxwell equations may exhibit singularities and is nonsmooth at sharp corners and material interfaces (Costabel *et al.*, 1999). An explicit low regularity bound of the Maxwell equations can be found, e.g., in (Bonito *et al.* (2013, Theorem 5.1).

There are several papers devoted to the analysis of FEMs for the time-harmonic Maxwell equations with low regularity solution. Ciarlet (2016) proposed an error estimate for low-regularity electromagnetic fields, where the fields are decomposed into a regular part and a gradient, which are approximated by the classical Nédélec interpolation (Monk, 2003, Section 5.5) and the Clément/Scott–Zhang interpolation (see, e.g., Ern & Guermond, 2004; Brenner & Scott, 2008), respectively. Ern & Guermond (2018b) presented optimal error estimates for a conforming FEM for low regularity Maxwell equations, which crucially employs recent results on the commuting quasi-interpolation (Ern & Guermond, 2016) defined on function spaces with low regularity index and their corresponding quasi-best approximation (Ern & Guermond, 2017).

The key difficulty in the error analysis of nonconforming FEMs for nonsmooth problems is that the classical trace theorems are not applicable, i.e., the exact solution does not have a sufficiently regular trace on mesh faces. Until now, only a few techniques have been developed to overcome this difficulty. One technique for the Maxwell equations relies on the definition of generalized traces (Buffa & Perugia, 2006, Proposition 7.3 and Assumption 4). In the spirit of Buffa & Perugia (2006), Bonito *et al.* (2016) proposed an interior-penalty method with C^0 finite elements for the Maxwell equations with minimal smoothness requirements. Recently, Ern & Guermond (2021) analyzed a nonconforming approximation of elliptic partial differential equations (PDEs) with minimal regularity by introducing a generalized normal derivative of the exact solution at the mesh faces. They also showed that this idea can be extended to solve the time-harmonic Maxwell equations with low regularity solutions by introducing a more general concept for the tangential trace. Another technique, which avoids the definition of generalized traces and was proposed by Gudi (2010) in the context of elliptic PDEs, is to use an enriching map to transform a nonconforming function into a conforming one.

In this paper we analyze a mixed DG formulation for the Maxwell equations with low regularity solutions, which modifies the method of Houston *et al.* (2004) by employing lifting operators as numerical fluxes, see also Bassi *et al.* (1997), Brezzi *et al.* (2000). Our main objective is to generalize the error analysis of Houston *et al.* (2004) to the nonsmooth case and to present locally optimal *a priori* error estimates for the low regularity solution in the broken Sobolev space $H^s(\mathcal{T}_h), s \in (0, 1/2)$ with $\mathcal{T}_h$ the finite element partition. One of the main issues addressed by Buffa & Perugia (2006), Bonito *et al.* (2013) and Ern & Guermond (2021) is to generalize the tangential trace of $\mu^{-1}\nabla \times \boldsymbol{u}$ on mesh faces to problems with minimal regularity where the standard trace operator is not applicable. The proof in our *a priori* error analysis instead only requires a notion of the tangential trace of $\boldsymbol{u}$ on mesh faces under low regularity assumptions, which is achieved by using the face-to-cell lifting operator from Ern & Guermond (2021). Local error estimates are then obtained by a careful combination of smoothed interpolation operators (Ern & Guermond, 2016) and new Scott–Zhang-type quasi-interpolation operators for low regularity functions that we present in the appendix. A further benefit of penalizing the lifting of the jumps is that we obtain an explicit expression for stabilization parameters, which, compared with Houston *et al.* (2004), improves the robustness of the DG method considerably, as is known for div-grad problems (see, e.g., Brezzi *et al.*, 2000).

The paper is organized as follows. We introduce notation and the variational formulation of the time-harmonic Maxwell equations in Section 2. The finite element spaces and the mixed DG method with the lifting operators as numerical fluxes are presented in Section 3. We state the main results of this paper in Section 4. An auxiliary variational formulation in the spirit of Houston *et al.* (2004) and some interpolation error estimates are presented in Section 5. Next, we first derive in Section 6 an error estimate for the Maxwell equations (1.1a) with $k = 0$, and subsequently, we show in Section 7 the well-posedness and error estimates of the mixed DG method for the indefinite Maxwell equations, i.e., $k \neq 0$. Auxiliary results, including the Scott–Zhang-type interpolations for low regularity functions, are given in the appendix.

2. Preliminaries

2.1 *Function spaces*

We introduce standard notation for Sobolev spaces. For a generic open domain $D \subset \mathbb{R}^3$, we denote by $W^{m,p}(D)$ the usual Sobolev spaces of integer order $m \geqslant 0$ with norms $\|\cdot\|_{W^{m,p}(D)}$, and write $L^p(D) = W^{0,p}(D)$. We also write $\|\cdot\|_{W^{m,p}(D)}$ for the norm of vector-valued function spaces $[W^{m,p}(D)]^3$. For $s \in (0, 1)$, we can define the fractional-order Sobolev spaces $W^{m+s,p}(D)$ with the Sobolev–Slobodeckij norm (see, e.g., Grisvard, 2011, p. 17; Tartar, 2007, p. 169), defined as

$$\|v\|^p_{W^{m+s,p}(D)} := \|v\|^p_{W^{m,p}(D)} + \sum_{|\alpha|=m} \int_D \int_D \frac{|D^\alpha v(\boldsymbol{x}) - D^\alpha v(\boldsymbol{y})|^p}{|\boldsymbol{x}-\boldsymbol{y}|^{3+sp}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}.$$

The space $W_0^{m+s,p}(D)$ is defined to be the completion of $C_0^\infty(D)$ with respect to the norm $\|\cdot\|_{W^{m+s,p}(D)}$. For $p=2$ and $s \in (0,1)$, we use $H^{m+s}(D) = W^{m+s,2}(D)$ with norm $\|\cdot\|_{m+s,D}$, and write $L^2(D) = H^0(D)$. We denote by $(\cdot,\cdot)_D$ the standard inner product in $[L^2(D)]^3$ and denote by $[L^2_\varepsilon(D)]^3$ the space $[L^2(D)]^3$ endowed with the ε-weighted inner product given by $(\boldsymbol{u},\boldsymbol{v})_{\varepsilon,D} := \int_D \varepsilon \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x$ for $\varepsilon(\boldsymbol{x})$ being symmetric and uniformly positive definite. If $D=\Omega$, we write $(\cdot,\cdot)_\varepsilon$ for $(\cdot,\cdot)_{\varepsilon,\Omega}$. We also use the following spaces:

$$\begin{aligned}
H(\mathrm{curl},\Omega) &= \{\boldsymbol{v} \in [L^2(\Omega)]^3 : \nabla\times\boldsymbol{v} \in [L^2(\Omega)]^3\},\\
H_0(\mathrm{curl},\Omega) &= \{\boldsymbol{v} \in H(\mathrm{curl},\Omega) : \boldsymbol{n}\times\boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega\},\\
H^s(\mathrm{curl},\Omega) &= \{\boldsymbol{v} \in [H^s(\Omega)]^3 : \nabla\times\boldsymbol{v} \in [H^s(\Omega)]^3\},\\
H(\mathrm{div},\Omega) &= \{\boldsymbol{v} \in [L^2(\Omega)]^3 : \nabla\cdot\boldsymbol{v} \in L^2(\Omega)\},\\
H_0(\mathrm{div},\Omega) &= \{\boldsymbol{v} \in H(\mathrm{div},\Omega) : \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}.
\end{aligned}$$

Here, $s \in (0,1)$, $\nabla\times\boldsymbol{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)^{\mathrm{T}}$ and $\nabla\cdot\boldsymbol{u} = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i}$. As usual, the tangential trace, $\boldsymbol{n}\times\boldsymbol{v}$, and the normal trace, $\boldsymbol{v}\cdot\boldsymbol{n}$, of functions $\boldsymbol{v} \in H(\mathrm{curl},\Omega)$ and $H(\mathrm{div},\Omega)$, respectively, have to be understood as bounded linear functionals acting on $H^{1/2}(\partial\Omega)$.

2.2 *Variational formulation*

Throughout this paper we assume that the domain and coefficients satisfy the following assumptions.

ASSUMPTION 2.1 (i) $\Omega \subset \mathbb{R}^3$ is a simply connected bounded Lipschitz domain with connected boundary Γ.
(ii) The matrix-valued functions $\mu, \varepsilon : \Omega \to \mathbb{R}^{3\times3}$ are piecewise constant with respect to some partition $\mathscr{T}_h$ of Ω into Lipschitz polyhedra such that there exist positive constants $\mu_*, \mu^*, \varepsilon_*, \varepsilon^*$ satisfying for a.e. $\boldsymbol{x} \in \overline{\Omega}$, and all $\boldsymbol{\xi} \in \mathbb{R}^3$,

$$\mu_*|\boldsymbol{\xi}|^2 \leqslant \boldsymbol{\xi}^{\mathrm{T}}\mu(\boldsymbol{x})\boldsymbol{\xi} \leqslant \mu^*|\boldsymbol{\xi}|^2 \quad\text{and}\quad \varepsilon_*|\boldsymbol{\xi}|^2 \leqslant \boldsymbol{\xi}^{\mathrm{T}}\varepsilon(\boldsymbol{x})\boldsymbol{\xi} \leqslant \varepsilon^*|\boldsymbol{\xi}|^2. \tag{2.1}$$

(iii) The source function satisfies $\boldsymbol{j} \in [L^2(\Omega)]^3$.
(iv) The squared wave number, k^2, is not an interior Maxwell eigenvalue; see Monk (2003, Section 1.4.2) or Boffi *et al.* (2013, (11.2.6)) for a definition.

Let $V := H_0(\mathrm{curl},\Omega)$ and $Q := H_0^1(\Omega)$. Define the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ as

$$\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v}) &= (\mu^{-1}\nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{v}) &\qquad& \forall\, \boldsymbol{u},\boldsymbol{v} \in V,\\
b(\boldsymbol{v},p) &= -(\varepsilon\boldsymbol{v}, \nabla p) && \forall\, \boldsymbol{v} \in V,\ p \in Q.
\end{aligned}$$

The mixed variational formulation of the time-harmonic Maxwell equations (1.1a)–(1.1d) is to find $\boldsymbol{u} \in V$ and $p \in Q$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) - k^2(\varepsilon\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = (\boldsymbol{j},\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in V, \tag{2.2}$$

$$b(\boldsymbol{u},q) = 0 \qquad \forall\, q \in Q. \tag{2.3}$$

Since $a(\cdot,\cdot)$ is continuous and coercive on the kernel of $b(\cdot,\cdot)$, and $b(\cdot,\cdot)$ is continuous and satisfies an inf-sup condition, see Houston *et al.* (2005b, Section 2.3) or Boffi *et al.* (2013, Theorem 11.2.1), the variational problem is well posed.

LEMMA 2.2 (Boffi *et al.*, 2013, Theorem 11.2.1). Under Assumption 2.1, the variational problem (2.2)–(2.3) has a unique solution $(\boldsymbol{u},p) \in V \times Q$, and there exists a constant $C > 0$ such that

$$\|\boldsymbol{u}\|_{H(\mathrm{curl},\Omega)} + \|p\|_{1,\Omega} \leqslant C\|\boldsymbol{j}\|_{0,\Omega}.$$

Our error analysis relies on the following stability and regularity result; see Bonito *et al.* (2013).

LEMMA 2.3 (Theorem 5.1, Bonito *et al.*, 2013). Let Assumption 2.1 hold. Then there exist constants $0 < \tau < 1/2$ and $C > 0$, which depend only on Ω, ε and μ, such that the weak solution $(\boldsymbol{u},p) \in V \times Q$ of the variational problem (2.2)–(2.3) satisfies $\boldsymbol{u} \in H^s(\mathrm{curl},\Omega)$ and

$$\|\boldsymbol{u}\|_{s,\Omega} + \|\nabla\times\boldsymbol{u}\|_{s,\Omega} \leqslant C\|\boldsymbol{j}\|_{0,\Omega} \quad \forall\, 0 \leqslant s < \tau,$$

$$\|\nabla\times(\mu^{-1}\nabla\times\boldsymbol{u})\|_{0,\Omega} + \|\nabla p\|_{0,\Omega} \leqslant C\|\boldsymbol{j}\|_{0,\Omega}.$$

We close this section by noting that both ε and μ induce multiplication operators that map $H^s(\Omega)$ boundedly into itself for $s < 1/2$ (Bonito *et al.*, 2013, Proposition 2.1). Thus, Lemma 2.3 also shows that $\varepsilon\boldsymbol{u} \in [H^s(\Omega)]^3$ and $\mu^{-1}\nabla\times\boldsymbol{u} \in [H^s(\Omega)]^3$.

3. Mixed DG discretization

3.1 *Finite element spaces*

Let $\mathscr{T}_h$ be a shape-regular and conforming partition of the domain Ω into tetrahedra such that the coefficients ε and μ are constant on each $K \in \mathscr{T}_h$. We denote by h_K the diameter of an element K and denote $h = \max_{K\in\mathscr{T}_h} h_K$. For an integer $\ell \geqslant 0$ and an element $K \in \mathscr{T}_h$, we define $\mathbb{P}_\ell(K)$ as the space of polynomials of total degree ℓ in K. Let $\mathscr{F}_h$ be the union of interior faces $\mathscr{F}_h^I$ and boundary faces $\mathscr{F}_h^B$. For piecewise smooth vector- or scalar-valued functions $\boldsymbol{v}$ and q, we define jumps and averages at faces in the mesh. Let $F \in \mathscr{F}_h^I$ be an interior face shared by two elements K^+ and K^- and let $\boldsymbol{n}^\pm$ be the unit outward normal vectors on the boundaries $\partial K^\pm$. We also define the normal vector on the face F via $\boldsymbol{n}_F = \boldsymbol{n}^+$. The tangential and normal jumps across F are, respectively, defined by $[\![\boldsymbol{v}]\!]_T := \boldsymbol{n}^+\times\boldsymbol{v}^+ + \boldsymbol{n}^-\times\boldsymbol{v}^-$, $[\![\boldsymbol{v}]\!]_N := \boldsymbol{v}^+\cdot\boldsymbol{n}^+ + \boldsymbol{v}^-\cdot\boldsymbol{n}^-$ and $[\![q]\!]_N := q^+\boldsymbol{n}^+ + q^-\boldsymbol{n}^-$. We also define the total jump by $[\![\boldsymbol{v}]\!] := \boldsymbol{v}^+ - \boldsymbol{v}^-$. The averages are defined via $\{\!\{\boldsymbol{v}\}\!\} := (\boldsymbol{v}^+ + \boldsymbol{v}^-)/2$ and $\{\!\{q\}\!\} := (q^+ + q^-)/2$. If $F \in \mathscr{F}_h^B$ is a boundary face, we set $[\![\boldsymbol{v}]\!]_T := \boldsymbol{n}\times\boldsymbol{v}$, $[\![\boldsymbol{v}]\!]_N := \boldsymbol{v}\cdot\boldsymbol{n}$, $[\![q]\!]_N := q\boldsymbol{n}$, $\{\!\{\boldsymbol{v}\}\!\} := \boldsymbol{v}$ and $\{\!\{q\}\!\} := q$. We denote by

$\omega_K := \mathrm{int}\big(\cup_{\bar{K}'\cap\bar{K}\neq\emptyset}\bar{K}'\big)$ the patch of neighbouring elements of K and by $\omega_F := \mathrm{int}\big(\cup_{F\subset\partial\bar{K}'}\bar{K}'\big)$ the neighbouring elements of face F.

We define, for real $s \geqslant 0$, the broken Sobolev spaces with respect to the partition $\mathscr{T}_h$ of Ω as

$$[H^s(\mathscr{T}_h)]^3 := \{\boldsymbol{v} \in [L^2(\Omega)]^3 : \boldsymbol{v}_{|K} \in [H^s(K)]^3, \quad \forall K \in \mathscr{T}_h\},$$

with norm

$$\|\boldsymbol{v}\|^2_{s,\mathscr{T}_h} := \sum_{K\in\mathscr{T}_h} \|\boldsymbol{v}\|^2_{s,K}.$$

Moreover, we define the finite element spaces without interelement continuity condition as

$$V_h := \{\boldsymbol{v} \in [L^2(\Omega)]^3 : \boldsymbol{v}_{|K} \in R_\ell(K)\ \forall K \in \mathscr{T}_h\}, \tag{3.1}$$

$$Q_h := \{q \in L^2(\Omega) : q_{|K} \in \mathbb{P}_\ell(K)\ \forall K \in \mathscr{T}_h\}, \tag{3.2}$$

where R_ℓ denotes the space of Nédélec functions of degree ℓ, i.e., $R_\ell = [\mathbb{P}_{\ell-1}]^3 \oplus S_\ell$, and $S_\ell = \{\boldsymbol{q} \in [\widetilde{\mathbb{P}}_\ell]^3 : \boldsymbol{x}\cdot\boldsymbol{q} = 0\}$ with $\widetilde{\mathbb{P}}_\ell = \mathbb{P}_\ell \setminus \mathbb{P}_{\ell-1}$ being the homogeneous polynomials of degree ℓ.

Since only problems with low regularity solutions will be considered in this paper, we focus on the lowest-order elements, i.e., $\ell = 1$, in the rest of the manuscript.

We also define $H(\mathrm{curl})$-conforming subspaces without and with vanishing tangential trace on the boundary, respectively, as

$$V_h^{\mathrm{c}} := V_h \cap H(\mathrm{curl}, \Omega) \quad \text{and} \quad V_{h0}^{\mathrm{c}} := V_h \cap H_0(\mathrm{curl}, \Omega). \tag{3.3}$$

Moreover, we introduce the Raviart–Thomas space, an $H(\mathrm{div})$-conforming finite element space (see, e.g., Monk, 2003, Section 5.4)

$$V_h^{\mathrm{d}} := \{\boldsymbol{v} \in [L^2(\Omega)]^3 : \boldsymbol{v}_{|K} \in [\mathbb{P}_{\ell-1}(K)]^3 \oplus [\widetilde{\mathbb{P}}_{\ell-1}(K)\boldsymbol{x}]\ \forall K \in \mathscr{T}_h\} \cap H(\mathrm{div}, \Omega), \tag{3.4}$$

and $V_{h0}^{\mathrm{d}} := V_h^{\mathrm{d}} \cap H_0(\mathrm{div}, \Omega)$ with vanishing normal trace on the boundary.

3.2 *Lifting operator*

In order to obtain a discretization with explicit parameters we employ lifting operators frequently used in DG settings (see, e.g., Brezzi *et al.*, 2000; Sármány *et al.*, 2010), to rewrite integrals over mesh faces in terms of integrals over adjacent elements. For the consistency analysis, we wish to apply the lifting operator to the tangential jump of the solution. For the nonsmooth solution $\boldsymbol{u}$ established in Lemma 2.3 we have $\boldsymbol{u} \in H^s(\mathrm{curl}, \Omega)$ with $s \in (0, 1/2)$, which requires us to define a proper meaning for the tangential jump $[\![\boldsymbol{u}]\!]_T$ on a single face $F \in \mathscr{F}_h$. Following Ern & Guermond (2021, Section 5.1), we start by introducing a weak definition of the tangential trace on mesh faces.

There exists a $p > 2$ such that the embedding $H^s(\Omega) \hookrightarrow L^p(\Omega)$ is continuous; see, e.g., Grisvard (2011, Theorem 1.4.4.1). Hence, there exists a $p > 2$ such that the following embedding is continuous:

$$H^s(\text{curl}, \Omega) \hookrightarrow \{\boldsymbol{w} \in [L^p(\Omega)]^3 : \nabla \times \boldsymbol{w} \in [L^p(\Omega)]^3\}. \tag{3.5}$$

Recall that $\boldsymbol{u} \in H^s(\text{curl}, \Omega)$ by Lemma 2.3. Let p' be such that $1/p' + 1/p = 1$. To proceed, we recall a simplified version of Ern & Guermond (2021, Lemma 3.1).

LEMMA 3.1 Let $p > 2$, $K \in \mathscr{T}_h$ and $F \in \mathscr{F}_h$ with $F \subset \partial K$. Then there exists a face-to-cell lifting operator $L_F^K : W^{\frac{1}{p},p'}(F) \to W^{1,p'}(K)$ such that

$$L_F^K(\eta)_{|F} = \eta \quad \text{and} \quad L_F^K(\eta)_{|\partial K \setminus F} = 0,$$

and there exists a constant $c > 0$ independent of the mesh size h_K such that

$$h_K |L_F^K(\eta)|_{W^{1,p'}(K)} + \|L_F^K(\eta)\|_{L^{p'}(K)} \leqslant c h_K^{\frac{1}{p'}} \left(\|\eta\|_{L^{p'}(F)} + h_K^{\frac{1}{p}} |\eta|_{W^{\frac{1}{p},p'}(F)} \right). \tag{3.6}$$

We apply L_F^K also to vector-valued functions, meaning a componentwise application. Let $K \in \mathscr{T}_h$ and $F \in \mathscr{F}_h$ with $F \subset \partial K$. Then, owing to Lemma 3.1 and embedding (3.5), any $\boldsymbol{u} \in H^s(\text{curl}, \Omega)$, $s \in (0, 1/2)$, defines a bounded linear functional on $[W^{\frac{1}{p},p'}(F)]^3$ via

$$\langle \boldsymbol{n} \times \boldsymbol{u}, \eta \rangle_F := \left(\nabla \times \boldsymbol{u}, L_F^K(\eta) \right)_K - \left(\boldsymbol{u}, \nabla \times L_F^K(\eta) \right)_K, \quad \eta \in [W^{\frac{1}{p},p'}(F)]^3; \tag{3.7}$$

see also Ern & Guermond (2021, Section 5.1). If $F \in \mathscr{F}_h^B$, we define the tangential jump $[\![\boldsymbol{u}]\!]_T := \boldsymbol{n} \times \boldsymbol{u}$ on F. For $F \in \mathscr{F}_h^I$ with $F = \partial K_1 \cap \partial K_2$, we define the tangential jump on F via

$$\langle [\![\boldsymbol{u}]\!]_T, \eta \rangle_F := \langle \boldsymbol{n}_1 \times \boldsymbol{u}_{|K_1}, \eta \rangle_F + \langle \boldsymbol{n}_2 \times \boldsymbol{u}_{|K_2}, \eta \rangle_F, \quad \eta \in [W^{\frac{1}{p},p'}(F)]^3. \tag{3.8}$$

Summarizing, in view of embedding (3.5) and Lemma 2.3, the solution of the Maxwell equations has a well-defined tangential jump $[\![\boldsymbol{u}]\!]_T$ in the sense of (3.8). In fact, the next result shows that $[\![\boldsymbol{u}]\!]_T$, and consequently also its lifting, vanishes on each face, which will be used in the consistency analysis.

LEMMA 3.2 (Vanishing tangential jump). For all $\boldsymbol{u} \in V \cap H^s(\text{curl}, \Omega)$, $s \in (0, 1/2)$, we have

$$\langle [\![\boldsymbol{u}]\!]_T, \eta \rangle_F = 0 \quad \forall \eta \in [W^{\frac{1}{p},p'}(F)]^3, \forall F \in \mathscr{F}_h.$$

Proof. (i) For $F \in \mathscr{F}_h^I$ with $F = \partial K_1 \cap \partial K_2$, recall $\omega_F = \text{int}(\bar{K}_1 \cup \bar{K}_2)$ and define

$$L_F(\eta) := \begin{cases} L_F^{K_1}(\eta) & \text{on } K_1, \\ L_F^{K_2}(\eta) & \text{on } K_2, \end{cases}$$

for any $\eta \in [W^{\frac{1}{p},p'}(F)]^3$. Since $L_F(\eta)$ is continuous across F and vanishes on $\partial\omega_F$ by Lemma 3.1, we obtain that $L_F(\eta) \in [W_0^{1,p'}(\omega_F)]^3$. Hence, definitions (3.7) and (3.8) yield

$$\langle [\![\boldsymbol{u}]\!]_T, \eta \rangle_F = \left(\nabla \times \boldsymbol{u}, L_F(\eta)\right)_{\omega_F} - \left(\boldsymbol{u}, \nabla \times L_F(\eta)\right)_{\omega_F} = 0.$$

(ii) For each boundary face $F \in \mathscr{F}_h^B$ with $F \subset \partial K \cap \partial\Omega$, we have

$$\langle [\![\boldsymbol{u}]\!]_T, \eta \rangle_F = \langle \boldsymbol{n} \times \boldsymbol{u}_{|K}, \eta \rangle_F = \left(\nabla \times \boldsymbol{u}, L_F^K(\eta)\right)_K - \left(\boldsymbol{u}, \nabla \times L_F^K(\eta)\right)_K.$$

Since $L_F^K(\eta)_{|\partial K \setminus F} = 0$ by Lemma 3.1, the zero extension of $L_F^K(\eta)$ from K to Ω, denoted also by $L_F^K(\eta)$, belongs to $W^{1,p'}(\Omega)$. Therefore,

$$\langle [\![\boldsymbol{u}]\!]_T, \eta \rangle_F = \left(\nabla \times \boldsymbol{u}, L_F^K(\eta)\right) - \left(\boldsymbol{u}, \nabla \times L_F^K(\eta)\right),$$

which together with $\boldsymbol{u} \in H_0(\mathrm{curl}, \Omega)$ and a density argument shows that the right-hand side of the previous equation equals zero. □

We are ready to introduce a lifting operator being used in the following DG discretization. Since $\{\!\{\boldsymbol{v}\}\!\} \in [W^{\frac{1}{p},p'}(F)]^3$ for all $F \in \mathscr{F}_h$ and $\boldsymbol{v} \in V_h$, we define $\mathscr{L}_F : ([W^{\frac{1}{p},p'}(F)]^3)' \to V_h$ by

$$\int_\Omega \mathscr{L}_F(f) \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \langle f, \{\!\{\boldsymbol{v}\}\!\} \rangle_F \qquad \forall\, \boldsymbol{v} \in V_h. \tag{3.9}$$

We know that $[\![\boldsymbol{u}]\!]_T$, $\boldsymbol{u} \in H^s(\mathrm{curl}, \Omega)$ defines a bounded functional on $[W^{\frac{1}{p},p'}(F)]^3$ via (3.8) for all $F \in \mathscr{F}_h$ and, thus, $\mathscr{L}_F\left([\![\boldsymbol{u}]\!]_T\right)$ is well defined. From (3.9), since the right-hand side is nonzero only when $\{\!\{\boldsymbol{v}\}\!\}$ has support on F, the support of the lifting $\mathscr{L}_F(f)$ is limited to the elements adjacent to face F.

REMARK 3.3 If $f \in [L^1(F)]^3$ for $F \in \mathscr{F}_h$, the right-hand side in (3.9) reduces to an integral over F, i.e.,

$$\int_\Omega \mathscr{L}_F(f) \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_F f \cdot \{\!\{\boldsymbol{v}\}\!\} \,\mathrm{d}s.$$

Thus, if $f = [\![\boldsymbol{w}]\!]_T$ for either $\boldsymbol{w} \in V_h$ or $\boldsymbol{w} \in [H^s(\mathscr{T}_h)]^3$, $s > 1/2$, $\mathscr{L}_F\left([\![\boldsymbol{w}]\!]_T\right)$ reduces to the usual definition of the local lifting operator employed in, e.g., Sármány *et al.* (2010) or Perugia & Schötzau (2003).

Sármány *et al.* (2010, Lemmas 1 and 2) showed the following stability property of the local lifting operator: there exist positive constants C_1 and C_2 such that for any $\boldsymbol{v} \in V_h$,

$$C_1 h_F^{-1/2} \left\| [\![\boldsymbol{v}]\!]_T \right\|_{0,F} \leqslant \left\| \mathscr{L}_F\left([\![\boldsymbol{v}]\!]_T\right) \right\|_{0,\Omega} \leqslant C_2 h_F^{-1/2} \left\| [\![\boldsymbol{v}]\!]_T \right\|_{0,F} \quad \forall\, F \in \mathscr{F}_h, \tag{3.10}$$

where h_F denotes the diameter of face F and C_1, C_2 are independent of $\boldsymbol{v}$.

The global lifting operator is now obtained by superposition,

$$\mathscr{L}(f) := \sum_{F\in\mathscr{F}_h} \mathscr{L}_F(f) \quad \forall f \text{ such that } f_{|F} \in \left([W^{\frac{1}{p},p'}(F)]^3\right)'. \tag{3.11}$$

3.3 *Mixed DG discretization*

The mixed DG discretization with lifting operators (see, e.g., Bassi *et al.*, 1997; Brezzi *et al.*, 2000; Sármány *et al.*, 2010) for the time-harmonic Maxwell equations is to find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}) - k^2(\varepsilon\boldsymbol{u}_h, \boldsymbol{v}) + b_h(\boldsymbol{v}, p_h) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in V_h, \tag{3.12}$$

$$b_h(\boldsymbol{u}_h, q) - c_h(p_h, q) = 0 \qquad \forall\, q \in Q_h, \tag{3.13}$$

where the bilinear forms are defined as

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}) &:= \left(\mu^{-1}\nabla_h \times \boldsymbol{u}, \nabla_h \times \boldsymbol{v}\right) - \left(\mathscr{L}(\llbracket\boldsymbol{u}\rrbracket_T), \mu^{-1}\nabla_h \times \boldsymbol{v}\right) \\ &\quad - \left(\mathscr{L}(\llbracket\boldsymbol{v}\rrbracket_T), \mu^{-1}\nabla_h \times \boldsymbol{u}\right) + \sum_{F\in\mathscr{F}_h} \alpha_F \left(\mu^{-1}\mathscr{L}_F(\llbracket\boldsymbol{u}\rrbracket_T), \mathscr{L}_F(\llbracket\boldsymbol{v}\rrbracket_T)\right), \\ b_h(\boldsymbol{v}, p) &:= -\left(\varepsilon\boldsymbol{v}, \nabla_h p\right) + \left(\mathscr{L}(\llbracket p\rrbracket_N), \varepsilon\boldsymbol{v}\right), \\ c_h(p, q) &:= \sum_{F\in\mathscr{F}_h} \gamma_F \left(\varepsilon\mathscr{L}_F(\llbracket p\rrbracket_N), \mathscr{L}_F(\llbracket q\rrbracket_N)\right), \end{aligned}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V_h + H^s(\text{curl}, \Omega)$, $s \in (0, 1/2)$ and $p, q \in Q_h + Q$. Here, ∇_h and $\nabla_h\times$ denote the elementwise action of the differential operators ∇ and $\nabla\times$, respectively. We set the parameter γ_F strictly positive for all $F \in \mathscr{F}_h$, and $\alpha_F > 0$ will be chosen later. The reader is referred to Houston *et al.* (2004) for the derivation of the DG formulation with the lifting operators as numerical flux replaced by the interior penalty numerical flux.

REMARK 3.4 Since the coefficients μ and ε are piecewise constant, we note from the definition of the global lifting operator $\mathscr{L}$ that the bilinear forms $a_h : V_h \times V_h \to \mathbb{R}$ and $b_h : V_h \times Q_h \to \mathbb{R}$ have the following equivalent expressions

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}) &= \left(\mu^{-1}\nabla_h \times \boldsymbol{u}, \nabla_h \times \boldsymbol{v}\right) - \int_{\mathscr{F}_h} \llbracket\boldsymbol{u}\rrbracket_T \cdot \{\!\{\mu^{-1}\nabla_h \times \boldsymbol{v}\}\!\}\, \mathrm{d}s \\ &\quad - \int_{\mathscr{F}_h} \llbracket\boldsymbol{v}\rrbracket_T \cdot \{\!\{\mu^{-1}\nabla_h \times \boldsymbol{u}\}\!\}\, \mathrm{d}s + \sum_{F\in\mathscr{F}_h} \alpha_F \left(\mu^{-1}\mathscr{L}_F(\llbracket\boldsymbol{u}\rrbracket_T), \mathscr{L}_F(\llbracket\boldsymbol{v}\rrbracket_T)\right), \\ b_h(\boldsymbol{v}, p) &= -\left(\varepsilon\boldsymbol{v}, \nabla_h p\right) + \int_{\mathscr{F}_h} \{\!\{\varepsilon\boldsymbol{v}\}\!\} \cdot \llbracket p\rrbracket_N\, \mathrm{d}s, \end{aligned}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V_h$ and $p \in Q_h$; see Remark 3.3. Here, $\int_{\mathscr{F}_h} \cdot\, \mathrm{d}s = \sum_{F\in\mathscr{F}_h} \int_F \cdot\, \mathrm{d}s$. Compared with the implementation of the DG discretization in Houston *et al.* (2004), we see that only the penalty terms need to be changed. A benefit of using $\mathscr{L}$ is that precise conditions for α_F can be computed explicitly from the mesh $\mathscr{T}_h$ that ensure stability, see Proposition 4.1, while in practice for interior-penalty formulations, depending on the computational mesh, the penalty parameters need to be frequently adjusted; see, e.g., Houston *et al.* (2004), Lu *et al.* (2017).

REMARK 3.5 The term $T := \int_{\mathscr{F}_h} [\![\boldsymbol{v}]\!]_T \cdot \{\!\{\mu^{-1}\nabla_h \times \boldsymbol{u}\}\!\}\, \mathrm{d}s$ in Remark 3.4 is not well defined for $\boldsymbol{u} \in V$ and $\boldsymbol{v} \in V_h$. Buffa & Perugia (2006), Bonito *et al.* (2016), Ern & Guermond (2018a, 2021) therefore generalized notions of the tangential trace of $\mu^{-1}\nabla \times \boldsymbol{u}$. Here, we overcome these difficulties in the analysis of T by using the terms $\big(\mathscr{L}\big([\![\boldsymbol{v}]\!]_T\big), \mu^{-1}\nabla_h \times \boldsymbol{u}\big)$ and $\big(\mathscr{L}\big([\![\boldsymbol{u}]\!]_T\big), \mu^{-1}\nabla_h \times \boldsymbol{v}\big)$ in (3.12), which only requires a notion of $[\![\boldsymbol{u}]\!]_T$ on a single mesh face. Note that $[\![\boldsymbol{u}]\!]_T$ is not properly defined on a single face for merely $\boldsymbol{u} \in V$. However, this can be remedied by using the slightly higher regularity of $\boldsymbol{u} \in H^s(\mathrm{curl}, \Omega)$ with $0 < s < 1/2$ and the definition of $[\![\boldsymbol{u}]\!]_T$ in (3.8), which uses the face-to-cell lifting of Ern & Guermond (2021).

REMARK 3.6 Let us compare the bilinear form $\big(\mathscr{L}\big([\![\boldsymbol{v}]\!]_T\big), \mu^{-1}\nabla_h \times \boldsymbol{u}\big)$, used in (3.12), to the corresponding bilinear form introduced in Ern & Guermond (2021, (5.10)):

$$n_\sharp^c(\boldsymbol{u},\boldsymbol{v}) := \sum_{F\in\mathscr{F}_h}\sum_{K\in\mathscr{T}_F} s_{K,F}\theta_{K,F}\langle((\mu^{-1}\nabla \times \boldsymbol{u})_{|K} \times \boldsymbol{n}_K)_{|F}, [\![\Pi_F \boldsymbol{v}]\!]\rangle_F \quad \forall \boldsymbol{u} \in \boldsymbol{V}_s + V_h,\ \boldsymbol{v} \in V_h. \tag{3.14}$$

Here, $s_{K,F} = \boldsymbol{n}_K \cdot \boldsymbol{n}_F = \pm 1$ and $\Pi_F \boldsymbol{v} = \boldsymbol{n}_K \times (\boldsymbol{v} \times \boldsymbol{n}_K)$. To simplify notation, we take $\theta_{K,F} = 1/2$ for interior faces and $\theta_{K,F} = 1$ for boundary faces. The set $\mathscr{T}_F$ denotes the set of elements adjacent to face F and

$$\boldsymbol{V}_s := \left\{\boldsymbol{w} \in H(\mathrm{curl}, \Omega) \mid \mu^{-1}\nabla \times \boldsymbol{w} \in [L^p(\Omega)]^3,\ \nabla \times \mu^{-1}\nabla \times \boldsymbol{w} \in [L^q(\Omega)]^3\right\}$$

for some $p > 2$ and $q \in (6/5, 2]$. Using $\boldsymbol{n} \times [\![\Pi_F \boldsymbol{v}]\!] = \boldsymbol{n} \times [\![\boldsymbol{v}]\!]$, Ern & Guermond (2021, (5.11a)) implies that in the fully discrete case,

$$n_\sharp^c(\boldsymbol{u},\boldsymbol{v}) = \int_{\mathscr{F}_h} [\![\boldsymbol{v}]\!]_T \cdot \{\!\{\mu^{-1}\nabla_h \times \boldsymbol{u}\}\!\}\, \mathrm{d}s = \Big(\mathscr{L}\big([\![\boldsymbol{v}]\!]_T\big), \mu^{-1}\nabla_h \times \boldsymbol{u}\Big) \quad \forall \boldsymbol{u},\boldsymbol{v} \in V_h,$$

where the second equality follows by definition (3.11) of $\mathscr{L}$; see also Remark 3.3. However, if we assume $\boldsymbol{u} \in [L^2(\Omega)]^3$ and $\nabla_h \times \boldsymbol{u} \in [L^2(\Omega)]^3$, due to $\mathscr{L}\big([\![\boldsymbol{v}]\!]_T\big) \in V_h$, there holds for all $\boldsymbol{v} \in V_h$,

$$\begin{aligned}
\Big(\mathscr{L}\big([\![\boldsymbol{v}]\!]_T\big), \mu^{-1}\nabla_h \times \boldsymbol{u}\Big) &= \Big(\mathscr{L}\big([\![\boldsymbol{v}]\!]_T\big), Q_h\big(\mu^{-1}\nabla_h \times \boldsymbol{u}\big)\Big) \\
&= \int_{\mathscr{F}_h} [\![\boldsymbol{v}]\!]_T \cdot \{\!\{Q_h(\mu^{-1}\nabla_h \times \boldsymbol{u})\}\!\}\, \mathrm{d}s \\
&= \sum_{F\in\mathscr{F}_h}\sum_{K\in\mathscr{T}_F} s_{K,F}\theta_{K,F}\big\langle(Q_h(\mu^{-1}\nabla_h \times \boldsymbol{u})_{|K} \times \boldsymbol{n}_K)_{|F}, [\![\Pi_F \boldsymbol{v}]\!]\big\rangle_F,
\end{aligned} \tag{3.15}$$

where Q_h is the elementwise L^2-projection onto V_h. The right-hand side of (3.15) with the additional Q_h is clearly different from $n_\sharp^c(\boldsymbol{u},\boldsymbol{v})$, defined in (3.14), for low regularity or even smooth $\boldsymbol{u}$. In fact, a simple example can illustrate the difference. Choose $\boldsymbol{u} = [b(\boldsymbol{x}), 0, 0]^{\mathrm{T}}$ on an element $K_0 \in \mathscr{T}_h$ and zero otherwise, where the bubble function $b(\boldsymbol{x})$, after transforming back to the reference element $\hat{K}$, is defined as $b(\boldsymbol{x}) = x_1^2 x_2^2 x_3^2 (1 - x_1 - x_2 - x_3)^2$. Clearly, $n_\sharp^c(\boldsymbol{u}, \boldsymbol{v}) = 0$ for all $v \in V_h$ by definition (3.14). On the other hand, take a special test function $v = 0$, except for K_0, where $v_{|\hat{K}} = [0, -x_3, x_2]$. It can be shown that for this special v, the right-hand side of (3.15) is nonzero.

The two bilinear forms require different regularities of $\boldsymbol{u}$: $n_\sharp^c$ needs $\boldsymbol{u} \in \mathbf{V}_s$, while only $\mu^{-1}\nabla \times \boldsymbol{u} \in [L^2(\Omega)]^3$ is required for $\big(\mathscr{L}([\![\boldsymbol{v}]\!]_T), \mu^{-1}\nabla_h \times \boldsymbol{u}\big)$. We note that despite the larger domain of $a_h(\cdot,\cdot)$ it leads to a nonconsistent finite element scheme. This property of using lifting operators has been indicated also in Di Pietro & Ern (2012, Remark 4.37) for the div-grad system.

4. The main results

First we will give explicit bounds on the stabilization parameter α_F that ensure well-posedness of (3.12)–(3.13). Subsequently, we present an *a priori* error estimate for low regularity solutions.

We start by defining mesh-dependent seminorms and norms as

$$
\begin{aligned}
|\boldsymbol{v}|_{V_h}^2 &:= \|\mu^{-\frac{1}{2}}\nabla_h \times \boldsymbol{v}\|_{0,\Omega}^2 + \sum_{F\in\mathscr{F}_h} \|\mu^{-\frac{1}{2}}\mathscr{L}_F\big([\![\boldsymbol{v}]\!]_T\big)\|_{0,\Omega}^2,\\
\|\boldsymbol{v}\|_{V_h}^2 &:= \|\varepsilon^{\frac{1}{2}}\boldsymbol{v}\|_{0,\Omega}^2 + |\boldsymbol{v}|_{V_h}^2,\\
\|q\|_{Q_h}^2 &:= \|\varepsilon^{\frac{1}{2}}\nabla_h q\|_{0,\Omega}^2 + \sum_{F\in\mathscr{F}_h} \|\varepsilon^{\frac{1}{2}}\mathscr{L}_F\big([\![q]\!]_N\big)\|_{0,\Omega}^2.
\end{aligned}
$$

Note that using the lifting operator defined in (3.9) the norm $\|\cdot\|_{V_h}$ is well defined for the exact solution $\boldsymbol{u}$ of (2.2)–(2.3) with the regularity established in Lemma 2.3, see Section 3.2, and $\|\cdot\|_{Q_h}$ is well defined for all functions in Q.

The following proposition shows that $a_h(\cdot,\cdot)$ is coercive on V_h with respect to the seminorm $|\cdot|_{V_h}$ for a simple and explicit choice of the parameter α_F, which facilitates the implementation of the DG method and is essential in the proof of the well-posedness of the mixed DG discretization (3.12)–(3.13).

PROPOSITION 4.1 (Discrete coercivity). The bilinear form $a_h(\cdot,\cdot)$ satisfies

$$
a_h(\boldsymbol{v},\boldsymbol{v}) \geqslant \frac{1}{2}|\boldsymbol{v}|_{V_h}^2 \quad \forall\, \boldsymbol{v} \in V_h
$$

provided $\alpha_F \geqslant \frac{1}{2} + 2n_K$ with $n_K = 4$ the number of faces of an element $K \in \mathscr{T}_h$.

Proof. The proof is in the same spirit as Brezzi *et al.* (2000, Proposition 1) or Di Pietro & Ern (2012, Lemma 5.17). From the definition of the lifting operator $\mathscr{L}_F$, we have for any $\boldsymbol{v} \in V_h$,

$$
a_h(\boldsymbol{v},\boldsymbol{v}) = \|\mu^{-\frac{1}{2}}\nabla_h \times \boldsymbol{v}\|_{0,\Omega}^2 - 2\sum_{F\in\mathscr{F}_h}\big(\mathscr{L}_F\big([\![\boldsymbol{v}]\!]_T\big), \mu^{-1}\nabla_h \times \boldsymbol{v}\big) + \sum_{F\in\mathscr{F}_h}\alpha_F\|\mu^{-\frac{1}{2}}\mathscr{L}_F\big([\![\boldsymbol{v}]\!]_T\big)\|_{0,\Omega}^2.
$$

Recall that the support of the local lifting operator $\mathscr{L}_F([\![\boldsymbol{v}]\!]_T)$, denoted by ω_F, consists of the element(s) adjacent to face F. By using the Cauchy–Schwarz and Young's inequality, there holds

$$
\begin{aligned}
2 \sum_{F\in\mathscr{F}_h} &(\mathscr{L}_F([\![\boldsymbol{v}]\!]_T), \mu^{-1}\nabla_h \times \boldsymbol{v}) \\
&\leqslant 2 \sum_{F\in\mathscr{F}_h} \left(\frac{1}{4\delta} \| \mu^{-\frac{1}{2}} \mathscr{L}_F([\![\boldsymbol{v}]\!]_T) \|_{0,\Omega}^2 + \delta \| \mu^{-\frac{1}{2}} \nabla_h \times \boldsymbol{v} \|_{0,\omega_F}^2 \right) \\
&\leqslant \sum_{F\in\mathscr{F}_h} \frac{1}{2\delta} \| \mu^{-\frac{1}{2}} \mathscr{L}_F([\![\boldsymbol{v}]\!]_T) \|_{0,\Omega}^2 + 2\delta n_K \| \mu^{-\frac{1}{2}} \nabla_h \times \boldsymbol{v} \|_{0,\Omega}^2,
\end{aligned}
$$

where $n_K = 4$ is the number of faces of an element $K \in \mathscr{T}_h$. Hence,

$$
\begin{aligned}
a_h(\boldsymbol{v},\boldsymbol{v}) &\geqslant (1 - 2\delta n_K) \| \mu^{-\frac{1}{2}} \nabla_h \times \boldsymbol{v} \|_{0,\Omega}^2 + \sum_{F\in\mathscr{F}_h} \left(\alpha_F - \frac{1}{2\delta} \right) \| \mu^{-\frac{1}{2}} \mathscr{L}_F([\![\boldsymbol{v}]\!]_T) \|_{0,\Omega}^2 \\
&\geqslant \min \left\{ 1 - 2\delta n_K, \alpha_F - \frac{1}{2\delta} \right\} |\boldsymbol{v}|_{V_h}^2.
\end{aligned}
$$

Hence, by setting $\delta = 1/(4n_K)$, we can get the coercivity with constant $1/2$ if $\alpha_F \geqslant 1/2 + 2n_K$. □

The following two theorems state the well-posedness of the mixed DG method (3.12)–(3.13) and *a priori* error estimates for low regularity solutions.

THEOREM 4.2 (Existence, uniqueness). Let Assumption 2.1 hold, $\alpha_F \geqslant 1/2+2n_K$ and $\gamma_F \geqslant 1/2$. For all mesh sizes h small enough, there exists a unique solution $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ to (3.12)–(3.13) satisfying

$$
\|\boldsymbol{u}_h\|_{V_h} + \|p_h\|_{Q_h} \leqslant C\|\boldsymbol{j}\|_0,
$$

where the constant $C > 0$ is independent of the mesh size and the solution $(\boldsymbol{u}_h, p_h)$.

THEOREM 4.3 (A priori error estimate). Let Assumption 2.1 hold, $\alpha_F \geqslant 1/2 + 2n_K$ and $\gamma_F \geqslant 1/2$. Let $(\boldsymbol{u}, p)$ be the solution of (2.2)–(2.3) with regularity as in Lemma 2.3. Further, assume that $p \in H^{1+s}(\mathscr{T}_h)$. Then, for all mesh sizes h small enough, there exists a unique solution $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ to (3.12)–(3.13), and it satisfies the *a priori* error estimate

$$
\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_h\|_{V_h} + \|p - p_h\|_{Q_h} \leqslant C\Bigg(&\sum_{K\in\mathscr{T}_h} h_K^{2s} \left(|\boldsymbol{u}|_{s,\omega_K}^2 + |\nabla \times \boldsymbol{u}|_{s,\omega_K}^2 \right) \\
&+ h_K^2 \|\nabla \times \boldsymbol{u}\|_{0,\omega_K}^2 + h_K^2 \|\nabla \times \mu^{-1} \nabla \times \boldsymbol{u}\|_{0,\omega_K}^2 + h_K^{2s} |p|_{1+s,\omega_K}^2 \Bigg)^{1/2},
\end{aligned}
$$

where the constant $C > 0$ is independent of the mesh size.

REMARK 4.4 We note that Theorems 4.2 and 4.3 also hold true with the choice of full polynomial spaces $[\mathbb{P}_\ell(\mathscr{T}_h)]^3$ for V_h, which are easier to construct but of a higher dimension than the corresponding local first family Nédélec spaces, and a full polynomial space of order $\ell + 1$ for Q_h; see, e.g., Buffa & Perugia (2006, Section 7.1) or Houston *et al.* (2005c) where such a construction has been used.

REMARK 4.5 The assumption on the regularity of p in Theorem 4.3 can be ensured by requiring $\nabla \cdot \boldsymbol{j} \in H^{s-1}(\Omega)$. In fact, by taking the divergence of both sides of (1.1a), we obtain an elliptic equation for p:

$$\begin{aligned} -\nabla \cdot \varepsilon \nabla p = \nabla \cdot \boldsymbol{j} \quad & \text{in } \Omega, \\ p = 0 \quad & \text{on } \partial\Omega, \end{aligned}$$

and we obtain that $p \in H^{1+s}(\Omega)$ for some $s \in (0, 1/2)$ from Bonito *et al.* (2013, Theorem 3.1).

Theorem 4.2 will first be proved for $k = 0$ in the next section using an auxiliary formulation in the spirit of Houston *et al.* (2004). The case $k \neq 0$ is treated in Section 7. Theorem 4.3 will be proven in Section 6 for $k = 0$ and in Section 7 for $k \neq 0$.

5. Auxiliary results

5.1 *Auxiliary mixed formulation*

The variational problem (3.12)–(3.13) is a saddle-point problem with penalty. To facilitate its analysis we follow Houston *et al.* (2004, Sections 4 and 5) and introduce an equivalent auxiliary mixed formulation, which is a saddle-point problem without penalty. To do so, let us introduce the discrete auxiliary space

$$M_h := \{\lambda \in [L^2(\mathscr{F}_h)]^3 : \lambda_{|F} \in [\mathbb{P}_1(F)]^3 \quad \forall F \in \mathscr{F}_h\},$$

with norm

$$\|\lambda\|^2_{M_h} := \sum_{F \in \mathscr{F}_h} \|\varepsilon^{\frac{1}{2}} \mathscr{L}_F(\lambda)\|^2_{0,\Omega},$$

and let $W_h = V_h \times M_h$ with seminorm and norm defined as

$$|(\boldsymbol{v}, \eta)|^2_{W_h} := |\boldsymbol{v}|^2_{V_h} + \|\eta\|^2_{M_h}, \quad \|(\boldsymbol{v}, \eta)\|^2_{W_h} := \|\boldsymbol{v}\|^2_{V_h} + \|\eta\|^2_{M_h}.$$

We state the auxiliary mixed formulation as follows: find $(\boldsymbol{u}_h, \lambda_h; p_h) \in W_h \times Q_h$ such that

$$A_h(\boldsymbol{u}_h, \lambda_h; \boldsymbol{v}, \eta) - k^2(\varepsilon \boldsymbol{u}_h, \boldsymbol{v}) + B_h(\boldsymbol{v}, \eta; p_h) = (\boldsymbol{j}, \boldsymbol{v}) \quad \forall (\boldsymbol{v}, \eta) \in W_h, \tag{5.1}$$

$$B_h(\boldsymbol{u}_h, \lambda_h; q) = 0 \qquad \forall q \in Q_h, \tag{5.2}$$

where the corresponding bilinear forms are defined as

$$A_h(\boldsymbol{u}_h, \lambda_h; \boldsymbol{v}, \eta) := a_h(\boldsymbol{u}_h, \boldsymbol{v}) + \sum_{F \in \mathscr{F}_h} \gamma_F \left(\varepsilon \mathscr{L}_F(\lambda_h), \mathscr{L}_F(\eta)\right),$$

$$B_h(\boldsymbol{v}, \eta; p_h) := b_h(\boldsymbol{v}, p_h) - \sum_{F \in \mathscr{F}_h} \gamma_F \left(\varepsilon \mathscr{L}_F([\![p_h]\!]_N), \mathscr{L}_F(\eta)\right).$$

Lemma 5.1 The mixed DG formulation (3.12)–(3.13) is equivalent to (5.1)–(5.2), i.e., if $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ solves (3.12)–(3.13), then $(\boldsymbol{u}_h, [\![p_h]\!]_N; p_h) \in W_h \times Q_h$ solves (5.1)–(5.2). If, on the other hand, $(\boldsymbol{u}_h, \lambda_h; p_h) \in W_h \times Q_h$ solves (5.1)–(5.2), then $(\boldsymbol{u}_h, p_h)$ solves (3.12)–(3.13) and $\lambda_h = [\![p_h]\!]_N$.

Proof. Suppose $(\boldsymbol{u}_h, \lambda_h; p_h)$ solves (5.1)–(5.2). By taking test function $\boldsymbol{v} = \boldsymbol{0}$ in (5.1), we have

$$\sum_{F \in \mathscr{F}_h} \gamma_F(\varepsilon \mathscr{L}_F(\lambda_h), \mathscr{L}_F(\eta)) = \sum_{F \in \mathscr{F}_h} \gamma_F(\varepsilon \mathscr{L}_F([\![p_h]\!]_N), \mathscr{L}_F(\eta)) \quad \forall \eta \in M_h.$$

Hence, $\lambda_h = [\![p_h]\!]_N$. This shows that (u_h, p_h) solves (3.12)–(3.13). The other direction follows immediately by setting $\lambda_h = [\![p_h]\!]_N$. □

Define the kernel of the form $B_h(\cdot, \cdot)$ as

$$\ker(B_h) := \{(\boldsymbol{v}, \eta) \in W_h : B_h(\boldsymbol{v}, \eta; q) = 0 \, \forall q \in Q_h\}. \tag{5.3}$$

The following three lemmas form the basis for the proof of Theorem 4.2 for $k = 0$.

Lemma 5.2 (Continuity). There exists a constant C independent of the mesh size and the coefficients μ and ε such that

$$|A_h(\boldsymbol{u}, \lambda; \boldsymbol{v}, \eta)| \leqslant C \|(\boldsymbol{u}, \lambda)\|_{W_h} \|(\boldsymbol{v}, \eta)\|_{W_h} \qquad \forall (\boldsymbol{u}, \lambda), (\boldsymbol{v}, \eta) \in W_h,$$

$$|B_h(\boldsymbol{v}, \eta; q)| \leqslant C \|(\boldsymbol{v}, \eta)\|_{W_h} \|q\|_{Q_h} \qquad \forall (\boldsymbol{v}, \eta) \in W_h, \ \forall q \in Q(h).$$

Note that these two bounds also hold for $\boldsymbol{u} \in H^s(\mathrm{curl}, \Omega), s \in (0, 1/2)$.

This lemma follows directly from an application of the Cauchy–Schwarz inequality.

Lemma 5.3 (Ellipticity on the kernel). For α_F given by Proposition 4.1 and $\gamma_F \geqslant \frac{1}{2}$, there holds

$$A_h(\boldsymbol{v}, \eta; \boldsymbol{v}, \eta) \geqslant \kappa_A \|(\boldsymbol{v}, \eta)\|^2_{W_h} \quad \forall (\boldsymbol{v}, \eta) \in \ker(B_h), \tag{5.4}$$

where $\kappa_A > 0$ depends on Ω and the coefficients μ and ε, but is independent of the mesh size.

Proof. Assume $(\boldsymbol{v}, \eta) \in \ker(B_h)$. Recalling the definition of A_h and using the discrete coercivity of a_h stated in Proposition 4.1, there holds

$$A_h(\boldsymbol{v}, \eta; \boldsymbol{v}, \eta) \geqslant \frac{1}{2} |\boldsymbol{v}|^2_{V(h)} + \sum_{F \in \mathscr{F}_h} \gamma_F \|\mathscr{L}_F(\eta)\|^2_{0,\Omega} \geqslant \frac{1}{2} |(\boldsymbol{v}, \eta)|^2_{W_h}. \tag{5.5}$$

From the discrete Friedrichs inequality in Lemma C.1, we have

$$\|\varepsilon^{\frac{1}{2}}\boldsymbol{v}\|_{0,\Omega} \leqslant c_F |\boldsymbol{v}|_{V_h},$$

which for $\delta > 0$ leads to

$$|\boldsymbol{v}|^2_{V_h} = (1-\delta)|\boldsymbol{v}|^2_{V_h} + \delta|\boldsymbol{v}|^2_{V_h} \geqslant (1-\delta)|\boldsymbol{v}|^2_{V_h} + \frac{\delta}{c_F^2}\|\varepsilon^{\frac{1}{2}}\boldsymbol{v}\|^2_{0,\Omega}.$$

By setting $\delta = c_F^2/(1+c_F^2)$ we have

$$|\boldsymbol{v}|^2_{V_h} \geqslant \frac{1}{1+c_F^2}\|\boldsymbol{v}\|^2_{V_h},$$

which, together with (5.5), completes the proof with $\kappa_A = 1/(2(1+c_F^2))$. □

The following stability result follows with similar arguments as in Houston *et al.* (2004, Proposition 5.4) and using the stability of the lifting operator (3.10).

LEMMA 5.4 (Inf-sup condition). There holds

$$\inf_{0\neq q\in Q_h} \sup_{0\neq(\boldsymbol{v},\eta)\in W_h} \frac{B_h(\boldsymbol{v},\eta;q)}{\|q\|_{Q_h}\|(\boldsymbol{v},\eta)\|_{W_h}} \geqslant \kappa_B > 0, \tag{5.6}$$

where $\kappa_B > 0$ depends on the coefficients μ and ε, but is independent of the mesh size.

Now we are ready to prove the well-posedness of (3.12)–(3.13) for $k = 0$.

Proof of Theorem 4.2 for $k = 0$. From the classical theory of mixed FEM (see, e.g., Boffi *et al.*, 2013), Lemmas 5.2, 5.3 and 5.4 imply that the auxiliary formulation (5.1)–(5.2) with $k = 0$ has a unique solution $(\boldsymbol{u}_h, \lambda_h; p_h) \in W_h \times Q_h$ and satisfies

$$\|(\boldsymbol{u}_h,\lambda_h)\|_{W_h} + \|p_h\|_{Q_h} \leqslant C\|\boldsymbol{j}\|_0, \tag{5.7}$$

where $C > 0$ is independent of the mesh size. From Lemma 5.1, $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ also solves (3.12)–(3.13) and the uniqueness of (3.12)–(3.13) follows from the *a priori* estimate (5.7). □

5.2 *Conforming approximation*

In the error analysis we shall use the conforming projection Π_h^c introduced in Houston *et al.* (2005a, Proposition 4.5), which states that the approximation of a discontinuous function in V_h by an $H(\mathrm{curl})$ averaging operator can be quantified in terms of certain jumps. Approximation by conforming projections has been used also in Bernardi & Verfürth (2000), Karakashian & Pascal (2003) and Houston *et al.* (2004, 2005c). The following lemma is a combination of the stability of the lifting operator (3.10) and the proof of Houston *et al.* (2005a, Proposition 4.5); see Houston *et al.* (2005a, appendix) for more details, see also Ern & Guermond (2017).

LEMMA 5.5 There exists an operator $\Pi_h^{\mathrm{c}} : V_h \to V_{h0}^{\mathrm{c}}$ such that for all $\boldsymbol{v} \in V_h$,

$$h_K^{-2}\|\boldsymbol{v} - \Pi_h^{\mathrm{c}}\boldsymbol{v}\|_{0,K}^2 + \|\nabla \times (\boldsymbol{v} - \Pi_h^{\mathrm{c}}\boldsymbol{v})\|_{0,K}^2 \leqslant C \sum_{F\in\mathscr{F}_h(K)} \|\mathscr{L}_F\big([\![\boldsymbol{v}]\!]_T\big)\|_{0,\Omega}^2,$$

where $\mathscr{F}_h(K) := \{F \in \mathscr{F}_h : \bar{F} \cap \partial K \neq \emptyset\}$ and the constant $C > 0$ depends on the shape regularity of the mesh, but not on the mesh size.

5.3 $H(\mathrm{curl})-$ *and* $H(\mathrm{div})$*-conforming quasi-interpolations*

In Ern & Guermond (2017), a sequence of quasi-interpolation operators that enjoys the best approximation property for low regularity functions is introduced. The error bounds for the quasi-interpolation operators that preserve boundary constraints are, however, stated for the whole domain. Here, we provide local error bounds by constructing new quasi-interpolations, defined in Appendices A and B, where the key ingredient is to combine the generalized tangential trace of Section 3.2 with a Scott–Zhang-type quasi-interpolation (Scott & Zhang, 1990). We have the following theorems, which are proven in Appendices A and B.

THEOREM 5.6 (Error bound $H(\mathrm{curl})$). Suppose that $\boldsymbol{v} \in H^s(\mathrm{curl}, \Omega)$, $s \in (0, 1/2)$ and let P_h^{c} be the $H(\mathrm{curl})$-conforming quasi-interpolation defined in (A.2). Then there exists a constant $c > 0$ depending on the mesh regularity, but independent of the mesh size h_K, such that for each $K \in \mathscr{T}_h$,

$$\|\boldsymbol{v} - P_h^{\mathrm{c}}\boldsymbol{v}\|_{0,K} \leqslant c\left(h_K^s|\boldsymbol{v}|_{s,\omega_K} + h_K\left(\|\nabla \times \boldsymbol{v}\|_{0,\omega_K} + h_K^s|\nabla \times \boldsymbol{v}|_{s,\omega_K}\right)\right). \tag{5.8}$$

THEOREM 5.7 (Error bound $H(\mathrm{div})$). Suppose that $\boldsymbol{v} \in [H^s(\Omega)]^3$, $\nabla \cdot \boldsymbol{v} \in H^s(\Omega)$, $s \in (0, 1/2)$ and let P_h^{d} be the $H(\mathrm{div})$-conforming quasi-interpolation defined in (B.3). Then there exists a constant $c > 0$ depending on the mesh regularity, but independent of the mesh size h_K, such that for each $K \in \mathscr{T}_h$,

$$\|\boldsymbol{v} - P_h^{\mathrm{d}}\boldsymbol{v}\|_{0,K} \leqslant c\left(h_K^s|\boldsymbol{v}|_{s,\omega_K} + h_K\left(\|\nabla \cdot \boldsymbol{v}\|_{0,\omega_K} + h_K^s|\nabla \cdot \boldsymbol{v}|_{s,\omega_K}\right)\right). \tag{5.9}$$

REMARK 5.8 The DG discretization is well defined for general nonconforming meshes. The conformity of the mesh $\mathscr{T}_h$ is only required for the interpolation operators we use in the error analysis, namely, (a) the $H(\mathrm{curl})$ and $H(\mathrm{div})$ Scott–Zhang-type interpolations presented in Appendices A and B, (b) the conforming projections defined in Section 5.2 and (c) the quasi-interpolations of Ern & Guermond (2016). The definition of corresponding interpolation operators suitable for nonconforming meshes is an interesting topic, but beyond the scope of the present article.

6. Definite Maxwell equations

The error estimates of the mixed DG discretization (3.12)–(3.13) for the definite Maxwell equations with $k = 0$ will greatly facilitate the analysis for the indefinite problem discussed in Section 7.

6.1 *Residual operators*

The error analysis of Houston *et al.* (2004) relies crucially on the smoothness $H^s(\mathscr{T}_h), s > 1/2$. In this section we will extend the analysis to $s \in (0, 1/2)$. Following Houston *et al.* (2004, Section 6.1) we introduce two consistency-related residual operators, which play a key role in deriving an *a priori* error estimate under minimal smoothness requirements.

In view of Lemma 2.3, let $(\boldsymbol{u}, p) \in H^s(\mathrm{curl}, \Omega) \times Q$ be the exact solution of the continuous variational problem (2.2)–(2.3). We define the residuals

$$\mathscr{R}_1(\boldsymbol{u}, p; \boldsymbol{v}, \eta) := A_h(\boldsymbol{u}, 0; \boldsymbol{v}, \eta) + B_h(\boldsymbol{v}, \eta; p) - (\boldsymbol{j}, \boldsymbol{v}) \quad \text{and} \quad \mathscr{R}_2(\boldsymbol{u}, q) := B_h(\boldsymbol{u}, 0; q)$$

for all $(\boldsymbol{v}, \eta) \in W_h$ and $q \in Q_h$. We also define norms of the residual operators as

$$\mathscr{R}_1(\boldsymbol{u}, p) := \sup_{0 \neq (\boldsymbol{v}, \eta) \in W_h} \frac{\mathscr{R}_1(\boldsymbol{u}, p; \boldsymbol{v}, \eta)}{\|(\boldsymbol{v}, \eta)\|_{W_h}}, \quad \mathscr{R}_2(\boldsymbol{u}) := \sup_{0 \neq q \in Q_h} \frac{\mathscr{R}_2(\boldsymbol{u}; q)}{\|q\|_{Q_h}}.$$

Lemma 6.1 Let $(\boldsymbol{u}, p) \in H^s(\mathrm{curl}, \Omega) \times Q$ be the solution of (2.2)–(2.3); then

$$\begin{aligned} \mathscr{R}_1(\boldsymbol{u}, p; \boldsymbol{v}, \eta) &= 0 \qquad \forall\, (\boldsymbol{v}, \eta) \in V_{h0}^{\mathrm{c}} \times M_h, \\ \mathscr{R}_2(\boldsymbol{u}, q) &= 0 \qquad \forall\, q \in Q_h^{\mathrm{c}} = Q_h \cap Q. \end{aligned}$$

Proof. The identities are actually direct results of (2.2)–(2.3). In fact, for all $\boldsymbol{v} \in V_{h0}^{\mathrm{c}}$, there holds from the definition of $\mathscr{R}_1$ and (2.2),

$$\mathscr{R}_1(\boldsymbol{u}, p; \boldsymbol{v}, \eta) = a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, q) - (\boldsymbol{j}, \boldsymbol{v}) = 0,$$

where we have used $\mathscr{L}([\![\boldsymbol{u}]\!]_T) = \boldsymbol{0}$; see Lemma 3.2 and the definition of $\mathscr{L}_F$ in (3.9). Similarly, the other identity follows from (2.3). □

Now we are ready to state our main results about the residuals.

Proposition 6.2 (Residual estimates). Let $(\boldsymbol{u}, p) \in H^s(\mathrm{curl}, \Omega) \times Q$ be the solution of (2.2)–(2.3); then

$$\mathscr{R}_1(\boldsymbol{u}, p) \leqslant C \left(\sum_{K \in \mathscr{T}_h} h_K^{2s} |\mu^{-1} \nabla \times \boldsymbol{u}|_{s,\omega_K}^2 + h_K^2 \|\nabla \times \mu^{-1} \nabla \times \boldsymbol{u}\|_{0,\omega_K}^2 \right)^{1/2},$$

$$\mathscr{R}_2(\boldsymbol{u}) \leqslant C \left(\sum_{K \in \mathscr{T}_h} h_K^{2s} |\varepsilon \boldsymbol{u}|_{s,\omega_K}^2 \right)^{1/2},$$

where the constant $C > 0$ is independent of the mesh size.

Proof. By using Lemma 3.2 and (3.9), we have $[\![\boldsymbol{u}]\!]_T = \boldsymbol{0}$ and $\mathscr{L}_F([\![\boldsymbol{u}]\!]_T) = \boldsymbol{0}$. In view of Lemma 2.3 we have $\sigma(\boldsymbol{u}) := \mu^{-1} \nabla \times \boldsymbol{u} \in [H^s(\Omega)]^3$ and

$$\nabla \times \sigma(\boldsymbol{u}) = \varepsilon \nabla p + \boldsymbol{j} \in [L^2(\Omega)]^3. \tag{6.1}$$

In the following, since $\sigma(\boldsymbol{u})$ and $\varepsilon \boldsymbol{u}$ have a nonzero trace at $\partial\Omega$, we shall use the smoothed interpolations $\mathscr{J}_h^{\mathrm{c}} : H(\mathrm{curl}, \Omega) \to V_h^{\mathrm{c}}$ and $\mathscr{J}_h^{\mathrm{d}} : H(\mathrm{div}, \Omega) \to V_h^{\mathrm{d}}$ without boundary constraints defined in Ern & Guermond (2016, Section 6.3).

Step 1: Estimate of $\mathscr{R}_1$. For any $(\boldsymbol{v}, \eta) \in W_h$, from the definition of $\mathscr{R}_1$ and (6.1), one gets

$$\begin{aligned}\mathscr{R}_1(\boldsymbol{u}, p; \boldsymbol{v}, \eta) &= (\sigma(\boldsymbol{u}), \nabla_h \times \boldsymbol{v}) - \left(\mathscr{L}\left([\![\boldsymbol{v}]\!]_T\right), \sigma(\boldsymbol{u})\right) - (\varepsilon \boldsymbol{v}, \nabla p) - (\boldsymbol{j}, \boldsymbol{v}) \\ &= -(\nabla \times \sigma(\boldsymbol{u}), \boldsymbol{v}) + (\sigma(\boldsymbol{u}), \nabla_h \times \boldsymbol{v}) - \left(\mathscr{L}\left([\![\boldsymbol{v}]\!]_T\right), \sigma(\boldsymbol{u})\right). \end{aligned} \tag{6.2}$$

To treat the last term, we employ $\mathscr{J}_h^{\mathrm{c}}$, i.e.,

$$\left(\mathscr{L}\left([\![\boldsymbol{v}]\!]_T\right), \sigma(\boldsymbol{u})\right) = \left(\mathscr{L}\left([\![\boldsymbol{v}]\!]_T\right), \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u})\right) + \left(\mathscr{L}\left([\![\boldsymbol{v}]\!]_T\right), \sigma(\boldsymbol{u}) - \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u})\right).$$

Using definition (3.9) of the lifting operator $\mathscr{L}_F$ and Remark 3.3 and integration by parts, we infer for $v \in V_h$,

$$\left(\mathscr{L}([\![\boldsymbol{v}]\!]_T), \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u})\right) = \sum_{F \in \mathscr{F}_h} \int_F [\![\boldsymbol{v}]\!]_T \cdot \{\!\{\mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u})\}\!\}\,\mathrm{d}s = (\mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u}), \nabla_h \times \boldsymbol{v}) - (\nabla \times \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u}), \boldsymbol{v}),$$

and obtain

$$\left(\mathscr{L}([\![\boldsymbol{v}]\!]_T), \sigma(\boldsymbol{u})\right) = (\mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u}), \nabla_h \times \boldsymbol{v}) - (\nabla \times \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u}), \boldsymbol{v}) + (\mathscr{L}([\![\boldsymbol{v}]\!]_T), \sigma(\boldsymbol{u}) - \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u})).$$

Substituting this identity into (6.2) we have

$$\begin{aligned}\mathscr{R}_1(\boldsymbol{u}, p; \boldsymbol{v}, \eta) = &- \left(\nabla \times \sigma(\boldsymbol{u}) - \nabla \times \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u}), \boldsymbol{v}\right) + (\sigma(\boldsymbol{u}) - \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u}), \nabla_h \times \boldsymbol{v}) \\ &- \left(\mathscr{L}([\![\boldsymbol{v}]\!]_T), \sigma(\boldsymbol{u}) - \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u})\right). \end{aligned} \tag{6.3}$$

Let $\boldsymbol{v}^{\perp} = \boldsymbol{v} - \Pi_h^{\mathrm{c}}\boldsymbol{v}$ with Π_h^{c} from Lemma 5.5; then $[\![\boldsymbol{v}]\!]_T = [\![\boldsymbol{v}^{\perp}]\!]_T$. Since $\Pi_h^{\mathrm{c}}\boldsymbol{v} \in V_{h0}^{\mathrm{c}} \subset V_h$, we deduce that (6.3) also holds with $\boldsymbol{v}$ replaced by $\Pi_h^{\mathrm{c}}\boldsymbol{v}$. Lemma 6.1 implies $\mathscr{R}_1(\boldsymbol{u}, p; \Pi_h^{\mathrm{c}}\boldsymbol{v}, \eta) = 0$, and hence

$$\begin{aligned}\mathscr{R}_1(\boldsymbol{u}, p; \boldsymbol{v}, \eta) &= \mathscr{R}_1(\boldsymbol{u}, p; \boldsymbol{v} - \Pi_h^{\mathrm{c}}\boldsymbol{v}, 0) \\ &= -(\nabla \times \sigma(\boldsymbol{u}) - \nabla \times \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u}), \boldsymbol{v}^{\perp}) + (\sigma(\boldsymbol{u}) - \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u}), \nabla_h \times \boldsymbol{v}^{\perp}) \\ &\quad - (\mathscr{L}([\![\boldsymbol{v}^{\perp}]\!]_T), \sigma(\boldsymbol{u}) - \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u})) \\ &\leqslant \sum_{K \in \mathscr{T}_h} \left(\|\nabla \times \sigma(\boldsymbol{u}) - \nabla \times \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u})\|_{0,K} \|\boldsymbol{v}^{\perp}\|_{0,K} + \|\sigma(\boldsymbol{u}) - \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u})\|_{0,K} \|\nabla \times \boldsymbol{v}^{\perp}\|_{0,K}\right) \\ &\quad + \sum_{F \in \mathscr{F}_h} \|\sigma(\boldsymbol{u}) - \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u})\|_{0,\omega_F} \|\mathscr{L}_F([\![\boldsymbol{v}^{\perp}]\!]_T)\|_{0,\omega_F} \\ &\leqslant C\left(\sum_{K \in \mathscr{T}_h} h_K^2 \|\nabla \times \sigma(\boldsymbol{u}) - \mathscr{J}_h^{\mathrm{d}}\nabla \times \sigma(\boldsymbol{u})\|_{0,K}^2 + \|\sigma(\boldsymbol{u}) - \mathscr{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u})\|_{0,K}^2\right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{F \in \mathscr{F}_h} \|\mathscr{L}_F([\![\boldsymbol{v}]\!]_T)\|_{0,\Omega}^2\right)^{\frac{1}{2}}, \end{aligned}$$

where we have used the conforming approximation stated in Lemma 5.5 and the commuting property of $\mathcal{J}_h^{\mathrm{c}}$ with respect to $\nabla\times$. By using the best approximation properties of $\mathcal{J}_h^{\mathrm{c}}$ and $\mathcal{J}_h^{\mathrm{d}}$, we obtain from the above inequality that

$$\mathscr{R}_1(\boldsymbol{u},p) \leqslant C\Big(\sum_{K\in\mathscr{T}_h} h_K^2 \|\nabla\times\sigma(\boldsymbol{u}) - \mathcal{J}_h^{\mathrm{d}}\nabla\times\sigma(\boldsymbol{u})\|_{0,K}^2\Big)^{\frac{1}{2}} + \|\sigma(\boldsymbol{u}) - \mathcal{J}_h^{\mathrm{c}}\sigma(\boldsymbol{u})\|_{0,\Omega}$$

$$\leqslant C \inf_{w\in V_h^{\mathrm{d}}}\Big(\sum_{K\in\mathscr{T}_h} h_K^2\|\nabla\times\sigma(\boldsymbol{u}) - \boldsymbol{w}\|_{0,K}^2\Big)^{\frac{1}{2}} + \inf_{\boldsymbol{v}\in V_h^{\mathrm{c}}}\|\sigma(\boldsymbol{u}) - \boldsymbol{v}\|_{0,\Omega}.$$

The estimate for $\mathscr{R}_1$ then follows from Ern & Guermond (2017, Theorem 5.2).

Step 2: Estimate of $\mathscr{R}_2$. Similarly, we have for $q \in Q_h$,

$$\begin{aligned}\mathscr{R}_2(\boldsymbol{u};q) &= -(\varepsilon\boldsymbol{u},\nabla_h q) + \big(\mathscr{L}([\![q]\!]_N),\varepsilon\boldsymbol{u}\big)\\ &= -(\varepsilon\boldsymbol{u},\nabla_h q) + \Big(\mathscr{L}([\![q]\!]_N),\mathcal{J}_h^{\mathrm{d}}\varepsilon\boldsymbol{u}\Big) + \Big(\mathscr{L}([\![q]\!]_N),\varepsilon\boldsymbol{u} - \mathcal{J}_h^{\mathrm{d}}\varepsilon\boldsymbol{u}\Big).\end{aligned} \tag{6.4}$$

By using the definition of lifting operator $\mathscr{L}$ and integrating by parts on each element, we obtain

$$\Big(\mathscr{L}([\![q]\!]_N),\mathcal{J}_h^{\mathrm{d}}\varepsilon\boldsymbol{u}\Big) = \int_{\mathscr{F}_h}[\![q]\!]_N\cdot\{\!\{\mathcal{J}_h^{\mathrm{d}}\varepsilon\boldsymbol{u}\}\!\}\,\mathrm{d}s = (\nabla\cdot\mathcal{J}_h^{\mathrm{d}}\varepsilon\boldsymbol{u},q) + (\mathcal{J}_h^{\mathrm{d}}\varepsilon\boldsymbol{u},\nabla_h q). \tag{6.5}$$

From the commuting property of $\mathcal{J}_h^{\mathrm{d}}$ and $\mathcal{J}_h^{\mathrm{b}}$ with respect to $\nabla\cdot$, see Ern & Guermond (2016, Section 6.3), $\nabla\cdot\varepsilon\boldsymbol{u} = 0$ implies

$$\nabla\cdot\mathcal{J}_h^{\mathrm{d}}\varepsilon\boldsymbol{u} = \mathcal{J}_h^{\mathrm{b}}\nabla\cdot\varepsilon\boldsymbol{u} = 0. \tag{6.6}$$

Substituting (6.5) and (6.6) into (6.4) leads to

$$\begin{aligned}\mathscr{R}_2(\boldsymbol{u};q) &= -(\varepsilon\boldsymbol{u} - \mathcal{J}_h^{\mathrm{d}}\varepsilon\boldsymbol{u},\nabla_h q) + \Big(\mathscr{L}([\![q]\!]_N),\varepsilon\boldsymbol{u} - \mathcal{J}_h^{\mathrm{d}}\varepsilon\boldsymbol{u}\Big)\\ &\leqslant C\Big(\sum_{K\in\mathscr{T}_h}\|\varepsilon\boldsymbol{u} - \mathcal{J}_h^{\mathrm{d}}\varepsilon\boldsymbol{u}\|_{0,K}^2\Big)^{1/2}\|q\|_{Q_h},\\ &\leqslant C\inf_{\boldsymbol{w}\in V_h^{\mathrm{d}}}\|\varepsilon\boldsymbol{u} - \boldsymbol{w}\|_{0,\Omega}\|q\|_{Q_h},\end{aligned}$$

which together with Ern & Guermond (2017, Theorem 5.2) completes the proof. □

6.2 *Error estimates*

The framework for the abstract error estimates (see, e.g., Ern & Guermond, 2004; Houston *et al.*, 2004), combined with the stability conditions given in Section 5.1, leads to the abstract error estimates for the auxiliary formulation (5.1)–(5.2) with $k = 0$.

THEOREM 6.3 For $k = 0$, let $(\boldsymbol{u}, p)$ be the solution of (2.2)–(2.3) and let $(\boldsymbol{u}_h, \lambda_h; p_h)$ be the solution of (5.1)–(5.2). Then there exists a constant $C > 0$, independent of the mesh size, such that

$$\|(\boldsymbol{u}-\boldsymbol{u}_h, -\lambda_h)\|_{W_h} \leqslant C\Big(\inf_{\boldsymbol{v}\in V_h}\|\boldsymbol{u}-\boldsymbol{v}\|_{V_h} + \inf_{q\in Q_h}\|p-q\|_{Q_h} + \mathscr{R}_1(\boldsymbol{u},p) + \mathscr{R}_2(\boldsymbol{u})\Big),$$

$$\|p-p_h\|_{Q_h} \leqslant C\Big(\inf_{q\in Q_h}\|p-q\|_{Q_h} + \|(\boldsymbol{u}-\boldsymbol{u}_h, -\lambda_h)\|_{W_h} + \mathscr{R}_1(\boldsymbol{u},p)\Big).$$

Proof. The estimates follow from an extension of the standard mixed finite element theory, see Houston *et al.* (2004, proof of Theorem 6.1) combined with the stability conditions in Lemmas 5.2–5.4. □

The next result quantifies the best-approximation error.

THEOREM 6.4 (Quasi-best approximation). Let $(\boldsymbol{u}, p) \in H^s(\mathrm{curl}, \Omega) \times H^{1+s}(\Omega)$ be the solution of (2.2)–(2.3), $s \in (0, 1/2)$. Then there exists a constant $C > 0$ independent of the mesh size such that

$$\inf_{\boldsymbol{v}\in \boldsymbol{V}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_{V_h} \leqslant C\Bigg(\sum_{K\in\mathscr{T}_h} h_K^{2s}\left(|\boldsymbol{u}|^2_{s,\omega_K} + |\nabla\times\boldsymbol{u}|^2_{s,\omega_K}\right) + h_K^2(\|\nabla\times\boldsymbol{u}\|_{0,\omega_K} + h_K^s|\nabla\times\boldsymbol{u}|_{s,\omega_K})^2\Bigg)^{1/2}, \tag{6.7}$$

$$\inf_{q\in Q_h}\|p-q\|_{Q_h} \leqslant C\Bigg(\sum_{K\in\mathscr{T}_h} h_K^{2s}|p|^2_{1+s,\omega_K}\Bigg)^{1/2}. \tag{6.8}$$

Proof. Recall V^{c}_{h0} defined in (3.3) and let V^{d}_{h0} be the corresponding $H(\mathrm{div})$ conforming polynomial space. Then, by using the commuting and quasi-best approximation properties of the smoothed interpolations $\mathscr{J}^{\mathrm{c}}_{h0}$ and $\mathscr{J}^{\mathrm{d}}_{h0}$ defined in Ern & Guermond (2016, Section 6.3) for $H_0(\mathrm{curl}, \Omega)$ and $H_0(\mathrm{div}, \Omega)$ functions, we obtain

$$\begin{aligned}\inf_{\boldsymbol{v}\in V_h}\|\boldsymbol{u}-\boldsymbol{v}\|_{V_h} &\leqslant c\|\boldsymbol{u}-\mathscr{J}^{\mathrm{c}}_{h0}\boldsymbol{u}\|_{V_h}\\ &= c\left(\|\varepsilon^{\frac{1}{2}}(\boldsymbol{u}-\mathscr{J}^{\mathrm{c}}_{h0}\boldsymbol{u})\|^2_{0,\Omega} + \|\mu^{-\frac{1}{2}}(\nabla\times\boldsymbol{u}-\mathscr{J}^{\mathrm{d}}_{h0}\nabla\times\boldsymbol{u})\|^2_{0,\Omega}\right)^{1/2}\\ &\leqslant c\left(\inf_{\boldsymbol{v}\in V^{\mathrm{c}}_{h0}}\|\boldsymbol{u}-\boldsymbol{v}\|^2_{0,\Omega} + \inf_{\boldsymbol{w}\in V^{\mathrm{d}}_{h0}}\|\nabla\times\boldsymbol{u}-\boldsymbol{w}\|^2_{0,\Omega}\right)^{1/2}\\ &\leqslant c\left(\|\boldsymbol{u}-P^{\mathrm{c}}_h\boldsymbol{u}\|^2_{0,\Omega} + \|\nabla\times\boldsymbol{u}-P^{\mathrm{d}}_h\nabla\times\boldsymbol{u}\|^2_{0,\Omega}\right)^{1/2}.\end{aligned}$$

Here, in the second step, we used that $\mathscr{L}_F([\![\boldsymbol{u} - \mathscr{I}^{\mathrm{c}}_{h0}\boldsymbol{u}]\!]_T) = 0$. In the last step we used the curl- and div-conforming interpolations P^{c}_h and P^{d}_h, which are defined in Appendices A and B, respectively. By using the error bounds (5.8) and (5.9), we arrive at

$$\inf_{\boldsymbol{v}\in V_h} \|\boldsymbol{u} - \boldsymbol{v}\|_{V_h} \leqslant c\Big(\sum_{K\in\mathscr{T}_h} \|\boldsymbol{u} - P^{\mathrm{c}}_h\boldsymbol{v}\|^2_{0,K} + \|\nabla\times\boldsymbol{u} - P^{\mathrm{d}}_h\nabla\times\boldsymbol{u}\|^2_{0,K}\Big)^{1/2}$$
$$\leqslant c\Big(\sum_{K\in\mathscr{T}_h} h^{2s}_K|\boldsymbol{u}|^2_{s,\omega_K} + h^2_K(\|\nabla\times\boldsymbol{u}\|_{0,\omega_K} + h^s_K|\nabla\times\boldsymbol{u}|_{s,\omega_K})^2 + h^{2s}_K|\nabla\times\boldsymbol{u}|^2_{s,\omega_K}\Big)^{1/2},$$

which proves the first estimate. The bound (6.8) follows from standard approximation results, e.g., Monk (2003, Section 5.6.1) or Scott & Zhang (1990). □

Now we are ready to prove Theorem 4.3 in the special case $k = 0$.

Proof of Theorem 4.3 *for* $k = 0$. From the abstract error estimates in Theorem 6.3, the polynomial approximation results (6.7)–(6.8) and the residual estimates in Proposition 6.2, we easily derive the *a priori* error bounds in Theorem 4.3 for $k = 0$. □

7. Indefinite Maxwell equations

We will first discuss existence and uniqueness properties of the mixed DG method (3.12)–(3.13) for the indefinite Maxwell equations (1.1a), and subsequently provide error estimates under minimal regularity requirements for $k \neq 0$, k^2 not a Maxwell eigenvalue. Instead of using the Fredholm alternative to show well-posedness of the mixed DG system (3.12)–(3.13), we shall prove an inf-sup condition for A_h on the kernel of B_h for $k \neq 0$.

7.1 *Uniform convergence and spectral theory*

The spectral properties of the solution operator are essential for the proof of the existence and uniqueness of the mixed DG method for the time-harmonic Maxwell equations. Using Lemma 2.2, we can define the bounded solution operators $T : [L^2_\varepsilon(\Omega)]^3 \to V$ and $T_p : [L^2_\varepsilon(\Omega)]^3 \to Q$ by

$$T\boldsymbol{j} := \boldsymbol{u}, \quad T_p\boldsymbol{j} := p, \tag{7.1}$$

where $(\boldsymbol{u}, p) \in V \times Q$ solves (2.2)–(2.3) with $\boldsymbol{j}$ replaced by $\varepsilon\boldsymbol{j}$. By Lemma 2.3, the image of T belongs to $V \cap H^s(\mathrm{curl}, \Omega)$, $s \in (0, 1/2)$, and thus $\|T\boldsymbol{j}\|_{V_h}$ is well defined. Similarly, using Theorem 4.2 for $k = 0$, we can define the bounded discrete solution operators $T_h : [L^2_\varepsilon(\Omega)]^3 \to V_h$ and $T_{p,h} : [L^2_\varepsilon(\Omega)]^3 \to Q_h$ by

$$T_h\boldsymbol{j} := \boldsymbol{u}_h, \quad T_{p,h}\boldsymbol{j} := p_h, \tag{7.2}$$

where $(\boldsymbol{u}_h, p_h)$ solves (3.12)–(3.13) with $k = 0$ and $\boldsymbol{j}$ replaced by $\varepsilon\boldsymbol{j}$. From the abstract estimates in Theorem 6.3 together with Proposition 6.2 and the consistency of the finite element spaces, i.e.,

$$\lim_{h\to 0}\inf_{\boldsymbol{v}\in V_h}\|\boldsymbol{u}-\boldsymbol{v}\|_{V_h}=0,\quad \lim_{h\to 0}\inf_{q\in Q_h}\|p-q\|_{Q_h}=0,\quad \forall\,\boldsymbol{u}\in V,\ p\in Q,$$

we obtain pointwise convergence of T_h to T: for any fixed $\boldsymbol{j}\in[L^2_\varepsilon(\Omega)]^3$, $\|T\boldsymbol{j}-T_h\boldsymbol{j}\|_{V_h}\to 0$ and $\|T_p\boldsymbol{j}-T_{p,h}\boldsymbol{j}\|_{Q_h}\to 0$ as $h\to 0$. The following proposition states the uniform convergence of T_h; see Appendix D for a proof.

PROPOSITION 7.1 (Uniform convergence).

$$\lim_{h\to 0}\sup_{\boldsymbol{j}_h\in V_h,\,\|\boldsymbol{j}_h\|_{0,\Omega}=1}\|(T-T_h)\boldsymbol{j}_h\|_{V_h}=0. \tag{7.3}$$

By using (7.3), one can derive that for any $z\in\rho(T)$, the resolvent set of T, the resolvent operator $R_z(T_h)=(z-T_h)^{-1}:V_h\to V_h$ exists and is bounded for all h sufficiently small. In fact, we have the following lemma (Descloux *et al.*, 1978, Lemma 1).

LEMMA 7.2 Let (7.3) hold and let $F\subset\rho(T)$ be closed. Then, for all h small enough, there holds

$$\sup_{\boldsymbol{v}\in V_h,\,\|\boldsymbol{v}\|_{V_h}=1}\|R_z(T_h)\boldsymbol{v}\|_{V_h}\leqslant C\quad\forall\,z\in F,$$

where $C>0$ depending on F is independent of the mesh size h.

7.2 *Existence and uniqueness*

By using the results of Lemma 7.2, it is straightforward to prove Theorem 4.2.

Proof of Theorem 4.2. We prove the uniqueness by proving the *a priori* bound in the theorem. From the definition of the solution operators T_h and $T_{p,h}$, we infer that

$$\boldsymbol{u}_h=T_h(\varepsilon^{-1}\boldsymbol{j})+k^2T_h\boldsymbol{u}_h,\quad p_h=T_{p,h}(\varepsilon^{-1}\boldsymbol{j})+k^2T_{p,h}\boldsymbol{u}_h. \tag{7.4}$$

With $z:=1/k^2$, the first equation becomes

$$(z-T_h)\boldsymbol{u}_h=zT_h(\varepsilon^{-1}\boldsymbol{j}).$$

Since k^2 is not a Maxwell eigenvalue by Assumption 2.1, i.e., z is not an eigenvalue of T, Lemma 7.2 shows that, for h small enough, the operator $R_z(T_h):V_h\to V_h$ exists and is bounded uniformly in h. Hence, $\boldsymbol{u}_h$ is uniquely determined by

$$\boldsymbol{u}_h=z(z-T_h)^{-1}T_h(\varepsilon^{-1}\boldsymbol{j}).$$

From definition (7.2) of T_h, it follows that $T_h : [L^2_\varepsilon(\Omega)]^3 \to V_h$ is also bounded and there exists a constant $C > 0$ independent of the mesh size such that

$$\|\boldsymbol{u}_h\|_{V_h} \leqslant C\|R_z(T_h)\|\|T_h\|\|\varepsilon^{-\frac{1}{2}}\boldsymbol{j}\|_{0,\Omega} \leqslant C\|\boldsymbol{j}\|_{0,\Omega}. \tag{7.5}$$

The uniqueness of p_h directly follows from the uniqueness of $\boldsymbol{u}_h$, and there exists a constant $C > 0$ independent of mesh size such that

$$\|p_h\|_{Q_h} \leqslant C\left(\|\varepsilon^{-\frac{1}{2}}\boldsymbol{j}\|_{0,\Omega} + \|\varepsilon^{\frac{1}{2}}\boldsymbol{u}_h\|_{0,\Omega}\right) \leqslant C\|\boldsymbol{j}\|_{0,\Omega}. \tag{7.6}$$

□

7.3 *Error estimates*

For the indefinite Maxwell equations ($k \neq 0$), we can similarly define the residuals

$$\begin{aligned}\mathscr{R}_{1,k}(\boldsymbol{u},p;\boldsymbol{v},\eta) &:= A_h(\boldsymbol{u},0;\boldsymbol{v},\eta) - k^2(\varepsilon\boldsymbol{u},\boldsymbol{v}) + B_h(\boldsymbol{v},\eta;p) - (\boldsymbol{j},\boldsymbol{v}),\\ \mathscr{R}_{2,k}(\boldsymbol{u},q) &:= B_h(\boldsymbol{u},0;q).\end{aligned}$$

As in Section 6, one can show that corresponding estimates for the residuals as in Proposition 6.2 hold true. The next proposition provides the inf-sup condition of A_h on the kernel of B_h for $k \neq 0$, which is a crucial ingredient for the error estimates. The idea of the proof is classical (see, e.g., Melenk, 1995; Buffa & Perugia, 2006).

PROPOSITION 7.3 (Inf-sup condition). For a mesh size h small enough, there holds

$$\inf_{0\neq(\boldsymbol{u}_h,\lambda_h)\in\ker(B_h)}\ \sup_{0\neq(\boldsymbol{v},\eta)\in\ker(B_h)} \frac{A_h(\boldsymbol{u}_h,\lambda_h;\boldsymbol{v},\eta) - k^2(\boldsymbol{u}_h,\boldsymbol{v})_\varepsilon}{\|(\boldsymbol{u}_h,\lambda_h)\|_{W_h}\|(\boldsymbol{v},\eta)\|_{W_h}} \geqslant \kappa_A \tag{7.7}$$

for a positive constant κ_A, which depends on the coefficients μ and ε, but is independent of h. Here, $\ker(B_h)$ is the kernel of B_h defined in (5.3).

Proof. For any $(\boldsymbol{v},\eta) \in \ker(B_h)$, let $(\boldsymbol{u}_h,\lambda_h) = (\boldsymbol{v},\eta) + k^2(\tilde{\boldsymbol{u}}_h,\tilde{\lambda}_h)$ with $(\tilde{\boldsymbol{u}}_h,\tilde{\lambda}_h;\tilde{p}_h)$ be the solution of auxiliary formulation (5.1)–(5.2) with $\boldsymbol{j} = \varepsilon\boldsymbol{v}$. Thus, by using the ellipticity of A_h on $\ker(B_h)$ in Lemma 5.3 and (5.1), we obtain

$$\begin{aligned}A_h(\boldsymbol{u}_h,\lambda_h;\boldsymbol{v},\eta) - k^2(\boldsymbol{u}_h,\boldsymbol{v})_\varepsilon &= A_h(\boldsymbol{v},\eta;\boldsymbol{v},\eta) - k^2(\boldsymbol{v},\boldsymbol{v})_\varepsilon + k^2\left(A_h(\tilde{\boldsymbol{u}}_h,\tilde{\lambda}_h;\boldsymbol{v},\eta) - k^2(\tilde{\boldsymbol{u}}_h,\boldsymbol{v})_\varepsilon\right)\\ &= A_h(\boldsymbol{v},\eta;\boldsymbol{v},\eta) \geqslant \kappa'_A\|(\boldsymbol{v},\eta)\|^2_{W_h}.\end{aligned}$$

From Lemma 5.1, we obtain $\tilde{\lambda}_h = [\![\tilde{p}_h]\!]_N$. Hence, by using the stability estimates (7.5)–(7.6), we have

$$\begin{aligned}\|(\tilde{\boldsymbol{u}}_h, \tilde{\lambda}_h)\|_{W_h} &= \|\tilde{\boldsymbol{u}}_h\|_{V_h} + \|\tilde{\lambda}_h\|_{M_h} \\ &\leqslant \|\tilde{\boldsymbol{u}}_h\|_{V_h} + \|\tilde{p}_h\|_{Q_h} \\ &\leqslant C\|\varepsilon^{\frac{1}{2}}\boldsymbol{v}\|_{0,\Omega} \leqslant C\|(\boldsymbol{v}, \eta)\|_{W_h}.\end{aligned}$$

Now we conclude from the above two inequalities,

$$\inf_{0\neq(\boldsymbol{v},\eta)\in\ker(B_h)} \sup_{0\neq(\boldsymbol{u}_h,\lambda_h)\in\ker(B_h)} \frac{A_h(\boldsymbol{u}_h, \lambda_h; \boldsymbol{v}, \eta) - k^2(\boldsymbol{u}_h, \boldsymbol{v})_\varepsilon}{\|(\boldsymbol{u}_h, \lambda_h)\|_{W_h}\|(\boldsymbol{v}, \eta)\|_{W_h}} \geqslant \kappa_A,$$

which is equivalent to (7.7) since $A_h(\cdot, \cdot)$ is symmetric. □

Now we are ready to prove Theorem 4.3.

Proof of Theorem 4.3. Since k^2 is not a Maxwell eigenvalue, the inf-sup condition (7.7) holds true for all h small enough. Together with the inf-sup condition (5.6) of B_h, we can get the corresponding abstract error estimates stated as in Theorem 6.3 for $k = 0$ also for the indefinite time-harmonic Maxwell equations (1.1). Thus, the *a priori* error bound follows directly from the polynomial approximation results (6.7)–(6.8) and residual estimates in Proposition 6.2. □

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A. Curl-conforming quasi-interpolation

The aim of this section is to construct an $H(\text{curl})$-conforming interpolation method for low regularity solutions and prove Theorem 5.6. The basic idea follows from the definition of Scott–Zhang interpolation (Scott & Zhang, 1990). For simplicity, we will only consider lowest-order curl-conforming Nédélec elements.

Let V_h^{c} be the curl-conforming space associated to $\mathscr{T}_h$ consisting of Nédélec elements of the lowest-order, see (3.3), and let $\boldsymbol{\phi}_i \in V_h^{\mathrm{c}},\ i = 1, 2, \ldots, N$ denote the corresponding global basis of V_h^{c}. We denote by $\{e_i\}_{i=1}^N$ the edges associated to the degrees of freedom σ_i for each $\boldsymbol{\phi}_i$. Next we associate to each edge e_i an arbitrary face F_i such that $e_i \subset \partial F_i$, and $F_i \subset \partial\Omega$ if $e_i \subset \partial\Omega$. Denote $e_{i,1} = e_i$ and let $\{e_{i,j}\}_{j=1}^3$ be the edges of face F_i. Similarly, we denote by $\{\boldsymbol{\phi}_{i,j}\}_{j=1}^3$ the restrictions to face F_i of the corresponding global basis function associated to the edges $\{e_{i,j}\}_{j=1}^3$. Let $\{\boldsymbol{\psi}_{i,k}\}_{k=1}^3 \subset \operatorname{span}\{\boldsymbol{n} \times \boldsymbol{\phi}_{i,j} : j \in \{1,2,3\}\}$ be defined by

$$\int_{F_i} \boldsymbol{n} \times \boldsymbol{\phi}_{i,j} \cdot \boldsymbol{\psi}_{i,k} \,\mathrm{d}s = \delta_{jk},$$

where δ_{jk} is the Kronecker delta. Note that $\{\boldsymbol{\psi}_{i,k}\}$ is well defined because $\{\boldsymbol{n} \times \boldsymbol{\phi}_{i,j}\}_j$ are linearly independent: in fact, suppose that for some $c_j \in \mathbb{R}$,

$$\sum_{j=1}^{3} c_j \boldsymbol{n} \times \boldsymbol{\phi}_{i,j} = 0.$$

Then $\sum_j c_j \boldsymbol{\phi}_{i,j}$ has zero tangential trace on F, so for each $\sigma_{i,k}$, we have $0 = \sigma_{i,k}(\sum_j c_j \boldsymbol{\phi}_{i,j}) = c_k$. To continue, let $\boldsymbol{\psi}_i = \boldsymbol{\psi}_{i,1}$ for $i = 1, \ldots, N$. Thus, there holds

$$\int_{F_i} \boldsymbol{n} \times \boldsymbol{\phi}_j \cdot \boldsymbol{\psi}_i \,\mathrm{d}s = \delta_{ij}, \quad i,j = 1, \ldots, N. \tag{A.1}$$

Define the quasi-interpolation $P_h^{\mathrm{c}} : H^s(\mathrm{curl}, \Omega) \to V_h^{\mathrm{c}}$ such that

$$P_h^{\mathrm{c}} \boldsymbol{u} := \sum_{i=1}^{N} \boldsymbol{\phi}_i \langle \boldsymbol{n} \times \boldsymbol{u}, \boldsymbol{\psi}_i \rangle_{F_i}, \tag{A.2}$$

where the duality pairing $\langle \cdot, \cdot \rangle_{F_i}$ is defined as in (3.7). This definition generalizes the Scott–Zhang interpolation (Scott & Zhang, 1990, (2.14)) in the sense that it applies to $H(\mathrm{curl})$ functions with low regularity.

LEMMA A1 (Properties of P_h^{c}). The quasi-interpolation $P_h^{\mathrm{c}} : H^s(\mathrm{curl}, \Omega) \to V_h^{\mathrm{c}}$, $s \in (0, 1/2)$ is well defined and satisfies the following properties:

(i) P_h^{c} preserves boundary condition, i.e.,

$$P_h^{\mathrm{c}} \boldsymbol{v} \in V_{h0}^{\mathrm{c}} \quad \forall \boldsymbol{v} \in V \cap H^s(\mathrm{curl}, \Omega).$$

(ii) V_h^{c} is invariant under P_h^{c}, i.e., $P_h^{\mathrm{c}} \boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in V_h^{\mathrm{c}}$.

Proof. (i) From Lemma 3.2, it holds that $\boldsymbol{n} \times \boldsymbol{v} = 0$ on all faces $F \in \mathscr{F}_h^B$. Thus, recalling the definition of P_h^{c}, we obtain that the tangential trace of $P_h^{\mathrm{c}} \boldsymbol{v}$ also vanishes on faces $F \in \mathscr{F}_h^B$.

(ii) Let $\boldsymbol{v} = \sum_j c_j \boldsymbol{\phi}_j \in V_h^{\mathrm{c}}$ for some coefficients c_j; then

$$P_h^{\mathrm{c}} \boldsymbol{v} = \sum_{i=1}^{N} \boldsymbol{\phi}_i \left\langle \boldsymbol{n} \times \left(\sum_{j=1}^{N} c_j \boldsymbol{\phi}_j \right), \boldsymbol{\psi}_i \right\rangle_{F_i} = \sum_{i=1}^{N} \boldsymbol{\phi}_i \int_{F_i} \boldsymbol{n} \times \left(\sum_{j=1}^{N} c_j \boldsymbol{\phi}_j \right) \cdot \boldsymbol{\psi}_i \, \mathrm{d}s = \sum_{i=1}^{N} c_i \boldsymbol{\phi}_i = \boldsymbol{v},$$

where the last identity follows from (A.1). □

LEMMA A2 Let $\boldsymbol{\psi}_i$ be defined as in (A.1) on some $F_i \subset \partial K, K \in \mathscr{T}_h$; then there exists a constant $c > 0$ independent of the mesh size h_K such that

$$\|\boldsymbol{\psi}_i\|_{L^\infty(F_i)} \leqslant c h_K^{-1}. \tag{A.3}$$

Proof. Using the affine map $T_K : \hat{K} \to K$ with Jacobian B_K, and adopting the transformation, see, e.g., Monk (2003, p. 77), for $H(\mathrm{curl})$ functions $\boldsymbol{\theta}_i \circ T_K = B_K^{-\mathrm{T}} \hat{\boldsymbol{\theta}}_i$ with $\hat{\boldsymbol{\theta}}_i$ being the shape function on the reference element, we have

$$\begin{aligned}
\int_{F_i} \boldsymbol{n} \times \boldsymbol{\theta}_i \cdot \boldsymbol{\psi}_i \, \mathrm{d}s &= \int_{\hat{F}} \frac{B_K^{-\mathrm{T}} \hat{\boldsymbol{n}}}{|B_K^{-\mathrm{T}} \hat{\boldsymbol{n}}|} \times B_K^{-\mathrm{T}} \hat{\boldsymbol{\theta}}_i \cdot \boldsymbol{\psi}_i \circ T_K \frac{|F_i|}{|\hat{F}|} \, \mathrm{d}\hat{s} \\
&= \int_{\hat{F}} B_K^{-\mathrm{T}} \hat{\boldsymbol{n}} \times B_K^{-\mathrm{T}} \hat{\boldsymbol{\theta}}_i \cdot \boldsymbol{\psi}_i \circ T_K \det(B_K) \frac{|F_i|}{\det(B_K) |B_K^{-\mathrm{T}} \hat{\boldsymbol{n}}| |\hat{F}|} \, \mathrm{d}\hat{s} \\
&= \int_{\hat{F}} B_K^{-\mathrm{T}} \hat{\boldsymbol{n}} \times B_K^{-\mathrm{T}} \hat{\boldsymbol{\theta}}_i \cdot \boldsymbol{\psi}_i \circ T_K \det(B_K) \, \mathrm{d}\hat{s},
\end{aligned} \tag{A.4}$$

where we have used that (Monk, 2003, p. 122)

$$\frac{|F_i|}{\det(B_K) |B_K^{-\mathrm{T}} \hat{\boldsymbol{n}}| |\hat{F}|} = 1.$$

By using the identity $(A\boldsymbol{x}) \times (A\boldsymbol{y}) = \det(A)A^{-\mathrm{T}}(\boldsymbol{x} \times \boldsymbol{y})$, with $A \in \mathbb{R}^{3\times 3}$ invertible, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, we have

$$\int_{F_i} \boldsymbol{n} \times \boldsymbol{\theta}_i \cdot \boldsymbol{\psi}_i \,\mathrm{d}s = \int_{\hat{F}} \det(B_K^{-\mathrm{T}}) B_K (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\theta}}_i) \cdot \boldsymbol{\psi}_i \circ T_K \det(B_K)\,\mathrm{d}\hat{s}$$
$$= \int_{\hat{F}} (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\theta}}_i) \cdot B_K^T \boldsymbol{\psi}_i \circ T_K \,\mathrm{d}\hat{s},$$

where the second identity follows from $\det(B_K^{-\mathrm{T}}) = 1/\det(B_K)$. From the uniqueness of the dual basis, we then obtain that the dual basis function $\hat{\boldsymbol{\psi}}_i$ defined on the reference element $\hat{K}$ is equal to

$$\hat{\boldsymbol{\psi}}_i = B_K^{\mathrm{T}} \boldsymbol{\psi}_i \circ T_K,$$

and by using the mesh regularity we have $\|B_K^{-1}\| \leqslant c h_K^{-1}$, see Monk (2003, Lemma 5.10), resulting in

$$\|\boldsymbol{\psi}_i\|_{L^\infty(F_i)} \leqslant c\|B_K^{-\mathrm{T}}\| \leqslant c h_K^{-1}. \qquad \square$$

Lemma A3 (Bound on tangential trace). Let $\boldsymbol{u}, \nabla \times \boldsymbol{u} \in [H^s(K)]^3$, $s \in (0, 1/2)$; then there exists a constant $c > 0$ independent of the mesh size such that for all $F \subset \partial K$,

$$\langle \boldsymbol{n} \times \boldsymbol{u}, \eta \rangle_F \leqslant c \left(h_K \left(\|\nabla \times \boldsymbol{u}\|_{0,K} + h_K^s |\nabla \times \boldsymbol{u}|_{s,K} \right) + \|\boldsymbol{u}\|_{0,K} + h_K^s |\boldsymbol{u}|_{s,K} \right) h_F^{-\frac{1}{2}} \|\eta\|_{0,F} \tag{A.5}$$

for all $\eta \in [\mathbb{P}_1(F)]^3$.

Proof. From the definition of the tangential trace (3.7) and bound (3.6), we have for $\eta \in [\mathbb{P}_\ell(F)]^3 \hookrightarrow [W^{1/p,p'}(F)]^3$ with $1/p + 1/p' = 1$ and $p > 2$,

$$\langle \boldsymbol{n} \times \boldsymbol{u}, \eta \rangle_F = \left(\nabla \times \boldsymbol{u}, L_F^K(\eta) \right)_K - \left(\boldsymbol{u}, \nabla \times L_F^K(\eta) \right)_K$$
$$\leqslant c \left(h_K \|\nabla \times \boldsymbol{u}\|_{L^p(K)} + \|\boldsymbol{u}\|_{L^p(K)} \right) h_K^{-\frac{1}{p}} \left(\|\eta\|_{L^{p'}(F)} + h_K^{\frac{1}{p}} |\eta|_{W^{\frac{1}{p},p'}(F)} \right).$$

On the reference element $\hat{K}$, there holds that the embedding $H^s(\hat{K}) \hookrightarrow L^p(\hat{K})$ is continuous for real number $s > 0$ and some $p > 2$; see, e.g., Grisvard (2011, Theorem 1.4.4.1). A standard scaling argument then leads to

$$\|\boldsymbol{u}\|_{L^p(K)} \leqslant c h_K^{3\left(\frac{1}{p} - \frac{1}{2}\right)} \left(\|\boldsymbol{u}\|_{0,K} + h_K^s |\boldsymbol{u}|_{s,K} \right), \tag{A.6}$$

where the constant $c > 0$ is independent of the mesh size h_K. The scaling argument and an inverse estimate imply that for any polynomial $\eta \in [\mathbb{P}_1(F)]^3$,

$$\|\eta\|_{L^{p'}(F)} \leqslant c h_F^{2\left(\frac{1}{p'} - \frac{1}{2}\right)} \|\eta\|_{L^2(F)}, \quad |\eta|_{W^{\frac{1}{p},p'}(F)} \leqslant c h_F^{2\left(\frac{1}{p'} - \frac{1}{2}\right)} h_F^{-\frac{1}{p}} \|\eta\|_{L^2(F)}. \tag{A.7}$$

Thus, by using (A.6) and (A.7) we conclude (A.5). $\square$

THEOREM A4 (Boundedness). For any $K \in \mathscr{T}_h$, suppose that $\boldsymbol{u}, \nabla \times \boldsymbol{u} \in [H^s(\omega_K)]^3$, $s \in (0, 1/2)$; then

$$\|P_h^{\mathrm{c}}\boldsymbol{u}\|_{0,K} \leqslant c\left(h_K\left(\|\nabla \times \boldsymbol{u}\|_{0,\omega_K} + h_K^s|\nabla \times \boldsymbol{u}|_{s,\omega_K}\right) + \|\boldsymbol{u}\|_{0,\omega_K} + h_K^s|\boldsymbol{u}|_{s,\omega_K}\right) \tag{A.8}$$

with the constant c depending only on the mesh regularity, but independent of the mesh size.

Proof. From the transformation for vector fields $\boldsymbol{\phi}_i \circ T_K = B_K^{-\mathrm{T}}\hat{\boldsymbol{\phi}}_i$ (Monk, 2003, p. 77), it holds that

$$\|\boldsymbol{\phi}_i\|_{0,K} \leqslant ch_K^{\frac{1}{2}}\|\hat{\boldsymbol{\phi}}_i\|_{0,\hat{K}} \leqslant ch_K^{\frac{1}{2}}.$$

Let I_K be all indices of global basis supported on K. Due to Lemma A3 and (A.3) and the mesh regularity $h_F \simeq h_K$, we obtain

$$\begin{aligned}
\|P_h^{\mathrm{c}}\boldsymbol{u}\|_{0,K} &\leqslant \sum_{i\in I_K}\|\boldsymbol{\phi}_i\|_{0,K}|\langle \boldsymbol{n}\times\boldsymbol{u}, \boldsymbol{\psi}_i\rangle_{F_i}| \\
&\leqslant \sum_{i\in I_K}\|\boldsymbol{\phi}_i\|_{0,K}\sum_{K'\in\omega_K}\left(h_{K'}\left(\|\nabla\times\boldsymbol{u}\|_{0,K'} + h_{K'}^s|\nabla\times\boldsymbol{u}|_{s,K'}\right) + \|\boldsymbol{u}\|_{0,K'} + h_{K'}^s|\boldsymbol{u}|_{s,K'}\right)h_F^{-\frac{1}{2}}\|\boldsymbol{\psi}_i\|_{0,F_i} \\
&\leqslant c\sum_{K'\in\omega_K}h_{K'}^{\frac{1}{2}}\left(h_{K'}\left(\|\nabla\times\boldsymbol{u}\|_{0,K'} + h_{K'}^s|\nabla\times\boldsymbol{u}|_{s,K'}\right) + \|\boldsymbol{u}\|_{0,K'} + h_{K'}^s|\boldsymbol{u}|_{s,K'}\right)h_K^{-\frac{1}{2}} \\
&\leqslant c\sum_{K'\in\omega_K}\left(h_{K'}\left(\|\nabla\times\boldsymbol{u}\|_{0,K'} + h_{K'}^s|\nabla\times\boldsymbol{u}|_{s,K'}\right) + \|\boldsymbol{u}\|_{0,K'} + h_{K'}^s|\boldsymbol{u}|_{s,K'}\right).
\end{aligned}$$

□

Proof of Theorem 5.6. Let $\bar{\boldsymbol{v}}$ be the average of $\boldsymbol{v}$ over ω_K; then $P_h^{\mathrm{c}}\bar{\boldsymbol{v}} = \bar{\boldsymbol{v}}$ on K and using the Poincaré inequality (Ern & Guermond, 2017, Lemma 7.1), we obtain

$$\begin{aligned}
\|\boldsymbol{v} - P_h^{\mathrm{c}}\boldsymbol{v}\|_{0,K} &\leqslant \|\boldsymbol{v} - \bar{\boldsymbol{v}}\|_{0,K} + \|P_h^{\mathrm{c}}(\boldsymbol{v} - \bar{\boldsymbol{v}})\|_{0,K} \\
&\leqslant \|\boldsymbol{v} - \bar{\boldsymbol{v}}\|_{0,K} + c\big(h_K\left(\|\nabla\times(\boldsymbol{v}-\bar{\boldsymbol{v}})\|_{0,\omega_K} + h_K^s|\nabla\times(\boldsymbol{v}-\bar{\boldsymbol{v}})|_{s,\omega_K}\right) \\
&\quad + \|\boldsymbol{v}-\bar{\boldsymbol{v}}\|_{0,\omega_K} + h_K^s|\boldsymbol{v}-\bar{\boldsymbol{v}}|_{s,\omega_K}\big) \\
&\leqslant ch_K^s|\boldsymbol{v}|_{s,\omega_K} + ch_K\left(\|\nabla\times\boldsymbol{v}\|_{0,\omega_K} + h_K^s|\nabla\times\boldsymbol{v}|_{s,\omega_K}\right).
\end{aligned}$$

□

B. Div-conforming quasi-interpolation

Similar to the previous section we next construct an $H(\mathrm{div})$-conforming quasi-interpolation operator. The construction process follows the lines of previous section. Let V_h^{d} be the lowest-order $H(\mathrm{div})$-conforming space associated to $\mathscr{T}_h$, see (3.4), and let $\boldsymbol{\phi}_i \in V_h^{\mathrm{d}}$, $i = 1, 2, \ldots, M$ denote the related global basis of V_h^{d} and denote by $\{F_i\}_{i=1}^M$ the faces associated to the degrees of freedom for $\{\boldsymbol{\phi}_i\}$. Then, on each face F_i, there is an $L^2(F_i)$-dual basis such that

$$\int_{F_i}\boldsymbol{\phi}_i\cdot\boldsymbol{n}\psi_i\,\mathrm{d}s = 1.$$

Thus, there holds that

$$\int_{F_i} \boldsymbol{\phi}_j \cdot \boldsymbol{n} \boldsymbol{\psi}_i \, \mathrm{d}s = \delta_{ij}, \quad i,j = 1, \ldots, M. \tag{B.1}$$

The following transformation for $H(\mathrm{div})$ functions relating $\boldsymbol{u}$ on an arbitrary element K to $\hat{\boldsymbol{u}}$ on the reference element $\hat{K}$,

$$\boldsymbol{u} \circ T_K = \frac{1}{\det(B_K)} B_K \hat{\boldsymbol{u}},$$

leads to

$$\begin{aligned}\int_{F_i} \boldsymbol{\phi}_i \cdot \boldsymbol{n} \psi_i \, \mathrm{d}s &= \int_{\hat{F}} \frac{1}{\det(B_K)} B_K \hat{\boldsymbol{\phi}}_i \cdot \frac{B_K^{-\mathrm{T}} \hat{\boldsymbol{n}}}{|B_K^{-\mathrm{T}} \hat{\boldsymbol{n}}|} \psi_i \circ T_K \frac{|F_i|}{|\hat{F}|} \, \mathrm{d}\hat{s} \\ &= \int_{\hat{F}} \hat{\boldsymbol{\phi}}_i \cdot \hat{\boldsymbol{n}} \psi_i \circ T_K \frac{|F_i|}{\det(B_K) |B_K^{-\mathrm{T}} \hat{\boldsymbol{n}}| |\hat{F}|} \, \mathrm{d}\hat{s} \\ &= \int_{\hat{F}} \hat{\boldsymbol{\phi}}_i \cdot \hat{\boldsymbol{n}} \psi_i \circ T_K \, \mathrm{d}\hat{s},\end{aligned}$$

which by the uniqueness of the dual basis defined on F implies that $\psi_i \circ T_K = \hat{\psi}_i$ and hence

$$\|\psi_i\|_{L^\infty(K)} \leqslant c. \tag{B.2}$$

Then define the $H(\mathrm{div})$-conforming quasi-interpolation as a map from space

$$H^s(\mathrm{div}, \Omega) = \left\{ \boldsymbol{u} \in [H^s(\Omega)]^3 : \nabla \cdot \boldsymbol{u} \in H^s(\Omega) \right\}, \; s \in (0, 1/2)$$

to V_h^{d} with $\ell = 1$ as

$$P_h^{\mathrm{d}} \boldsymbol{u} := \sum_{i=1}^{M} \boldsymbol{\phi}_i \langle \boldsymbol{u} \cdot \boldsymbol{n}, \psi_i \rangle_{F_i}, \tag{B.3}$$

where we define analogously to (3.7) the duality pairing $\langle \cdot, \cdot \rangle_{F_i}$ for the normal trace as

$$\langle \boldsymbol{u} \cdot \boldsymbol{n}, \psi_i \rangle_{F_i} := \left(\nabla \cdot \boldsymbol{u}, L_F^K(\psi_i) \right)_K + \left(\boldsymbol{u}, \nabla L_F^K(\psi_i) \right)_K.$$

As above, P_h^{d} is a projection onto V_h^{d} by using the definition of ψ_i and the orthogonality (B.1).

Lemma B1 Let $\boldsymbol{u} \in [H^s(K)]^3$, $\nabla \cdot \boldsymbol{u} \in H^s(K)$, $s \in (0, 1/2)$; then there exists a constant $c > 0$ independent of the mesh size such that for any $F \subset \partial K$,

$$\langle \boldsymbol{u} \cdot \boldsymbol{n}, \eta \rangle_F \leqslant c \left(h_K \left(\|\nabla \cdot \boldsymbol{u}\|_{0,K} + h_K^s |\nabla \cdot \boldsymbol{u}|_{s,K} \right) + \|\boldsymbol{u}\|_{0,K} + h_K^s |\boldsymbol{u}|_{s,K} \right) h_F^{-\frac{1}{2}} \|\eta\|_{0,F} \tag{B.4}$$

for all $\eta \in \mathbb{P}_1(F)$.

Proof.

$$\begin{aligned}\langle \boldsymbol{u}\cdot\boldsymbol{n},\eta\rangle_F &= \left(\nabla\cdot\boldsymbol{u}, L_F^K(\eta)\right)_K + \left(\boldsymbol{u},\nabla L_F^K(\eta)\right)_K \\ &\leqslant c\left(h_K\|\nabla\cdot\boldsymbol{u}\|_{L^p(K)} + \|\boldsymbol{u}\|_{L^p(K)}\right) h_K^{-\frac{1}{p}}\left(\|\eta\|_{L^{p'}(F)} + h_K^{\frac{1}{p}}|\eta|_{W^{\frac{1}{p},p'}(F)}\right).\end{aligned}$$

By using embedding inequalities (A.6) and (A.7), we arrive at (B.4). □

THEOREM B2 (Boundedness). For $K\in\mathscr{T}_h$, suppose that $\boldsymbol{u}\in[H^s(\omega_K)]^3,\ \nabla\cdot\boldsymbol{u}\in H^s(\omega_K),\ s\in(0,1/2)$; then

$$\|P_h^{\mathrm{d}}\boldsymbol{u}\|_{0,K}\leqslant c\left(h_K\left(\|\nabla\cdot\boldsymbol{u}\|_{0,\omega_K}+h_K^s|\nabla\cdot\boldsymbol{u}|_{s,\omega_K}\right)+\|\boldsymbol{u}\|_{0,\omega_K}+h_K^s|\boldsymbol{u}|_{s,\omega_K}\right). \tag{B.5}$$

Proof. The proof is along the lines of the proof of Theorem 5.6. In view of the transformation $\boldsymbol{\phi}_i\circ T_K=\frac{1}{\det(B_K)}B_K\hat{\boldsymbol{\phi}}_i$, it holds that

$$\|\boldsymbol{\phi}_i\|_{0,K}\leqslant ch_K^{-\frac{1}{2}}\|\hat{\boldsymbol{\phi}}_i\|_{0,\hat{K}}\leqslant ch_K^{-\frac{1}{2}},$$

and due to (B.2) we obtain

$$\begin{aligned}\|P_h^{\mathrm{d}}\boldsymbol{u}\|_{0,K} &\leqslant \sum_{i=1}^{N_K}\|\boldsymbol{\phi}_i\|_{0,K}|\langle\boldsymbol{u}\cdot\boldsymbol{n},\boldsymbol{\psi}_i\rangle_{F_i}| \\ &\leqslant c\sum_{i=1}^{N_K}\|\boldsymbol{\phi}_i\|_{0,K}\sum_{K'\in\omega_K}\left(h_{K'}\left(\|\nabla\cdot\boldsymbol{u}\|_{0,K'}+h_{K'}^s|\nabla\cdot\boldsymbol{u}|_{s,K'}\right)+\|\boldsymbol{u}\|_{0,K'}+h_{K'}^s|\boldsymbol{u}|_{s,K'}\right)h_F^{-\frac{1}{2}}\|\boldsymbol{\psi}_i\|_{0,F} \\ &\leqslant c\sum_{K'\in\omega_K}h_K^{-\frac{1}{2}}\left(h_{K'}\left(\|\nabla\cdot\boldsymbol{u}\|_{0,K'}+h_{K'}^s|\nabla\cdot\boldsymbol{u}|_{s,K'}\right)+\|\boldsymbol{u}\|_{0,K'}+h_{K'}^s|\boldsymbol{u}|_{s,K'}\right)h_F^{\frac{1}{2}} \\ &\leqslant c\sum_{K'\in\omega_K}\left(h_{K'}\left(\|\nabla\cdot\boldsymbol{u}\|_{0,K'}+h_{K'}^s|\nabla\cdot\boldsymbol{u}|_{s,K'}\right)+\|\boldsymbol{u}\|_{0,K'}+h_{K'}^s|\boldsymbol{u}|_{s,K'}\right).\end{aligned}$$ □

The proof of Theorem 5.7 is similar to Theorem 5.6. The details are omitted for brevity.

C. Discrete Friedrichs inequality

To prove the ellipticity of A_h on the kernel of B_h (see Lemma 5.3) we employ the following discrete Friedrichs inequality, which is essentially stated in Buffa & Perugia (2006, Lemma 7.6).

LEMMA C1 (Discrete Friedrichs inequality). There holds

$$\|\varepsilon^{\frac{1}{2}}\boldsymbol{v}\|_{0,\Omega}\leqslant C|\boldsymbol{v}|_{V_h}\quad \forall\,\boldsymbol{v}\text{ such that }(\boldsymbol{v},\eta)\in\ker(B_h).$$

Here, the positive constant C is independent of the mesh size.

Proof. Suppose $(\boldsymbol{v},\eta)\in\ker(B_h)$. Then, there holds

$$0=B_h(\boldsymbol{v},\eta;q)=-\int_\Omega\varepsilon\boldsymbol{v}\cdot\nabla_h q\,\mathrm{d}\boldsymbol{x}\qquad\forall\,q\in Q_h^{\mathrm{c}}=Q_h\cap Q,$$

that is, $\boldsymbol{v}$ belongs to the set

$$K_h^{\perp} := \{\boldsymbol{w} \in V_h : (\varepsilon \boldsymbol{w}, \nabla q) = (\boldsymbol{w}, \nabla q)_{V_h} = 0 \ \forall \boldsymbol{q} \in Q_h^{c}\}. \tag{C.1}$$

Here, $(\cdot,\cdot)_{V_h}$ is the natural inner product which induces $\|\cdot\|_{V_h}$. As stated in Buffa & Perugia (2006, Lemma 7.6) we have

$$\|\varepsilon^{\frac{1}{2}}\boldsymbol{v}\|_{0,\Omega} \leqslant C|\boldsymbol{v}|_{V_h} \quad \forall \boldsymbol{v} \in K_h^{\perp}.$$

Hence, we can directly conclude our result. □

D. Uniform convergence

We prove Proposition 7.1, i.e., the uniform convergence of T_h to T. To this end, we follow the procedure of Buffa & Perugia (2006, Section 4.2). Define the space

$$Z_h := \left\{\boldsymbol{v} \in V_h : (\boldsymbol{v}, \eta) \in \ker(B_h) \ \forall \eta \in M_h\right\}.$$

It is easy to see that Z_h contains the range of T_h. We define $Q_h^{c} := Q_h \cap Q$ and introduce an orthogonal decomposition of V_h:

$$V_h = K_h \oplus K_h^{\perp},$$

where $K_h = \nabla Q_h^{c}$ and $K_h^{\perp}$ is defined by (C.1). It follows from the proof of Lemma C1 that $Z_h \subseteq K_h^{\perp}$.

Furthermore, from the definition of the solution operators T, T_p and T_h, $T_{p,h}$ (7.1)–(7.2), we easily obtain the following lemma.

Lemma D1 For all $\boldsymbol{f}_h^0 \in K_h$, there hold

$$T\boldsymbol{f}_h^0 = T_h\boldsymbol{f}_h^0 = \boldsymbol{0} \quad \text{and} \quad T_p\boldsymbol{f}_h^0 = T_{p,h}\boldsymbol{f}_h^0 = -\boldsymbol{f}_h^0.$$

Proof. Since $\boldsymbol{f}_h^0 \in K_h$, there exists $q \in Q_h^{c}$ such that $\boldsymbol{f}_h^0 = \nabla q$. It is easy to check that $(\boldsymbol{0}, -q)$ solves (2.2)–(2.3) and (3.12)–(3.13) simultaneously with $\boldsymbol{j} = \varepsilon\nabla q$. □

Thus, the uniform convergence (7.3) of T_h to T follows directly from Buffa & Perugia (2006, Proposition 4.4), in combination with the regularity estimates given in Lemma 2.3 and the error estimate given in Theorem 4.3.