Infill asymptotics for logistic regression estimators for spatio-temporal point processes

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Abstract
This paper discusses infill asymptotics for logistic regression estimators for spatio-temporal point processes whose intensity functions are of log-linear form. We establish strong consistency and asymptotic normality for the parameters of a Poisson point process model and demonstrate how these results can be extended to general point process models. Additionally, under proper conditions, we also extend our central limit theorem to other unbiased estimating equations that are based on the Campbell–Mecke theorem.

Keywords & Phrases: Campbell–Mecke theorem, infill asymptotics, logistic regression estimator, spatio-temporal point process, unbiased estimating equation.

2010 Mathematics Subject Classification: 60G55, 62M30.

1 Introduction
Spatial and spatio-temporal point processes have been widely used to model, for example, earthquakes (Bray and Schoenberg, 2013), fires (Møller and Díaz-Avalos, 2010; Lu et al., 2021) and tropical rain forests (Waagepetersen, 2008).

Usually, the modelling procedure of a spatial or spatio-temporal point pattern starts with the intensity function which characterizes the probability of a point occurring in an infinitesimal ball centred at a given location and time. In many applications, the intensity function is defined as a log-linear parametric function of certain covariates (e.g., Møller and Waagepetersen, 2007; Waagepetersen, 2008; Guan and Shen, 2010; Coeurjolly and Møller, 2014). To estimate the parameters, classical options include maximum likelihood estimation (e.g., Ogata and Tanemura, 1981; Møller and Waagepetersen, 2004), maximum pseudo-likelihood estimation (e.g., Besag, 1977; Baddeley and Turner, 2000), logistic regression estimation (e.g., Baddeley et al., 2010; Baddeley et al., 2014) and minimum contrast estimation (e.g., Guyon, 1995).

Among them, maximum likelihood estimation is the most computationally intensive method, since for many point process models the likelihood function involves an intractable normalizing constant that must be approximated by Markov chain Monte Carlo simulations (e.g., Møller and Waagepetersen, 2004). Instead, maximum pseudo-likelihood estimation and logistic regression estimation are based on the well-known Campbell–Mecke and Nguyen–Zessin theorems (see, e.g., Daley and Vere-Jones, 2009) and are easy to implement using standard software for generalized linear models (e.g., Baddeley et al., 2015).

From a theoretical perspective, it is important to analyze the asymptotic properties of the
parameter estimators mentioned above. For spatial and spatio-temporal point processes, two asymptotic regimes can be formulated: increasing-domain and infill asymptotics [Ripley, 1988]. In the former regime, the observation window grows; whereas in the latter, the window remains fixed but contains more and more points.

In the literature, Waagepetersen (2008) studied infill asymptotics for maximum pseudo-likelihood estimators for Poisson and cluster point process models with log-linear parametric intensity functions using various weighting schemes to approximate the integral involved. Again, assuming log-linear parametric intensity functions, Thurman et al. (2015) and Choiruddin et al. (2018) developed increasing-domain asymptotics for regularized versions of pseudo-likelihood and logistic regression estimators. In addition, Baddeley et al. (2014) proved increasing-domain asymptotics for logistic regression estimators for stationary Gibbs point processes. In this paper, we study consistency and asymptotic normality for logistic regression estimators for spatio-temporal point processes in the infill asymptotic regime and extend the central limit theorem to other unbiased estimating equations that are based on the Campbell–Mecke theorem.

The remainder of the paper is organized as follows. Section 2 introduces some background and notation regarding spatio-temporal point processes: parametric intensity function, logistic regression estimation, computations of the first two moments of the first-order \(U\)-statistics and infill asymptotic regime. Section 3 provides proofs for strong consistency and asymptotic normality for logistic regression estimators in the case of Poisson point process models. Section 4 discusses the extension of the asymptotic results to general point process models and general unbiased estimating equations. Finally, the paper finishes with some conclusions.

2 Background and notation

Let \(X\) be a spatio-temporal point process on a bounded non-empty open set \(W \times T \subset \mathbb{R}^2 \times \mathbb{R}\) equipped with the Borel \(\sigma\)-algebra (see, e.g., Daley and Vere-Jones, 2009). Suppose that the first-order moment measure of \(X\), which is defined as

\[
\Lambda (B) = \mathbb{E} \left\{ \sum_{x \in X} 1 (x \in B) \right\}
\]

for any Borel subset \(B \subset W \times T\), exists as a \(\sigma\)-finite measure that is absolutely continuous with respect to Lebesgue measure with Radon–Nikodym derivative \(\lambda\). Here, \(1(\cdot)\) denotes the indicator function. Then \(\lambda : W \times T \to [0, \infty)\) is called the intensity function of \(X\).

Higher order moment measures are defined analogously. For instance, the second-order factorial moment measure of \(X\) is

\[
\Lambda^{(2)} (B_1 \times B_2) = \mathbb{E} \left\{ \sum_{x,y \in X} 1 (x \in B_1, y \in B_2) \right\}
\]

for any Borel subsets \(B_1, B_2 \subset W \times T\). Suppose this measure also exists as a \(\sigma\)-finite measure that is absolutely continuous with respect to Lebesgue measure with Radon–Nikodym derivative \(\lambda^{(2)}\). Then \(\lambda^{(2)} : (W \times T)^2 \to [0, \infty)\) is called the second-order product density function of \(X\). Upon scaling, one obtains the pair correlation function \(g(x, y) = \lambda^{(2)} (x, y) / (\lambda(x) \lambda(y))\) of \(X\), provided that \(\lambda(x) \lambda(y) > 0\).
2.1 Parametric intensity function

In this paper, we assume that $X$ admits an intensity function $\lambda$ that is of log-linear form offset by a measurable function $b$ and parameterized by a vector $\theta$ in some parameter space $\Theta \subset \mathbb{R}^m$. Specifically,

$$\lambda(u; \theta) = b(u) \exp(\theta^T z(u)), \quad (1)$$

where $u = (w, t) \in W \times T$ denotes a location and time combination, $b : W \times T \rightarrow [0, \infty)$ is a measurable function that serves as the baseline or reference intensity, $z = [z(1), \ldots, z(m)]^T : W \times T \rightarrow \mathbb{R}^m$ is an $m$-dimensional measurable vector of spatio-temporal covariates and $\theta = [\theta(1), \ldots, \theta(m)]$ is the parameter vector. The gradient vector of $\lambda(u; \theta)$ with respect to $\theta$ then takes the form

$$\nabla \lambda(u; \theta) = \lambda(u; \theta) z(u).$$

Conditions must be imposed on $b$ and $z$ to ensure that $\lambda$ is absolutely integrable. In the sequel, it will sometimes also be necessary to assume that $b$, and thus $\lambda$, is strictly positive. When this is the case, we will state it explicitly.

2.2 Logistic regression estimation

Estimating equations for the parameters of a spatio-temporal point process model in general and the logistic regression estimator in particular are based on the Campbell–Mecke theorem (see, e.g., Daley and Vere-Jones, 2009).

Consider a spatio-temporal point process $X$ on $W \times T$ with intensity function $\lambda$. For any real-valued measurable function $f$ defined on $W \times T$ such that $f \lambda$ is absolutely integrable, the Campbell–Mecke theorem reads

$$\mathbb{E} \left\{ \sum_{x \in X} f(x) \right\} = \int_{W \times T} f(u) \lambda(u) \, du, \quad (2)$$

where $x$ runs through the points of $X$. When $\lambda$ is parameterized by a vector $\theta$ as $\lambda(u; \theta)$, (2) provides a basis for estimating $\theta$.

The logistic regression estimator is based on the components of the vector function

$$f(u; \theta) = \nabla \log \left[ \frac{\lambda(u; \theta)}{\lambda(u; \theta) + \rho(u)} \right] = \frac{\rho(u) / \lambda(u; \theta)}{\lambda(u; \theta) + \rho(u)} \nabla \lambda(u; \theta). \quad (3)$$

Here, one assumes that $\lambda(u; \theta)$ is a positive-valued differentiable function such that its gradient vector $\nabla \lambda(u; \theta)$ is absolutely integrable, and that $\rho(u)$ is a positive-valued measurable function also defined on $W \times T$. The idea is then to estimate both sides of (2) and solve the equations for $\theta$. In order to approximate the right-hand side, one may use a ‘dummy’ point process $D$ on $W \times T$ which is independent of $X$ and has absolutely integrable intensity function $\rho$. Applying the Campbell–Mecke theorem to $D$, one finds that

$$\sum_{x \in D} \frac{1}{\lambda(x; \theta) + \rho(x)} \nabla \lambda(x; \theta)$$
is an unbiased estimator for the right-hand side of (2) with \( f \) as in (3). Hence,

\[
s(X, D; \theta) = \sum_{x \in X} \frac{\rho(x)}{\lambda(x; \theta)} \frac{1}{\lambda(x; \theta) + \rho(x)} \nabla \lambda(x; \theta) - \sum_{x \in D} \frac{1}{\lambda(x; \theta) + \rho(x)} \nabla \lambda(x; \theta) = 0
\] (4)

is an unbiased estimating equation. It is interesting to observe that the middle part of (4) is exactly the gradient of a logistic log-likelihood function

\[
l(X, D; \theta) = \sum_{x \in X} \log \left[ \frac{\lambda(x; \theta)}{\lambda(x; \theta) + \rho(x)} \right] + \sum_{x \in D} \log \left[ \frac{\rho(x)}{\lambda(x; \theta) + \rho(x)} \right].
\] (5)

Thus, (4) can also be interpreted as a score function and solved using standard software such as the R-package \textit{stats} \cite{Venables2002}. The existence and uniqueness of a maximizer of the log-likelihood function (5) are ensured under proper conditions \cite{Silvapulle1981}.

For simplicity of writing, in the remainder of this paper, we write \( s(\theta) \) and \( l(\theta) \) for \( s(X, D; \theta) \) and \( l(X, D; \theta) \) to suppress the dependence on the point patterns \( X \) and \( D \). Moreover, we use \( \theta_0 \) and \( \hat{\theta} \) to respectively denote the true value and the estimator of \( \theta \).

### 2.3 First two moments of first-order \( U \)-statistics

Following \cite{Reitzner2013}, we call random vectors of the form

\[
H = \sum_{x \in X} \begin{bmatrix} h^{(1)}(x), \ldots, h^{(m)}(x) \end{bmatrix}^\top
\]

first-order \( U \)-statistics of \( X \) if \( h^{(l)} \lambda \) is absolutely integrable for every \( 1 \leq l \leq m \).

Suppose that \( X \) has intensity function \( \lambda^{(1)} = \lambda \) and second-order product density function \( \lambda^{(2)} \). By the Campbell–Mecke theorem and in analogy to (2),

\[
\mathbb{E} \left\{ \sum_{x_1, \ldots, x_r \in X} f(x_1, \ldots, x_r) \right\} = \int_{(W \times T)^r} f(u_1, \ldots, u_r) \lambda^{(r)}(u_1, \ldots, u_r) \, du_1 \ldots du_r
\]

for \( r = 1, 2 \) and any real-valued measurable function \( f \) defined on \( W \times T \) such that \( f \lambda \) and \( f \lambda^{(2)} \) are absolutely integrable. Then, it can be used to derive the first two moments of \( H \). Indeed,

\[
\mathbb{E} \{ H \} = \left[ \int_{W \times T} h^{(l)}(u) \lambda(u) \, du \right]_{l=1}^m
\]

and

\[
\mathbb{E} \{ HH^\top \} = \left[ \int_{W \times T} h^{(k)}(u) h^{(l)}(u) \lambda(u) \, du + \int_{W \times T} \int_{W \times T} h^{(k)}(u) h^{(l)}(v) \lambda^{(2)}(u, v) \, dudv \right]_{k,l=1}^m .
\]

If \( \mathbb{E} \{ HH^\top \} \) is finite and \( \lambda > 0 \), the entries in the covariance matrix of \( H \) are finite and can be expressed in terms of the pair correlation function \( g(u, v) \) as

\[
\text{Cov} \{ H \} = \left[ \int_{W \times T} h^{(k)}(u) h^{(l)}(u) \lambda(u) \, du \right]_{k,l=1}^m + \left[ \int_{W \times T} \int_{W \times T} h^{(k)}(u) h^{(l)}(v) (g(u, v) - 1) \lambda(u) \lambda(v) \, dudv \right]_{k,l=1}^m .
\]
2.4 Infill asymptotic regime

The infill asymptotic regime considered in this paper is as defined in Waagepetersen (2008) and Lieshout (2020).

Let \( \{Y_i\} \) and \( \{E_i\} \), with \( i \in \mathbb{N}^+ \), be two sequences of independent and identically distributed spatio-temporal point processes with intensity functions \( \lambda \) and \( \rho \), respectively. Set

\[
X_n = \bigcup_{i=1}^n Y_i, \quad D_n = \bigcup_{i=1}^n E_i.
\]

Write \( \lambda_n \) for the intensity function of \( X_n \) and \( \rho_n \) for that of the ‘dummy’ point process \( D_n \). Thus, \( \lambda_n = n\lambda \) and \( \rho_n = n\rho \), and we assume that the intensity functions of \( X_n \) and \( D_n \) increase at the same rate. For any \( n \in \mathbb{N}^+ \), the score function (4) based on \( X_n \) and \( D_n \) then becomes

\[
s_n(\theta) = \sum_{x \in X_n} \frac{\rho_n(x)}{\lambda_n(x; \theta)} \nabla \lambda_n(x; \theta) - \sum_{x \in D_n} \frac{1}{\lambda_n(x; \theta) + \rho_n(x)} \nabla \lambda_n(x; \theta)
\]

\[
= \sum_{x \in X_n} \frac{\rho(x)}{\lambda(x; \theta) + \rho(x)} \nabla \lambda(x; \theta) - \sum_{x \in D_n} \frac{1}{\lambda(x; \theta) + \rho(x)} \nabla \lambda(x; \theta) = 0.
\]

Note that the terms in the two sums above do not depend on \( n \) after simplification while the subscripts \( x \) run through the points of \( X_n \) and \( D_n \). Taking the limit as \( n \to \infty \), one obtains an asymptotic regime that Ripley (1988) calls ‘infill asymptotics’.

Intuitively, under this regime, the estimate for the parameter vector \( \theta \) will become more precise when the points observed in the fixed domain \( W \times T \) become more dense.

Our aim in the remainder of this paper is to analyze the asymptotic behaviour of the estimator \( \hat{\theta}_n \) defined by (6) as \( n \to \infty \).

3 Infill asymptotics for Poisson point processes

For the sake of completeness, we first recall the definition of Poisson point processes (see, e.g., Lieshout, 2000; Daley and Vere-Jones, 2009).

**Definition 3.1.** A spatio-temporal point process \( X \) defined as in the beginning of Section 2 is a Poisson point process if it satisfies the following properties:

(i) for any bounded Borel set \( B \subset W \times T \), the number of points that fall in \( B \) is Poisson distributed with mean \( \Lambda(B) \);

(ii) in disjoint bounded Borel sets \( B_1, B_2 \subset W \times T \), the numbers of points that fall in \( B_1, B_2 \) are independent.

The probability distribution of a Poisson point process \( X \) is completely specified by its first-order moment measure \( \Lambda \). Thus, the assumption of a log-linear intensity function parameterized as (1) is quite natural and constitutes an exponential family with the components of \( \sum_{x \in X} z(x) \) as sufficient statistics. Moreover, the pair correlation function of \( X \) is always equal to one. The
infill asymptotic regime (cf., Section 2.4) is then particularly appropriate for a Poisson point process model because of the conditional independence of the points given a fixed number.

In this section, we establish strong consistency and asymptotic normality for logistic regression estimators in the case of spatio-temporal Poisson point processes. For ease of referencing, we list here the conditions (C.1)–(C.8) that are required to derive the asymptotic results:

(C.1) \{Y_i\} and \{E_i\} with \(i \in \mathbb{N}^+\) are two independent sequences of independent and identically distributed spatio-temporal point processes on some bounded open set \(W \times T \subset \mathbb{R}^2 \times \mathbb{R}\) and defined on the same underlying probability space \(\Omega\). Set \(X_n = \bigcup_{i=1}^n Y_i\) and \(D_n = \bigcup_{i=1}^n E_i\).

(C.2) \(Y_i\) is a Poisson point process with intensity function \(\lambda(u; \theta)\) as (1), where \(b > 0\) is an absolutely integrable function, \(z\) is a measurable vector of covariates and the parameter vector \(\theta\) lies in an open set \(\Theta \subset \mathbb{R}^m\); \(E_i\) has absolutely integrable intensity function \(\rho(u) > 0\).

(C.3) \(E_i\) has bounded pair correlation function \(g\), that is, \(\sup_{(u,v) \in (W \times T)^2} g(u,v) < \infty\).

(C.4) For every \(\theta \in \Theta\), there exist \(\epsilon_1(\theta), \epsilon_2(\theta) > 0\) such that \(\epsilon_1(\theta) < \inf_{u \in W \times T} \rho(u)/\lambda(u; \theta)\) and \(\sup_{u \in W \times T} \rho(u)/\lambda(u; \theta) < \epsilon_2(\theta)\).

(C.5) The elements of the measurable covariate vector \(z\) are bounded, that is, \(\sup_{u \in W \times T} \|z(u)\| < \infty\).

(C.6) The parameter space \(\Theta\) is convex.

(C.7) The parametric model for \(\lambda\) is identifiable, that is, \(\lambda(u; \theta) = \lambda(u; \tilde{\theta})\) almost everywhere on \(W \times T\) implies \(\theta = \tilde{\theta}\).

(C.8) The matrix \(U\), whose \((k,l)\)-th entry reads \(\int_{W \times T} \lambda(u; \theta_0) \rho(u) z^{(k)}(u) z^{(l)}(u) / [\lambda(u; \theta_0) + \rho(u)] du\) with \(\theta_0 \in \Theta\) and \(1 \leq k, l \leq m\), is positive definite.

A few remarks on some of the conditions are appropriate. In condition (C.2), \(Y_i\) is assumed to be a Poisson point process, however, it may be relaxed (cf., Section 3.1). In condition (C.3), \(E_i\) is not required to be a Poisson point process, as its realisations are only used to approximate an integral. Condition (C.4) is reasonable, recalling the rule of thumb recommended by Baddeley et al. (2014) for selecting \(\rho(u)\) that \(\rho = 4\lambda\). Condition (C.7) is necessary for strong consistency (cf., Theorem 3.3) and condition (C.8) helps ensure the attainment of the logistic regression estimator \(\hat{\theta}_n\) as \(n \to \infty\) (cf., Theorem 3.11).

In the remainder of this paper, we will use \(P_{\theta_0}\) to denote the distribution of \((X_n, D_n)\) under the true parameter value \(\theta_0\).

### 3.1 Strong consistency

To establish strong consistency, we start our investigations with the asymptotic behaviour of the scaled log-likelihood function.

**Lemma 3.2.** Assume that the conditions (C.1)–(C.2) and (C.4)–(C.5) hold. Define \(l_n(\theta) = l(X_n, D_n; \theta)\) by (5) with \(\theta \in \Theta\). Then, as \(n \to \infty\), \(l_n(\theta)/n\) converges \(P_{\theta_0}\)-almost surely to

\[
\int_{W \times T} \left\{ \lambda(u; \theta_0) \log \left[ \frac{\lambda(u; \theta)}{\lambda(u; \theta) + \rho(u)} \right] + \rho(u) \log \left[ \frac{\rho(u)}{\lambda(u; \theta) + \rho(u)} \right] \right\} du.
\]
Proof. Under conditions (C.1)–(C.2) and recalling the logistic log-likelihood function \( l_n(\theta) \),

\[
\frac{l_n(\theta)}{n} = \frac{1}{n} \sum_{x \in X_n} \log \left( \frac{\lambda(x; \theta)}{\lambda(x; \theta) + \rho(x)} \right) + \frac{1}{n} \sum_{x \in D_n} \log \left( \frac{\rho(x)}{\lambda(x; \theta) + \rho(x)} \right),
\]

which consists of two first-order \( U \)-statistics defined on \( X_n \) and \( D_n \). We shall derive their strong convergence separately.

Write the first-order \( U \)-statistic defined on \( X_n \) as

\[
\frac{1}{n} \sum_{x \in X_n} \log \left( \frac{\lambda(x; \theta)}{\lambda(x; \theta) + \rho(x)} \right) = \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{x \in X_i} \log \left( \frac{\lambda(x; \theta)}{\lambda(x; \theta) + \rho(x)} \right) \right).
\]

The sum is the average of independent and identically distributed real-valued random variables. By the Campbell–Mecke theorem,

\[
\mathbb{E}_{\theta_0} \left\{ \sum_{x \in Y_i} \log \left( \frac{\lambda(x; \theta)}{\lambda(x; \theta) + \rho(x)} \right) \right\} = \int_{W \times T} \log \left( \frac{\lambda(u; \theta)}{\lambda(u; \theta) + \rho(u)} \right) \lambda(u; \theta_0) \, du. \tag{7}
\]

Conditions (C.2) and (C.5) imply that the intensity function \( \lambda(u; \theta_0) \) is absolutely integrable on \( W \times T \). Furthermore, by conditions (C.2) and (C.4),

\[
\log \left( \frac{1}{1 + \epsilon_2(\theta)} \right) < \log \left( \frac{1}{1 + \rho(u)/\lambda(u; \theta)} \right) = \log \left( \frac{\lambda(u; \theta)}{\lambda(u; \theta) + \rho(u)} \right) < \log \left( \frac{1}{1 + \epsilon_1(\theta)} \right).
\]

Thus, the \( \mathbb{P}_{\theta_0} \)-mean in (7) is finite for all \( \theta \). Then, Kolmogorov’s strong law of large numbers implies that the first-order \( U \)-statistic defined on \( X_n \) converges \( \mathbb{P}_{\theta_0} \)-almost surely to the integral in the right-hand side of equation (7).

Similarly, for the first-order \( U \)-statistic defined on the dummy point process \( D_n \), as \( n \to \infty \),

\[
\frac{1}{n} \sum_{x \in D_n} \log \left( \frac{\rho(x)}{\lambda(x; \theta) + \rho(x)} \right) \xrightarrow{\mathbb{P}_{\theta_0}} \mathbb{E}_{\theta_0} \left\{ \sum_{x \in E_i} \log \left( \frac{\rho(x)}{\lambda(x; \theta) + \rho(x)} \right) \right\}
\]

\[
= \int_{W \times T} \log \left( \frac{\rho(u)}{\lambda(u; \theta) + \rho(u)} \right) \rho(u) \, du
\]

because \( \rho(u) \) is absolutely integrable by condition (C.2) and

\[
\log \left( \frac{\epsilon_1(\theta)}{1 + \epsilon_1(\theta)} \right) < \log \left( \frac{\rho(u)/\lambda(u; \theta)}{1 + \rho(u)/\lambda(u; \theta)} \right) = \log \left( \frac{\rho(u)}{\lambda(u; \theta) + \rho(u)} \right) < \log \left( \frac{\epsilon_2(\theta)}{1 + \epsilon_2(\theta)} \right)
\]

by conditions (C.2) and (C.4).

The proof is then completed by combining these two strong convergence results. \( \square \)

Recall that the logistic regression estimator minimizes \( U_n(\theta) = -l_n(\theta)/n \) and thus can be considered as a minimum contrast estimator. The next theorem is then concerned with strong consistency.
Theorem 3.3. Assume that the conditions (C.1)–(C.2) and (C.4)–(C.7) hold. Define \( l_n(\theta) = l(X_n, D_n; \theta) \) by (5) with \( \theta \in \Theta \) and set \( \hat{\theta}_n = \arg \max_{\theta \in \Theta} l_n(\theta) \). If \( \hat{\theta}_n \) is attained, then as \( n \to \infty \), \( \hat{\theta}_n \) converges \( P_{\theta_0} \)-almost surely to \( \theta_0 \).

Proof. Conditions (C.2) and (C.6) ensure that the parameter space \( \Theta \) is open and convex.

Firstly, we prove that, for every \( \omega \) in the underlying probability space \( \Omega \), the realisations of the function \( \theta \mapsto U_n(\theta) \) are convex. By condition (C.2), the intensity function \( \lambda(x; \theta) \) has the log-linear form (1) and is thus twice differentiable with respect to the parameter vector \( \theta \in \Theta \). Then, the Hessian matrix of \(-l_n(\theta)\) reads

\[
\sum_{x \in X_n \cup D_n} \frac{\lambda(x; \theta)\rho(x)}{\lambda(x; \theta) + \rho(x)} z^{(k)}(x)z^{(l)}(x) \left[ \begin{array}{c} k,l=1 \\ \end{array} \right].
\]

By decomposition, it can be written into the product of a matrix \( M \) and its transpose as \( M^\top M \), where \( M \) is a \([|X_n \cup D_n| \times m]\)-matrix given by

\[
M = \left[ \begin{array}{c} (\lambda(x_i; \theta)\rho(x_i))^{1/2} \\ \lambda(x_i; \theta) + \rho(x_i) \end{array} \right]_{i=1,k=1}^{X_n \cup D_n,m}.
\]

Here, \(|X_n \cup D_n|\) denotes the number of points in \( X_n \cup D_n \), the subscript \( i \) runs through all points in \( X_n \cup D_n \) and \( k \) runs through the \( m \) covariates. One can readily obtain that the Hessian matrix of \(-l_n(\theta)\) is positive semi-definite for every \( \theta \in \Theta \), which implies that the function \( \theta \mapsto -l_n(\theta) \), and thus \(-l_n(\theta)/n\), is convex.

Secondly, we prove that, as \( n \to \infty \), \( U_n(\theta) - U_n(\theta_0) \) converges \( P_{\theta_0} \)-almost surely to a function \( K(\theta, \theta_0) \) that is non-negative and vanishes only at \( \theta = \theta_0 \). By Lemma 3.2, under conditions (C.1)–(C.2) and (C.4)–(C.5), \( U_n(\theta) - U_n(\theta_0) \) converges \( P_{\theta_0} \)-almost surely to

\[
\int_{W \times T} \left\{ \lambda(u; \theta_0) \log \left( \frac{\lambda(u; \theta_0)}{\lambda(u; \theta)} \right) - (\lambda(u; \theta_0) + \rho(u)) \log \left( \frac{\lambda(u; \theta_0) + \rho(u)}{\lambda(u; \theta) + \rho(u)} \right) \right\} du.
\]

Denote this limit by \( K(\theta, \theta_0) \) and the integrand by \( k(u; \theta, \theta_0) \). Then, clearly \( K(\theta_0, \theta_0) = 0 \). Furthermore,

\[
k(u; \theta, \theta_0) = \lambda(u; \theta) \left\{ \frac{\lambda(u; \theta_0)}{\lambda(u; \theta)} \log \left( \frac{\lambda(u; \theta_0)}{\lambda(u; \theta)} \right) - \left( \frac{\lambda(u; \theta_0)}{\lambda(u; \theta)} + \frac{\rho(u)}{\lambda(u; \theta)} \right) \log \left( \frac{\lambda(u; \theta_0) + \rho(u)}{\lambda(u; \theta) + \rho(u)} \right) \right\}.
\]

By condition (C.2), \( \lambda(u; \theta) > 0 \) as \( b(u) > 0 \). Consider the function \( a \mapsto a \log a - (a + b) \log[(a + b)/(a + b)] \) with \( a, b > 0 \). Its derivative with respect to \( a \) is \( a \log[a(1+b)/(a+b)] \). Hence, the function is strictly decreasing when \( a \in (0,1) \) and strictly increasing when \( a \in (1, +\infty) \). For \( a = 1, a \log a - (a + b) \log[(a + b)/(1 + b)] = 0 \). Thus, \( k(u; \theta, \theta_0) \) is non-negative, and is strictly positive when \( \lambda(u; \theta) \neq \lambda(u; \theta_0) \).

Under condition (C.7), strong consistency follows from an appeal to the Proposition below Guyon (1995, Theorem 3.4.4).
3.2 Asymptotic normality

To establish asymptotic normality, we start our investigations with the Taylor series of the score function \( \hat{g} \).

For every component of the score function, denoted by \( s_n^{(i)}(\theta) \) with \( 1 \leq i \leq m \), the second-order Taylor expansion of \( s_n^{(i)}(\hat{\theta}_n) \) with respect to \( \theta \) at \( \theta_0 \) reads

\[
0 = s_n^{(i)}(\hat{\theta}_n) = s_n^{(i)}(\theta_0) + \nabla s_n^{(i)}(\theta_0)(\hat{\theta}_n - \theta_0) + \frac{1}{2}(\hat{\theta}_n - \theta_0)\nabla^2 s_n^{(i)}(\theta')(\hat{\theta}_n - \theta_0),
\]

where \( \nabla s_n^{(i)}(\theta) \) is the \([1 \times m]\)-vector containing the first-order partial derivatives of \( s_n^{(i)}(\theta) \) with respect to \( \theta \) and \( \nabla^2 s_n^{(i)}(\theta) \) is the \([m \times m]\)-matrix containing the second-order partial derivatives of \( s_n^{(i)}(\theta) \). Moreover, \( \theta'(\cdot) \) is a convex combination of \( \hat{\theta}_n \) and \( \theta_0 \) which, by condition (C.6), lies in \( \Theta \) as well.

Write \( \nabla s_n(\theta_0) \) for the matrix whose \( i \)-th row is \( s_n^{(i)}(\theta_0) \) with \( 1 \leq i \leq m \) and assume that its inverse is well-defined. Heuristically, the idea is then to ignore the error term which is the quadratic form in \( \Theta \) and rearrange the remaining terms to obtain

\[
n^{1/2}((\hat{\theta}_n - \theta_0) \approx \left[ -\nabla s_n(\theta_0) n \right]^{-1} s_n(\theta_0)/n^{1/2}.
\]

In the next two lemmas, we first study the asymptotic behaviours of the terms \(-\nabla s_n(\theta_0)/n\) and \( s_n(\theta_0)/n^{1/2} \), respectively.

**Lemma 3.5.** Assume that the conditions (C.1)–(C.2) and (C.5) hold. Define \( s_n(\theta) = s(X_n, D_n; \theta) \) by (8) with \( \theta \in \Theta \). Then, as \( n \to \infty \), \(-\nabla s_n(\theta_0)/n\) converges \( P_{\theta_0} \)-almost surely to the matrix \( U \) given by

\[
U = \left[ \int_{W \times T} \frac{\lambda(u; \theta_0)\rho(u)}{\lambda(u; \theta_0) + \rho(u)} z^{(k)}(u)z^{(l)}(u)du \right]_{k,l=1}^{m}.
\]

**Proof.** Under conditions (C.1)–(C.2) and recalling the score function (6),

\[
\frac{-\nabla s_n(\theta_0)}{n} = \left[ \sum_{x \in X_n \cup D_n} \frac{\lambda(x; \theta_0)\rho(x)}{n(\lambda(x; \theta_0) + \rho(x))} z^{(k)}(x)z^{(l)}(x) \right]_{k,l=1}^{m}.
\]

We shall prove component-wise strong convergence.

Consider the \((k,l)\)-th entry of the matrix above which, recalling condition (C.1), is given by

\[
\frac{1}{n} \sum_{i=1}^{n} \left\{ \sum_{x \in Y_i \cup E_i} \frac{\lambda(x; \theta_0)\rho(x)}{(\lambda(x; \theta_0) + \rho(x))^2} z^{(k)}(x)z^{(l)}(x) \right\}.
\]

The sum is the average of independent and identically distributed real-valued random variables.

By the Campbell–Mecke theorem,

\[
\mathbb{E}_{\theta_0} \left\{ \sum_{x \in Y_i \cup E_i} \frac{\lambda(x; \theta_0)\rho(x)}{(\lambda(x; \theta_0) + \rho(x))^2} z^{(k)}(x)z^{(l)}(x) \right\} = \int_{W \times T} \frac{\lambda(u; \theta_0)\rho(u)}{\lambda(u; \theta_0) + \rho(u)} z^{(k)}(u)z^{(l)}(u)du.
\]
By condition (C.5), the covariate terms in the integrand above are all bounded. Furthermore, by condition (C.2),

\[ 0 < \frac{\lambda(u; \theta_0)\rho(u)}{\lambda(u; \theta_0) + \rho(u)} < \rho(u) \]

and \( \rho(u) \) is absolutely integrable. Kolmogorov’s strong law of large numbers implies the claimed \( P_{\theta_0} \)-almost sure convergence.

\[ \square \]

**Lemma 3.6.** Assume that the conditions (C.1)–(C.3) and (C.5) hold. Define \( s_n(\theta) = s(X_n, D_n; \theta) \) by (5) with \( \theta \in \Theta \). Then, as \( n \to \infty \), \( s_n(\theta_0)/n^{1/2} \) converges under \( P_{\theta_0} \) in distribution to an \( m \)-dimensional normally distributed random vector with mean zero and covariance matrix

\[
V = \left[ \int_{W \times T} \frac{\lambda(u; \theta_0)\rho(u)}{\lambda(u; \theta_0) + \rho(u)} z^{(k)}(u)z^{(l)}(u)du \right]_{k,l=1}^m
+ \left[ \int_{W \times T} \lambda(u; \theta_0)\lambda(v; \theta_0)\rho(u)\rho(v)z^{(k)}(u)z^{(l)}(v) \left( \frac{g(u,v) - 1}{u\rho(u)} \right) du dv \right]_{k,l=1}^m.
\]

**Proof.** Under conditions (C.1)–(C.2) and recalling the score function (6),

\[
\frac{s_n(\theta_0)}{n^{1/2}} = \sum_{x \in X_n} \frac{\rho(x)}{n^{1/2}(\lambda(x; \theta_0) + \rho(x))} z(x) - \sum_{x \in D_n} \frac{\lambda(x; \theta_0)}{n^{1/2}(\lambda(x; \theta_0) + \rho(x))} z(x).
\]

It consists of two first-order \( U \)-statistics defined on \( X_n \) and \( D_n \). We write it as

\[
\frac{s_n(\theta_0)}{n^{1/2}} = \frac{s_n(X_n; \theta_0)}{n^{1/2}} - \frac{s_n(D_n; \theta_0)}{n^{1/2}}
= n^{1/2} \left\{ \frac{1}{n} \sum_{i=1}^n \sum_{x \in Y_i} \frac{\rho(x)}{\lambda(x; \theta_0) + \rho(x)} z(x) - \frac{1}{n} \sum_{i=1}^n \sum_{x \in E_i} \frac{\lambda(x; \theta_0)}{\lambda(x; \theta_0) + \rho(x)} z(x) \right\}
\]

and discuss the two first-order \( U \)-statistics separately.

Consider \( n^{-1/2}s_n(X_n; \theta_0) \). Note that the first term between the curly brackets above is again the average of independent and identically distributed real-valued random vectors. By the Campbell–Mecke theorem and recalling Section 2.3,

\[
M_{\theta_0} = \mathbb{E}_{\theta_0} \left( \sum_{x \in Y_i} \frac{\rho(x)z(x)}{\lambda(x; \theta_0) + \rho(x)} \right) = \left[ \int_{W \times T} \frac{\rho(u)\lambda(u; \theta_0)z^{(l)}(u)}{\lambda(u; \theta_0) + \rho(u)} du \right]_{l=1}^m
\]

and, because \( Y_i \) is a Poisson point process under condition (C.2),

\[
\text{Cov}_{\theta_0} \left( \sum_{x \in Y_i} \frac{\rho(x)z(x)}{\lambda(x; \theta_0) + \rho(x)} \right) = \left[ \int_{W \times T} \frac{\lambda(u; \theta_0)\rho^2(u)z^{(k)}(u)z^{(l)}(u) du}{(\lambda(u; \theta_0) + \rho(u))^2} \right]_{k,l=1}^m.
\]

By condition (C.5), the covariate terms in the integrand above are all bounded. Furthermore, by condition (C.2),

\[ 0 < \frac{\lambda(u; \theta_0)\rho^2(u)}{(\lambda(u; \theta_0) + \rho(u))^2} < \lambda(u; \theta_0), \quad 0 < \frac{\lambda(u; \theta_0)\lambda(v; \theta_0)\rho(u)\rho(v)}{(\lambda(u; \theta_0) + \rho(u))(\lambda(v; \theta_0) + \rho(v))} < \lambda(u; \theta_0)\lambda(v; \theta_0) \]

\[ \quad \]
and \( \lambda(u; \theta_0), \lambda(v; \theta_0) \) are absolutely integrable. Then, the multi-variate Lindeberg–Lévy central limit theorem implies that \( n^{-1/2}(s_n(X_n; \theta_0) - nM_{\theta_0}) \) converges under \( P_{\theta_0} \) in distribution to an \( m \)-dimensional normally distributed random vector with mean zero and covariance matrix given by (10).

Analogously, by condition (C.3), \( n^{-1/2}(s_n(D_n, \theta_0) - nM_{\theta_0}) \) converges under \( P_{\theta_0} \) in distribution to an \( m \)-dimensional normally distributed random vector with mean zero and covariance matrix given by

\[
\text{Cov}_{\theta_0} \left\{ \sum_{x \in E_i} \frac{\lambda(x; \theta_0)z(x)}{\lambda(x; \theta_0) + \rho(x)} \right\} = \left[ \int_{W \times T} \frac{\lambda^2(u; \theta_0)\rho(u)z(u)z(u)}{(\lambda(u; \theta_0) + \rho(u))^2} \right]^{m}_{k,l=1}
\]

\[
+ \left[ \int_{W \times T} \int_{W \times T} \frac{\lambda(u; \theta_0)\lambda(v; \theta_0)\rho(u)\rho(v)z(u)z(v)}{(\lambda(u; \theta_0) + \rho(u))(\lambda(v; \theta_0) + \rho(v))} (g(u, v) - 1) du dv \right]^{m}_{k,l=1}.
\]

Applying Lévy’s continuity theorem and using the independence of \( X_n \) and \( D_n \) under condition (C.1), the weak limits for the two first-order \( U \)-statistics can be combined, which completes the proof.

**Remark 3.7.** The pair correlation function \( g(u, v) \) of \( E_i \) can be tuned to control the covariance matrix \( \mathbf{V} \). When \( E_i \), and thus \( D_n \), is a Poisson point process, \( g(u, v) \equiv 1 \) and the second term in \( \mathbf{V} \) vanishes. Moreover, regular point processes of \( E_i \), whose \( g(u, v) < 1 \), may be useful to reduce the variance \( \text{(Gautier et al., 2014)} \).

Recalling (2), we are now in a position to conjecture a central limit theorem for the logistic regression estimator. For a formal proof, though, we need to analyze the error term in the Taylor series (5). The next theorem is then concerned with asymptotic normality.

**Theorem 3.8.** Assume that the conditions (C.1)–(C.8) hold. Define \( s_n(\theta) = s(X_n, D_n; \theta) \) by (6) with \( \theta \in \Theta \) and let \( \hat{\theta}_n \) be the logistic regression estimator for which \( s_n(\hat{\theta}_n) = 0 \). If \( \hat{\theta}_n \) is attained, then as \( n \to \infty, n^{1/2}(\hat{\theta}_n - \theta_0) \) converges under \( P_{\theta_0} \) in distribution to an \( m \)-dimensional normally distributed random vector with mean zero and covariance matrix \( \mathbf{U}^{-1}\mathbf{V}(\mathbf{U}^{-1})^\top \), where \( \mathbf{U}, \mathbf{V} \) are as defined in Lemma 3.5 and Lemma 3.6.

**Proof.** Consider the error term in the Taylor series (5) which, under conditions (C.1) and (C.2) and recalling the score function (6), reads

\[
\frac{1}{2}(\hat{\theta}_n - \theta_0)^\top \nabla^2 s_n(\theta)(\hat{\theta}_n - \theta_0) = \frac{1}{2} \sum_{k=1}^{m} \sum_{l=1}^{m} (\hat{\theta}_n - \theta_0)^{(k)}(\hat{\theta}_n - \theta_0)^{(l)} \times
\]

\[
\sum_{x \in X_n \cup D_n} \frac{\lambda(x; \theta^{(i)}, \theta^{(j)}) - \rho(x)}{(\lambda(x; \theta^{(i)}, \theta^{(j)}))^{k}} \left( \frac{\lambda(x; \theta^{(i)}, \theta^{(j)})^3}{\lambda(x; \theta^{(i)}, \theta^{(j)})^{3}} \right)
\]

for all choices of \( \theta^{(i)} \) among the convex combinations of \( \hat{\theta}_n \) and \( \theta_0 \).

Firstly, note that

\[
\sum_{x \in X_n \cup D_n} \frac{\lambda(x; \theta^{(i)}, \theta^{(j)}) - \rho(x)}{(\lambda(x; \theta^{(i)}, \theta^{(j)}))^{k}} \left( \frac{\lambda(x; \theta^{(i)}, \theta^{(j)})^3}{\lambda(x; \theta^{(i)}, \theta^{(j)})^{3}} \right) = n \left( \lambda(x; \theta^{(i)}, \theta^{(j)})^3 \right)
\]

for all choices of \( \theta^{(i)} \) among the convex combinations of \( \hat{\theta}_n \) and \( \theta_0 \).
is bounded in absolute value by
\[ G_{i,k,l}^{(n)} = \sum_{x \in X_n \cup D_n} \frac{|z^{(i)}(x)z^{(k)}(x)z^{(l)}(x)|}{n}. \]

This bound does not depend on \( \theta^{i,i} \) and thus does not depend on \( \hat{\theta}_n \). By the Campbell–Mecke theorem,
\[ \mathbb{E}_{\theta_0} \left\{ G_{i,k,l}^{(n)} \right\} = \int_{W \times T} (\lambda(u; \theta_0) + \rho(u)) \left| z^{(i)}(u)z^{(k)}(u)z^{(l)}(u) \right| du. \]

Under conditions (C.2) and (C.5), \( \mathbb{E}_{\theta_0} \{ G_{i,k,l}^{(n)} \} \) is a non-negative constant. If it is zero, the product of the covariate terms inside has to be zero almost everywhere on \( W \times T \), which contradicts condition (C.8). Thus, \( \mathbb{E}_{\theta_0} \{ G_{i,k,l}^{(n)} \} \) is strictly positive. Then, by Markov’s inequality, for any \( \delta > 0 \), there always exists a finite \( H_{i,k,l}(\delta) = \mathbb{E}_{\theta_0} \{ G_{i,k,l}^{(n)} \}/\delta \) such that
\[ \mathbb{P}_{\theta_0} \left\{ \left| G_{i,k,l}^{(n)} \right| \geq H_{i,k,l}(\delta) \right\} \leq \delta. \]

Since \( H_{i,k,l}(\delta) \) depends only on \( \delta \) but not on \( n \), \( G_{i,k,l}^{(n)} \), and thus \( [12] \), converges under \( P_{\theta_0} \) in probability to \( O(1) \). In addition, note that this result applies to every component of \( [12] \) with \( 1 \leq i, k, l \leq m \).

Now, move back to the Taylor expansion [8]. By Lemma 3.5, \( -\nabla s_n(\theta_0)/n \) converges almost surely, and thus in probability, under \( P_{\theta_0} \) to \( U \). Then, collecting all \( i \) with \( 1 \leq i \leq m \) and recalling the results obtained above, [8] can be rewritten as
\[ \left[ U + o_P(1) - \frac{1}{2} (\hat{\theta}_n - \theta_0) \top O_P(1) \right] (\hat{\theta}_n - \theta_0) = \frac{s_n(\theta_0)}{n}. \]

Next, we appeal to strong consistency (cf., Theorem 3.3) to conclude that \( (\hat{\theta}_n - \theta_0) \top O_P(1) \) converges under \( P_{\theta_0} \) in probability to zero. Furthermore, by condition C.8, the matrix \( U \) is invertible. Thus, \( U + o_P(1) \) is also invertible with a probability tending to one as \( n \to \infty \).

Multiplication by this inverse then yields that
\[ n^{1/2}(\hat{\theta}_n - \theta_0) = \left[ U + o_P(1) \right]^{-1} \frac{s_n(\theta_0)}{n^{1/2}}. \]

The remainder of the proof is then a straightforward application of Slutsky’s theorem. Obviously, \( [U + o_P(1)]^{-1} \) converges under \( P_{\theta_0} \) in probability to \( U^{-1} \). By Lemma 3.6, \( s_n(\theta_0)/n^{1/2} \) converges under \( P_{\theta_0} \) in distribution to an \( m \)-dimensional normally distributed random vector with mean zero and covariance matrix \( V \). Thus, as \( n \to \infty \), \( n^{1/2}(\hat{\theta}_n - \theta_0) \) converges under \( P_{\theta_0} \) in distribution to an \( m \)-dimensional normally distributed random vector with mean zero and covariance matrix \( U^{-1} V (U^{-1})^\top \).

\( \square \)

Remark 3.9. In Theorem 3.3, we have proved that the logistic regression estimator \( \hat{\theta}_n \) is a strongly consistent estimator. Here, observing that \( U \) and \( V \) are constant matrices, we thus further provide the convergence rate, which is \( (\hat{\theta}_n - \theta_0)l = O_P(n^{-1/2}) \) for every \( 1 \leq l \leq m \).
Remark 3.10. The approximate variance $n^{-1}U^{-1}V(U^{-1})^\top$ of $\hat{\theta}_n$ decreases, as $n$ increases. The covariance matrix depends on the unknown intensity function $\lambda(u; \theta_0)$. In practice, a plug-in estimator of $\lambda(u; \theta_0)$ would be used and the dummy point process $D_n$ can be employed to approximate the integrals inside based on the Campbell–Mecke theorem. For instance,

$$nV \approx \left[ \sum_{x \in D_n} \frac{\lambda(x; \hat{\theta}_n)z^{(k)}(x)z^{(l)}(x)}{\lambda(x; \hat{\theta}_n) + \rho(x)} \right]_{k,l=1}^m + \left[ \sum_{x,y \in D_n} \frac{\lambda(x; \hat{\theta}_n)z^{(k)}(x)\lambda(y; \hat{\theta}_n)z^{(l)}(y)}{\lambda(x; \hat{\theta}_n) + \rho(x)\lambda(y; \hat{\theta}_n) + \rho(y)} \right]_{k,l=1}^m$$

Similarly, $nU$ can be approximated by the first of the three terms in the right hand side of the equation above. When $E_i$ and thus $D_n$, is a Poisson point process, $U = V$ and $U^{-1}V(U^{-1})^\top$ reduces to $U^{-1}$. The construction of approximate confidence intervals for the components of $\theta_0$ is now straightforward.

Note that both strong consistency and asymptotic normality of the logistic regression estimator rely on the attainment of $\hat{\theta}_n$. Hence, for the sake of completeness, we also formulate the asymptotic existence of the logistic regression estimator in the next theorem.

Theorem 3.11. Assume that the conditions (C.1)–(C.8) hold. Define $s_n(\theta) = s(X_n; D_n; \theta)$ by (1) with $\theta \in \Theta$. Then, for every $n$, an estimator $\hat{\theta}_n$ exists that solves $s_n(\hat{\theta}_n) = 0$ with a probability tending to one as $n \to \infty$.

Proof. The proof follows from an appeal to Sørensen (1999, Corollary 2.6). By condition (C.2), $s_n(\theta)$ is continuously differentiable with respect to $\theta$ for all $\theta \in \Theta$. Thus, we only need to verify Sørensen (1999, Condition 2.5). For elegance of writing, we omit the proof here but provide a proof in more general cases in Lemma 4.5. \qed

4 Generalized infill asymptotic results

In this section, we discuss the extensions of our infill asymptotic results to general point processes and to other unbiased estimating equations that are based on the Campbell–Mecke theorem.

4.1 Extensions to general point processes

By releasing the Poisson assumption of $X_n$ and modifying corresponding conditions, the infill asymptotic results for logistic regression estimators obtained in Section 3 can be extended to other point processes, although a log-linear form (11) for the intensity function may sometimes not be that natural.

In the remainder of this section, we first demonstrate the extended infill asymptotics for general spatio-temporal point processes. Afterwards, we discuss the applicability for some specific families of point processes.

To extend the asymptotic results to general point processes, we shall need the following modified conditions:
(C.9) $Y_i$ is a point process with intensity function $\lambda(u; \theta)$ given by (1), where $b > 0$ is an absolutely integrable function, $z$ is a measurable vector of covariates and the parameter vector $\theta$ lies in an open set $\Theta \subset \mathbb{R}^m$; $E_i$ has absolutely integrable intensity function $\rho(u) > 0$.

(C.10) Both $E_i$ and $Y_i$ have bounded pair correlation functions, $g$ and $h$, that is, $\sup_{(u,v) \in (W \times T)^2} g(u,v) < \infty$ and $\sup_{(u,v) \in (W \times T)^2} h(u,v) < \infty$.

The theorems concerned with strong consistency, asymptotic normality and existence of the logistic regression estimator are then as follows.

**Theorem 4.1.** Assume that the conditions (C.1), (C.4)–(C.7) and (C.9) hold. Define $l_n(\theta) = l(X_n, D_n; \theta)$ by (1) with $\theta \in \Theta$ and set $\hat{\theta}_n = \arg \max_{\theta \in \Theta} l_n(\theta)$. If $\hat{\theta}_n$ is attained, then as $n \to \infty$, $\hat{\theta}_n$ converges $P_{\theta_0}$-almost surely to $\theta_0$.

**Proof.** The same proofs as for Lemma 3.2 and Theorem 3.3 can be applied straightforwardly. $\Box$

**Theorem 4.2.** Assume that the conditions (C.1), (C.4)–(C.10) hold. Define $s_n(\theta) = s(X_n, D_n; \theta)$ by (2) with $\theta \in \Theta$ and let $\hat{s}_n$ be the logistic regression estimator for which $s_n(\hat{\theta}_n) = 0$. If $\hat{\theta}_n$ is attained, then as $n \to \infty$, $n^{1/2}(\hat{\theta}_n - \theta_0)$ converges under $P_{\theta_0}$ in distribution to an $m$-dimensional normally distributed random vector with mean zero and covariance matrix $U^{-1}V(U^{-1})^T$, where $U$ is as defined in Lemma 3.5 and $V$ is now given by

$$
V = \left[ \int_{W \times T} \lambda(u; \theta_0)\rho(u) \lambda\left(\frac{\rho(u)}{\lambda(u; \theta_0)}\right) z^{(k)}(u)z^{(l)}(u) du \right]_{k,l=1}^m + \left[ \int_{W \times T} \int_{W \times T} \frac{\lambda(u; \theta_0)\lambda(v; \theta_0)\rho(u)\rho(v)z^{(k)}(u)z^{(l)}(v)}{(\lambda(u; \theta_0) + \rho(u))(\lambda(v; \theta_0) + \rho(v))} (g(u,v) + h(u,v) - 2) dudv \right]_{k,l=1}^m.
$$

**Proof.** Similar proofs as for Lemma 3.5, Lemma 3.6 and Theorem 3.8 can be used. Note that, by condition (C.10), the pair correlation function of $Y_i$, denoted by $h(u,v)$, is bounded, which ensures the entries of $V$ to be finite. $\Box$

**Remark 4.3.** The covariance matrix now depends not only on the unknown intensity function $\lambda(u; \theta_0)$ but also on the pair correlation function $h(u,v)$. Similar to Remark 3.10, in practice, a plug-in estimator of $\lambda(u; \theta_0)$ would be used and the point processes $X_n$ and $D_n$ can be employed to approximate the integrals. Compared to Remark 3.10, the approximation of $nV$ needs the two extra terms below

$$
\left[ \sum_{x,y \in X_n} \frac{\rho(x)z^{(k)}(x)}{\lambda(x; \theta_n) + \rho(x)} \frac{\rho(y)z^{(l)}(y)}{\lambda(y; \theta_n) + \rho(y)} \right]_{k,l=1}^m - \left[ \sum_{x \in D_n} \lambda(x; \theta_n)z^{(k)}(x) \sum_{y \in D_n} \lambda(y; \theta_n)z^{(l)}(y) \right]_{k,l=1}^m,
$$

while the approximation of $nU$ stays the same. The construction of approximate confidence intervals for the components of $\theta_0$ is then straightforward again.

**Theorem 4.4.** Assume that the conditions (C.1), (C.4)–(C.10) hold. Define $s_n(\theta) = s(X_n, D_n; \theta)$ by (2) with $\theta \in \Theta$. Then, for every $n$, an estimator $\widehat{\theta}_n$ exists that solves $s_n(\widehat{\theta}_n) = 0$ with a probability tending to one as $n \to \infty$. 

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Proof. A similar proof as for Theorem 3.11 can be applied as well.

In application, for general point processes, the strength of interactions can sometimes enter into the intensity functions.

For log-Gaussian Cox processes (Coles and Jones, 1991; Møller et al., 1998), their intensity functions are the exponentials of Gaussian random variables driven by associated Gaussian random fields with mean function \((\theta^T z(u))\) and covariance function \(\sigma^2 c(u, v)\). Here, \(c(u, v)\) is some correlation function and \(\sigma^2 > 0\). Then, the intensity functions read \(\lambda(u; \theta) \exp(\sigma^2/2)\), which conforms with the log-linear form (1) upon adding a component with the entry one to \(z\). Should \(c(u, v)\) depend on further parameters, e.g. the decay rate of interactions, additional estimating equations are required.

For Markov point processes (Lieshout, 2000), in general, their intensity functions are not known explicitly. However, in the vector function (3), one can replace the intensity function \(\lambda(u; \theta)\) by the Papangelou conditional intensity function \(\lambda(u; \theta | x)\) (Baddeley et al., 2014). The latter characterizes the probability of a new point occurring in an infinitesimal ball centred at a location and time combination \(u \in W \times T\) given the current point realisations \(x\) of \(X\). Similar infill asymptotics in such cases may be obtained upon adding appropriate conditions on the Papangelou conditional intensity functions.

4.2 Extensions to other unbiased estimating equations

The Campbell–Mecke theorem provides the theoretical foundation for a wide range of unbiased estimating equations. Although strong consistency of these estimators may not hold due to the loss of the likelihood interpretation, asymptotic normality and existence hold under appropriate conditions.

In the remainder of this section, we extend the central limit theorem (cf., Theorems 4.2 and 4.4) to more general unbiased estimating equations by introducing the necessary background and providing full proofs.

Recalling (2), a more general unbiased estimating function based on the Campbell–Mecke theorem can take the form

\[
    s(\theta) = \sum_{x \in X} f(x; \theta) - \sum_{x \in D} f(x; \theta) \frac{\lambda(x; \theta)}{\rho(x)} = 0,
\]

for some test vector function \(f : W \times T \to \mathbb{R}^m\) such that every component of \(f\), denoted by \(f^{(i)}\) with \(1 \leq i \leq m\), has the property that \(f^{(i)} \lambda\) is absolutely integrable. Note that, the test function \(f\) can contain some non-negative quadrature weight functions (e.g., Waagepetersen, 2008; Guan and Shen, 2010) depending on the first- and second-order characteristics of \(X\) to reduce the bias caused by deterministic numerical approximation.

To extend the central limit theorem to such unbiased estimating equations, we shall need the following additional conditions:

(C.11) The component functions \(f^{(i)}(u, \theta)\) are twice continuously differentiable with respect to \(\theta\).

Its components and the first- and second-order partial derivatives, denoted by \(\frac{\partial}{\partial \theta} f^{(i)}(u; \theta)\) and \(\frac{\partial^2}{\partial \theta^2} f^{(i)}(u; \theta)\), in a neighbourhood of \(\theta_0\) are bounded in absolute value by some
functions \(d^{(i)}(u), d^{(i)}_l(u)\) and \(e^{(i)}_{k,l}(u)\) respectively, that are absolutely integrable with respect to \(\lambda(u; \theta_0)\).

\((C.12)\) The second-order partial derivatives satisfy an adapted Hölder condition in a neighbourhood of \(\theta_0\): there exist some \(\alpha \in (0, 1]\) and some functions \(e^{(i)}_{k,l}(u)\) which are absolutely integrable with respect to \(\lambda(u; \theta_0)\) such that

\[
\left| \frac{\partial^2}{\partial \theta^{(k)} \partial \theta^{(l)}} f^{(i)}(u; \theta) - \frac{\partial^2}{\partial \theta^{(k)} \partial \theta^{(l)}} f^{(i)}(u; \theta_0) \right| \leq e^{(i)}_{k,l}(u) \| \theta - \theta_0 \|^\alpha.
\]

\((C.13)\) The matrix \(\tilde{U}\), whose \((k,l)\)-th entry reads \(\int_{W \times T} f^{(k)}(u; \theta_0) \lambda(u; \theta_0) z^{(l)}(u) du\) with \(\theta_0 \in \Theta\) and \(1 \leq k, l \leq m\), is invertible.

\((C.14)\) The product of any two component functions, denoted by \(f^{(k)}(u; \theta_0) f^{(l)}(u; \theta_0)\) with \(1 \leq k, l \leq m\), is absolutely integrable with respect to \(\lambda(u; \theta_0)\).

A few remarks on the conditions above are appropriate. Condition \((C.11)\) and \((C.12)\) restrict the smoothness of \(f\) and its first-order and second-order derivatives. Condition \((C.13)\) provides a statement corresponding to condition \((C.8)\) but in the case of general unbiased estimating equations. Condition \((C.14)\) complements necessary technical constraints to ensure that the entries in the variances of these estimators are finite.

To establish the central limit theorem, we start our investigations with the following lemma.

**Lemma 4.5.** Assume that the conditions \((C.1)\), \((C.5)\), \((C.9)\) and \((C.11)\) hold. Define \(s_n(\theta) = s(X_n, D_n; \theta)\) by \((12)\) with \(\theta \in \Theta\). Then, as \(n \to \infty\), for all \(\beta > 0\),

\[
\sup_{\theta^i, k \in C^{(n)}_\beta} \left\| -\frac{1}{n} \nabla s_n(\theta^i, \ldots, \theta^i, m) - \tilde{U} \right\|
\]

converges under \(P_{\theta_0}\) in probability to zero, where \(C^{(n)}_\beta = \{ \theta \in \Theta : \| \theta - \theta_0 \| \leq \beta / n^{1/2} \}\) and

\[
\tilde{U} = \left[ \int_{W \times T} f^{(k)}(u; \theta_0) \lambda(u; \theta_0) z^{(l)}(u) du \right]_{k,l=1}^m.
\]

**Proof.** Under conditions \((C.1)\), \((C.9)\) and \((C.11)\) and recalling the unbiased estimating equation \((13)\), \(-\nabla s_n(\theta^{i_1}, \ldots, \theta^{i_m})/n\) reads

\[
\frac{1}{n} \left[ \sum_{x \in X_n} -\frac{\partial}{\partial \theta^{(i)}} f^{(k)}(x; \theta^{i,k}) + \sum_{x \in D_n} \frac{\lambda(x; \theta^{i,k})}{\rho(x)} \left( \frac{\partial}{\partial \theta^{(i)}} f^{(k)}(x; \theta^{i,k}) + f^{(k)}(x; \theta^{i,k}) z^{(l)}(x) \right) \right]_{k,l=1}^m.
\]

Fix \(\beta > 0\). We then prove the convergence of the supremum component-wisely.

Firstly, by condition \((C.11)\), one can obtain that, for large enough \(n\) and every \(1 \leq i, k, l \leq m\),

\[
\left| f^{(i)}(u; \theta) \right| \leq d^{(i)}(u), \quad \left| \frac{\partial}{\partial \theta^{(i)}} f^{(i)}(u; \theta) \right| \leq d^{(i)}_l(u), \quad \left| \frac{\partial^2}{\partial \theta^{(k)} \partial \theta^{(l)}} f^{(i)}(u; \theta) \right| \leq d^{(i)}_{k,l}(u), \quad \text{(14)}
\]

for all \(u \in W \times T\) and \(\theta \in C^{(n)}_\beta\). Note that, taking \(n\) large enough, \(C^{(n)}_\beta\) lies entirely within \(\Theta\), as the parameter space \(\Theta\) is open by condition \((C.9)\).
Furthermore, under condition (C.9), the first-order Taylor expansion of \( \lambda(u; \theta) \) with respect to \( \theta \) at \( \theta_0 \) reads

\[
\lambda(u; \theta) - \lambda(u; \theta_0) = b(u) \exp \left( \theta^\top z \right) \sum_{i=1}^{m} (\theta - \theta_0)^{(i)} z^{(i)}(u),
\]

where \( \theta \) is a convex combination of \( \theta \) and \( \theta_0 \). By condition (C.5), the covariate terms in the inner product \( \theta^\top z \) and the sum above are all bounded. Thus, recalling the definition of \( \mathcal{C}_\beta^{(n)} \), when \( n \to \infty \), one obtains that \( \lambda(u; \theta) = \lambda(u; \theta_0) + b(u) o(1) \) for all \( u \in W \times T \) and \( \theta \in \mathcal{C}_\beta^{(n)} \).

Now, consider \(-\nabla s_n(\theta^{t,1}, \ldots, \theta^{t,m})/n\). Denotes its \((k,l)\)-th entry by \([-\nabla s_n^{(k)}(\theta^{t,k})/n]_i \) in the remainder of the proof. The difference between \([-\nabla s_n^{(k)}(\theta^{t,k})/n]_i \) and \([-\nabla s_n(\theta_0)/n]_{k,l} \) reads

\[
\begin{align*}
- \frac{1}{n} \sum_{x \in X_n} & \frac{\partial}{\partial \theta^{(l)}} f^{(k)}(x; \theta^{t,k}) \bigg|_{\theta = \theta_0} + \frac{1}{n} \sum_{x \in X_n} \frac{\partial}{\partial \theta^{(l)}} f^{(k)}(x; \theta_0) \\
+ & \frac{1}{n} \sum_{x \in D_n} \frac{\lambda(x; \theta^{t,k})}{\rho(x)} \frac{\partial}{\partial \theta^{(l)}} f^{(k)}(x; \theta^{t,k}) - \frac{1}{n} \sum_{x \in D_n} \frac{\lambda(x; \theta_0)}{\rho(x)} \frac{\partial}{\partial \theta^{(l)}} f^{(k)}(x; \theta_0) \\
+ & \frac{1}{n} \sum_{x \in D_n} \frac{\lambda(x; \theta^{t,k})}{\rho(x)} f^{(k)}(x; \theta^{t,k}) z^{(l)}(x) - \frac{1}{n} \sum_{x \in D_n} \frac{\lambda(x; \theta_0)}{\rho(x)} f^{(k)}(x; \theta_0) z^{(l)}(x).
\end{align*}
\]

Below, we specifically analyze the asymptotic behavior of the second line in (15). The analysis of the other two lines proceeds along similar lines.

Under condition (C.11), the first-order Taylor expansion of \( \frac{\partial}{\partial \theta^{(l)}} f^{(k)}(x; \theta^{t,k}) \) with respect to \( \theta \) at \( \theta_0 \) reads

\[
\frac{\partial}{\partial \theta^{(l)}} f^{(k)}(x; \theta^{t,k}) - \frac{\partial}{\partial \theta^{(l)}} f^{(k)}(x; \theta_0) = \sum_{i=1}^{m} (\theta^{t,k} - \theta_0)^{(i)} \frac{\partial^2}{\partial \theta^{(i)} \partial \theta^{(l)}} f^{(k)}(x; \theta^{t,k}),
\]

where \( \theta^{t,k} \) is a convex combination of \( \theta^{t,k} \) and \( \theta_0 \), and thus \( \theta^{t,k} \in \mathcal{C}_\beta^{(n)} \). Then, recalling that \( \lambda(x; \theta^{t,k}) = \lambda(x; \theta_0) + b(x) o(1) \), the second line in (15) is bounded in absolute value by

\[
\frac{1}{n} \sum_{x \in D_n} \left\{ \sum_{i=1}^{m} \frac{\lambda(x; \theta_0)}{\rho(x)} \left| (\theta^{t,k} - \theta_0)^{(i)} \right| \left| \frac{\partial^2}{\partial \theta^{(i)} \partial \theta^{(l)}} f^{(k)}(x; \theta^{t,k}) \right| + \frac{b(x) o(1)}{\rho(x)} \left| \frac{\partial}{\partial \theta^{(l)}} f^{(k)}(x; \theta^{t,k}) \right| \right\},
\]

which, recalling the definition of \( \mathcal{C}_\beta^{(n)} \) and the bound functions in (14), is further bounded by

\[
\sum_{i=1}^{m} \sum_{x \in D_n} \frac{\beta}{n^{3/2}} \frac{\lambda(x; \theta_0)}{\rho(x)} d^{(k)}_i(x) + \sum_{x \in D_n} \frac{1}{n} \frac{b(x) o(1)}{\rho(x)} d^{(k)}_i(x)
\]

for large enough \( n \). Note that this bound depends only on \( n \) and not on \( \theta^{t,k} \). Furthermore, by condition (C.11), one can apply the Campbell-Meck theorem and Markov’s inequality as in the proof of Theorem 3.8 to obtain that the inner sum in the first term of (16) converges under
Thus in probability, under P\textsubscript{θ₀}, to obtain that 

\[
\sup_{\theta^r \in C_{\beta}^{(n)}} \left| \frac{1}{n} \sum_{x \in D_n} \frac{\lambda(x; \theta^r)}{\rho(x)} \frac{\partial}{\partial \theta^r} f^{(k)}(x; \theta^r) - \frac{1}{n} \sum_{x \in D_n} \frac{\lambda(x; \theta_0)}{\rho(x)} \frac{\partial}{\partial \theta_0} f^{(k)}(x; \theta_0) \right| = o_P(1).
\]

Applying similar proofs to the other two lines in (15) and combining the obtained results as above, by the triangle inequality, one obtains that

\[
\sup_{\theta^r \in C_{\beta}^{(n)}} \left| \left[ -\frac{1}{n} \nabla s^{(k)}_n(\theta^r) \right]_{k,l} - \left[ -\frac{1}{n} \nabla s_n(\theta_0) \right]_{k,l} \right| = o_P(1). \tag{17}
\]

Finally, consider the analogue of Lemma 3.5 for the unbiased estimating equation (13). Under conditions (C.1), (C.9) and (C.11), \(-\nabla s_n(\theta_0)/n\) now reads

\[
\frac{1}{n} \sum_{i=1}^{n} \left[ \sum_{x \in Y_i} \frac{\partial}{\partial \theta^r} f^{(k)}(x; \theta_0) \right. \left. \frac{\lambda(x; \theta_0)}{\rho(x)} \left( \frac{\partial}{\partial \theta^r} f^{(k)}(x; \theta_0) + f^{(k)}(x; \theta_0) z^{(l)}(x) \right) \right]_{k,l=1}^{m}.
\]

Again, by conditions (C.5) and (C.11), one can apply the Campbell–Mecke theorem and Kolmogorov’s strong law of large numbers to obtain that \(-\nabla s_n(\theta_0)/n\) converges almost surely, and thus in probability, under \(P_{\theta_0}\) to \(\hat{U}\). Accordingly,

\[
\left| \left[ -\frac{1}{n} \nabla s_n(\theta_0) \right]_{k,l} - \hat{U}_{k,l} \right| = o_P(1). \tag{18}
\]

The proof is completed by applying the triangle inequality to combine (17) and (18).

The next theorem is then concerned with the central limit theorem.

**Theorem 4.6.** Assume that the conditions (C.1), (C.4)–(C.5) and (C.9)–(C.14) hold. Define \(s_n(\theta)\) by (13) with \(\theta \in \Theta\). Then, as \(n \to \infty\), for every \(n\), an estimator \(\hat{\theta}_n\) exists that solves \(s_n(\theta) = 0\) with a probability tending to one. Moreover, \(\hat{\theta}_n\) converges under \(P_{\theta_0}\) in probability to \(\theta_0\) and \(n^{1/2}(\hat{\theta}_n - \theta_0)\) converges under \(P_{\theta_0}\) in distribution to an \(m\)-dimensional normally distributed random vector with mean 0 and covariance matrix \(\tilde{U}^{-1}V(\tilde{U}^{-1})^\top\), where \(\tilde{U}\) as is defined in Lemma 4.3 and \(V\) is given by

\[
\tilde{V} = \left[ \int_{W \times T} f^{(k)}(u; \theta_0)f^{(l)}(u; \theta_0)\lambda(u; \theta_0) \left( 1 + \frac{\lambda(u; \theta_0)}{\rho(u)} \right) du \right]_{k,l=1}^{m} \]

\[
+ \left[ \int_{W \times T} f^{(k)}(u; \theta_0)\lambda(u; \theta_0)f^{(l)}(v; \theta_0)\lambda(v; \theta_0)(g(u, v) + h(u, v) - 2)dv \right]_{k,l=1}^{m}.
\]

**Proof.** First of all, by condition (C.11), \(s_n(\theta)\) is twice continuously differentiable with respect to \(\theta\). Then, we shall verify the analogues of Lemmas 3.5 and 3.6 for the unbiased estimating equation (13).

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The analogue of Lemma 3.5 was established in Lemma 4.3. For Lemma 3.6 under conditions (C.1) and (C.9), \( s_n(\theta)/n^{1/2} \) now reads

\[
\frac{s_n(\theta_0)}{n^{1/2}} = n^{1/2} \left\{ \frac{1}{n} \sum_{i=1}^{n} \sum_{x \in V_i} f(x; \theta_0) - \frac{1}{n} \sum_{i=1}^{n} \sum_{x \in E_i} f(x; \theta_0) \frac{\lambda(x; \theta_0)}{\rho(x)} \right\}.
\]

The two terms between the curly brackets above are both averages of independent and identically distributed real-valued random vectors. By conditions (C.9)–(C.11), their means are finite and identical. Additionally by conditions (C.4) and (C.14), their variances are also finite and given by the entries of \( V \). Then, similar to the proof of Lemma 3.6 one can apply the Campbell–Mecke theorem and multi-variate Lindeberg–Lévy’s central limit theorem to obtain that \( s_n(\theta_0)/n^{1/2} \) converges under \( P_{\theta_0} \) in distribution to an \( m \)-dimensional normally distributed random vector with mean zero and covariance matrix \( V \).

From Sørensen (1999, Condition 2.7, Corollary 2.8 and Theorem 2.9) and by condition (C.13), we thus only need to prove that

\[
\sup_{\theta \in C^{(n)}_{\beta}} \left\| \frac{1}{n} s_n(\theta) \right\|, \quad \sup_{\theta^r, k \in C^{(n)}} \left\| -\frac{1}{n} \nabla s_n(\theta^r, \ldots, \theta^r, k) - \bar{U} \right\|
\]

and

\[
\sup_{\theta^r, k \in C^{(n)}_{\beta}} \left\| -\frac{1}{n} \nabla^2 s_n^{(i)}(\theta^r, \ldots, \theta^r, m) - \bar{Q}^{(i)} \right\|
\]

all converge under \( P_{\theta_0} \) in probability to zero, where \( \bar{Q}^{(i)} \) is given by

\[
\bar{Q}^{(i)} = \left[ \int_{W \times T} f^{(i)}(u, \theta_0) \lambda(u, \theta_0) z^{(k)}(u) z^{(l)}(u) du \right]_{k,l=1}^m + \left[ \int_{W \times T} \left( \partial_{\theta^r(k)} f^{(i)}(u, \theta_0) z^{(l)}(u) + \partial_{\theta^r(l)} f^{(i)}(u, \theta_0) z^{(k)}(u) \right) \lambda(u, \theta_0) du \right]_{k,l=1}^m.
\]

In Lemma 4.5, we already proved that

\[
\sup_{\theta^r, k \in C^{(n)}_{\beta}} \left\| -\frac{1}{n} \nabla s_n(\theta^r, \ldots, \theta^r, m) - \bar{U} \right\| \xrightarrow{P_{\theta_0}} 0.
\]

The remainder of the proof mostly proceeds along similar lines for the other two cases.

Note that condition (C.12) is needed in proving the convergence of some terms in the \((k, l)\)-th entry of \(-\nabla^2 s_n^{(i)}(\theta^r, \ldots, \theta^r, m)/n\). For instance, we need to prove that

\[
\sup_{\theta^r, k \in C^{(n)}_{\beta}} \left| \frac{1}{n} \sum_{x \in D_n} \frac{\lambda(x; \theta^r, k)}{\rho(x)} - \frac{\partial^2}{\partial \theta^r(j) \partial \theta^r(l)} f^{(k)}(x; \theta^r, k) - \frac{1}{n} \sum_{x \in D_n} \frac{\lambda(x; \theta_0)}{\rho(x)} - \frac{\partial^2}{\partial \theta^r(j) \partial \theta^r(l)} f^{(k)}(x; \theta_0) \right| = o_P(1).
\]

Since \( f^{(k)}(u, \theta) \) is only twice continuously differentiable with respect to \( \theta \), the proof in Lemma 4.5 does not apply here. However, recalling from the proof of Lemma 4.3 that \( \lambda(x; \theta^r, k) = \frac{\lambda(x; \theta_0)}{\rho(x)} \)
\( \lambda(x; \theta_0) + b(x) o(1) \), the absolute value term above is bounded by

\[
\frac{1}{n} \sum_{x \in D_n} \left\{ \frac{\lambda(x; \theta_0)}{\rho(x)} \left| \frac{\partial^2 f^{(k)}(x; \theta^{t,k})}{\partial \theta^{(i)} \partial \theta^{(l)}} \right| - \frac{\partial^2 f^{(k)}(x; \theta_0)}{\partial \theta^{(i)} \partial \theta^{(l)}} \right| + \left| \frac{b(x) o(1)}{\rho(x)} \right| \left| \frac{\partial^2 f^{(k)}(x; \theta^{t,k})}{\partial \theta^{(i)} \partial \theta^{(l)}} \right| \right\},
\]

which, under the adapted Hölder condition in condition (C.12), is further bounded by

\[
\sum_{x \in D_n} \left\{ \frac{\beta}{n^{\alpha/2+1}} \frac{\lambda(x; \theta_0)}{\rho(x)} e^{(k)}_{i,l}(x) + \frac{1}{n} \frac{b(x)}{\rho(x)} |\rho(1)| d^{(k)}_{i,l}(x) \right\}
\]

for large enough \( n \). This bound depends only on \( n \) and not on \( \theta^{t,k} \). By conditions (C.11) and (C.12), one can again apply the Campbell-Mecke theorem and Markov’s inequality as in the proof of Theorem 3.8 to obtain that the bound converges under \( P_{\theta_0} \) in probability to zero.

**Remark 4.7.** Looking back on the logistic regression estimator given by (3), its component functions are smooth with respect to \( \theta \) and thus the second-order derivatives satisfy the adapted Hölder condition. Moreover, its component functions and the first- and second-order derivatives are bounded under condition (C.5).

## 5 Conclusion

In this paper, we established strong consistency and asymptotic normality for logistic regression estimators for spatio-temporal point processes under the regime of infill asymptotics. We also extended the asymptotic results to other unbiased estimating functions that are based on the Campbell–Mecke theorem. We demonstrated how to construct approximate confidence intervals in practice. Compared to increasing-domain asymptotics, infill asymptotics provide the theoretical foundation on the asymptotic behaviour of parameter estimates where identical point patterns are only valid within a fixed region.

In addition, we assumed that spatio-temporal point processes of interest have log-linear parametric intensity functions, which simplifies the computation. However, we believe that the asymptotic results can be proved for a wide class of intensity functions. Moreover, motivated by Baddeley et al. (2014), it would also be interesting to extend infill asymptotics to the Gibbs domain by studying associated background and conditions.

## Acknowledgements

This research was funded by the Dutch Research Council (NWO) for the project ‘Data driven risk management for fire services’ (18004). We thank Maurits de Graaf and Clément Lezane for valuable discussions and comments.

## References


