Generic evolutionarily stable strategies are DSC-stable

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Abstract

An evolutionary equilibrium is DSC-stable if it is (a) **D**ynamically stable, i.e., if the system at equilibrium is slightly disturbed, it returns to it, (b) **S**tructurally stable, i.e., preserves defining properties for small perturbations of the underlying structure of the system, (c) **C**onceptually stable, i.e., equivalent to at least one other evolutionary equilibrium (concept), for a non-singleton class of dynamics.

Attractiveness is a minor *refinement* of the defining properties of certain evolutionary equilibria. We show that attractive evolutionarily stable strategies, attractive evolutionarily stable equilibria and attractive truly evolutionarily stable states are DSC-stable for specific ('dense') classes of dynamics, and that each strict saturated (Nash) equilibrium is DSC-stable for a vast class of evolutionary dynamics.

So, *generically* neither the exact specification of the dynamic system, nor the equilibrium concept matter for qualitative conclusions about the system's behavior nearby.
Key words: attractive evolutionary equilibria; evolutionary dynamics; dynamic, structural & conceptual stability.
JEL-Codes: C62; C72; C73.

1 Introduction

The evolutionarily stable strategy (or evolutionarily stable state, *ESS*) of Maynard Smith & Price [1973], probably the best-known concept from evolutionary game theory, was introduced almost 50 years ago. Although meant for and originally applied to a strictly biological framework, its appeal has lead to applications of evolutionary game theory to various topics beyond mathematical biology such as social dilemmas, the evolution of language,

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mating behavior of animals, computer science, engineering and control theory (see also e.g., Sigmund [2011], Preface).

Game theory was in use in biology at the time but it is fair to say that biologists drew inspiration from game theory, and developments lagged somewhat behind those in game theory. Lewontin [1961] played an important role in making biologists aware of game theory and of the concepts of (*playing the field*) strategies and optimality. The paper mentions Von Neumann & Morgenstern [1944] and Luce & Raiffa [1957]. The latter however, presumably in a desire to pay tribute to the then recently deceased John Von Neumann, attribute a somewhat modest role to the work of Nash. Admittedly, Nash may not have been recognized sufficiently yet, and he was under attack by Von Neumann and Morgenstern (cf., e.g., Kuhn *et al.* [1996]). This may explain why Lewontin [1961] does not cite Nash [1950, 1951].

Maynard Smith and Price[1] may not have been aware of the relationship between their *ESS* and the Nash equilibrium initially (cf., e.g., Kuhn *et al.* [1996]). Even Maynard Smith [1982] in an overview of developments since Maynard Smith & Price [1973], cites Lewontin [1961] and the game-theoretical classics mentioned there repeatedly, but still fails to mention Nash. An indication that novel concepts, ideas and approaches tend to diffuse slowly even within a discipline, let alone across disciplinary boundaries.

Maynard Smith & Price [1973] altered the picture quite drastically, as biology inspired game theory instead of the other way around, and diffusion of this new concept was uncharacteristically rapid. Though the success of evolutionary game theory is to be attributed to the invention of the *ESS* (cf., e.g., Hofbauer [2000], Sigmund [2011]), Taylor & Jonker [1978] deserve credit too. The latter added the arguably second-best-known concept from evolutionary game theory, the replicator dynamics, to the framework and proved that the (essentially static) *ESS* is an attractor under these dynamics.

The idea of low-cognition driven adaptive dynamics potentially inducing a refinement of a Nash equilibrium (asymptotically) stable under the same dynamics, proved too attractive to ignore by various other scientific disciplines. Ambitions to extend evolutionary theorizing to economics,[2] to finance,[3] to game theory and mathematics,[4] and to the social sciences,[5] as well as a surge of micro-foundation approaches for aggregate adjustments[6]

[1] Maynard Smith & Price [1973] thank RC Lewontin for suggestions at the end.

[2] See e.g., Dow [1986], Silverberg [1988], Saviotti & Mani [1996], Saviotti [1996], Metcalfe [1994,1998].

[3] See e.g., Amir *et al.* [2005], Amir *et al.* [2011], Amir *et al.* [2013], Amir *et al.* [2021], Evstigneev *et al.* [2008,2016], Hens & Schenk-Hoppé [2008,2009].

[4] See e.g., Hofbauer & Sigmund [1998], Fudenberg & Levine [1998], Sigmund [2011], Samuelson [1998].

[5] See e.g., Bicchieri et al. [1997], Skyrms [1996,2004], Axelrod & Hamilton [1981], Axelrod [1984], Sigmund [2010].

[6] See e.g., Börgers & Sarin [1990], Schlag [1998,1999], Björnerstedt &Weibull [1996], Sethi [1998], Ritzberger & Weibull [1995], Sandholm [2010].

led to a proliferation of dynamics eligible for the formalization of all kinds of 'evolutionary' changes (cf., also Section 3).

Remarkably, Kuhn *et al.* [1996] reporting from a Nobel Seminar in honor of John Nash, devote nearly one-third of the contents to applications of the Nash equilibrium which without any doubt may be labelled evolutionary game theory, thereby implicitly giving tribute to Maynard Smith & Price [1973], too. Evolutionary game theory today is a rich, perhaps even independent, field with roots in biology and game theory proper, branching out into multiple scientific disciplines integrating or unifying these to some degree.

The introduction of Chapter 2 of Maynard Smith [1982], starts with the author taking a rather modest and almost apologetic position:

> This chapter aims to make clear the assumptions lying behind evolutionary game theory. I will be surprised if it is fully successful. When I first wrote on the applications of game theory to evolution (Maynard Smith & Price, 1973), I was unaware of many of the assumptions being made and of many of the distinctions between different kinds of games which ought to be drawn. No doubt many confusions and obscurities remain, but at least they are fewer than they were.

Several of these obscurities/confusions induced interesting developments, one, the inception of a static concept lacking dynamics to make sense, has already been mentioned. Another one is how to cement the *ESS* in a formula, as several versions have been around for decades (cf., e.g., Vickers & Cannings [1987], Lessard [1990]). Distinctions are subtle, e.g., most see the unbeatable strategy of Hamilton [1967] as a stricter concept than the *ESS* (e.g., Kojima [2006], Amir *et al.* [2020]), whereas Maynard Smith & Price [1973] and Maynard Smith [1982] see it as essentially equivalent. To be clear about the definition employed, we define the *ESS* in the ensuing subsection, before comparing it to related evolutionary equilibrium concepts.

1.1 Technical introduction

The state $y \in \Delta^n$ is an *ESS* if and only if an open neighborhood $U \subset \Delta^n$ containing y exists such that $x \in U\backslash\{y\}$ implies

$$(y - x) \cdot f(x) > 0. \tag{1}$$

Here, x, y are vectors of shares subgroups have in a population representing the latter's composition, or alternatively interpreted, mixed strategies (e.g., Weibull [1996], Fudenberg & Levine [1998]); therefore the set of all possible vectors of population shares (or weights) is $\Delta^n = \{x \in \mathbb{R}^{n+1}_+ | \sum_i x_i = 1\}$, the n-dimensional unit simplex; the *relative fitness function* f (Joosten [1996]) attributes for every composition of the population to every subgroup its fitness relative to the population-share-weighted-average fitness.[7]

Although the concept is meant to capture 'if the dynamical system while being in or sufficiently near an *ESS* is disturbed by an invasion of a small group, the system returns to it', the formalization by Eq. (1) yields a thoroughly static concept (Zeeman [1980]) as nothing in it deals with dynamics. Plausible candidates for the latter may be motivated by an alternative narrative inspired by Darwinian thoughts (Darwin [1859]) namely that 'fitter groups grow faster than less fit groups'. For systems with two subgroups, the *ESS* narrative is readily satisfied under the Darwinian one, as all dynamics plausible in view of the latter, coincide. However, for environments with more subgroups, matters are much more involved as evolutionary dynamics on a higher dimensional unit simplex may, even for bilinear relative fitness functions, display rather complex behavior.

Efforts to reconcile the *ESS* narrative with a mathematical formalization (for suitable dynamics in higher dimensions) started early. Taylor & Jonker [1978] introduced the replicator dynamics, and presented conditions guaranteeing that the *ESS* is an asymptotically stable fixed point under the replicator dynamics. Zeeman [1981] showed that the eigenvalue conditions of the dynamics closely related to the underlying payoff structure of the game assumed and subsequently exploited by Taylor & Jonker [1978], are in fact rather weak. This implies that each *generic ESS* is asymptotically stable under the replicator dynamics.

Another way of dealing with the discrepancy between narrative and formalization is to find alternatives for (1) such that both concur. In this line of thought, dynamics should enter the defining part of any useful concept, and several such concepts have been proposed. For instance, the state $y \in \Delta^n$ is an evolutionarily stable equilibrium (*ESE,* Joosten [1996]) if and only if an open neighborhood $U \subset \Delta^n$ containing y exists such that $x \in U \backslash \{y\}$ implies

$$(y - x) \cdot h(x) > 0, \tag{2}$$

where $h : \Delta^n \rightarrow \mathcal{O}^{n+1} = \{x \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i = 0\}$ represents relevant evolutionary dynamics, yielding a system of $n + 1$ differential equations

$$\frac{dx}{dt} = h(x) \text{ for all } x \in \Delta^n.$$

Here, $\frac{dx}{dt}$ is the continuous-time change of composition of the population; function h should be connected to f in a manner that makes sense in an evolutionary framework (see also Section 3). Eq. (2) implies that along any trajectory starting at $x_0 \in U \backslash \{y\}$, the Euclidean distance to y decreases

[7] We discuss the advantages of our notation in Section 7. Important here is that f is motivated in settings considerably more general than the usual bi-matrix games (cf., e.g., Hofbauer & Sigmund [1998]).

monotonically in time (cf., Joosten [1996]). Beyond the optical similarity between Eqs. (1) and (2), if one defines dynamics based on f extended to the positive orthant by means of a function homogeneous of degree zero in x, then (1) implies that any trajectory starting at $x_0 \in U\backslash\{y\}$ converges monotonically to $\frac{||x_0||}{||y||}y$.

Joosten [2013] introduced *TESS* and *GESE:* the truly evolutionarily stable state and the generalized evolutionarily stable equilibrium. *TESS* places restrictions on dynamics similar to those on the replicator dynamics near an *ESS*;[8] *GESE* relaxes *ESE* requirements to monotone convergence towards equilibrium under a function homothetic to some metric.

We are guided here by the following. Can we find conditions such that several types of evolutionary stability concur for **classes** of dynamics (instead of isolated instances), and can we regain[9] a hold on structural stability?

Let evolutionary equilibrium $y \in \Delta^n$ satisfy $f(y) \leq 0^{n+1} = h(y)$, let property P induce consequence C, and let us write this as a chain of implications $(y, f, h, P) \to C$. Now, imagine perturbations to (y, f, h, P). We distinguish three types of stability related to three types of perturbations of this tuple: dynamic (to y), structural (to f or h) and conceptual stability (to P). For instance, let y be an *ESE*, i.e., P is (2), and C be $\{x_t\}_{t\geq 0} \overset{t\to\infty}{\to} y$ under h if $x_0 \in U$. Now, if the system is perturbed slightly to $x_0 \in U\backslash\{y\}$, consequence C is immediate. This is the usual issue of **dynamic stability**, also known as asymptotic stability (cf., e.g., Perko [1991]).

Next, let y, f, C as before, yet P be (1), and h the replicator dynamics, clearly $(y, f, h, P) \to C$ is covered by e.g., Taylor & Jonker [1978]. Now, let $\widetilde{h}$ be the ray-projection dynamics of Joosten & Roorda [2008, 2011]. Sufficiently near y these can be regarded as a perturbation of h. Joosten & Roorda [2011] prove that C is preserved under $\widetilde{h}$, i.e., the *ESS* is asymptotically stable under $\widetilde{h}$ as well. Part of the literature implicitly focusses on **structural stability**, too, by finding dynamics for which C holds if P is (1).

Continuing, we interpret the *TESS* and the *GESE* as perturbations of some P: the one defining *TESS* is a variant of (1), the one defining *GESE* is a modification of (2). We call an evolutionary equilibrium **conceptually stable** if in the chain of implications $(y, f, h, P) \to C$, we can substitute P by a non-trivial alternative $\widetilde{P}$ for a non-singleton class of dynamics.

An evolutionary equilibrium is **DSC-stable** if it is dynamically, structurally and conceptually stable for a class of evolutionary dynamics. As a

[8] See Fryer [2012], Harper & Fryer [2015] for a related concept: incentive stable state.

[9] Early analysis on the dynamic stability of *ESS* yielded structural stability as a by-product: if all eigen values of the matrix of first derivatives of the dynamics have negative real parts at equilibrium, asymptotical as well as structural stability are guaranteed (cf., Perko [1991] and more specifically Zeeman [1980]). Lyapunov's second method used by e.g., Hofbauer *et al.* [1979], gives answers on dynamic stability (even if eigen value analysis on stability is inconclusive) at the price of losing grip on structural stability.

way of obtaining DSC-stability we came up with the notion of attractiveness, to be introduced next.

For given evolutionary equilibrium $y \in \Delta^n$, let $z = \left(z^1, z^2\right)$ with $z^1, z^2 : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $z^1(y) \cdot z^2(y) = 0$. To make sense here, z *must be connected* to the mathematics defining the evolutionary equilibrium. More precisely, we connect the first to the present location relative to the equilibrium, and the second to a possible direction at the present location. Then, y is **attractive** with respect to z iff an $\varepsilon > 0$ and open neighborhood $U \subset \Delta^n$ containing y exist such that

$$\frac{z^1(x) \cdot z^2(x)}{||z^1(x)|| \cdot ||z^2(x)||} > \varepsilon \text{ for all } x \in U \backslash \{y\}. \tag{ATT}$$

Here, for $a \in \mathbb{R}^{n+1}$, $||a|| = \sqrt{\sum_i a_i{}^2}$. Note that taking $z = \left(z^1, z^2\right) = (y - x, f(x))$ in the inequality above with $\varepsilon = 0$, induces the *ESS*-condition.

We show that attractive evolutionary equilibria satisfy **structural stability,** i.e., preserve their defining properties for perturbations of the payoff system or the dynamics. So, slight miss-specifications of f or h are harmless for conclusions regarding 'stability' of the equilibrium at hand.

Attractive evolutionary equilibria may also satisfy **conceptual stability**, i.e., consequences C in the above may be robust against slight discrepancies in specifications of the equilibrium concept at hand, i.e., perturbations of P, as well. Each attractive *ESS* concurs with an attractive *ESE* under subclasses of the barycentric projection dynamics (Joosten & Roorda [2008]) and of the q-deformed replicator dynamics (Harper [2011]), and with an attractive *TESS* under a subclass of the latter dynamics, as well as under a subclass of the dynamics of Sethi [1998]. So, we not only know that (y, f, h, P) and $(y, f, h, \widetilde{P})$ imply C for certain attractive evolutionary equilibria, but we may take any $\widetilde{h} \neq h$ from a set of evolutionary dynamics $H \ni h$, such that the chain of implications holds for both.

Remarkably, strict saturated (or Nash) equilibria satisfy DSC-stability. Every such state is an *ESS*, and both a *TESS* and an *ESE* for weakly sign compatible dynamics, a class containing the vast majority of evolutionary dynamics proposed in the literature (cf., Joosten [2013]). We show that each strict saturated equilibrium is structurally stable and an attractive *ESS*. Furthermore, if an attractive *ESS* is located sufficiently near the barycenter of the unit simplex, then similar results can be obtained easily, as the dynamics used in the proofs throughout this paper (almost) coincide.

The viability of any refinement hinges on the value added of its properties not necessarily shared by the original, i.e., structural and conceptual stability. A drawback might be that a refinement is too strict. The three equilibrium concepts examined however, *generically* satisfy attractiveness as well, as $\varepsilon > 0$ may be chosen arbitrarily small in the above.

2 Evolutionary equilibria and attractiveness

Let $x \in \Delta^n$ denote a vector of shares $n+1$ distinguishable, interacting subgroup have in a population; recall, Δ^n is the n-dimensional unit simplex. The interaction of the subgroups has consequences on their respective abilities to reproduce, and 'fitness' may be seen as a measure of this ability to reproduce depending only on the state of the system, i.e., the composition of the population x.

Let $F : \Delta^n \to \mathbb{R}^{n+1}$ be a **fitness function**, i.e., a continuous function attributing to every subgroup its fitness at each state $x \in \Delta^n$. Then, the **relative fitness function** $f : \Delta^n \to \mathbb{R}^{n+1}$ is given by:

$$f_i(x) = F_i(x) - \textstyle\sum_j x_j F_j(x), \text{ for all } i \in I^{n+1} = \{1, ..., n+1\}.$$

So, a relative fitness function attributes to each subgroup the difference between its fitness and the population share weighted average fitness, given the composition of the population. Note that f satisfies complementarity, i.e., $x \cdot f(x) = 0$ for all $x \in \Delta^n$.

Let $h : \Delta^n \to \mathcal{O}^{n+1}$ represent the dynamics in a system of $n+1$ autonomous differential equations:

$$\dot{x} = \tfrac{dx}{dt} = h(x) \text{ for all } x \in \Delta^n. \tag{3}$$

A **trajectory** under the dynamics h is a solution, $\{x(t)\}_{t \geq 0}$, to $x(0) = x_0 \in \Delta^n$ and Eq. (3) for all $t \geq 0$. We 'merely' require existence and uniqueness of trajectories under (3). For the former, continuity of h suffices, for the latter, Lipschitz continuity of h suffices.

The state $y \in \Delta^n$ is a **saturated equilibrium** (*SAT*, Hofbauer & Sigmund [1988]) if $f(y) \leq \mathbf{0}^{n+1}$, y is a **strict saturated equilibrium** (*SSAT*) if $\max_{k \neq j} f_k(y) < f_j(y) = 0$ for some subgroup j (cf., Joosten [1996]); a **fixed point** if $h(y) = \mathbf{0}^{n+1}$. A fixed point y is **(asymptotically) stable** if, for any neighborhood $U \subset \Delta^n$ of y, there exists an open neighborhood $V \subset U$ such that any trajectory starting in V stays in U (converges to y).

The fixed point $y \in \Delta^n$ is a **generalized evolutionarily stable state** (*GESS*, Joosten [1996]) if and only if there exists an open neighborhood $U \subset \Delta^n$ of y such that (1) holds.[10] The fixed point $y \in \Delta^n$ is a **truly evolutionarily stable state** (*TESS*, Joosten [2013]) if and only if an open neighborhood $U \subset \Delta^n(C(y))$ of y exists such that

$$\textstyle\sum_{i \in C(y)} \frac{(y_i - x_i) h_i(x)}{x_i} - \sum_{i \notin C(y)} h_i(x) > 0, \tag{4}$$

where $C(y)$ is the carrier of y, and $\Delta^n(C(y))$ is the corresponding face of the unit simplex. Joosten [2013] shows that (4) guarantees asymptotic stability (cf., Fryer [2012], Harper & Fryer [2015]).

[10] *GESS* allows *arbitrary* relative fitness functions. If *restricted* to $f(x) = Ax - (x \cdot Ax) \cdot \mathbf{1}^{n+1}$ for some square matrix A, *GESS* and *ESS* coincide, so do Nash equilibrium and *SAT*.

Joosten [2013] also generalized the idea behind the *ESE*. Let $d : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be a distance function, and $V : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}_{-} \cup \{0\}$ be differentiable, $-V$ homothetic to d, and $V(x,x) = 0$ for all possible x. Then, $y \in \Delta^n$ is a **generalized evolutionarily stable equilibrium** if and only if a nonempty open neighborhood $U \subseteq \Delta^n$ containing y, exists such that for all $x \in U \backslash \{y\}$ it holds that $\dot{V}(x,y) = \sum_i \frac{\partial V}{\partial x_i} h_i(x) > 0$. So, each trajectory sufficiently near a *GESE* converges such that *at least one homothetic transformation of at least one distance* decreases monotonically.

Now, we are ready to present the attractive variants of the concepts mentioned in the preceding sections. As highlighted in the introduction, $z = (z^1, z^2)$ should be connected to the mathematics defining the original concepts. Therefore, let $y \in \Delta^n$, $U \subset \Delta^n$ be a nonempty open neighborhood of y, and $\frac{y-x}{x} \equiv [\frac{y_1-x_1}{x_1}, ..., \frac{y_{n+1}-x_{n+1}}{x_{n+1}}]^\top \in \mathbb{R}^{n+1}$ with $\frac{0}{0} \equiv 1$, then we say

- y is an **attractive (G)ESS** iff (ATT) holds for all $x \in U \backslash \{y\}$, with $z^1(x) = (y-x), z^2(x) = f(x)$,
- y is an **attractive ESE** iff (ATT) holds for all $x \in U \backslash \{y\}$, with $z^1(x) = (y-x)$, $z^2(x) = h(x)$,
- y is an **attractive TESS** iff (ATT) holds for all $x \in U \backslash \{y\}$, with $z^1(x) = \frac{y-x}{x}$, $z^2(x) = h(x)$ for all $i \in I^{n+1}$.

To avoid inessential technicalities, we focus on equilibria either in the interior of the unit simplex, or at its vertices. Equilibria on faces or facets of the state space can be easily dealt with, but notational burdens increase (too much), all kinds of exceptions must be formulated regarding the boundary. Such technicalities distract from the main issues which are our extended notion of stability and the minor refinement that gets the job done.

3 Connections among dynamics and equilibria

Technically speaking, the evolution of the composition of the population is represented by system (3). To make sense in an evolutionary framework h is assumed to be connected to the relative fitness function f in one of the many ways proposed in the literature, cf., e.g., Nachbar [1990], Friedman [1991], Swinkels [1993], Joosten [1996], Ritzberger & Weibull [1995].

For so-called **sign-compatible** (*SC*) dynamics, the change in population share of each subgroup with positive population share corresponds in sign[11] with its relative fitness, i.e., $sign\ f_i(x) = sign\ h_i(x)$ for all $i \in I^{n+1}$; for **weakly sign-compatible** (*WSC*) dynamics, at least one subgroup with positive relative fitness grows, i.e., $sign\ f_i(x) = sign\ h_i(x) > 0$ for at least

[11] Here, $sign\ x = +1$ iff $x > 0$, $sign\ x = -1$ iff $x < 0$, and $sign\ x = 0 = x$ for all $x \in \mathbb{R}$.

one $i \in I^{n+1}$. Weibull [1995] calls this property *weak payoff positivity.* Dynamics are **one-sided sign-compatible** ($OSSC$) if one of two cases hold everywhere: either (i) all subgroups having above-average fitness grow, *sign* $h_i(x) = +1$ whenever *sign* $f_i(x) = +1$, or (ii) all non-extinct subgroups having below-average fitness shrink, i.e., *sign* $h_i(x) = -1$ whenever *sign* $f_i(x) = -1$ & $x_i > 0$. Friedman [1991] calls dynamics **weakly compatible** (WC) if $f(x) \cdot h(x) \geq 0$ (with strict inequality if x is not an equilibrium), **order compatible** (OC) if $f_i(x) < f_j(x) \Longleftrightarrow h_i(x) < h_j(x)$ for all $x \in int$ Δ^n. Figure 1 visualizes relations between these classes of dynamics.

Let the following functions from the interior of the n-dimensional unit simplex to $\mathcal{O}^{n+1}$, be componentwise, i.e., for all $i \in I^{n+1}$, given by:

$$
\begin{aligned}
h_i^{REP}(x) &= x_i f_i(x); \\
h_i^{q-REP}(x) &= x_i^q \left[f_i(x) - \frac{\sum_j x_j^q f_j(x)}{\sum_j x_j^q} \right]; \\
h_i^{BR}(x) &= [e(j^*) - x]_i; \\
h_i^{BN}(x) &= [f_i(x)]_+ - x_i \cdot \textstyle\sum_j [f_j(x)]_+; \\
h_i^{L}(x) &= e^{f_i(x)} - x_i \cdot \left(\textstyle\sum_j e^{f_j(x)} \right); \\
h_i^{WL}(x) &= x_i \left[e^{f_i(x)} - \textstyle\sum_j x_j e^{f_j(x)} \right]; \\
h_i^{OPD}(x) &= f_i(x) - \tfrac{1}{n+1} \textstyle\sum_j f_j(x); \\
h_i^{RPD}(x) &= f_i(x) - x_i \cdot \left(\textstyle\sum_j f_j(x) \right); \\
h_i^{\alpha}(x) &= f_i(x) - (x_i - \alpha) \tfrac{\sum_j f_j(x)}{1-(n+1)\alpha} \text{ for some } \alpha \leq 0; \\
h_i^{SE}(x) &= x_i \left[\textstyle\sum_j \lambda_i x_j [f_i(x) - f_j(x)]_+ - \textstyle\sum_j \lambda_j x_j [f_j(x) - f_i(x)]_+ \right] \\
&\quad \text{..........with } \lambda_k \in [0,1] \text{ for all } k \in I^{n+1}.
\end{aligned}
$$

All functions above are well-defined and continuous, except h^{BR} for which $j^* \in \{k \in I^{n+1} | f_k(x) = \max_{h \in I^{n+1}} f_h(x)\}$ is to be uniquely determined in order to make the time-derivative of its solutions continuous 'from the right'; $e(k)$ is the k-th unit vector in Δ^n; $[y]_+ = \max\{0, y\}$. Superscripts *REP*, *q-REP*, *BR*, *BN*, *L*, *WL OPD*, *RPD*, α and *SE* above refer to replicator, q-deformed replicator (cf., Harper [2011]), best-response (Gilboa & Matsui [1991], Matsui [1992], Rosenmüller [1971]), Brown-von-Neumann- (Brown & Von Neumann [1950]), logit (Fudenberg & Levine [1998]), weighted logit (Björnerstedt & Weibull [1996]), orthogonal-projection (Lahkar & Sandholm [2008]), ray-projection (Joosten & Roorda [2011]), α-barycentric ray-projection dynamics (Joosten & Roorda [2014]), and Sethi (*ibid* [1998]), respectively. Continuity of dynamics guarantees 'existence of a solution', Lipschitz continuity guarantees 'uniqueness'. All functions above are assumed Lipschitz continuous on the unit simplex.

Quite strikingly, the replicator dynamics are in each of the larger classes of dynamics mentioned. Replicator and orthogonal-projection dynamics are q-deformed replicator dynamics for $q = 1$ and $q = 0$ respectively; α-

barycentric ray-projection dynamics connect ray- ($\alpha = 0$) and orthogonal-projection dynamics ($\alpha \to -\infty$).

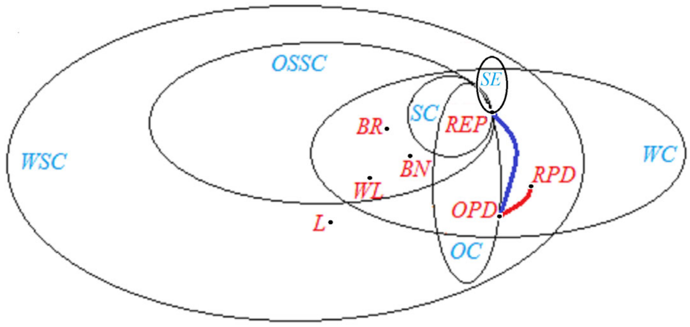

Figure 1: Links between dynamics: classes are ovals indicated in light blue, and specific ones are indicated as dots, all relevant abbreviations are in the text. The dark red (resp. blue) curve represents α-barycentric projection (resp. q-deformed replicator) dynamics.

Figure 2 treats links between equilibrium concepts under different dynamics. $(S)SAT$, $(G)ESE$, $(G)ESS$, $TESS$, $(A)SFP$, LP and FP denote the sets[12] of (strictly) saturated equilibria, (generalized) evolutionarily stable equilibria, (generalized) evolutionarily stable states, truly evolutionary stable states, (asymptotically) stable fixed points, limit points of at least one interior trajectory **not** starting in it, and fixed points respectively. QRE denotes the set of quantal response equilibria, i.e., limit points of logit dynamics (McKelvey & Palfrey [1995]). SSS denotes the set of socially stable states, i.e., states out of which deterministic best response dynamics do not leave (Gilboa & Matsui [1991], Matsui [1992]).

4 Structural stability

Our first results pertain to perturbations of f or h in the chain $(y, f, h, P) \to C$. **Unless mentioned otherwise**, C means $\{x_t\}_{t\geq 0} \overset{t\to\infty}{\to} y$ under h if x_0 sufficiently close to equilibrium y, so simply standard convergence to equilibrium. Next, we examine whether equivalences of certain attractive evolutionary equilibria, i.e., perturbations of P, can be shown to hold for certain classes of dynamics.

To examine the topic of structural stability we use the following notion. Given $z^2 : \Delta^n \to \mathbb{R}^{n+1}$, $\theta \in (0, 1)$ and $\varepsilon > 0$, let $Z^\theta(z^2, \varepsilon)$ be the set of

[12] Using the same letters for a point and a set should not be confusing in context.

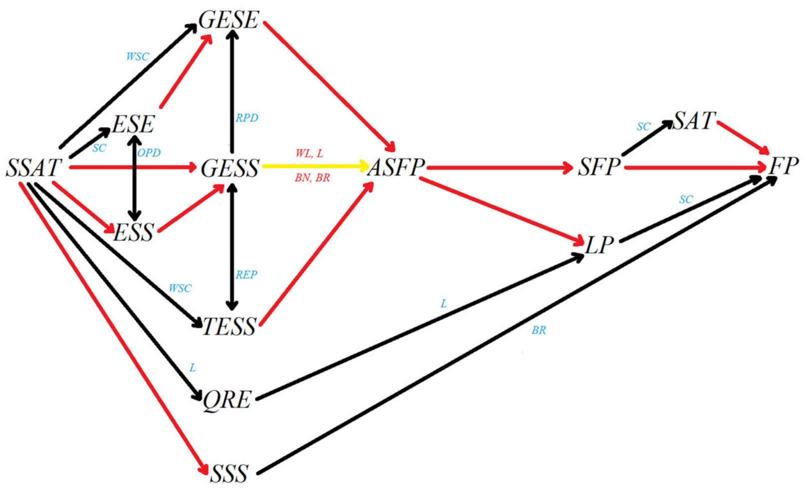

Figure 2: Evolutionary equilibrium or fixed point concepts (in large black caps) are connected by arrows indicating inclusions: red in general, black for special classes of dynamics; yellow under extra conditions on the system.

perturbations of z^2 given by

$$Z^{\theta}(z^2,\varepsilon)=\left\{z:\Delta^n\backslash\{y\}\to\mathbb{R}^{n+1}|\tfrac{z^2(x)\cdot z(x)}{||z^2(x)||\cdot||z(x)||}\geq\sqrt{1-(\theta\varepsilon)^2}\right\}.$$

Obviously, $Z(z^2,\varepsilon)=\cup_{\theta\in(0,1)}Z^{\theta}(z^2,\varepsilon)$ is always nonempty as it contains z^2. The next result links the ε above to the ε in the definition of an attractive equilibrium and gives a lower bound for the cosine between z^1 and a continuous perturbation taken from $Z^{\theta}(z^2,\varepsilon)$.

Proposition 1 *Let $y\in$ int Δ^n be attractive with respect to $z=(z^1,z^2)$. Then, $U=\{x\in\Delta^n|\ \frac{z^1(x)\cdot z^2(x)}{||z^1(x)||\cdot||z^2(x)||}>\varepsilon$ and $0<||y-x||<\delta\}\neq\emptyset$. Moreover, for $\theta\in(0,1)$, $\varepsilon>0$ and a continuous perturbation $z\in Z^{\theta}(z^2,\varepsilon)$ with $z(y)=\mathbf{0}^{n+1}$, if $x\in U$ then $\frac{z^1(x)\cdot z(x)}{||z^1(x)||\cdot||z(x)||}>\varepsilon\left(\sqrt{1-(\theta\varepsilon)^2}-\sqrt{\theta^2-(\theta\varepsilon)^2}\right)$.*

The closer ε is to unity, the more slack can be offered to the perturbations of the dynamics; θ is necessary to specify the part behind the inequality sign.

Corollary 1 *For every attractive ESS, ESE and TESS, a set of sufficiently small continuous perturbations of the payoffs or dynamics exists such that the equilibrium is attractive under these perturbations, too.*

For *ESE* and *TESS*, the proposition and its corollary pertain to continuous perturbations of the function h, i.e., the one representing the dynamics, in the chain $(y, f, h, P) \to C$, whereas for *ESS* these results pertain to continuous perturbations to f, i.e., the relative fitness function in the chain.

5 Conceptual stability

As a starting point regarding the topic of conceptual stability, i.e., perturbations of P in the chain $(y, f, h, P) \to C$, we focus on the attractive variants of the *ESS* and the *ESE*. Our natural ally in this endeavor is the *OPD* of Lahkar & Sandholm [2008] for which Hofbauer & Sandholm [2009] implicitly show that interior *ESS* and *ESE* concur. This can be (informally) checked by examining Figures 1 & 2. However, we want to establish a *class* of dynamics for which said equivalence holds. Next, we reveal potential of α-barycentric ray-projection dynamics h^α of Section 3 in this respect.

Lemma 2 *For given relative fitness function f and α-barycentric ray-projection dynamics h^α and $x \in int\ \Delta^n$:*

- $\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} \geq \sqrt{\frac{1-2\alpha(n+1)}{(1-(n+1)\alpha)^2}}\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||} - \frac{\sqrt{n+1}}{1-(n+1)\alpha}$;
- $\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||} \geq \sqrt{1+\frac{(n+1)}{(1-(n+1)\alpha)^2}}^{-1}\left[\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} - \frac{\sqrt{n+1}}{1-(n+1)\alpha}\right]$.

The right hand sides of both inequalities must be made strictly positive to show equivalence between attractive *ESS* and attractive *ESE* for certain barycentric projection dynamics. The following hinges on the possibility to do so for both inequalities at the same time.

Proposition 3 *Given relative fitness function f, let $y \in int\ \Delta^n$. Then, $\alpha_0 \in \mathbb{R}$ exists such that for all $\alpha \leq \alpha_0$ the following statements are equivalent:*

- *y is an attractive ESE for the α-barycentric ray-projection dynamics;*
- *y is an attractive ESS.*

For the next proposition, we focus on the attractive variants of the *TESS*, *ESE* and *ESS* for which we draw support from q-deformed replicator dynamics of Harper [2011], as well as from the following lemma's inequalities.

Lemma 4 *For given relative fitness function f, open $U \subset int\ \Delta^n$ with $y \in U$, $0 < m_U < M_U \leq 1$, $0 < m_\angle \leq 1$ exist such that for all $x \in U\backslash\{y\}$:*

- $\frac{1}{M_U}||y-x|| \leq ||\frac{y-x}{x}|| \leq \frac{1}{m_U}||y-x||$;
- $m_U||f(x)|| \leq ||h^{REP}(x)|| \leq M_U||f(x)||$;
- $m_\angle||f(x)|| \leq ||h^{OPD}(x)|| \leq ||f(x)||$.

As it is hard to prove conceptual stability for all variants at once, we pair the *TESS* and the *ESS* up for one range of the parameter defining the dynamics of Harper [2011], and the *ESE* and the *ESS* for another.

Proposition 5 *For given relative fitness function f, $y \in int\ \Delta^n$, there exist a strictly increasing sequence Q converging to* 1*, and a strictly decreasing sequence $\widetilde{Q}$ converging to* 0*, such that the statement 'y is an attractive ESS' is equivalent to*

- *y is an attractive TESS for h^{q-REP} with $q \in Q$;*
- *y is an attractive ESE for h^{q-REP} with $q \in \widetilde{Q}$.*

So, interior attractive *ESS*, *ESE* and *TESS* are conceptually stable in the sense that they concur for the classes of dynamics specified above. This means in terms of the chain of implications used before that a specific class of dynamics H is shown to exist such that $(y, f, h, P) \to C$ is equivalent to $\left(y, f, h, \widetilde{P}\right) \to C$ for several $h \in H$. Moreover, each $h \in H$ mentioned in the respective statements must have a set of perturbations of either the relative fitness function, or the dynamics at hand such that the chains $(y, f, h, P) \to C$ and $\left(y, f, h, \widetilde{P}\right) \to C$ remain valid for the respective equilibria.

Now, we focus on conceptual stability for an attractive *ESS* and an attractive *ESE* for a subclass of the dynamics of Sethi [1998].

Lemma 6 *(Sethi [1998], Proposition 3) For arbitrary relative fitness functions and for $\lambda = (1, ..., 1)$, $h^{SE} = h^{REP}$.*

This intermediate result is a stepping stone to the following.

Proposition 7 *For given relative fitness function f and $y \in int\ \Delta^n$, $\lambda^* < 1$ exists such that the following two statements are equivalent:*

- *y is an attractive ESS;*
- *y is an attractive TESS for $h^{SE,\lambda}$ for all λ satisfying $\lambda_j \geq \lambda^*$ for all $j \in I^{n+1}$.*

6 Special states and DSC stability

Each strict saturated equilibrium is asymptotically stable for weakly sign compatible dynamics. If we regard two different dynamics in this class as mutual perturbations, it follows immediately that such an equilibrium is also structurally stable with respect to perturbations of h in the chain $(y, f, h, P) \to C$. For the same class of dynamics each strict saturated equilibrium is a *GESE* and a *TESS*, and for a smaller class of dynamics it is also an *ESE*, cf., Figure 2, hence it is also conceptually stable with respect to perturbations of P in the same chain. Our next result covers further stability properties of strict saturated equilibria.

Proposition 8 *Every strict saturated equilibrium is structurally stable with respect to perturbations of the relative fitness function. Furthermore, each strict saturated equilibrium is an attractive evolutionarily stable strategy.*

Another special state is the barycenter of the unit simplex, i.e., $b = \frac{1}{n+1}\mathbf{1}^{n+1}$. The following lemma is a stepping stone to the ensuing result.

Lemma 9 *Let* $b = \frac{1}{n+1}\mathbf{1}^{n+1}$*, then for relative fitness function* f *,* $y \in int\ \Delta^n \backslash \{b\}$*: if* $f(b) \neq \mathbf{0}^{n+1}$*, then for all* $q \in [0,1]$ *and for all* $\alpha \leq 0$ *:*

$$\left[\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||}\right]_{x=b} = \left[\frac{(y-x)\cdot h^{\alpha}(x)}{||y-x||\cdot||h^{\alpha}(x)||}\right]_{x=b} = \left[\frac{(y-x)\cdot h^{q-REP}(x)}{||y-x||\cdot||h^{q-REP}(x)||}\right]_{x=b} = \left[\frac{\frac{y-x}{x}\cdot h^{q-REP}(x)}{||(y-x)||\cdot||h^{q-REP}(x)||}\right]_{x=b} = \left[\frac{\frac{y-x}{x}\cdot h^{\alpha}(x)}{||\frac{y-x}{x}||\cdot||h^{\alpha}(x)||}\right]_{x=b}.$$

If $f(b) = \mathbf{0}^{n+1}$, the equality signs hold by $\frac{0}{0} \equiv 1$. At b, the barycentric ray-projection dynamics collapse into the orthogonal projection dynamics. The deformed replicator dynamics collapse into dynamics that differ from the relative fitness function only in speed. So, the next result follows easily.

Proposition 10 *For all* $q \in [0,1]$ *and all* $\alpha \leq 0$*, the following statements are equivalent for* $y = b$ *(or* y *sufficiently close to the barycenter* b*):*

- *y is an attractive ESS;*
- *y is an attractive ESE and an attractive TESS for the q-deformed replicator dynamics;*
- *y is an attractive ESE and an attractive TESS for the α-barycentric ray-projection dynamics.*

7 Discussions

We use the relative fitness function f (cf., Joosten [1996], Sandholm [2005]), rather than the fitness function F. Although for all $x, y \in \Delta^n$ we have

$$(y-x)\cdot f(x) = (y-x)\cdot F(x),$$

an analogy regarding attractive *ESS* with respect to F can not be constructed for an interior saturated equilibrium y as

$$\frac{(y-x)\cdot F(x)}{||y-x||\cdot||F(x)||} \overset{x\to y}{\to} 0.$$

Clearly, $F(y) = c \cdot 1^{n+1}$ is well defined for $c \neq 0$ and perpendicular to Δ^n and therefore perpendicular to $(y-x)$ which implies the arrow in the statement.

An attractive version of the generalized evolutionarily stable equilibrium (cf., Section 3), *GESE*, is easily formulated abstractly as follows

$$\frac{DV(y,x) \cdot h(x)}{||DV(y,x)|| \cdot ||h(x)||} > \epsilon,$$

where $(z_1(x), z_2(x)) = (DV(y,x), h(x))$, $DV(y,x) = \left[\frac{\partial V(y,x)}{\partial x_1}, ..., \frac{\partial V(y,x)}{\partial x_{n+1}}\right]$. So, the 'normalized' time-derivative of the distance-related Lyapunov function V is strictly bounded away from zero, too. However, given the generality of V, connections of an attractive *GESE* with the attractive equilibria treated in the previous section must unfortunately be shown case-by-case.

7.1 System restrictions inducing attractive equilibria

Hofbauer & Sandholm [2009] introduce strict monotonicity formulated in terms of our relative fitness function f as

$$(y-x) \cdot (f(y) - f(x)) < 0 \text{ for all } x, y \in \Delta^n, \; x \neq y. \tag{5}$$

Monotonicity has a weak inequality sign. Hofbauer & Sandholm [2009] call games for which (5) holds 'strictly stable games'. Strict monotonicity implies that each interior saturated equilibrium is unique and an *ESS,* since (5) implies (1) in that case. Under strict monotonicity of f an interior ESS seems to satisfy the criterion of an unbeatable strategy (cf., Hamilton [1967], Kojima [2006]). Strict monotonicity applied to the dynamics h **excluding** the boundary of Δ^n, yields the defining condition for an interior *ESE* as (5) implies (2) in that case. Monotonicity of f or h implies that the set of equilibria is connected and convex.

Strict monotonicity of f is sufficient for convergence of various adaptive processes to equilibrium and if the latter is interior, it is unique (cf., e.g., Joosten [2006], Harker & Pang [1990], Nikaidô [1959], Nikaidô & Uzawa [1960]). So, strict monotonicity ($\widetilde{P}$ given by (5)) implies that an interior equilibrium (y) is an asymptotically stable fixed point (consequence C) of a class of evolutionary dynamics (H) as well, i.e., $\left(y, f, h, \widetilde{P}\right) \to C$ for several $h \in H$. In other words, strict monotonicity implies structural stability of an interior equilibrium for a large class of plausible evolutionary dynamics. The yellow arrow in Figure 2 is motivated by strict monotonicity applied to the dynamics indicated (cf., e.g., Hofbauer & Sandholm [2009]).

Attractive monotonicity of f or h may be used similarly to induce the corresponding attractive interior evolutionary equilibrium ($\varepsilon > 0$), i.e.,

$$(y-x) \cdot (f(y) - f(x)) < -\varepsilon \cdot ||y-x|| \cdot ||f(y) - f(x)|| \text{ or}$$
$$(y-x) \cdot (h(y) - h(x)) < -\varepsilon \cdot ||y-x|| \cdot ||h(y) - h(x)||,$$

for all $x, y \in D \subseteq int\ \Delta^n$, $x \neq y$. Under attractive monotonicity of f or h, the statement $f(y) = h(y) = 0^{n+1}$ for $y \in D$, implies for all $x \in D \backslash \{y\}$:

$$\begin{array}{l} -\frac{(y-x)\cdot(f(y)-f(x))}{||y-x||\cdot||f(y)-f(x)||} = -\frac{-(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} = \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} > \varepsilon, \\ -\frac{(y-x)\cdot(h(y)-h(x))}{||y-x||\cdot||h(y)-h(x)||} = -\frac{-(y-x)\cdot h(x)}{||y-x||\cdot||h(x)||} = \frac{(y-x)\cdot h(x)}{||y-x||\cdot||h(x)||} > \varepsilon. \end{array}$$

Attractive monotonicity is clearly more stringent than strict monotonicity. If the system satisfies attractive monotonicity, then DSC-stability of the three evolutionary equilibria examined in this paper, is immanent.

7.2 Recruiting dynamics for conceptual stability

One-parameter variants of dynamics presented in Section 3 are used elsewhere to form families of dynamics, among which these prominent ones:

$$\begin{array}{lcl} h_i^{L,\beta}(x) & = & e^{\beta\cdot f_i(x)} - x_i \cdot \left(\sum_j e^{\beta\cdot f_j(x)}\right); \\ h_i^{WL,\beta}(x) & = & x_i \left[e^{\beta\cdot f_i(x)} - \sum_j x_j e^{\beta\cdot f_j(x)}\right]; \\ h_i^{BN,\gamma}(x) & = & [f_i(x)]_+^\gamma - x_i \cdot \sum_j [f_j(x)]_+^\gamma . \end{array}$$

The first and third yield dynamics similar to the best response dynamics h^{BR} (sufficiently far away from any saturated equilibrium) for $\beta, \gamma \to \infty$, cf., e.g., Hofbauer [2000]. Note that for $\beta \to 0$, the dynamics from the second family are quite similar to the replicator dynamics, i.e., $\lim_{\beta\to 0} h^{WL,\beta} \to h^{REP}$ and dynamics taken from the first are quite similar to the ray-projection dynamics, i.e., $\lim_{\beta\to 0} h^{L,\beta} \to h^{RPD}$. However, for the desired proofs of conceptual stability the third dynamics have very limited use, we suspect.

Joosten & Roorda [2008] introduced generalized projection dynamics, and, as shown, many dynamics in the literature are of the generalized ray-projection type with close relatives in a generalized orthogonal-projection variant. This gives rise to a series of two-parameter families for which further conceptual stability results may be obtained

$$\begin{array}{lcl} h_i^{L,\alpha,\beta}(x) & = & e^{\beta\cdot f_i(x)} - \frac{(x_i-\alpha)}{1-(n+1)\alpha} \sum_j e^{\beta\cdot f_j(x)}; \\ h_i^{WL,\alpha,\beta}(x) & = & x_i e^{\beta\cdot f_i(x)} - \frac{(x_i-\alpha)}{1-(n+1)\alpha} \sum_j x_j e^{\beta\cdot f_j(x)}; \\ h_i^{L,q,\beta}(x) & = & x_i^q e^{\beta\cdot f_i(x)} - x_i \left(\sum_j x_j^q e^{\beta\cdot f_j(x)}\right). \end{array}$$

Here, $\beta \geq 0$, and as before $q \in [0,1]$, $\alpha \leq 0$. For $\beta \to 0$, $h^{L,\alpha,\beta} \to h^\alpha$ and $h^{WL,\alpha,\beta} = h^{REP}$. Recall $h^0 = h^{RPD}$ and $h^{-\infty} = h^{OPD}$. Observe that $\lim_{q\downarrow 0} \lim_{\beta\to\infty} h^{L,q,\beta} = h^{BR}$. So, BR-dynamics may be approximated for appropriate limits ($\beta \to \infty$, $\alpha \to 0$) and the logit ($q = 0, \beta = 1$) and weighted logit ($q = 1, \beta = 1$) dynamics may be captured as special cases.

8 Conclusion

An evolutionary equilibrium is **DSC-stable** if it is (a) Dynamically stable, i.e., if the system at equilibrium is slightly disturbed, it returns to it, (b) Structurally stable, i.e., preserves defining properties for small perturbations of the underlying structure of the system, i.e., the payoffs or dynamics (c) Conceptually stable, i.e., equivalent to at least one other attractive evolutionary equilibrium (concept), for a non-singleton class of dynamics.

We presented **attractiveness**, a refinement criterion for equilibria in evolutionary game theory meant to be applied to the mathematics defining them. One interpretation of this criterion is geometric and intrinsically linked to Lyapunov's second method, the dominant mode for proving asymptotic stability of evolutionary equilibria nowadays. In case namely, that a pair of appropriately chosen vectors with a positive inner product for a neighborhood of the equilibrium exists linked to its defining properties, we bound the cosine between these vectors strictly away from zero.

We proved **structural stability** of attractive evolutionary equilibria. We proved **conceptual stability** under attractiveness for the evolutionarily stable state (*ESS*, Maynard Smith & Price [1973]), the evolutionarily stable equilibrium (*ESE*, Joosten [1996]), and the truly evolutionarily stable state (*TESS*, Joosten [2013]). We showed that interior attractive *ESE* and *ESS* concur for subclasses of two distinct one-parameter families of evolutionary dynamics, the barycentric projection dynamics (Joosten & Roorda [2008]) and q-deformed replicator dynamics (Harper [2011]). Also, equivalence was shown of interior attractive *ESS* and *TESS* for a subclass of the q-deformed replicator dynamics and the dynamics of Sethi [1998]. This means that the attractive variants of these three concepts are dynamically, structurally and conceptually stable for a non-singleton class of dynamics.

Our findings add a layer of robustness to results as neither the complete specification of the dynamics and payoff structure, nor the equilibrium concept used, matter for qualitative conclusions about the dynamics nearby. After all, the system or the underlying payoff structure might be known imprecisely, or how the latter translates into fitness or utilities, or how the latter guide micro-adjustments of agents. Heterogeneity of learning rules is likely in large populations (cf., Hommes [2006], Kirman [2006]), and mixes of rules may induce complicated convergence behavior (e.g., Golman [2011]). Furthermore, aggregate dynamics remain deterministic approximations of very complex underlying stochastic processes (cf., e.g., Sandholm [2010]). Our results indicate that DSC-stability offers the kind of resilience to cope with various kinds of ambiguities inevitable in the framework of evolutionary game theory. Attractive *ESS*, *ESE* and *TESS* fit the bill for DSC-stability for certain classes of evolutionary dynamics.

Maynard Smith & Price [1973] were *almost* 'on the money' as generic *ESS* are attractive, hence DSC-stable, i.e., almost all *ESS* are DSC-stable.

We see a parallel here to the difference between Taylor & Jonker [1978] and Zeeman [1981] on the dynamical stability of *ESS*, i.e., the latter points out that generically the matrix of first derivatives of the replicator dynamics of an *ESS* have eigen values with strictly negative real parts stated as a condition in the first. Moreover, interior attractive *ESS* share desirable stability criteria met only by strict saturated (Nash) equilibria.

Strict saturated (i.e., Nash) equilibria are attractive *ESS*, and they are also attractive *ESE*, as well as attractive *TESS* for all weakly sign compatible dynamics, a class much larger than the classes of dynamics for which interior saturated equilibria are DSC stable. For saturated equilibria 'close enough' to the barycenter of the unit simplex, the sets of dynamics used in the proofs of DSC-stability for interior evolutionary equilibria examined here, make the same angle with the vector pointing to equilibrium, i.e., all fulfill attractiveness simultaneously, or not. So, conceptual stability is immediate for a larger set of dynamics than for interior equilibria located farther away.

We took single perturbations to the tuple (y, f, h, P) leading to consequences C. A higher notion of stability involves robustness to multiple perturbations. To prove conceptual stability of certain equilibria, we focussed on replicator and orthogonal-projection dynamics, where distinct one-parameter families of evolutionary dynamics meet. This can be extended to k-parameter ($k \geq 2$) families of dynamics meeting at these dynamics.

9 Appendix

Proposition 1 Let $y \in int\ \Delta^n$ be attractive for $z = (z^1, z^2)$ and let U be as stipulated by (ATT), take $\widetilde{z}(x) \in Z^\theta(z^2, \varepsilon)$, $\theta \in (0, 1)$. We use the trigonometric identity $\cos(\beta + \gamma) = \cos\beta \cdot \cos\gamma - \sin\beta \cdot \sin\gamma$, as follows: $\beta \equiv \sup_{x \in U \setminus \{y\}} \beta(x)$ and $\gamma \equiv \sup_{x \in U \setminus \{y\}} \gamma(x)$ where $\beta(x)$ is the angle between $z^1(x)$ and $z^2(x)$, and $\gamma(x)$ is the angle between $z^2(x)$ and $\widetilde{z}(x)$. Then, y is also attractive for $(z^1, \widetilde{z})$ since $\cos\ \beta \geq \varepsilon$, $\cos\gamma \geq \sqrt{1 - (\theta\varepsilon)^2}$, and $\inf_{x \in U \setminus \{y\}} \frac{z^1(x) \cdot \widetilde{z}(x)}{||z^1(x)|| \cdot ||z(x)||} \geq \cos(\beta + \gamma) \geq \varepsilon\sqrt{1 - (\theta\varepsilon)^2} - \sqrt{1 - \varepsilon^2}\theta\varepsilon = \varepsilon\left(\sqrt{1 - (\theta\varepsilon)^2} - \sqrt{\theta^2 - (\theta\varepsilon)^2}\right) > 0.$ ■

Lemma 2 For $a = \alpha \cdot 1^{n+1}$ and $C(x) = (1 - (n+1)\alpha)^{-1} \sum_i f_i(x)$,

$$\begin{aligned}
\frac{(y-x) \cdot f(x)}{||y-x|| \cdot ||f(x)||} &= \frac{(y-x) \cdot f(x) - (y-x) \cdot C(x) \cdot (x-a) + (y-x) \cdot C(x) \cdot (x-a)}{||y-x|| \cdot ||f(x)||} \\
&= \frac{(y-x) \cdot f(x) - (y-x) \cdot C(x) \cdot (x-a)}{||y-x|| \cdot ||f(x)||} + C(x) \frac{(y-x) \cdot (x-a)}{||y-x|| \cdot ||f(x)||} \\
&= \frac{||h^\alpha(x)||}{||f(x)||} \frac{(y-x) \cdot h^\alpha(x)}{||y-x|| \cdot ||h^\alpha(x)||} + \frac{C(x)}{||f(x)||} \frac{(y-x) \cdot x}{||y-x||} \\
&= \frac{||h^\alpha(x)||}{||f(x)||} \frac{(y-x) \cdot h^\alpha(x)}{||y-x|| \cdot ||h^\alpha(x)||} + \frac{\sum_i f_i(x)}{||f(x)||} \frac{(y-x) \cdot x}{||y-x|| \cdot ||x||} \frac{||x||}{1-(n+1)\alpha}.
\end{aligned}$$

Observe furthermore that $\frac{\sum_i f_i(x)}{||f(x)||} \leq \frac{||f(x)||_1}{||f(x)||} \leq \frac{\sqrt{n+1}||f(x)||}{||f(x)||} = \sqrt{n+1}$. So,

$$-\frac{\sqrt{n+1}}{1-(n+1)\alpha} \leq \frac{\sum_i f_i(x)}{||f(x)||}\frac{(y-x)\cdot x}{||y-x||\cdot||x||}\frac{||x||}{1-(n+1)\alpha} \leq \frac{\sqrt{n+1}}{1-(n+1)\alpha}.$$

Now, several calculations yield

$$||h^\alpha(x)||^2 = (f(x)+C(x)(a-x))\cdot(f(x)+C(x)(a-x)) =$$

$$||f(x)||^2 + 2\alpha(1-\alpha(n+1))C(x)^2 + C(x)^2||a-x||^2 =$$
$$||f(x)||^2 + C(x)^2\left(||x||^2 - \alpha^2(n+1)\right).$$

This in turn leads to the following series of implications

$$1-\alpha^2(n+1)\frac{C(x)^2}{||f(x)||^2} \leq \frac{||h^\alpha(x)||^2}{||f(x)||^2} \leq 1+\frac{C(x)^2}{||f(x)||^2} \Longrightarrow$$
$$1-\alpha^2(n+1)\frac{(n+1)}{(1-(n+1)\alpha)^2} \leq \frac{||h^\alpha(x)||^2}{||f(x)||^2} \leq 1+\frac{(n+1)}{(1-(n+1)\alpha)^2} \Longrightarrow$$
$$\frac{1-2\alpha(n+1)}{(1-(n+1)\alpha)^2} \leq \frac{||h^\alpha(x)||^2}{||f(x)||^2} \leq 1+\frac{(n+1)}{(1-(n+1)\alpha)^2}.$$

Here, the first implication follows from (5). So, this yields the first inequality of the statement of the lemma as:

$$\begin{aligned}\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} &\geq \frac{||h^\alpha(x)||}{||f(x)||}\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||} - \frac{\sqrt{n+1}}{1-(n+1)\alpha} \\ &\geq \sqrt{\frac{1-2\alpha(n+1)}{(1-(n+1)\alpha)^2}}\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||} - \frac{\sqrt{n+1}}{1-(n+1)\alpha}.\end{aligned}$$

Moreover, the second inequality of the lemma is proven by

$$\frac{||h^\alpha(x)||}{||f(x)||}\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||} \geq \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} - \frac{\sqrt{n+1}}{1-(n+1)\alpha} \Longrightarrow$$
$$\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||} \geq \sqrt{1+\frac{(n+1)}{(1-(n+1)\alpha)^2}}^{-1}\left(\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} - \frac{\sqrt{n+1}}{1-(n+1)\alpha}\right). \quad \blacksquare$$

Proposition 3 Let $y \in int\ \Delta^n$. To prove the statement of the proposition we show equivalence of the following two inequalities

$$\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||} > \varepsilon_{ESE} \quad \text{and} \quad \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} > \varepsilon_{ESS}.$$

for all $\alpha < \alpha_0 \equiv \min\{-\frac{1}{2\varepsilon_{ESE}^2}, -\sqrt{\frac{1}{(n+1)\varepsilon_{ESS}^2}}\}$.
Let y be an attractive *ESE*, i.e., for all $x \neq y$ sufficiently near y :

$$\frac{(y-x)\cdot h^\alpha(x)}{||y-x||\cdot||h^\alpha(x)||} > \varepsilon_{ESE}.$$

Then by Lemma 2

$$\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} \geq \frac{||h^\alpha(x)||}{||f(x)||}\varepsilon_{ESE} - \frac{\sqrt{n+1}}{1-(n+1)\alpha} \geq \sqrt{\frac{1-2\alpha(n+1)}{(1-(n+1)\alpha)^2}}\varepsilon_{ESE} - \frac{\sqrt{n+1}}{1-(n+1)\alpha}.$$

Hence, if $\sqrt{1-2\alpha(n+1)}\varepsilon_{ESE} - \sqrt{n+1} > 0$, then y must be an *ESS* as well, i.e., we have

$$\sqrt{1-2\alpha(n+1)}\varepsilon_{ESE} > \sqrt{n+1} \Longrightarrow 1-2\alpha(n+1) > \frac{1}{\varepsilon_{ESE}^2}(n+1) \Longrightarrow$$
$$\alpha < -\frac{1}{2\varepsilon_{ESE}^2} + \frac{1}{2(n+1)}.$$

If $\alpha < \alpha_0$, the right-hand inequality above is satisfied.
Now, let y be an attractive *ESS*, , i.e., for all $x \neq y$ sufficiently near y :

$$\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} > \varepsilon_{ESS}.$$

Then Lemma 2 implies

$$\frac{(y-x)\cdot h^{\alpha}(x)}{||y-x||\cdot||h^{\alpha}(x)||} \geq \sqrt{1+\frac{(n+1)}{(1-(n+1)\alpha)^2}}^{-1} \left[\varepsilon_{ESS} - \frac{\sqrt{n+1}}{1-(n+1)\alpha}\right].$$

As the term before the brackets is strictly positive for $\alpha \leq 0$, for y to be an attractive *ESE* for these dynamics, it suffices that

$$\begin{aligned} &1-(n+1)\alpha > \sqrt{n+1}\frac{1}{\varepsilon_{ESS}} \implies \\ &\alpha < -\sqrt{\frac{1}{n+1}}\frac{1}{\varepsilon_{ESS}} + \frac{1}{n+1} = -\sqrt{\frac{1}{(n+1)\varepsilon_{ESS}^2}} + \frac{1}{(n+1)}. \end{aligned}$$

If $\alpha \leq \alpha_0$ the left inequality above is satisfied. ■

Lemma 4 Let $m_U \equiv \inf_{x\in U}\min_i x_i$ and $M_{U'} \equiv \sup_{x\in U}\max_i x_i$. Then, rewriting the first two inequalities yields for all $x, y \in U$:

$$\begin{aligned} &\sqrt{\sum_{i=1}^{n+1}\left(\frac{y_i-x_i)}{M_{U'}}\right)^2} \leq \sqrt{\sum_{i=1}^{n+1}\left(\frac{y_i-x_i)}{x_i}\right)^2} \leq \sqrt{\sum_{i=1}^{n+1}\left(\frac{y_i-x_i)}{m_{U'}}\right)^2}, \\ &\sqrt{\sum_{i=1}^{n+1} m_{U'}^2 f_i(x)^2} \leq \sqrt{\sum_{i=1}^{n+1} x_i^2 f_i(x)^2} \leq \sqrt{\sum_{i=1}^{n+1} M_{U'}^2 f_i(x)^2} \end{aligned}$$

To prove the validity of the statement for the third inequality: as $x \cdot f(x) = 0 = 1^{n+1} \cdot h^{OPD}(x)$, we have $f(x) \in H_1 = \{y \in \mathbb{R}^{n+1} | \; y \cdot x = 0\}$ and $h^{OPD}(x) \in H_2 = \{y \in \mathbb{R}^{n+1} | \; y \cdot 1^{n+1} = 0\}$. Let $\theta(x) = \angle(f(x), h^{OPD}(x))$ denote the angle between $f(x)$ and $h^{OPD}(x)$. It is well known that for $x \in int\ \Delta^n$, $h^{OPD}(x)$ is the orthogonal projection of $f(x)$ on H_2 (cf., e.g., Joosten & Roorda [2011]). So, $||h^{OPD}(x)|| = \cos\theta(x) \cdot ||f(x)||$. For what follows, we are interested in finding more restrictive lower bound for $\cos\theta(x)$ than 0. Let

$$\widehat{\theta}(x) = \max\{\angle(z_1, z_2) | \; z_1 \in H_1, \; z_2 = z_1 - \tfrac{1}{n+1}\left(z_1 \cdot 1^{n+1}\right) \cdot 1^{n+1} \in H_2\}$$

denote the largest angle between any vector in H_1 and its orthogonal projection on the hyperplane H_2. By definition this is the angle between hyperplanes H_1 and H_2, which in turn equals the angle between their normal vectors x and 1^{n+1} (cf., e.g., Golub & Van Loan [1996]), hence

$$\cos\widehat{\theta}(x) = \frac{x\cdot 1^{n+1}}{||x||\cdot||1^{n+1}||} = \frac{1}{||x||\cdot||1^{n+1}||} \leq \cos\theta(x) \leq 1.$$

So, aiming to find a global upper and lower bound by varying x over all possible elements of the unit simplex, we obtain for an arbitrary $\widetilde{x} \in \Delta^n$

$$\frac{1}{\sqrt{n+1}} \leq \cos\widehat{\theta}(\widetilde{x}) \leq 1,$$

where the lower (upper) bound is reached at any of the vertices (at the barycenter) of Δ^n. So, $\frac{1}{\sqrt{n+1}} \leq \cos\widehat{\theta}(x) \leq \cos\theta(x) \leq 1$. Let $m_\angle \equiv \frac{1}{\sqrt{n+1}}$, then for $x \in \Delta^n$

$$m_\angle ||f(x)|| \leq ||h^{OPD}(x)|| \leq ||f(x)||.$$

These upper and lower bounds clearly also hold for $x \in U' \subset int\ \Delta^n$. ■

Proposition 5 Observe that for an interior state y and x sufficiently nearby

$$\begin{aligned} \lim_{q\uparrow 1} \tfrac{y-x}{x} \cdot h^{q-REP}(x) &= \tfrac{y-x}{x} \cdot h^{REP}(x) = (y-x)\cdot f(x) \\ \lim_{q\uparrow 1} ||h^{q-REP}(x)||^2 &= ||h^{REP}(x)||^2 = \textstyle\sum_j x_j^2 f_j(x)^2. \end{aligned}$$

As $h(q,x) \equiv h^{q-REP}(x)$ is continuous in q, $q>0$, the above and Lemma 4 (noting $h^{0-REP} = h^{OPD}$) imply that a monotonically increasing sequence $Q = \{q_k\}_{k=1}^{\infty}$ converging to 1 and numbers $0 < \alpha, \beta < 1$ exist such that

$$\begin{array}{ccccc} \frac{1}{\alpha}(y-x)\cdot f(x) & \geq & \frac{y-x}{x}\cdot h^{q_k-REP}(x) & \geq & \alpha\,(y-x)\cdot f(x) \qquad \text{and} \\ \frac{M_{U'}}{\beta}||f(x)|| & \geq & ||h^{q_k-REP}(x)|| & \geq & \beta\cdot m_{U'}||f(x)|| \quad \text{for all } x \in U'. \end{array}$$

Let $y \in int\ \Delta^n$ be an attractive ESS, i.e., for some open neighborhood $U \ni y : x \in U\backslash\{y\}$ implies

$$\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} > \varepsilon_{ESS} > 0.$$

Let $U' \ni y$ be a closed subset of U and let $U' \subset int\ \Delta^n$. Then, for $q_k \in Q$ and for all $x \in U'\backslash\{y\}$:

$$\begin{aligned} \frac{\frac{y-x}{x}\cdot h^{q_k-REP}(x)}{||\frac{y-x}{x}||\cdot||h^{q_k-REP}(x)||} &\geq \alpha\frac{(y-x)\cdot f(x)}{||\frac{y-x}{x}||\cdot||h^{q_k-REP}(x)||} \\ &= \alpha\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||}\frac{||y-x||\cdot||f(x)||}{||\frac{y-x}{x}||\cdot||h^{q_k-REP}(x)||} \\ &> \alpha\cdot\varepsilon_{ESS}\cdot\frac{||y-x||}{||\frac{y-x}{x}||}\cdot\frac{||f(x)||}{||h^{q_k-REP}(x)||} \\ &\geq \alpha\cdot\varepsilon_{ESS}\cdot m_{U'}\cdot\frac{\beta}{M_{U'}} > 0. \end{aligned}$$

Hence, y is an attractive $TESS$.
Conversely, let $y \in int\ \Delta^n$ be an interior attractive $TESS$ for h^{q_k-REP} with $q_k \in Q$, i.e.,

$$\frac{\frac{y-x}{x}\cdot h^{q_k-REP}(x)}{||\frac{y-x}{x}||\cdot||h^{q_k-REP}(x)||} > \varepsilon_{TESS} > 0.$$

for some open neighborhood $U \ni y$, then for all $x \in U' \ni y$ being a closed subset of U in the interior of the state space,

$$\begin{aligned} \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} &\geq \frac{\alpha\cdot\frac{y-x}{x}\cdot h^{q_k-REP}(x)}{||y-x||\cdot||f(x)||} \\ &= \alpha\cdot\frac{\frac{y-x}{x}\cdot h^{q_k-REP}(x)}{||\frac{y-x}{x}||\cdot||h^{q_k-REP}(x)||}\frac{||\frac{y-x}{x}||\cdot||h^{q_k-REP}(x)||}{||y-x||\cdot||f(x)||} \\ &> \alpha\cdot\varepsilon_{TESS}\cdot\frac{||\frac{y-x}{x}||}{||y-x||}\cdot\frac{||h^{q_k-REP}(x)||}{||f(x)||} \\ &\geq \alpha\cdot\varepsilon_{TESS}\cdot\frac{1}{M_{U'}}\cdot\beta\cdot m_{U'} > 0. \end{aligned}$$

So, y is an attractive *ESS*. This ends the proof of the part connecting *ESS* and *TESS*.
Along the same line, to prove the part connecting attractive *ESS* and *ESE, note* that $h^{0-REP} = h^{OPD}$ and

$$\begin{aligned}(y-x)\cdot h^{0-REP}(x) &= \sum_i (y_i - x_i)\, x_i^0 \left[f_i(x) - \frac{\sum_j x_j^0 f_j(x)}{\sum_j x_j^0} \right] \\ &= \sum_i (y_i - x_i) \left[f_i(x) - \frac{1}{n+1} \sum_j f_j(x) \right] \\ &= \sum_i (y_i - x_i)\, f_i(x) = (y-x)\cdot f(x).\end{aligned}$$

Let $y \in int\ \Delta^n$ be an attractive *ESS*, i.e., for some open neighborhood $U \ni y$:

$$\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} > \varepsilon_{ESS} > 0.$$

By continuity of $h(q,x) = h^{q-REP}(x)$ a monotonically decreasing sequence $\widetilde{Q} = \{\widetilde{q}_k\}_{k=1}^{\infty}$ converging to 0 and numbers $0 < \gamma, \delta < 1$ exist such that for all $x \in U$

$$\begin{aligned}\tfrac{1}{\gamma}(y-x)\cdot f(x) \geq (y-x)\cdot h^{\widetilde{q}_k-REP}(x) \geq \gamma(y-x)\cdot f(x). \\ \tfrac{1}{\delta}||h^{OPD}(x)|| \geq ||h^{\widetilde{q}_k-REP}(x)|| \geq \delta||h^{OPD}(x)||\end{aligned}$$

Then, if $\widetilde{q}_k \in \widetilde{Q}$ for all $x \in U\backslash\{y\}$, we obtain in view of Lemma 4:

$$\begin{aligned}\frac{(y-x)\cdot h^{\widetilde{q}_k-REP}(x)}{||y-x||\cdot||h^{\widetilde{q}_k-REP}(x)||} &\geq \frac{\gamma(y-x)\cdot f(x)}{||y-x||\cdot||h^{\widetilde{q}_k-REP}(x)||} \\ &\geq \gamma\cdot\varepsilon_{ESS}\frac{||f(x)||}{||h^{\widetilde{q}_k-REP}(x)||} \\ &\geq \gamma\cdot\varepsilon_{ESS}\frac{||f(x)||}{||h^{0-REP}(x)||}\cdot\frac{||h^{OPD}(x)||}{||h^{\widetilde{q}_k-REP}(x)||} \\ &\geq \gamma\cdot\delta\cdot\varepsilon_{ESS} > 0.\end{aligned}$$

So, y is an attractive ESE, too.
Conversely, let $y \in int\ \Delta^n$ be an attractive ESE for dynamics $h^{\widetilde{q}_k-REP}(x)$ with $\widetilde{q}_k \in \widetilde{Q}$ then an open set $U \ni y$, $U \subseteq \Delta^n$ exists such that for all $x \in U\backslash\{y\}$

$$\frac{(y-x)\cdot h^{\widetilde{q}_k-REP}(x)}{||y-x||\cdot||h^{\widetilde{q}_k-REP}(x)||} > \varepsilon_{ESE} > 0$$

then in view of Lemma 4

$$\begin{aligned}\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} &\geq \frac{\gamma(y-x)\cdot h^{\widetilde{q}_k-REP}(x)}{||y-x||\cdot||f(x)||} \geq \gamma\cdot\varepsilon_{ESE}\frac{||h^{\widetilde{q}_k-REP}(x)||}{||f(x)||} \\ &\geq \gamma\cdot\varepsilon_{ESE}\frac{||h^{OPD}(x)||}{||f(x)||}\frac{||h^{\widetilde{q}_k-REP}(x)||}{||h^{OPD}(x)||} \geq \gamma\cdot\delta\cdot\varepsilon_{ESE}\cdot m_{\angle} > 0.\end{aligned}$$

So, y is an attractive ESS, too. ■

Proposition 7 For given relative fitness function f, $\lambda = (\lambda_1, ..., \lambda_{n+1}) \in D \equiv [0,1]^{n+1}\backslash\{\mathbf{0}^{n+1}\}$, the Sethi dynamics for all $i \in I^{n+1}$ are given by

$$h_i^{SE,\lambda}(x) = x_i \left(\sum_j \lambda_i x_j \left[f_i(x) - f_j(x)\right]_+ - \sum_j \lambda_j x_j \left[f_j(x) - f_i(x)\right]_+ \right).$$

Let $h^{SE}(\lambda, x) \equiv h^{SE,\lambda}(x)$ then clearly this function is continuous on the domain $D \times \Delta^n$. Define for $U \subseteq \Delta^n$, $\lambda_{\min} \in [0,1]$

$$\begin{aligned} M_{\lambda,U} = \quad & \inf\{X \in [1,\infty)|\ \tfrac{1}{X}||h^{REP}(x)|| \leq ||h^{SE}(\lambda,x)|| \leq X||h^{REP}(x)|| \\ & \quad x \in U, \lambda \in D\}, \\ M_{\lambda_{\min},U} = \quad & \sup\{M_{\lambda,U}|\ \lambda \in D \text{ and } \min_{i\in I^{n+1}} \lambda_i = \lambda_{\min}\}. \end{aligned}$$

Let $C(\lambda,x) = (C_1(\lambda,x),...,C_{n+1}(\lambda,x))^\top$ be for each $i \in I^{n+1}$ be given by

$$\begin{aligned} C_i(\lambda,x) \quad = \quad & x_i \textstyle\sum_{j=1}^{n+1}(1-\lambda_i)\cdot x_j\,[f_i(x) - f_j(x)]_+ - \\ & x_i \textstyle\sum_{j=1}^{n+1}(1-\lambda_j)\cdot x_j\,[f_j(x) - f_i(x)]_+ . \end{aligned}$$

Then, by Lemma 6, for all $i \in I^{n+1}$

$$h_i^{SE,\lambda}(x) = h_i^{REP}(x) - C_i(\lambda,x) = x_i f_i(x) - C_i(\lambda,x).$$

Clearly, $\frac{y-x}{x} = \mathbf{0}^{n+1}$ and if $f(y) = \mathbf{0}^{n+1}$, then $C(\lambda,y) = \mathbf{0}^{n+1}$. Both functions are continuous, and therefore for arbitrary $M_{f,y,U'}, M_{U',y} > 0$ an open neighborhood U' of an interior saturated equilibrium y exists such that for all $x \in U'$

$$\begin{aligned} \max_{i,j\in I^{n+1}}\ |f_i(x) - f_j(x)|\ & \leq M_{f,y,U'}||f(x)|| \\ \max_{i\in I^{n+1}} |\tfrac{y-x}{x}\cdot e_i|\ & \leq \tfrac{1}{n+1}M_{U',y}||y-x||. \end{aligned}$$

Therefore, for all $x \in U'$

$$-(1-\lambda_{\min})M_{f,y,U'}||f(x)|| \leq C_i(\lambda,x) \leq (1-\lambda_{\min})M_{f,y,U'}||f(x)||,$$

where $\lambda_{\min} = \min(\lambda_1,...,\lambda_{n+1})$. So, we have

$$|x_i f_i(x) - h_i^{SE,\lambda}(x)| = |C(\lambda,x)|\ \leq (1-\lambda_{\min})M_{f,y,U'}||f(x)||.$$

Note $\frac{y-x}{x}\cdot h^{REP}(x) = (y-x)\cdot\left(\widehat{x}^{-1}\widehat{x}\right)\cdot f(x) = (y-x)\cdot f(x)$ which leads to

$$\begin{aligned} \tfrac{|\frac{y-x}{x}\cdot C(\lambda,x)|}{||f(x)||\cdot||y-x||} &= \tfrac{|\frac{y-x}{x}\cdot h^{SE,\lambda}(x) - (y-x)\cdot\left(\widehat{x}^{-1}\widehat{x}\right)\cdot f(x)|}{||y-x||\cdot||f(x)||} \\ &= \tfrac{|\frac{y-x}{x}\cdot h^{SE,\lambda}(x) - (y-x)\cdot f(x)|}{||y-x||\cdot||f(x)||} \\ &\leq \tfrac{(1-\lambda_{\min})M_{f,y,U'}||f(x)||\cdot\left(\frac{y-x}{x}\cdot\mathbf{1}^{n+1}\right)}{||y-x||\cdot||f(x)||} \\ &\leq \tfrac{(1-\lambda_{\min})M_{f,y,U'}\cdot M_{U',y}||f(x)||\cdot||y-x||}{||y-x||\cdot||f(x)||} \\ &= (1-\lambda_{\min})M_{f,y,U'}\cdot M_{U',y}. \end{aligned}$$

If y is an interior attractive *ESS*, $\varepsilon_{ESS} > 0$ and open neighborhood U exist such that for all $x \in U\backslash\{y\}$:

$$\tfrac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} > \varepsilon_{ESS}.$$

We choose the numbers $M_{f,y,U'}, M_{U',y}$ and the associated neighborhood U' of y such that

$$(1-\lambda_{\min})M_{f,y,U'}\cdot M_{U',y} \leq \tfrac{1}{4}\varepsilon_{ESS}.$$

Next, we obtain

$$\begin{aligned}
& \frac{\frac{y-x}{x}h^{SE,\lambda}(x)}{||\frac{y-x}{x}||\cdot||h^{SE,\lambda}(x)||} = \frac{||y-x||\cdot||f(x)||}{||\frac{y-x}{x}||\cdot||h^{SE,\lambda}(x)||}\frac{(y-x)\cdot f(x)-\frac{y-x}{x}\cdot C(\lambda,x)}{||y-x||\cdot||f(x)||} \\
= & \frac{||y-x||\cdot||f(x)||}{||\frac{y-x}{x}||\cdot||h^{SE,\lambda}(x)||}\left[\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||}-\frac{\frac{y-x}{x}\cdot C(\lambda,x)}{||y-x||\cdot||f(x)||}\right] \\
\geq & \frac{||y-x||\cdot||f(x)||}{||\frac{y-x}{x}||\cdot||h^{SE,\lambda}(x)||}\left[\frac{3}{4}\varepsilon_{ESS}\right] \\
\geq & \frac{||y-x||\cdot||f(x)||}{\frac{1}{m_U}||y-x||\cdot M_{\lambda_{\min},U}||h^{REP}(x)||}\left[\frac{3}{4}\varepsilon_{ESS}\right] \\
\geq & \frac{||y-x||\cdot||f(x)||}{\frac{1}{m_U}||y-x||\cdot M_{\lambda_{\min},U}\cdot M_U||f(x)||}\left[\frac{3}{4}\varepsilon_{ESS}\right] = \frac{3}{4}\frac{m_U}{M_{\lambda_{\min},U}\cdot M_U}\varepsilon_{ESS}.
\end{aligned}$$

The first inequality is implied by the inequalities immediately above, the second one requires Lemma 4 and the inequalities derived at the beginning of this proof, the third only requires Lemma 4. Then, λ^* exists such that for all $x \in U\backslash\{y\}$ and λ with $\lambda_{\min} \geq \lambda^*$:

$$\frac{\frac{y-x}{x}h^{SE,\lambda}(x)}{||\frac{y-x}{x}||\cdot||h^{SE,\lambda}(x)||} > \frac{1}{2}\frac{m_U}{M_{\lambda_{\min},U}\cdot M_U}\varepsilon_{ESS}.$$

Hence, y is an attractive *TESS* for the dynamics of Sethi [1998].

Similarly, if y is an interior attractive *TESS*, $\varepsilon_{TESS} > 0$ and open neighborhood U exist such that for all $x \in U\backslash\{y\}$:

$$\frac{\frac{y-x}{x}h^{SE,\lambda}(x)}{||\frac{y-x}{x}||\cdot||h^{SE,\lambda}(x)||} > \varepsilon_{TESS}.$$

We choose the numbers $M_{f,y,U''}, M_{U'',y}$ and the associated neighborhood $U'' \subseteq U$ of y such that

$$(1-\lambda_{\min})M_{f,y,U''}\cdot M_{U'',y} \leq \frac{1}{4}\frac{m_U}{M_U\cdot M_{\lambda_{\min},U}}\varepsilon_{TESS}.$$

Then, we obtain

$$\begin{aligned}
& \frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} = \frac{\frac{y-x}{x}\cdot h^{SE,\lambda}(x)-\frac{y-x}{x}\cdot C(\lambda,x)}{||y-x||\cdot||f(x)||} \\
= & \frac{\frac{y-x}{x}\cdot h^{SE,\lambda}(x)-\frac{y-x}{x}\cdot C(\lambda,x)}{||y-x||\cdot||f(x)||} - \frac{\frac{y-x}{x}\cdot C(\lambda,x)}{||y-x||\cdot||f(x)||} \\
\geq & \frac{\frac{y-x}{x}\cdot h^{SE,\lambda}(x)}{||\frac{y-x}{x}||\cdot||h^{SE,\lambda}(x)||}\frac{||\widetilde{z}(x)||\cdot||h^{SE,\lambda}(x)||}{||y-x||\cdot||f(x)||} - \frac{1}{4}\frac{m_U}{M_U\cdot M_{\lambda_{\min},U}}\varepsilon_{TESS} \\
\geq & \frac{\frac{y-x}{x}\cdot h^{SE,\lambda}(x)}{||\frac{y-x}{x}||\cdot||h^{SE,\lambda}(x)||}\frac{\frac{1}{M_U}||y-x||\cdot\frac{1}{M_{\lambda_{\min},U}}||h^{REP}(x)||}{||y-x||\cdot||f(x)||} - \frac{1}{4}\frac{m_U}{M_U\cdot M_{\lambda_{\min},U}}\varepsilon_{TESS} \\
\geq & \frac{(y-x)\cdot\widehat{x}^{-1}h^{SE,\lambda}(x)}{||\frac{y-x}{x}||\cdot||h^{SE,\lambda}(x)||}\frac{\frac{1}{M_U}||f(x)||\cdot\frac{1}{M_{\lambda_{\min},U}}m_U||f(x)||}{||y-x||\cdot||f(x)||} - \frac{1}{4}\frac{m_U}{M_U\cdot M_{\lambda_{\min},U}}\varepsilon_{TESS} \\
= & \frac{m_U}{M_U\cdot M_{\lambda_{\min},U}}\left[\frac{\frac{y-x}{x}h^{SE,\lambda}(x)}{||\frac{y-x}{x}||\cdot||h^{SE,\lambda}(x)||}-\frac{1}{4}\varepsilon_{TESS}\right] \geq \frac{3}{4}\frac{m_U}{M_U\cdot M_{\lambda_{\min},U}}\varepsilon_{TESS}.
\end{aligned}$$

Then λ^{**} exists such that for all $x \in U''\backslash\{y\}$ and λ with $\lambda_{\min} \geq \lambda^{**}$:

$$\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} > \frac{1}{2}\frac{m_U}{M_U\cdot M_{\lambda_{\min},U}}\varepsilon_{TESS}.$$

Hence, y is an attractive *ESS* for the dynamics of Sethi [1998]. ■

Proposition 8 Let y be a strict saturated equilibrium, then $y = e_j$ for some $j \in \{1, ..., n+1\}$. Let $f_j(y) = 0 > -\delta \equiv \max_{k \neq j} f_k(y) \geq -\xi \equiv \min_k f_k(y)$. Let $\varepsilon = \frac{\delta}{2}$ and consider a perturbation $\widetilde{f}$of f such that $\max_{x \in \Delta^n} \max_h |f_h(x) - \widetilde{f}_h(x)| < \varepsilon$. Since $y = e_j$ it follows immediately that $\widetilde{f}_j(y) = 0$, furthermore $\max_{k \neq j} \widetilde{f}_k(y) < -\frac{\delta}{3}$. This implies that y is also a strict saturated equilibrium with respect to $\widetilde{f}$. This proves the first part.

Let $U = \{x \in \Delta^n | -\frac{\delta}{2} > \max_{k \neq j} f_k(x) \text{ and } -\frac{3}{2}\xi < \min_k f_k(x)\}$, then clearly $y \in U$. By continuity of f there exist $r > 0$ and an open neighborhood U' of y satisfying that $U' = \{x \in \Delta^n | \ 0 < ||y - x|| < r\} \cap U \neq \emptyset$. Observe that $x \cdot f(x) = 0$ for all $x \in \Delta^n$, hence for all $x \in U' \backslash \{y\}$:

$$f_j(x) = -\frac{1}{x_j} \sum_{k \neq j} x_k f_k(x) \Longrightarrow \frac{1-x_j}{x_j} \frac{\delta}{2} < f_j(x) < \frac{1-x_j}{x_j} \frac{3\xi}{2}.$$

Then for $x \in U' \backslash \{y\}$

$$\begin{aligned} &\frac{(y-x)\cdot f(x)}{||y-x|| \cdot ||f(x)||} = \frac{f_j(x)}{||y-x|| \cdot ||f(x)||} > \frac{1-x_j}{x_j} \frac{\delta}{2} \frac{1}{||y-x|| \cdot ||f(x)||} > \\ &\frac{\delta}{2} \frac{(1-x_j)}{||y-x|| \cdot ||f(x)||} = \frac{\delta}{2} \frac{||y-x||_\infty}{||y-x|| \cdot ||f(x)||} \geq \frac{\delta}{2} \frac{||y-x||_\infty}{\sqrt{n+1}||y-x||_\infty} \frac{1}{||f(x)||} = \\ &\frac{\delta}{2} \frac{1}{\sqrt{n+1}} \frac{1}{||f(x)||} \geq \frac{\delta}{2} \frac{1}{\sqrt{n+1}} \frac{1}{\sqrt{n+1}||f(x)||_\infty} > \frac{\frac{\delta}{2}}{\frac{3\xi}{2}} \frac{1}{n+1} = \frac{\delta}{3\xi} \frac{1}{n+1} > 0. \end{aligned}$$

This implies that y is an attractive evolutionarily stable strategy. ■

Lemma 9 Let $c(x) = \sum_{j=1}^{n+1} f_j(x)$, since $x \cdot f(x) \equiv 0$ for all $x \in \Delta^n$ we have

$$c(b) = (n+1)\,(b \cdot f(b)) = 0.$$

Therefore, we obtain for all $\alpha \leq 0$, $i \in I^{n+1}$,

$$h_i^\alpha(b) \ = \left[f_i(x) - (x_i - \alpha) \frac{c(x)}{1-(n+1)\alpha} \right]_{x=b} = [f_i(x)]_{x=b} = f_i(b).$$

Hence, $h^\alpha(b) = f(b)$ which in turn of course implies

$$\left[\frac{(y-x)\cdot h^\alpha(x)}{||y-x|| \cdot ||h^\alpha(x)||} \right]_{x=b} = \left[\frac{(y-x)\cdot f(x)}{||y-x|| \cdot ||f(x)||} \right]_{x=b}$$

Furthermore, for all $q \geq 0$

$$\begin{aligned} h_i^{q-REP}(b) \ &= \frac{1}{n+1}^q \left[f_i(b) - \frac{\sum_j \frac{1}{n+1}^q f_j(b)}{\sum_j \frac{1}{n+1}^q} \right] = \frac{1}{n+1}^q \left[f_i(b) - \frac{1}{n+1} \sum_j f_j(b) \right] \\ &= \frac{1}{n+1}^q \left[f_i(b) - \frac{1}{n+1} c(b) \right] = \frac{1}{n+1}^q \left[f_i(b) \right]. \end{aligned}$$

This implies

$$\begin{aligned} \left[\frac{(y-x)\cdot h^{q-REP}(x)}{||y-x|| \cdot ||h^{q-REP}(x)||} \right]_{x=b} &= \left[\frac{(y-x)\cdot \left[\frac{1}{n+1}^q f(x) \right]}{||y-x|| \cdot || \frac{1}{n+1}^q f(x)||} \right]_{x=b} = \\ \left[\frac{\frac{1}{n+1}^q [(y-x)\cdot f(x)]}{\frac{1}{n+1}^q [||y-x|| \cdot ||f(x)||]} \right]_{x=b} &= \left[\frac{(y-x)\cdot f(x)}{||y-x|| \cdot ||f(x)||} \right]_{x=b}. \end{aligned}$$

Additionally, this implies

$$\left[\frac{\frac{y-x}{x}\cdot h^{q-REP}(x)}{||\frac{y-x}{x}||\cdot||h^{q-REP}(x)||}\right]_{x=b} = \left[\frac{\frac{y-x}{x}\cdot\left[\frac{1}{n+1}^{q}f(x)\right]}{||\frac{y-x}{x}||\cdot||\frac{1}{n+1}^{q}f(x)||}\right]_{x=b} =$$
$$\left[\frac{\frac{1}{n+1}^{q-1}[(y-x)\cdot f(x)]}{\frac{1}{n+1}^{q-1}[||y-x||\cdot||f(x)||]}\right]_{x=b} = \left[\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||}\right]_{x=b}. \qquad \blacksquare$$

Proposition 10 Let $y=b$ be an attractive *ESS*, then an open neighborhood U containing y exists such that

$$\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} > \epsilon_{ESS} > 0,$$

for all $x \in U\backslash\{y\}$. Then, by the preceding lemma and continuity in x some $\delta_1 > 0$ exists such that for all $x \in B_{\delta_1} = \{z \in \Delta^n |\ 0 < ||b-z|| \leq \delta_1\}$:

$$\begin{aligned}
\left|\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} - \frac{(y-x)\cdot h^{\alpha}(x)}{||y-x||\cdot||h^{\alpha}(x)||}\right| &\leq \frac{\epsilon_{ESS}}{6},\\
\left|\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} - \frac{(y-x)\cdot h^{q-REP}(x)}{||y-x||\cdot||h^{q-REP}(x)||}\right| &\leq \frac{\epsilon_{ESS}}{6},\\
\left|\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} - \frac{\frac{y-x}{x}\cdot h^{\alpha}(x)}{||\frac{y-x}{x}||\cdot||h^{\alpha}(x)||}\right| &\leq \frac{\epsilon_{ESS}}{6},\\
\left|\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} - \frac{\frac{y-x}{x}\cdot h^{q-REP}(x)}{||\frac{y-x}{x}||\cdot||h^{q-REP}(x)||}\right| &\leq \frac{\epsilon_{ESS}}{6}.
\end{aligned}$$

Take $\epsilon = \frac{1}{2}\epsilon_{ESS}$, then $\frac{(y-x)\cdot h^{\alpha}(x)}{||y-x||\cdot||h^{\alpha}(x)||}$, $\frac{(y-x)\cdot h^{q-REP}(x)}{||y-x||\cdot||h^{q-REP}(x)||}$, $\frac{\frac{y-x}{x}\cdot h^{\alpha}(x)}{||\frac{y-x}{x}||\cdot||h^{\alpha}(x)||}$ and $\frac{\frac{y-x}{x}\cdot h^{q-REP}(x)}{||\frac{y-x}{x}||\cdot||h^{q-REP}(x)||}$ are greater than ϵ. Moreover, the difference between any pair of these four expressions taken in absolute values is smaller than ϵ. This means that y is an attractive *ESE* and *TESS* for all q-deformed replicator dynamics and α-barycentric projection dynamics. The statement between brackets follows furthermore by realizing that continuity in y guarantees that the above must also hold for all $y \neq b$ with $||y-b||$ sufficiently small, since for all $x \in B_{\delta_1}\backslash\{y\}$ such that $f(x) \neq \mathbf{0}^{n+1}$:

$$\begin{aligned}
&\left|\frac{(y-x)\cdot f(x)}{||y-x||\cdot||f(x)||} - \frac{(y-x)\cdot h^{\alpha}(x)}{||y-x||\cdot||h^{\alpha}(x)||}\right|\\
=\ &\left|\frac{(y-b)\cdot f(x)}{||y-x||\cdot||f(x)||} - \frac{(y-b)\cdot h^{\alpha}(x)}{||y-x||\cdot||h^{\alpha}(x)||} - \frac{||b-x||}{||y-x||}\left(\frac{(b-x)\cdot f(x)}{||b-x||\cdot||f(x)||} + \frac{(b-x)\cdot h^{\alpha}(x)}{||b-x||\cdot||h^{\alpha}(x)||}\right)\right|\\
\leq\ &\left|\frac{(y-b)\cdot f(x)}{||y-x||\cdot||f(x)||} - \frac{(y-b)\cdot h^{\alpha}(x)}{||y-x||\cdot||h^{\alpha}(x)||}\right| + \frac{1}{3}\frac{||b-x||}{||y-x||}\epsilon_{ESS}\\
=\ &\frac{1}{3}\frac{||b-x||}{||y-x||}\epsilon_{ESS} + \frac{||y-b||}{||y-x||\cdot||f(x)||}\left|\frac{(y-b)\cdot f(b)}{||y-b||\cdot||f(b)||} - \frac{(y-b)\cdot h^{\alpha}(b)}{||y-b||\cdot||h^{\alpha}(b)||}\right|.
\end{aligned}$$

Since, $\lim_{y\to b}\frac{||b-x||}{||y-x||} = 1$ and $\lim_{y\to b}||y-b|| = 0$ we have

$$\frac{(y-x)\cdot h^{\alpha}(x)}{||y-x||\cdot||h^{\alpha}(x)||} \geq \epsilon,$$

for any y sufficiently close to the barycenter and all x sufficiently nearby. As the same can be done by taking any other defining property of an attractive evolutionary equilibrium among the two remaining candidates and the two classes of dynamics, we may consider the proof complete.

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