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Nonlinear inner-outer factorization

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Abstract

It is shown how the method for inner-outer factorization of stable nonlinear state space systems as put forward in [11] may be extended to the non-invertible case by replacing a Hamilton-Jacobi equation by a dissipation inequality. The construction of the outer factor is based on the factorization of this inequality.

In linear control theory inner-outer factorization (or more generally J-inner-outer factorization) of rational matrices has played an important role e.g. in the theory of $\mathcal{H}_\infty$ optimal control. In linear as well as in nonlinear theory [10], [16], [6] it has been argued that the control design of non-minimum phase stable systems can be based upon the inverse of the minimum phase (outer) factor, with the inner factor remaining as a limiting element in the closed-loop system. In a series of papers, see e.g. [2], [3], [1], Ball and Helton have investigated inner-outer factorization of nonlinear input-output operators and of nonlinear state space systems in discrete time. In the present note we will study inner-outer factorization of nonlinear state space systems in continuous time, using a quite different approach. Indeed our method will be a kind of "nonlinear spectral factorization" and concentrates on finding first the *outer* factor (instead of starting with the *inner* factor). The present paper is a continuation of [12] where the *invertible* case has been studied, and concentrates on the *non-invertible* case. More details will appear in [4].

Consider a (smooth) nonlinear system

$$\Sigma : \begin{cases} \dot{x} &= a(x) + b(x)u, \quad u \in \mathbf{R}^m \\ y &= c(x) + d(x)u, \quad y \in \mathbf{R}^p \end{cases} \tag{1}$$

where $x = (x_1, \cdots, x_n)$ are local coordinates for the state space manifold M, with globally asymptotically stable equilibrium 0 (thus $a(0) = 0$). Without loss of generality we assume $c(0) = 0$. The problem of inner-outer factorization consists in finding a *lossless* nonlinear system Θ (the *inner* factor) and an *asymptotically stable* and *minimum phase* nonlinear system R (the *outer* factor), such that symbolically

$$\Sigma = \Theta \cdot R. \tag{2}$$

By this we mean that for every initial condition of Σ there exist initial conditions of Θ and R such that the input-output behavior of Σ equals the input-output behavior of the series interconnection of R followed by Θ.

Let us recall [14] that a nonlinear system (1) is called *lossless* with respect to the *supply rate* $\frac{1}{2} \parallel u \parallel^2 - \frac{1}{2} \parallel y \parallel^2$ if there exists a function $V(x) \geq 0$ (the *storage* function) such that

$$V(x(t_1)) - V(x(t_0)) = \frac{1}{2}\int_{t_0}^{t_1} (\parallel u(t) \parallel^2 - \parallel y(t) \parallel^2) dt \tag{3}$$

for all t_0, t_1 and $u(\cdot)$, or equivalently, if V is C^1,

$$\begin{aligned} & V_x(x)\left[a(x) + b(x)u\right] = \\ & \tfrac{1}{2}u^T u - \tfrac{1}{2}\left[c(x) + d(x)u\right]^T \left[c(x) + d(x)u\right] \end{aligned} \tag{4}$$

for all x, u. ($V_x(x)$ denotes the row vector of partial derivatives of $V(x)$.) Taking $t_0 = 0$ and $t_1 = \infty$ in (3), it follows that (1) is L_2*-norm preserving*. Furthermore, a nonlinear (1) is called *minimum phase* if 0 is a Lyapunov stable equilibrium of its zero-dynamics [8].

Our approach for constructing the outer factor R runs as follows. First we consider the *Hamiltonian extension* of Σ, see [5]

$$\begin{cases} \dot{x} &= a(x) + b(x)u \\ \dot{p} &= -\left[\frac{\partial a}{\partial x}(x) + \frac{\partial b}{\partial x}(x)u\right]^T p \\ & \quad -\frac{\partial^T c}{\partial x}(x)u_a - u^T \frac{\partial^T d}{\partial x}(x)u_a, \quad u_a \in \mathbf{R}^p \end{cases} \tag{5}$$
$$\begin{cases} y &= c(x) + d(x)u, \\ y_a &= b^T(x)p + d^T(x)u_a, \quad y_a \in \mathbf{R}^m \end{cases}$$

which is Hamiltonian system living on T^*M (with coordinates (x,p)), having inputs (u, u_a) and outputs (y, y_a). Imposing the interconnection $u_a = y$ to (5) leads to the Hamiltonian system

$$\Sigma^*\Sigma : \begin{aligned} \dot{x} &= \tfrac{\partial H}{\partial p}(x,p,u) \\ \dot{p} &= -\tfrac{\partial H}{\partial x}(x,p,u) \\ y_a &= \tfrac{\partial H}{\partial u}(x,p,u) \end{aligned} \tag{6}$$

with Hamiltonian function

$$\begin{aligned} H(x,p,u) &= p^T\left[a(x) + b(x)u\right] + \\ & \tfrac{1}{2}\left[c(x) + d(x)u\right]^T \left[c(x) + d(x)u\right] \end{aligned} \tag{7}$$

Note that for a *linear* system (1) $\Sigma^*\Sigma$ reduces to the series interconnection of Σ and its adjoint linear system Σ^*, having transfer matrix $G^T(-s)G(s)$ ($G(s)$ being the transfer matrix of Σ). In [12] we have shown how to obtain the outer factor R by "spectral factorization" of the Hamiltonian system $\Sigma^*\Sigma$, assuming the invertibility condition

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$$E(x) := d^T(x)d(x) \quad \text{is invertible for all } x \tag{8}$$

In fact, if (8) is satisfied then we may directly compute the inverse system $(\Sigma^*\Sigma)^{-1}$. The outer factor R is now obtained by computing the stable invariant manifold of the inverse system via the Hamilton-Jacobi equation in $P(x)$

$$\begin{aligned} &P_x(x)[a(x) - b(x)E^{-1}(x)d^T(x)c(x)] + \\ &\tfrac{1}{2}c^T(x)[I_p - d(x)E^{-1}(x)d^T(x)]c(x) - \\ &\tfrac{1}{2}P_x(x)b(x)E^{-1}(x)b^T(x)P_x^T(x) = 0, \quad P(0) = 0, \end{aligned} \tag{9}$$

with $P_x(x) = \left(\frac{\partial P}{\partial x_1}(x), \cdots, \frac{\partial P}{\partial x_n}(x)\right)$. Factorizing $E(x) = d^T(x)d(x)$ as

$$d^T(x)d(x) = \bar{d}^T(x)\bar{d}(x) \tag{10}$$

for some $m \times m$ matrix $\bar{d}(x)$ (this may be always done; however for $\bar{d}$ to depend *smoothly* on x we need to invoke Morse's Lemma), the outer factor is now given as

$$R : \begin{cases} \dot{x} = a(x) + b(x)u & , u \in \mathbf{R}^m \\ \bar{y} = \bar{c}(x) + \bar{d}(x)u & , \bar{y} \in \mathbf{R}^m \end{cases} \tag{11}$$

$$\bar{c}(x) = \bar{d}(x)E^{-1}(x)\left[d^T(x)c(x) + b^T(x)P_x^T(x)\right]$$

In this paper we will concentrate on the non-invertible case, i.e. if (8) is not satisfied. First of all, we note that (9) is the Hamilton-Jacobi-Bellman equation corresponding to the cost-functional (with x_0 the initial condition)

$$J(x_0, u) = \int_0^\infty \| y(t) \|^2 \, dt, \tag{12}$$

for Σ, and that H as given in (7) is the pseudo-Hamiltonian of the Maximum Principle. If (8) is not satisfied then this optimal control problem is *singular*. Our approach will be heavily motivated by the work of Hill and Moylan [7], and the work of Willems [15] and Schumacher [13] on singular LQ-control where it is shown that the Riccati-equation for the *regular LQ* optimal control problem may be replaced by a *matrix inequality* in the singular case. We define the optimal cost for any x_0 as

$$P^+(x_0) = \inf\{J(x_0, u) \mid u \text{ admissible}, x(\infty) = 0\} \tag{13}$$

Assumption 1 $P^+(x_0)$ exists for every x_0, and P^+ is a smooth function on M.

We now consider the *dissipation inequality* corresponding to the pseudo-Hamiltonian (7)

$$\begin{aligned} &P_x^T(x)[a(x) + b(x)u] + \\ &\tfrac{1}{2}[c(x) + d(x)u]^T[c(x) + d(x)u] \geq 0, \quad P(0) = 0, \end{aligned} \tag{14}$$

which should hold for every x and u. It immediately follows from (13) that P^+ satisfies (14). Furthermore (compare [15], [13])

Proposition 2 Let P satisfy (14), then $P(x) \leq P^+(x)$, for all x.

Proof Let P be any solution to (14). Consider any input function u on the time-interval $[0, T]$, and integrate (14) from $t = 0$ to $t = T$ for this particular u to obtain

$$P(x(T)) - P(x(0)) + \frac{1}{2}\int_0^T \| y \|^2 \, dt \geq 0 \tag{15}$$

Now let u be defined on $[0, \infty)$ such that $\lim_{t \to \infty} x(t) = 0$. Then it follows that

$$\frac{1}{2}\int_0^\infty \| y \|^2 \, dt \geq P(x(0)) \tag{16}$$

and thus by definition of P^+ we obtain $P^+(x(0)) \geq P(x(0))$ for all $x(0) \in M$. □

Thus P^+ is completely characterized as the maximal solution to (14), and, in principle, may be *computed* this way.
Now consider the following smooth function of x and u

$$\begin{aligned} K^+(x, u) \;:=\; &P_x^+(x)\,[a(x) + b(x)u] + \\ &\tfrac{1}{2}\,[c(x) + d(x)u]^T\,[c(x) + d(x)u] \end{aligned} \tag{17}$$

Clearly, $K^+(0, 0) = 0$ and $K^+(x, u) \geq 0$. Our next main assumption is

Assumption 3 There exists a smooth mapping $\bar{G} : M \times \mathbf{R}^m \to \mathbf{R}^{\bar{p}}$ for some $\bar{p} \in \mathbf{N}$, such that

$$K^+(x, u) = \frac{1}{2}\bar{G}^T(x, u)\bar{G}(x, u) \tag{18}$$

Note that *without* the smoothness assumption Assumption 3 is trivially satisfied since we may take $\bar{p} = 1$ and $\bar{G}(x, u) = \sqrt{K^+(x, u)}$.
Sufficient conditions for the *local* existence of a smooth $\bar{G}$ satisfying (18) are provided by the following generalization of Morse's Lemma:

Lemma 4 Suppose the Hessian matrix of K^+, i.e.,

$$\begin{bmatrix} \frac{\partial^2 K^+}{\partial x^2}(x, u) & \frac{\partial^2 K^+}{\partial x \partial u}(x, u) \\ \frac{\partial^2 K^+}{\partial x \partial u}(x, u) & \frac{\partial^2 K^+}{\partial u^2}(x, u) \end{bmatrix} \tag{19}$$

has constant rank, say $\bar{p}$, on a neighborhood of $(x, u) = (0, 0)$. Then locally near $(0, 0)$ there exists a C^∞ mapping $\bar{G} : M \times \mathbf{R}^m \to \mathbf{R}^{\bar{p}}$ such that (18) is satisfied.

Proof Can be based on [9]. □

Now let us define the *new system* $\bar{\Sigma}$, defined as

$$\bar{\Sigma} : \begin{cases} \dot{x} = a(x) + b(x)u & , u \in \mathbf{R}^m \quad , x \in M \\ \bar{y} = \bar{G}(x, u) & , \bar{y} \in \mathbf{R}^{\bar{p}} \end{cases} \tag{20}$$

It can be readily checked that in the *invertible* case (i.e. $E(x) = d^T(x)d(x)$ being invertible) $\bar{\Sigma}$ coincides with R given in (11). We claim that also in the non-invertible case $\bar{\Sigma}$ is the outer factor of Σ. In order to prove this we first consider the dissipation in equality (14) for $\bar{\Sigma}$, i.e.,

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$$\bar{P}_x(x)\,[a(x)+b(x)u] + \tfrac{1}{2}\bar{G}^T(x,u)\bar{G}(x,u) \geq 0, \qquad (21)$$
$$\bar{P}(0) = 0$$

Lemma 5 The maximal solution $\bar{P}^+$ to (21) is $\bar{P}^+ = 0$.

Proof Clearly $\bar{P} = 0$ satisfies (21). Let now $\bar{P} \geq 0$ satisfy (21). Then by adding (21) and (14) for $P = P^+$, and using (17), (18) we conclude that $P^+ + \bar{P}$ is a solution to (14). By Proposition 2 this implies $\bar{P} = 0$. □

This lemma is instrumental in proving the main result:

Theorem 6 The zero-dynamics of $\bar{\Sigma}$ is not exponentially unstable.

For the proof, based on Lemma 5 and a linearization idea (making use of the linear results described in [13], [15]) we refer to [4]. It follows that if the zero-dynamics of $\bar{\Sigma}$ does not have imaginary eigenvalues, then it will be actually (locally) asymptotically stable, and thus $\bar{\Sigma}$ is an outer factor of Σ!

The *inner* factor Θ of Σ is now easily obtained, at least in the following "right factorization" format:

$$\Theta : \begin{cases} \dot{x} &= a(x) + b(x)u \\ y &= c(x) + d(x)u \\ \bar{y} &= \bar{G}(x,u) \end{cases} \qquad (22)$$

(with driving variables u). Indeed, by considering (14) for $P = P^+$ and (18), we obtain

$$P^+(x(t_1)) - P^+(x(t_0)) + \tfrac{1}{2}\int_{t_0}^{t_1} \| y(t) \|^2\, dt = \tfrac{1}{2}\int_{t_0}^{t_1} \| \bar{y}(t) \|^2\, dt \qquad (23)$$

implying that Θ is lossless (from $\bar{y}$ to y), with storage function P^+.

An explicit input-output representation of Θ, however, may not be easily obtainable, due to non-invertibility of $E(x) = d^T(x)d(x)$.

A useful property of the inner factor $\bar{\Sigma}$ is that $\bar{\Sigma}$ and Σ have the same *static gains*, in the following sense. Consider the set of all *controlled* equilibria for Σ, i.e.,

$$E_c = \{(x,u) \in M \times \mathbf{R}^m \,|\, a(x) + b(x)u = 0\} \qquad (24)$$

Lemma 7 Consider Σ and $\bar{\Sigma}$. For every $(x,u) \in E_c$

$$\| c(x) + d(x)u \| = \| \bar{G}(x,u) \| \qquad (25)$$

(or equivalently $\| y \| = \| \bar{y} \|$).

Proof Consider the equality

$$P_x^+(x)[a(x)+b(x)u] + \tfrac{1}{2}[c(x)+d(x)u]^T[c(x)+d(x)u] = \tfrac{1}{2}\bar{G}^T(x,u)\bar{G}(x,u) \qquad (26)$$

on E_c. □

Thus, if we compare the *step responses* of Σ and $\bar{\Sigma}$ for every constant input u, then the static gains of Σ and $\bar{\Sigma}$ (assuming that the corresponding controlled equilibrium (x,u) of $\dot{x} = a(x) + b(x)u$ is (globally) asymptotically stable) are equal. Thus for output set-point control of Σ one may also consider its outer factor $\bar{\Sigma}$, which is asymptotically equivalent to Σ. The control of Σ thus can be based on $\bar{\Sigma}$, and since $\bar{\Sigma}$ is minimum phase, *inversion* techniques can be applied. This idea, which generalizes an old idea in linear control theory (see e.g. [10]), is discussed in [16], [6].

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