Received: 14 October 2020 | Revised: 23 May 2022 | Accepted: 15 June 2022

DOI: 10.1112/jlms.12667

RESEARCH ARTICLE

Journal of the London Mathematical Society

Patchworking oriented matroids

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Funding information
Deutsche Forschungsgemeinschaft; Horizon 2020, Grant/Award Number: ScaleOpt-757481; Trond Mohn Stiftelse, Grant/Award Number: Algebraic-and-topological-cycles-in-tropical-and-complex-geometry; Croucher Foundation; Nederlandse Organisatie voor Wetenschappelijk Onderzoek, Grant/Award Number: Vici-639.033.514

Abstract

In a previous work, we gave a construction of (not necessarily realisable) oriented matroids from a triangulation of a product of two simplices. In this follow-up paper, we use a combinatorial analogue of Viro's patchworking to derive a topological representation of the oriented matroid directly from the polyhedral structure of the triangulation. This provides a combinatorial manifestation of patchworking besides tropical algebraic geometry. We achieve this by defining a general homeomorphism-preserving operation on regular cell complexes which acts by merging adjacent cells in the complex together. We then rephrase the patchworking procedure in terms of this process using the theory of tropical oriented matroids.

MSC 2020

52C40 (primary), 05E45, 14T15, 52C30, 57N60, 57Q99 (secondary)

1 | INTRODUCTION

Oriented matroids are ordinary matroids equipped with extra sign data, which capture and extend the combinatorics of directed graphs, real hyperplane arrangements, and more generally linear dependence over $\mathbb{R}$. They appear in many subjects in mathematics and related areas, from discrete geometry and optimisation algorithms to algebraic geometry and topology; we refer the reader to [6, Chapter 1 & 2] for more examples. Besides having various equivalent combinatorial axiom systems, a major result in the theory of oriented matroids is the *Topological Representation Theorem* of Folkman and Lawrence [13], which states that every oriented matroid can be represented by a *pseudosphere arrangement* (a topological generalisation of real hyperplane arrangements) and vice versa.

Motivated by a number of interrelated topics, including the work of Sturmfels and Zelevinsky on maximal minors, connections with tropical geometry, and optimisation questions surrounding the tropical simplex method, the authors of the current paper introduced in [8] a construction of (uniform) oriented matroids from triangulations of $\triangle_{d-1} \times \triangle_{n-1}$ with suitable sign data. When such triangulations are regular, the construction is implicit in the literature using the idea of *signed tropicalisation*. Considering general triangulations, however, allows us to additionally obtain non-realisable examples, including Ringel's classical line arrangement (see [8, Section 4.2]). The construction given in [8] expresses an oriented matroid using a *chirotope* defined using certain bipartite matchings that encode the triangulation. However, in view of the polyhedral structure of a triangulation, it is natural to ask whether the topological realisation of such oriented matroids can be related to the triangulation directly. In this paper, we provide a construction to achieve this.

To do so, we adapt the method of patchworking, which goes back to Viro in the 1980s [34]. Viro's method has numerous applications in real algebraic geometry and tropical geometry (see the survey by Viro [35][†]), and is related to the Gelfand–Kapranov–Zelevinsky theory [15]. The idea of combinatorial patchworking is that one can construct piecewise linear (PL) objects isotopic to real algebraic varieties by some 'cut and paste' procedure, starting with a *regular* subdivision of a Newton polytope with sign data. Sturmfels used this idea in [33] to study complete intersections, where *coherent* mixed subdivisions play the crucial role to derive the structure of the intersecting hypersurfaces. While the latter are focused on the study of the intersections, we deal with the whole cellular complex cut out by them.

Theorem A. *Given a (not necessarily coherent) fine mixed subdivision of $n\triangle_{d-1}$ and a sign matrix, we can construct a pseudosphere arrangement representing the oriented matroid in* [8] *via a patchworking procedure.*

Most previous works on patchworking use *tropicalisation* or *dequantisation* of polynomials or similar techniques, and they study the topology of the combinatorially defined loci by comparing them with actual (real) algebro-geometric objects. Beyond these techniques, we note the work of De Loera and Wicklin [11] which studies general patchworking in dimension two to give a combinatorial version of Hilbert's lemma, and the work of Itenberg and Shustin [21] which shows that patchworking with arbitrary subdivisions of $\triangle_2$ produces real pseudoholomorphic curves. However, we are interested in understanding patchworking with general subdivisions in higher dimension, and in particular on giving concrete meaning to the resulting loci. To our knowledge, little work has been carried out at this level of generality in the literature. The aforementioned example of Ringel's non-realisable arrangement shows that our patchworking setting can indeed produce meaningful non-algebro-geometric objects beyond the tropicalisation paradigm.

Furthermore, just as matroids and their fans comprise the local theory of tropical geometry (particularly *smooth tropical manifolds*) [27, Chapter 7], oriented matroids provide the local theory for *real tropical geometry* as illustrated in the recent work of Rau, Renaudineau and Shaw [29]. In particular, they defined the real part of a tropical variety equipped with a *real phase structure* [5, 30], which is locally (the topological realisation of) an oriented matroid. From the interplay of subdivisions and sign matrices, we also implicitly obtain a real phase structure here, and our work can be interpreted as patchworking these more general tropical objects (not necessarily from tropicalisation) at least in a local and generic sense. We expect that our construction can be generalised

[†] The title of our paper is inspired by the title of this survey.

to even more general settings, and potentially extend the fruitful area of 'geometry of matroids' [1] to oriented matroids.

Our proof uses a combination of combinatorial and topological methods, building on the insights of Horn in [19] on topologically representing a (generic) tropical oriented matroid by tropical pseudohyperplanes. Roughly speaking, we show that it is possible to interpolate between the dual complex of what we call a patchworking complex, which may be regarded as a cell decomposition of the boundary of the sphere, and a pseudosphere arrangement representing our oriented matroid. In fact, our results imply a 'Topological Representation Theorem' for each interpolation step between Horn's result and the result of Folkman and Lawrence; we elaborate further on the relation with Horn's work in Section 3.4.

More precisely, in each of the interpolation steps, we merge two equidimensional cells that share a common (relative) facet, the pair of cells to merge is chosen carefully to ensure that the combinatorics and the topology are controlled. We actually study such a strategy at a larger generality, and formulate the following theorem. Here we start with a general regular cell complex Δ, which is determined up to homeomorphism by its face poset P. We encode the merging step in an equivalence relation $\sim$ on the ground set of P, where each non-singleton equivalence class consists of two cells to be merged together with their common facet. The resulting poset $\mathrm{P}/\sim$, which we call an *elementary quotient* (cf. Definition 4.3), inherits the poset structure of P. Given an equivalence class $\tilde{\sigma} \in \mathrm{P}/\sim$, let $\bigcup \tilde{\sigma}$ denote the union of the cells in $\tilde{\sigma}$. Let $\Delta/\sim$ denote the set of these unions.

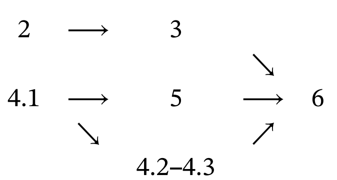
Theorem B. *Suppose that:*

(i) *the poset $\mathrm{P}/\sim$ is an elementary quotient,*
(ii) *the augmented poset $\mathcal{L}(\mathrm{P}) := \mathrm{P} \cup \{\hat{\mathbf{0}}, \hat{\mathbf{1}}\}$ is a lattice, and*
(iii) *each $\sigma \in \Delta$ is a PL ball.*

Then $\Delta/\sim$ is a regular cell complex with face poset $\mathrm{P}/\sim$, such that each $\bigcup \tilde{\sigma} \in \Delta/\sim$ is a PL ball.

The advantage of the theorem is that, as long as we guarantee a minimal amount of initial topological conditions, we can perform the simplification of a cell complex in a purely combinatorial manner with the face poset. Such a paradigm was pioneered by Hersh in her work on total positivity [18], see Section 3.3 for more discussion on her work and ours. As such, Theorem B gives a potential new tool to study quotients arising from other problems in combinatorial topology, for example, for more complicated combinatorial patchworking of polyhedral complexes with sign information, as mentioned above.

The paper is organised as follows. In Section 2, we collect essential definitions and background for the central objects in this paper, as well as a summary of results from [8] which are needed in this part. Section 3 is devoted to stating the main theorem, Theorem A, and contains an illustration of the rank 3 case. Sections 4 and 5 elaborate on the two main ingredients in the proof of Theorem A, namely *quotients of regular cell complexes* and *elimination systems*. The main theorem itself is proved in Section 6. The dependencies of each of the sections on each other are shown below:



2 BACKGROUND

Throughout the paper, we fix a ground set E of size n and a set R of size $d \leqslant n$. We often identify E, R with $[n] = \{1, 2, \ldots, n\}$ and $[d]$, hence fixing an ordering for them. We use $\{+, -, 0\}$ and $\{1, -1, 0\}$ for signs interchangeably, and we adopt the ordering $+, - > 0$ of signs. This is extended componentwise to a partial order on sign vectors. For a sign vector X and a sign $s \in \{+, -, 0\}$, we denote by X^s the set of all indices e such that $X_e = s$. We say two sign vectors X, Y are *conformal* if $X(e)Y(e) \neq -, \forall e$.

2.1 Oriented matroids

We refer the reader to [6] for a comprehensive survey on oriented matroids.

Definition 2.1. A *chirotope* on E of rank d is a non-zero, alternating map $\chi : \mathsf{E}^d \to \{+, -, 0\}$ that satisfies the *Grassmann–Plücker (GP) relation*:

For any $x_1, \ldots, x_{d-1}, y_1, \ldots, y_{d+1} \in \mathsf{E}$, the $d+1$ expressions

$$(-1)^k \chi(x_1, \ldots, x_{d-1}, y_k)\chi(y_1, \ldots, \widehat{y_k}, \ldots, y_{d+1}), \qquad k = 1, \ldots, d+1,$$

either contain both a positive and a negative term, or are all zeros. Here $\widehat{y_k}$ means that we remove y_k from the list.

By the alternating property, we can specify a chirotope by its values over all d-tuples of strictly increasing elements, which are identified with d-subsets of E.

For the purpose of topological constructions, we switch to an alternative axiom system. We recall that the *composition* $X \circ Y$ of two signed vectors agrees with X in all positions $e \in \mathsf{E}$ with $X_e \neq 0$, and agrees with Y otherwise.

Definition 2.2. A collection of sign vectors $\mathcal{L} \subset \{+, -, 0\}^{\mathsf{E}}$ is the collection of *covectors* of an oriented matroid

(i) if $\mathbf{0} \in \mathcal{L}$;
(ii) if $X \in \mathcal{L}$, then $-X \in \mathcal{L}$;
(iii) if $X, Y \in \mathcal{L}$, then $X \circ Y \in \mathcal{L}$;
(iv) for any $X, Y \in \mathcal{L}$ and $e \in X^+ \cap Y^-$, there exists $Z \in \mathcal{L}$ such that $Z_e = 0$, and $Z_f = (X \circ Y)_f = (Y \circ X)_f$ for all f for which the latter equality holds.

Example 2.3. Let M be an oriented matroid realised by the real matrix $A \in \mathbb{R}^{d \times n}$ with columns $a^{(1)}, \ldots, a^{(n)}$, that is,

$$\chi(j_1, j_2, \ldots, j_d) = \operatorname{sign} \det \left(a^{(j_1)}, a^{(j_2)}, \ldots, a^{(j_d)}\right).$$

The covectors of M are precisely the sign patterns of the vectors in the row space of A.

Finally, we give the definition of pseudosphere arrangements in the statement of the Topological Representation Theorem mentioned in the introduction.

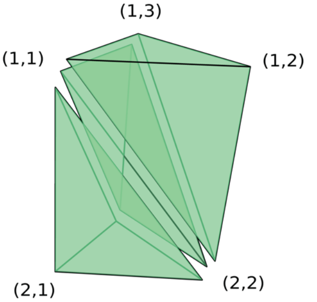


FIGURE 1 A triangulation of $\triangle_1 \times \triangle_2$. The vertices are labelled by the corresponding edges in $K_{2,3}$. This picture was created with `polymake` [14]

Definition 2.4. A *pseudosphere arrangement* of rank d is a collection $(S_e : e \in \mathsf{E})$ of $(d-2)$-spheres PL, central symmetrically embedded on S^{d-1} together with sign data, that is, for each S_e, specify a positive and a negative side for the two connected components of $S^{d-1} \setminus S_e$. Furthermore, we require that for any $\mathsf{E}' \subset \mathsf{E}$, $S_{\mathsf{E}'} := \bigcap_{e \in \mathsf{E}'} S_e$ is also a PL sphere, and that for every other S_e, either $S_{\mathsf{E}'} \subset S_e$ or $S_{\mathsf{E}'} \cap S_e$ is a PL sphere of codimension 1 within $S_{E'}$.

The face lattice of such an arrangement is isomorphic to the *covector lattice* of the oriented matroid; we again refer the reader to [6, Chapter 5] for details.

2.2 | Triangulations of $\triangle_{d-1} \times \triangle_{n-1}$ and polyhedral matching fields

We refer the reader to [10], respectively, [8, 25], for details in polyhedral geometry and matching fields. We denote the $(k-1)$-simplex, respectively, the product of a $(d-1)$-simplex and an $(n-1)$-simplex, by $\triangle_{k-1}$ and, respectively, $\triangle_{d-1} \times \triangle_{n-1}$. We fix their embeddings in $\mathbb{R}^k$ (respectively, $\mathbb{R}^d \times \mathbb{R}^n$) as $\operatorname{conv}\{\mathbf{e}_i : i \in [k]\}$ (respectively, $\operatorname{conv}\{(\mathbf{e}_i, \mathbf{e}_j) : i \in [d], j \in [n]\}$).

A collection $\mathcal{T}$ of simplices is a *triangulation* of $\triangle_{d-1} \times \triangle_{n-1}$ if

(1) the vertices of each simplex is a subset of the vertices of $\triangle_{d-1} \times \triangle_{n-1}$,
(2) for any simplex in $\mathcal{T}$, each of its faces is also in $\mathcal{T}$,
(3) the union of all simplices in $\mathcal{T}$ is $\triangle_{d-1} \times \triangle_{n-1}$,
(4) the intersection of any two simplices in $\mathcal{T}$ is a common face of them.

By identifying the vertices of $\triangle_{d-1} \times \triangle_{n-1}$ with the edges of the complete bipartite graph $K_{\mathsf{R},\mathsf{E}}$, that is, identify each vertex $(\mathbf{e}_i, \mathbf{e}_j)$ with the edge between i and j, each full-dimensional simplex satisfying (1) gives rise to a spanning tree of $K_{\mathsf{R},\mathsf{E}}$ (Figure 1). Hence we can identify $\mathcal{T}$ as a collection of spanning trees and their spanning subforests without losing information, in particular, it is enough to consider the set of spanning trees. A combinatorial characterisation of these trees is given in [2, Proposition 7.2].

Definition 2.5 [8, Section 2.3]. A *polyhedral matching field* is the collection of all R-saturating matchings (those covering all nodes in R) that are subgraphs of the spanning trees encoding a triangulation of $\triangle_{d-1} \times \triangle_{n-1}$. It was proven in [25] that the collection consists of exactly one perfect matching M_σ between R and σ for every d-subset $\sigma \subset \mathsf{E}$.

We describe another (larger) matching field induced from a triangulation, which comprises the full information of the original triangulation. We augment the ground set E by a copy $\widetilde{\mathsf{R}}$ of R to obtain a ground set $\widetilde{\mathsf{E}}$ of size $n+d$, and we set all elements of $\widetilde{\mathsf{R}}$ to be smaller than all elements of E. The collection $\widetilde{\mathcal{T}}$ contains, for every tree T in $\mathcal{T}$, the tree on $\mathsf{R} \sqcup \widetilde{\mathsf{E}}$ obtained from T by adding an edge between i and its copy for every $i \in \mathsf{R}$, together with all their spanning subforests.

Definition 2.6. The *pointed polyhedral matching field* associated with a triangulation $\mathcal{T}$ of $\triangle_{d-1} \times \triangle_{n-1}$ is the collection of R-saturating matchings on $\mathsf{R} \sqcup \widetilde{\mathsf{E}}$ that are subgraphs of the spanning trees in $\widetilde{\mathcal{T}}$.

The *Cayley trick* [32] establishes a bijective correspondence between triangulations of $\triangle_{d-1} \times \triangle_{n-1}$ and *fine mixed subdivisions* of the dilated simplex $n\triangle_{d-1}$ as follows: For each spanning tree G in the triangulation $\mathcal{T}$, we form the Minkowski sum

$$\sum_{j \in \mathsf{E}} \operatorname{conv}\{\mathbf{e}_i : i \in \mathcal{N}_G(j)\},$$

where $\mathcal{N}_G(j)$ is the neighbourhood of an element $j \in \mathsf{E}$ in G. The collection of these Minkowski sums tiles $n\triangle_{d-1}$. We denote by S the collection of these cells and their faces (which are also cells); all these cells correspond to spanning forests of $K_{\mathsf{R},\mathsf{E}}$ without isolated nodes in E.

In [3], Ardila and Develin studied the dual of these mixed subdivisions as *tropical pseudohyperplane arrangements*, which generalises tropical hyperplane arrangements as the dual of coherent mixed subdivisions [12]. In [19, 28], it was shown that these objects are equivalent to *tropical oriented matroids*, defined by purely combinatorial axioms back in [3].

2.3 | Oriented matroids from triangulations of $\triangle_{d-1} \times \triangle_{n-1}$

We recall the results of [8] which are needed in this paper. For the rest of this paper, we fix a sign matrix $A \in \{-,+\}^{\mathsf{R}\times\mathsf{E}}$ and a polyhedral matching field (M_σ) extracted from a triangulation of $\triangle_{d-1} \times \triangle_{n-1}$. We also denote by $\widetilde{(M_\sigma)}$ the pointed polyhedral matching field encoding the starting triangulation, and by $\widetilde{A}$ the sign matrix $(I_{d,d}|A)$.

Using the ordering on R and $\sigma \subset \mathsf{E}$, we can interpret a matching M_σ as a permutation, and we define the sign of the matching by the sign of the permutation.

Theorem 2.7 [8, Theorem A]. *The sign map* $\chi : \binom{\mathsf{E}}{d} \to \{+,-\}$ *given by*

$$\sigma \mapsto \operatorname{sign}(M_\sigma) \prod_{Edge(i,j) \in M_\sigma} A_{ij}, \tag{2.1}$$

is the chirotope of an oriented matroid.

We denote the oriented matroid described by χ as $\mathcal{M}$. Similarly, $\widetilde{(M_\sigma)}$ induces an oriented matroid $\widetilde{\mathcal{M}}$ on $\widetilde{\mathsf{E}}$.

Now, we describe how to convert cells of a fine mixed subdivision of $n\triangle_{d-1}$, as special subgraphs of $K_{\mathsf{R},\mathsf{E}}$, into a covector of $\mathcal{M}$ (respectively, $\widetilde{\mathcal{M}}$).

Definition 2.8 [8, Definition 3.26]. Given $S \in \{-1,0,1\}^{\mathsf{R}}$ and $F \subseteq \mathsf{R} \times \mathsf{E}$, the sign matrix $SA_F \in \{-1,0,1\}^{\mathsf{R}\times\mathsf{E}}$ is defined as

$$(SA_F)_{i,j} = \begin{cases} S_i A_{i,j}, & (i,j) \in F, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.9 [8, Definition 3.27]. Given a subgraph F of $K_{\mathsf{R},\mathsf{E}}$ and a sign vector $S \in \{-1,0,1\}^{\mathsf{R}}$, the sign vector $\psi_A(S,F) = Z \in \{-1,0,1\}^{\mathsf{E}}$ is given by

$$Z_j = \begin{cases} 0, & \text{column } j \text{ of } SA_F \text{ contains both positive and negative entries, or all zeros} \\ 1, & \text{column } j \text{ of } SA_F \text{ is non-zero and contains only non-negative entries} \\ -1, & \text{column } j \text{ of } SA_F \text{ is non-zero and contains only non-positive entries.} \end{cases}$$

Proposition 2.10 [8, Proposition 3.32]. *Let F be a subgraph of a tree in $\widetilde{\mathcal{T}}$ without isolated nodes in $\mathsf{E} \subset \widetilde{\mathsf{E}}$, and such that a node in $\widetilde{\mathsf{R}} \subset \widetilde{\mathsf{E}}$ is isolated only if the corresponding node in R is isolated as well. Let $S \in \{-1,0,1\}^{\mathsf{R}}$ be a sign vector whose support contains the set of non-isolated nodes of F in R.*

Then the sign vector $\psi_{\widetilde{A}}(S,F)$ is a covector of $\widetilde{\mathcal{M}}$.

Corollary 2.11 [8, Corollary 3.33]. *For a subgraph F of a tree in $\mathcal{T}$ with no isolated node in E, the sign vector $\psi_A(S,F)$ is a covector of $\mathcal{M}$ for every sign vector S.*

Note that the latter subgraphs are called *covector pd-graphs* in [8].

3 | PATCHWORKING PSEUDOSPHERE ARRANGEMENTS

We start by giving an explicit example of a pseudoline arrangement arising from a planar fine mixed subdivision to motivate the geometric constructions used in Theorem A. This leads to an overview of its proof. We finish by relating our work with *tropical oriented matroids* and *real Bergman fans*.

3.1 | Patchworking pseudolines on an example

The classical theory of patchworking states that the structure of the real zero set of a polynomial in one orthant, parameterised by $t > 0$, is captured for sufficiently small t by the regular triangulation of its Newton polytope induced by the exponents of t. Hence, one can recover the structure of the real zero set by gluing the triangulations for all orthants. This uses an appropriate assignment of signs to the vertices of the Newton polytope. By considering coherent fine mixed subdivisions, see Section 2.2, this was extended to complete intersections in [33].

We use patchworking of not-necessarily coherent fine mixed subdivisions of $n\triangle_{d-1}$ to derive a representation theorem for the oriented matroids induced from polyhedral matching fields. This can be seen as a generalisation of the linear case of [33, Theorem 4] for generic hyperplane arrangements. While a complete intersection for generic hyperplanes would only yield one specific cell, the oriented matroid captures the information of all intersections in a generic hyperplane arrangement.

Example 3.1 is a toy example that illustrates our construction, which is generalised to larger E and higher rank in this section.

Example 3.1. We start with the regular triangulation of $\triangle_2 \times \triangle_2$ induced by the height matrix (see [23, § 1.2] for the general construction of a regular subdivision)

$$H = \begin{pmatrix} 0 & 3 & 2 \\ 0 & 0 & 0 \\ 1 & 3 & 0 \end{pmatrix}.$$

This gives rise to the following height function on the lattice points of $3\triangle_2$ (where the first component denotes the lattice point and the second component the function value):

$$(300;5),(201;6),(210;5),(102;6),(111;5),(120;3),(003;4),(012;4),(021;3),(030;0).$$

Here, the height of the lattice point $p_1p_2p_3$ is the weight of the maximal matching on $K_{3,3}$ for which the weight function is obtained from H by taking p_ℓ copies of the ℓth row of H. For example, the height 6 of the lattice point 201 is given by the matching with weights

$$\begin{pmatrix} 0 & 3 & 2 \\ 0 & 3 & 2 \\ 1 & 3 & 0 \end{pmatrix}.$$

Note that, alternatively, the latter height function of the mixed subdivision can be obtained by multiplying the max-tropical linear polynomials $(x_0 \oplus 3 \odot x_1 \oplus 3 \odot x_2) \odot (x_0 \oplus x_1 \oplus 2 \odot x_2) \odot (x_0 \oplus 1 \odot x_1 \oplus x_2)$. We refer the reader further interested in this connection to [23].

Additionally, we equip the subdivision by the sign matrix

$$\begin{pmatrix} - & + & - \\ + & + & - \\ - & - & - \end{pmatrix}.$$

The fine mixed subdivision of $3\triangle_2$ induced by H is shown in the upper-right quartile of Figure 2. The cells are labelled by their Minkowski summands (cf. the Cayley trick in Section 2.2) as follows. The elements of E (as the three columns from left to right) are represented by the red, green and blue simplices, respectively; the elements of R (as the three rows from top to bottom) are represented by the top, lower-left and right vertices of each simplex, respectively. Furthermore, the vertices of these simplices are labelled with signs coming from the sign matrix.

The faces of a top-dimensional mixed cell correspond to the subgraphs of its spanning tree without isolated nodes in E. In particular, a vertex of a mixed cell can be specified by choosing a vertex from each coloured simplex; thus, it encodes a sign vector $\{+,-\}^{\mathsf{E}}$ from the corresponding entries in the sign matrix. Such a forest associated with a vertex is independent of the mixed cell containing it. Hence, we have a well-defined assignment of sign vectors to the vertices of the mixed subdivision. For example, the lower left vertex v of the square in the upper-right quartile

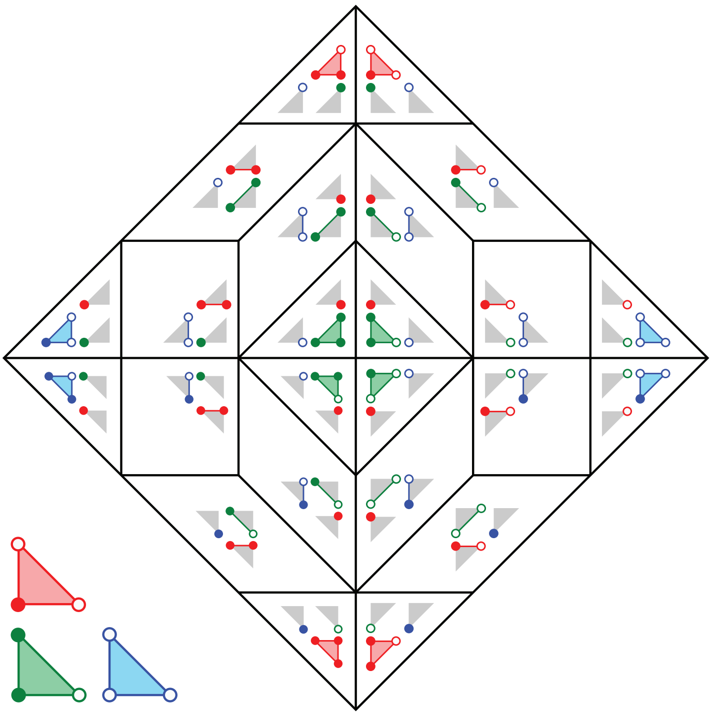


FIGURE 2 Fine mixed subdivision of $3\triangle_2$ with cells labelled by their summands and sign; filled vertices denote '+', empty ones '−'

of Figure 2 is the Minkowski sum of a filled red vertex, an empty green and an empty blue vertex. Therefore, it encodes the sign vector $(+,-,-)$.

Next we reflect the dilated simplex across the coordinate hyperplanes in $\mathbb{R}^3$ so that there is a copy in every octant†. We keep the same subdivision in all copies and label the vertices of these copies with sign vectors similar to the above, but instead of the original sign matrix, we negate a row of it if the corresponding coordinate in the octant is negative. For the vertex v, for example, this yields $(+,+,-)$ for its reflection in the upper-left quartile of Figure 2. Again, the sign vector assigned to a vertex that appears in multiple copies of the dilated simplex is independent of the copy chosen: whenever a vertex lies on a hyperplane $\{x_i = 0\}$, the ith node of R must be an isolated one in the forest corresponding to the vertex; thus, the negation of the ith row does not affect the sign vector.

As our example is of rank 3, we obtain a subdivision of the boundary of a dilated octahedron (which is PL homeomorphic to S^2), with vertices of the subdivision labelled by sign vectors.

Finally, we define a 'zero locus' for each element $e \in \mathsf{E}$ as a subset of the patchworking complex. This zero locus is dual to the cells which have a Minkowski summand with vertices of different sign. Given a cell of the subdivision, select the edges (one-dimensional faces) of the cell in which the sign vectors of their endpoints disagree on the eth coordinate, and take the convex hull of the midpoints of them. Take the union of all such convex hulls, it can be seen from Figure 3 that each of such 'zero loci' is a pseudosphere on the patchworking complex.

† We only show the upper half of that *patchworking complex* in Figure 2 as the construction is centrally symmetric.

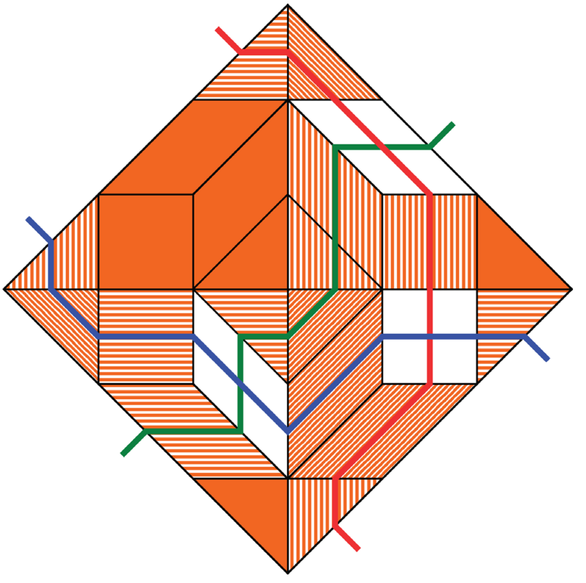


FIGURE 3 Pseudohyperplane arrangement derived from a fine mixed subdivisions of $3\triangle_2$ with signs indicated in Figure 2

Note that the boundary of $3\triangle_2$ in $\mathbb{R}^3$ can be seen as the intersection with the three hyperplanes bounding the non-negative orthant. Extending these through the reflections of $3\triangle_2$ yields three further pseudospheres. This gives rise to an interpretation of Figure 3 as an arrangement of six pseudospheres. By 'fattening' the latter three coordinate pseudospheres we arrive at the *extended patchworking complex* introduced in the next section.

Remark 3.2. Since oriented matroids coming from regular triangulations are all realisable, the pseudosphere arrangements constructed from a coherent fine mixed subdivision are all stretchable, which gives some non-trivial structural constraints on coherence.

We recall the example from [8] that treats Ringel's non-realisable uniform oriented matroid **R** of rank 3 on 9 elements. It can be realised by patchworking a suitable non-coherent fine mixed subdivision of $6\triangle_2$ and choosing appropriate signs as depicted in Figure 4. It is not clear to the authors, if the oriented matroid **R** can be constructed from another non-coherent fine mixed subdivision of $6\triangle_2$.

It is an interesting experimental question of which of the 24 (non-isomorphic) non-realisable oriented matroids of rank 4 on 8 elements arise from the non-regular triangulation constructed by de Loera in [9] or its modifications by choosing appropriate signs.

Remark 3.3. It is also an interesting problem to interpret our construction here as a limit with respect to some one-dimensional family of (meaningful) geometric objects; the regular (and non-singular) case is closely related to the theory of amoebas [26]. The rank 3 case is worth to put emphasis on, not only because it is already combinatorially rich enough, but the work of Ruberman–Starkton indicates that every pseudoline arrangement can be complexified into an arrangement of symplectic spheres [31], hence suggesting a symplectic flavoured answer here (see also the aforementioned work of Itenberg–Shustin [21]).

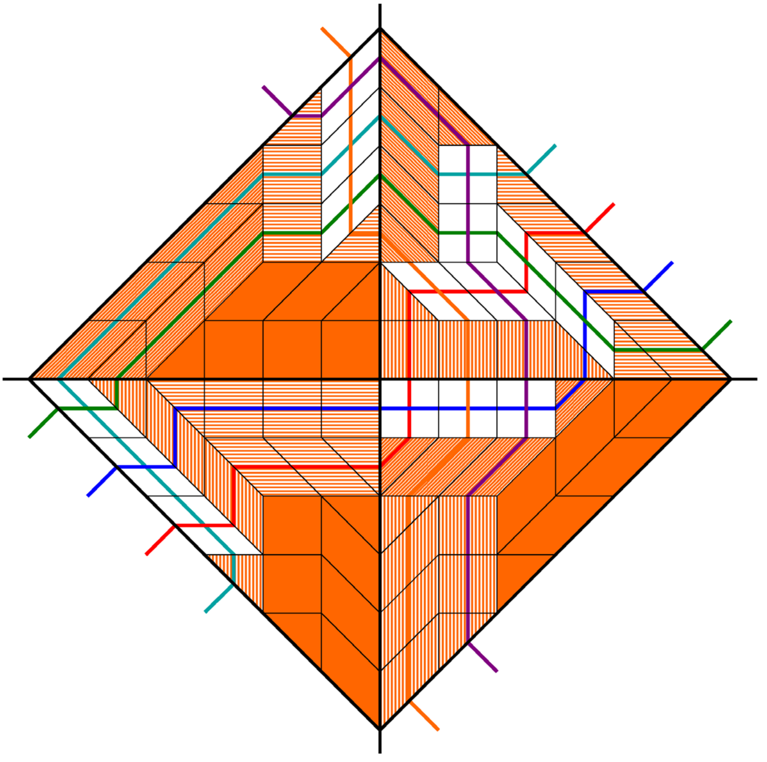


FIGURE 4 Non-coherent fine mixed subdivision of $6\Delta_2$ patchworking the Ringel arrangement discussed in Remark 3.2

3.2 | From fine mixed subdivisions to pseudosphere arrangements

We now state precisely our method for constructing a pseudosphere arrangement representing an oriented matroid associated to a polyhedral matching field. For this, we fix a fine mixed subdivision S of $n\triangle_{d-1}$. By the Cayley trick, this corresponds to a triangulation $\mathcal{T}$ of $\triangle_{d-1} \times \triangle_{n-1}$. By means of Definition 2.6, it gives rise to a pointed polyhedral matching field $(\widetilde{M_\sigma})$ on $\mathsf{R} \sqcup \widetilde{\mathsf{E}}$ with $\widetilde{\mathsf{E}} = \widetilde{\mathsf{R}} \cup \mathsf{E}$. For an arbitrary matrix $A \in \{+,-\}^{\mathsf{R}\times\mathsf{E}}$, we consider the augmented matrix $\widetilde{A} = (I_{\mathsf{R}} \mid A)$ as a sign matrix for $(\widetilde{M_\sigma})$. Let $\widetilde{\mathcal{M}}$ denote the oriented matroid on $\widetilde{\mathsf{E}} = \widetilde{\mathsf{R}} \cup \mathsf{E}$ associated to the pointed polyhedral matching field $(\widetilde{M_\sigma})$ with the sign matrix $\widetilde{A}$, and let $\mathcal{M}$ be its restriction to E.

Recall from Section 2.2 that we may identify the simplices in $\mathcal{T}$ with forests of $K_{\mathsf{R},\mathsf{E}}$. The cells σ_F of S are then in 1-1 correspondence with the forests F contained in a spanning tree of $\mathcal{T}$ for which $\deg_F(j) \geqslant 1$ for all $j \in \mathsf{E}$.

We denote the cube $[-1,1]^d \subset \mathbb{R}^d$ and its polar dual, the crosspolytope, by $\square_d$ and $\diamondsuit_d$, respectively. For a sign vector $S \in \{-1,0,1\}^d$ and a set K contained in the coordinate subspace $\mathbb{R}^{\mathrm{supp}(S)} \times \{0\}^{\mathrm{supp}(S)^c}$ of $\mathbb{R}^d$, define

$$S \cdot K := \left\{(S_1x_1, \dots, S_dx_d) \in \mathbb{R}^d \,:\, (x_1, \dots, x_d) \in K\right\}$$

$$\square_S := \left\{x \in \square_d \,:\, x_i = S_i \text{ for all } i \in \mathrm{supp}(S)\right\}.$$

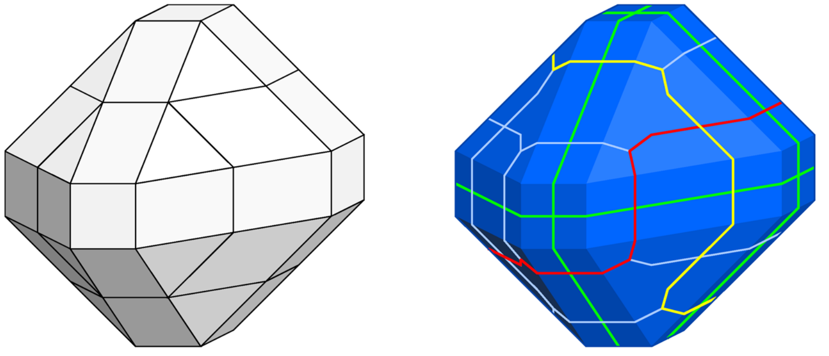


FIGURE 5 The complex $\octagon_{\mathsf{S}}$ (left) and its dual $\Delta = \octagon_{\mathsf{S}}^{\vee}$ (right). The five subcomplexes of Δ that yield pseudospheres are highlighted: there are three of the form Δ_i, $i \in \mathsf{R}$ (shown in green) and two of the form Δ_j, $j \in \mathsf{E}$ (shown in red and yellow)

Hence, $S \cdot K$ denotes the reflections of K to the orthant indicated by S, and $\square_S$ comprises the sign patterns of orthants containing the sign vector S. For a subgraph F of $K_{\mathsf{R},\mathsf{E}}$, let $\mathrm{supp}_{\mathsf{R}}(F) := \{i \in \mathsf{R} : \deg_F(i) \geqslant 1\}$. This set encodes the unique minimal face of $n\triangle_{d-1}$ containing F.

Proposition 3.4. *The subdivision* $\mathbb{S}$ *of* $n\triangle_{d-1}$ *gives rise to the subdivision*

$$\octagon_{\mathsf{S}} := \left\{\sigma_{(S,F)} : \sigma_F \in \mathbb{S},\ S \in \{-1,0,1\}^{\mathsf{E}},\ \mathrm{supp}(S) \supseteq \mathrm{supp}_{\mathsf{R}}(F)\right\}$$

of the boundary of $\octagon_d := \square_d + n\diamond_d$, *where* $\sigma_{(S,F)} := \square_S + S \cdot \sigma_F$.

We call the complex arising in the latter Proposition the *extended patchworking complex*; we prove the statement together with more technical properties of the extended patchworking complex in Section 6.1. This complex is analogous to the complex Δ'_{ω} defined in [33, Theorem 5], with additional cells that are dual to the coordinate hyperplanes of $\mathbb{R}^d$. The extended patchworking complex subdivides the boundary of a polytope, and is therefore a PL sphere. Hence, we may consider its dual complex (cf. Proposition 4.17)

$$\Delta := \octagon_{\mathsf{S}}^{\vee} := \{\sigma^{\vee}_{(S,F)} : \sigma_{(S,F)} \in \octagon_{\mathsf{S}}\}.$$

The realisation of the poset as a polyhedral cell complex is further explained in Section 4.2. For $i \in \widetilde{\mathsf{R}}$ and $j \in \mathsf{E}$, define the subcomplexes

$$\Delta_i := \{\sigma^{\vee}_{(S,F)} \in \Delta : i \notin \mathrm{supp}(S)\}, \tag{3.1}$$

$$\Delta_j := \{\sigma^{\vee}_{(S,F)} \in \Delta : \text{there exist edges } (i,j),(\ell,j) \text{ in } F \text{ such that } S_i A_{i,j} = -S_\ell A_{\ell,j} \neq 0\}. \tag{3.2}$$

This is depicted in Figure 5. Recall the notion of a pseudosphere arrangement from Definition 2.4.

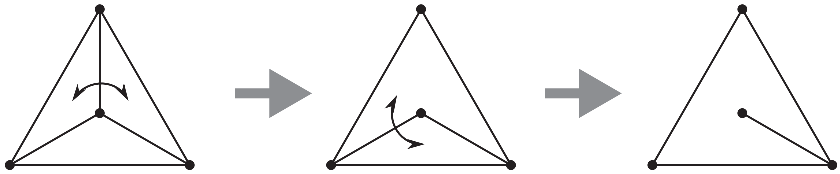


FIGURE 6 Two cell-merging steps of a planar embedding of the complete graph K_4 into the plane. The first merging step results in a regular CW complex; however, the second does not

Theorem 3.5. *The spaces $\|\Delta_k\|$ ranging over all $k \in \widetilde{\mathsf{E}}$ form an arrangement of pseudospheres within $\|\Delta\|$ representing the oriented matroid $\widetilde{\mathcal{M}}$.*

Deleting the pseudospheres $\|\Delta_i\|$ for $i \in \widetilde{\mathsf{R}}$ yields the following.

Corollary 3.6. *The spaces $\|\Delta_j\|$ ranging over all $j \in \mathsf{E}$ form an arrangement of pseudospheres representing the oriented matroid $\mathcal{M}$.*

Note that we intentionally include the additional hyperplanes which correspond to the boundary of $n\triangle_{d-1}$ as an analogue of coordinate hyperplanes even if it might seem cumbersome at first glance. This allows to capture the full information of the fine mixed subdivision $n\triangle_{d-1}$ as only the extended pointed matching field encodes the triangulation $\triangle_{d-1} \times \triangle_{n-1}$. Furthermore, it simplifies the transition between orthants as this boundary then has a natural interpretation in terms of a separating pseudohyperplane.

3.3 | Overview of the Proof of Theorem 3.5

Before getting into the technical details of our proof, we explain the overall picture. Using the results from [8] (as summarised in Section 2.3 here), we show that the face poset of the arrangement of functions of Δ_k within $\|\Delta\|$ is precisely captured by the covector lattice of $\widetilde{\mathcal{M}}$. This arrangement forms a subcomplex of the codimension one skeleton of Δ, and hence, we would like to remove all facets from Δ which do not participate in this subcomplex. Our approach is to formulate this process as a stepwise cell merging process, using the formalism of regular cell complexes. The removal of a facet determines an equivalence relation on the cells of the complex: two cells are equivalent if their interiors intersect the interior of a common cell once the facet is removed. By taking the union of the cells in each equivalence class, we show that we get another regular cell complex with the same underlying topological space. Iterating this procedure, we show that we end up at a regular cell complex. Since the face poset of a regular cell complex determines the complex up to cellular homeomorphism, this completes the proof. Our main challenge here is to make sure that the facet removal process does not create pathologies, so the topological structure reflects the combinatorial structure. Figure 6 depicts a two-dimensional example where naïve cell merging does not preserve regularity, hence justifying the technical work here.

Hersh describes a similar step-by-step process in [18, § 4] to simplify a regular cell complex while preserving homeomorphism type. Her single step involves collapsing a single cell σ to a cell γ on its boundary and then filling up the 'hole' by stretching the neighbourhood of σ; from a poset point of view, the step identifies every element τ of $\overline{\sigma} \subset \mathsf{P}$ to an element in $\overline{\gamma} \subsetneq \overline{\sigma}$ whose rank is no larger than that of τ. In contrast, our step involves merging two neighbouring cells σ, τ along a facet γ, that is, identifying γ and τ with σ, whose rank is at least those of γ, τ. Hence, our step is conceptually dual to hers as an operation to reduce the number of elements in P. Nevertheless, as the two operations are not formally dual to each other and they are based on completely different topological lemmas, mathematically our method is not a corollary of her method nor the other way around.

Summarising, we obtain a combinatorial proof of this topological representation of the oriented matroid arising from a fine mixed subdivision of $n\triangle_{d-1}$.

3.4 | Relation to tropical pseudohyperplane arrangements

The codimension one skeleton of Δ in each orthant is a tropical pseudohyperplane arrangement in the sense of [3]. Including the sign data, each Δ_k restricted to an orthant is either empty or is (the boundary of) a *tropical (pseudo)halfspace* in the sense of [22]. This allows us to use the intuition from Horn's work on tropical pseudohyperplane arrangements, in particular [19, Chapter 6]. In this work, she establishes the equivalence between (d, n)-tropical oriented matroids as defined in [3], and polyhedral subdivisions of $\triangle_{d-1} \times \triangle_{n-1}$. This result generalises the coherent case in [3] and the generic case in [28].

In order to establish this correspondence, Horn defines the notion of an *arrangement of tropical pseudohyperplanes* topologically. A single tropical pseudohyperplane is defined by a PL-embedding of the normal fan of the standard simplex $\triangle_{d-1}$ into the tropical torus $\mathbb{R}^d/\mathbb{R} \cdot \mathbf{1}$. This embedding divides the tropical torus into d open regions, called *sectors*. If these sectors are further partitioned into positive and negative sectors, then the closure of the positive sectors meets the closure of the negative sectors at a topological hyperplane, or an *affine pseudohyperplane*. Now, given a collection of n tropical pseudohyperplanes, a $d \times n$ sign matrix A assigns a sign to each sector cut out by each tropical pseudohyperplane, and hence determines an collection of affine pseudohyperplanes. Horn's point of view is that a collection of tropical pseudohyperplanes meets the definition of an arrangement if and only if, for each sign matrix A, the corresponding collection of affine pseudohyperplanes satisfy certain topological axioms which are analogous to those found in Definition 2.4 defining an arrangement of pseudospheres.

Our contribution in relation to this work is as follows: a single $d \times n$ sign matrix A gives rise to the 2^d sign matrices obtained by multiplying subsets of the rows by -1. We implicitly show that the resulting 2^d affine pseudohyperplane arrangements fit together to form a classical pseudosphere arrangement representing the oriented matroid of Theorem 2.7. To relate the structure of the patchworking complex directly with the structure of the oriented matroid, we only use the local properties of a fine mixed subdivision from [19, Chapter 6] in our proof of Theorem 3.5. In particular, we show how the *elimination axiom* of tropical oriented matroids enables our cell merging process to work by making use of the correspondence between fine mixed subdivisions and generic tropical oriented matroids.

3.5 | Relation to real Bergman fan and complex

To close this section, we sketch the relation of our construction with the *real Bergman fan* of the oriented matroid, considered by the first author in [7, Chapter 2]. The real Bergman fan generalises to oriented matroids the more well-known *Bergman fan* of a matroid, which is itself the tropical analogue of a linear subspace. In particular, the Bergman fan of a matroid is the union of cones taken over all flags of flats, whereas the real Bergman fan is the union of cones over all flags of *conformal covectors*:

Definition 3.7. Let $\mathcal{L}$ be the collection of non-zero covectors of an oriented matroid $\mathcal{M}$. Identify each sign vector $X \in \mathcal{L}$ with its associated lattice point $\mathbf{e}_X \in [-1,1]^E \cap \mathbb{Z}^E$. Then the *real Bergman fan* $\Sigma^*_{\mathcal{M}}$ of $\mathcal{M}$ is the collection of all cones of the form

$$\text{cone}\{\mathbf{e}_{X_1}, \dots, \mathbf{e}_{X_k}\} \subset \mathbb{R}^E,$$

where $X_1 < \dots < X_k$ and each $X_i \in \mathcal{L}$. The *real Bergman complex* $\Delta^*_{\mathcal{M}}$ of $\mathcal{M}$ is the intersection of $\Sigma^*_{\mathcal{M}}$ with the boundary of the hypercube $[-1,1]^E$.

We note that, after taking the component-wise logarithm, the real Bergman fan (respectively, complex) restricted to the positive orthant coincides with the positive Bergman fan (respectively, complex) considered by Ardila, Klivans and Williams [4]. Conversely, the real Bergman fan can be recovered from the positive Bergman fans of all reorientations of $\mathcal{M}$. Such fans were used in the realisable setting by Jürgens in [24]. See [7, Chapter 2.4] for further combinatorial properties of the real Bergman fan.

The complex $\Delta^*_{\mathcal{M}}$ is a geometric realisation of the order complex of $\mathcal{L}$. Hence, a direct consequence of the Topological Representation Theorem is that this complex is PL homeomorphic to a sphere of dimension $d-1$, and its intersections with the coordinate hyperplanes of $\mathbb{R}^E$ are the pseudospheres representing the elements. It is therefore natural to ask if there is a PL map from the extended patchworking complex $\Delta = \bigcirc^\vee_{\mathrm{S}}$ defined in Section 3.2 to the real Bergman complex of the associated oriented matroid $\widetilde{\mathcal{M}}$, one which respects the pseudosphere arrangement structure. This can indeed be carried out; we omit the details as they are routine.

Proposition 3.8. *Define the following map on the vertices $\sigma_{(S,F)}$ of $\bigcirc_{\mathrm{S}}$ into $\mathbb{R}^{\widetilde{\mathrm{E}}}$:*

$$\sigma_{(S,F)} \mapsto \left(S, \mathbf{1}^\top(SA_F)\right) \in \mathbb{R}^{\widetilde{\mathrm{E}}}.$$

Here SA_F is the matrix as in Definition 2.8, and $\mathbf{1}$ denotes the vector of all ones. Extend this map linearly on each maximal cell of $\bigcirc_{\mathrm{S}}$, to get a map

$$\|\Delta\| = \|\bigcirc_{\mathrm{S}}\| \to \mathbb{R}^{\widetilde{\mathrm{E}}}.$$

Then this map is well defined, and the image of this map is precisely $\Delta^*_{\widetilde{\mathcal{M}}}$.

Furthermore, the choices implicit in the construction of $\bigcirc^\vee_{\mathrm{S}}$ can be made so that this map respects the cellular structure of the pseudosphere arrangement $\|\Delta_k\|$ over all $k \in \widetilde{\mathrm{E}}$ as given by Theorem 3.5, and the pseudosphere arrangement obtained by intersecting $\Delta^*_{\widetilde{\mathcal{M}}}$ with each of coordinate hyperplane of $\mathbb{R}^{\widetilde{\mathrm{E}}}$.

Example 3.9. In Figure 4, the map can be visualised as contracting each shaded cell to a point, and each striped cell into a segment by contracting each stripe to a point on that segment.

4 | QUOTIENTS OF REGULAR CELL COMPLEXES

This section is devoted to the understanding of cell merging in regular cell complexes leading to Theorem B.

4.1 | Poset and lattice quotients

The following definition is due to Hallam and Sagan [16], which proved useful in their work on factorising characteristic polynomials of lattices. This definition turns out to be the right one for us as well.

Definition 4.1. Let P be a finite poset. An equivalence relation $\sim$ on the ground set of P is P-*homogeneous,* provided that the following condition holds: if $\tau \leqslant \sigma$ in P, then for every $u \in \tilde{\tau}$ there exists $v \in \tilde{\sigma}$ such that $u \leqslant v$ in P. We denote by either $\tilde{\sigma}$ or $\sigma/\sim$ the equivalence class of σ in $\sim$.

Proposition 4.2 [16, Lemma 5]. *Suppose that $\sim$ is P-homogeneous. Then we have a well-defined poset $\mathrm{P}/\sim$ on the classes of $\sim$ defined as follows: $\tilde{\tau} \leqslant \tilde{\sigma}$ in $\mathrm{P}/\sim$ if and only if there exists $u \in \tilde{\tau}$ and $v \in \tilde{\sigma}$ such that $u \leqslant v$ in P. Equivalently, for every $u \in \tilde{\tau}$ there exists $v \in \tilde{\sigma}$ such that $u \leqslant v$ in P.*

We call the poset $\mathrm{P}/\sim$ the *homogeneous quotient of* P *by* $\sim$. We are particularly interested in homogeneous quotients which have nice factorisations in the following sense.

Definition 4.3. A homogeneous quotient $\mathrm{P}/\sim$ is an *elementary quotient* if every equivalence class of $\sim$ is either a singleton, or consists of exactly three elements $\sigma, \tau, \gamma \in \mathrm{P}$ such that σ and τ both cover γ in P.

Definition 4.4. We say that $\mathrm{P}/\sim$ *admits a factorisation into elementary quotients* if there exist posets $\mathrm{P} = \mathrm{P}_0, \mathrm{P}_1, \dots, \mathrm{P}_k = \mathrm{P}/\sim$ such that $\mathrm{P}_i = \mathrm{P}_{i-1}/\sim_i$ is an elementary quotient of P_{i-1} for all $i = 1, 2, \dots, k$.

Since the following notion appears several times in this paper, we give the definition here.

Definition 4.5. The *augmented poset* of a poset P is the poset $\mathcal{L}(\mathrm{P}) := \mathrm{P} \cup \{\hat{\mathbf{0}}, \hat{\mathbf{1}}\}$, where $\hat{\mathbf{0}}$ and $\hat{\mathbf{1}}$ are two additional elements such that $\hat{\mathbf{0}} < \sigma < \hat{\mathbf{1}}$ for all $\sigma \in \mathrm{P}$.

4.2 | Background: Regular cell complexes and PL topology

We quickly review the key aspects of combinatorial topology we wish to use. The main reference here is [6, Section 4.7].

4.2.1 | Regular cell complexes

Definition 4.6. A *regular cell complex* Δ is a Hausdorff space $\|\Delta\|$ together with a finite collection of balls Δ such that:

(i) the interiors of the balls in Δ partition the space: $\|\Delta\| = \bigcup_{\sigma \in \Delta} \sigma^\circ$;
(ii) the boundary of any $\sigma \in \Delta$ is a union of members of Δ: $\mathrm{bd}(\sigma) = \bigcup_{\tau \subset \sigma} \tau$.

Definition 4.7. An important special case of the above definition is a *polyhedral cell complex.* This is a regular cell complex Δ such that each $\sigma \in \Delta$ is a polytope in $\mathbb{R}^d$, and for each $\sigma, \tau \in \Delta$ we have

$\sigma \cap \tau$ is a face of both σ and τ. If every polytope in Δ is a simplex, we call Δ a *geometric simplicial complex*. A *triangulation* of a set $Q \subset \mathbb{R}^d$ is a geometric simplicial complex with underlying space Q.

Definition 4.8. The *face poset* $\mathrm{P}(\Delta)$ of a regular cell complex Δ is the poset whose underlying set is the set of balls Δ, and whose ordering is given by inclusion.

Definition 4.9. The *order complex* $\Delta(\mathrm{P})$ of a poset P is the simplicial complex whose vertices are the elements of P and whose simplices are the chains of P. We denote by $\|\mathrm{P}\|$ the topological space $\|\Delta(\mathrm{P})\|$.

Proposition 4.10. *Every abstract simplicial complex (i.e. set system closed under taking subsets) can be realised as a geometric simplicial complex in some Euclidean space.*

4.2.2 | PL balls and spheres

Definition 4.11. Given $P \subset \mathbb{R}^k$, $Q \subset \mathbb{R}^\ell$, a map $f : P \to Q$ is PL if there is a triangulation Δ of P into simplices such that f restricted to each simplex of Δ is an affine function. That is, if $\sigma = \mathrm{conv}(v_0, \ldots, v_k) \in \Delta$, then $f|_\sigma$ satisfies

$$f(\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_k v_k) = \lambda_0 f(v_0) + \lambda_1 f(v_1) + \cdots + \lambda_k f(v_k)$$

for all convex combinations $\sum_i \lambda_i v_i$ of the vertices $v_0, \ldots, v_k$ of σ. We call a PL map that is also a homeomorphism a *PL homeomorphism*.

Definition 4.12. Let $P \subset \mathbb{R}^k$ be the underlying space of a polyhedral cell complex. Then P is a *PL d-sphere* (respectively, *PL d-ball*) if there is a PL homeomorphism from P to the boundary of the standard d-simplex (respectively, to the standard d-simplex).

Proposition 4.13.

(i) [6, *Theorem* 4.7.21(i)] *The union of two PL d-balls, whose intersection is a PL $(d-1)$-ball lying in the boundary of each, is a PL d-ball.*
(ii) [6, *Theorem* 4.7.21(ii)] *The union of two PL d-balls, which intersect along their entire boundaries, is a PL d-sphere.*
(iii) [6, *Theorem* 4.7.21(iii)] *(Newman's Theorem) The closure of the complement of a PL d-ball embedded in a PL d-sphere is itself a PL d-ball.*

Lemma 4.14. *Let σ, τ be two PL d-balls, such that $\sigma \cap \tau$ is a PL $(d-1)$-ball contained in the boundaries of both σ and τ. Then the interior of $\sigma \cup \tau$ is equal to $\sigma^\circ \cup \tau^\circ \cup (\sigma \cap \tau)^\circ$.*

Proof. By Proposition 4.13 (i), $\sigma \cup \tau$ is a PL d-ball. Since $(\sigma \cup \tau)^\circ \backslash \tau \subset \sigma$ and that $(\sigma \cup \tau)^\circ \backslash \tau$ is open, we conclude $(\sigma \cup \tau)^\circ \backslash \tau \subset \sigma^\circ$.

From this containment we immediately get

$$\partial\sigma \subseteq \sigma \backslash ((\sigma \cup \tau)^\circ \backslash \tau) = (\sigma \cap \tau) \cup (\sigma \cap \partial(\sigma \cup \tau)),$$

and in particular

$$U := \partial\sigma \backslash (\sigma \cap \tau) \subseteq \partial(\sigma \cup \tau).$$

Since boundaries are closed, $V := \partial(\sigma \cup \tau) \cap \partial\sigma$ is closed inside $\partial\sigma$. Now $W := \partial(\sigma \cap \tau)$ is the boundary of U in $\partial\sigma$; thus, it is contained in $\overline{U} \subset V \subset \partial(\sigma \cup \tau)$. Similarly, $U' := \partial\tau \setminus (\sigma \cap \tau) \subset \partial(\sigma \cup \tau)$. By Proposition 4.13 (iii) and the assumption that $\sigma \cap \tau$ is a PL $(d-1)$-ball, both $U \cup W, U' \cup W$ are PL $(d-1)$-balls with common boundary W, so by Proposition 4.13 (ii), $U \cup W \cup U'$ is a PL $(d-1)$-sphere contained in $\partial(\sigma \cup \tau)$. Invariance of domain implies that the containment is an equality, see, for example, [17, Corollary 2B.4]. After taking the complement with respect to $\sigma \cup \tau$, this equality yields an expression for $(\sigma \cup \tau)^\circ$ which simplifies to $\sigma^\circ \cup \tau^\circ \cup (\sigma \cap \tau)^\circ$. □

4.2.3 | Regular cell complexes that are PL spheres

Definition 4.15. We say that a regular cell complex Δ with face poset P is a *PL sphere* if some realisation of the order complex $\Delta(\mathrm{P})$ in some Euclidean space is a PL sphere.

Proposition 4.16 [6, Proposition 4.7.26(iii)]. *Let Δ be a regular cell complex that is a PL sphere. Then every $\sigma \in \Delta$ is a PL ball.*

An important fact about PL spheres is that they admit a dual cell structure:

Proposition 4.17 [6, Proposition 4.7.26(iv)]. *Let Δ be a regular cell complex that is a PL sphere. Then there exists a regular cell complex $\Delta^\vee$, also a PL sphere, such that $\|\Delta\| = \|\Delta^\vee\|$ and $\mathrm{P}(\Delta^\vee) \simeq \mathrm{P}(\Delta)^\vee$.*

Here $\mathrm{P}^\vee$ denotes the dual poset of P. In the special case when Δ is a polyhedral cell complex, there is a non-canonical way to construct this $\Delta^\vee$:

Definition 4.18. Let Δ be a polyhedral cell complex. A *first derived subdivision* Δ^1 is a subdivision of Δ obtained as follows: choose a point x_σ in the relative interior of each $\sigma \in \Delta$. Then, Δ^1 is given by

$$\Delta^1 := \left\{\mathrm{conv}(x_{\sigma_1}, \dots, x_{\sigma_k}) : \sigma_1 \subsetneq \sigma_2 \subsetneq \cdots \subsetneq \sigma_k,\ \text{each } \sigma_i \in \Delta\right\}.$$

Theorem 4.19 [20, § 1.6]. *If Δ is a polyhedral cell complex, then $\Delta^\vee$ may be constructed as follows: Choose a first derived subdivision Δ^1 of Δ. For each cell $\sigma \in \Delta$, define*

$$\sigma^\vee := \bigcap_{v \text{ vertex of } \sigma} \left\| \overline{\mathrm{star}}(v; \Delta^1) \right\|,$$

where $\overline{\mathrm{star}}(\sigma; \Delta) := \{\tau \in \Delta : \tau$ is contained in a cell containing $\sigma\}$. Then let

$$\Delta^\vee := \left\{\sigma^\vee : \sigma \in \Delta\right\}.$$

4.3 | Quotients of regular cell complexes

Our next goal is to develop a notion of a quotient of a regular cell complex Δ, in which cells are merged together according to a given equivalence relation on the cells of Δ.

Let Δ be a regular cell complex with face poset $\mathtt{P}$, so that $\|\Delta\| \subseteq \mathbb{R}^d$. Given a homogeneous quotient $\mathtt{P}/\!\sim$ of $\mathtt{P}$, define the set

$$\Delta/\!\sim \; := \left\{ \bigcup \tilde{\sigma} : \tilde{\sigma} \in \mathtt{P}/\!\sim \right\},$$

where $\bigcup \tilde{\sigma}$ denotes the union $\bigcup_{\tau \in \tilde{\sigma}} \tau$. Note that homogeneity of $\sim$ implies that $\bigcup \tilde{\sigma} \subseteq \bigcup \tilde{\tau}$ as sets if and only if $\bigcup \tilde{\sigma} \leqslant \bigcup \tilde{\tau}$ in $\mathtt{P}/\!\sim$.

Under certain conditions, $\Delta/\!\sim$ is again a regular cell complex.

Theorem 4.20. *Suppose that:*

(i) *the poset* $\mathtt{P}/\!\sim$ *is an elementary quotient,*
(ii) *the augmented poset* $\mathcal{L}(\mathtt{P})$ *is a lattice and*
(iii) *each* $\sigma \in \Delta$ *is a PL ball.*

Then $\Delta/\!\sim$ *is a regular cell complex with face poset* $\mathtt{P}/\!\sim$*, such that each* $\bigcup \tilde{\sigma} \in \Delta/\!\sim$ *is a PL ball.*

Corollary 4.21. *Suppose that* $\mathtt{P}$ *admits a factorisation* $\mathtt{P} = \mathtt{P}_0, \mathtt{P}_1, \dots, \mathtt{P}_k = \mathtt{P}/\!\sim$ *into elementary quotients, such that* $\mathcal{L}(\mathtt{P}_i)$ *is a lattice for each* $i = 0, 1, 2, \dots, k-1$*. Suppose further that each* $\sigma \in \Delta$ *is a PL ball. Then* $\Delta/\!\sim$ *is a regular cell complex with face poset* $\mathtt{P}/\!\sim$*.*

In the remainder of this section we prove Theorem 4.20. The main ingredient is a topological criterion for $\Delta/\!\sim$ to be a regular cell complex.

Lemma 4.22. *Suppose that each* $\bigcup \tilde{\sigma}$ *in* $\Delta/\!\sim$ *is a ball whose interior equals the union of the interiors of the cells of* $\tilde{\sigma}$*. Then* $\Delta/\!\sim$ *is a regular cell complex with face poset* $\mathtt{P}/\!\sim$*.*

Proof. We firstly show that $\Delta/\!\sim$ is a regular cell complex. It is clear that the underlying topological spaces of Δ and $\Delta/\!\sim$ are the same. To see that the interiors of the balls in $\Delta/\!\sim$ are disjoint, let $\bigcup \tilde{\sigma}_1$ and $\bigcup \tilde{\sigma}_2$ be two balls in $\Delta/\!\sim$ such that

$$\left(\bigcup \tilde{\sigma}_1\right)^\circ \cap \left(\bigcup \tilde{\sigma}_2\right)^\circ = \left(\bigcup_{\tau_1 \in \tilde{\sigma}_1} \tau_1^\circ\right) \cap \left(\bigcup_{\tau_2 \in \tilde{\sigma}_2} \tau_2^\circ\right) = \bigcup_{\substack{\tau_1 \in \tilde{\sigma}_1 \\ \tau_2 \in \tilde{\sigma}_2}} \tau_1^\circ \cap \tau_2^\circ$$

is non-empty. In particular, there must exist $\tau_1 \in \tilde{\sigma}_1$ and $\tau_2 \in \tilde{\sigma}_2$ such that τ_1° and τ_2° intersect. This can only happen if $\tau_1 = \tau_2$, and hence $\tilde{\sigma}_1 = \tilde{\sigma}_2$. To see that the boundary of each $\bigcup \tilde{\sigma}$ in $\Delta/\!\sim$ is a union of members of $\Delta/\!\sim$, let $\bigcup \tilde{\sigma}$ be an element of $\Delta/\!\sim$. Then

$$\bigcup_{\tilde{\tau} < \tilde{\sigma}} \left(\bigcup \tilde{\tau}\right) = \bigcup_{\delta \in \tilde{\sigma}} \bigcup_{\substack{\tau < \delta \\ \tau \notin \tilde{\sigma}}} \tau = \bigcup_{\delta \in \tilde{\sigma}} \bigcup_{\substack{\tau < \delta \\ \tau \notin \tilde{\sigma}}} \tau^\circ. \tag{4.1}$$

We justify the last equality. We may write $\tau = \bigcup_{\gamma \leqslant \tau} \gamma^\circ$ for every $\tau \in \Delta$. Hence, the last equality holds, provided that we can show the following statement: whenever we have $\gamma \leqslant \tau < \delta \in \tilde{\sigma}$ where $\tau \notin \tilde{\sigma}$, we must also have $\gamma \notin \tilde{\sigma}$. The condition $\gamma \leqslant \tau$ implies $\tilde{\gamma} \leqslant \tilde{\tau}$. The condition $\tau < \delta$ implies $\tilde{\tau} \leqslant \tilde{\delta} = \tilde{\sigma}$. On the other hand, the condition $\tau \notin \tilde{\sigma}$ implies $\tilde{\tau} \neq \tilde{\sigma}$, and therefore $\tilde{\tau} < \tilde{\sigma}$. We conclude

$\tilde{\gamma} \leqslant \tilde{\tau} < \tilde{\sigma}$, and in particular $\gamma \notin \tilde{\sigma}$. Note that this argument uses the fact that $\mathrm{P}/\!\sim$ is a poset, which follows from homogeneity of $\sim$. Now, since the interiors of cells of Δ partition $\|\Delta\|$, we have by (4.1) that

$$\bigcup_{\tilde{\tau}<\tilde{\sigma}}\Big(\bigcup\tilde{\tau}\Big) = \bigcup_{\delta\in\tilde{\sigma}}\left(\left(\bigcup_{\tau\leqslant\delta}\tau^\circ\right)\setminus\bigcup_{\gamma\in\tilde{\sigma}}\gamma^\circ\right) = \bigcup_{\delta\in\tilde{\sigma}}\left(\delta\setminus\bigcup_{\gamma\in\tilde{\sigma}}\gamma^\circ\right) = \Big(\bigcup\tilde{\sigma}\Big)\setminus\bigcup_{\gamma\in\tilde{\sigma}}\gamma^\circ.$$

We therefore conclude

$$\mathrm{bd}\Big(\bigcup\tilde{\sigma}\Big) = \Big(\bigcup\tilde{\sigma}\Big)\setminus\Big(\bigcup\tilde{\sigma}\Big)^\circ = \Big(\bigcup\tilde{\sigma}\Big)\setminus\bigcup_{\gamma\in\tilde{\sigma}}\gamma^\circ = \bigcup_{\tilde{\tau}<\tilde{\sigma}}\Big(\bigcup\tilde{\tau}\Big).$$

The proof that the face poset of $\Delta/\!\sim$ is $\mathrm{P}/\!\sim$ is straightforward. If $\bigcup\tilde{\tau}\subseteq\bigcup\tilde{\sigma}$, then this means in particular that $\tau\subseteq\sigma$, hence $\tau\leqslant\sigma$ in P, hence $\tilde{\tau}\leqslant\tilde{\sigma}$ in $\mathrm{P}/\!\sim$. Conversely, if $\tilde{\tau}\leqslant\tilde{\sigma}$ in $\mathrm{P}/\!\sim$, then there exists a cell of $\tilde{\tau}$ contained in some cell of $\tilde{\sigma}$. By homogeneity, then, every cell of $\tilde{\tau}$ in contained in some cell of $\tilde{\sigma}$. Hence $\bigcup\tilde{\tau}\subseteq\bigcup\tilde{\sigma}$. □

Proposition 4.23 [6, Section 4.7, p. 204]. *Let Δ be a regular cell complex with face poset* P. *Then the augmented poset $\mathcal{L}(\mathrm{P}) := \mathrm{P}\cup\{\hat{\mathbf{0}},\hat{\mathbf{1}}\}$ is a lattice if and only if Δ is closed under non-empty intersections: for all $\sigma,\tau\in\Delta$ such that $\sigma\cap\tau$ is non-empty, we have $\sigma\cap\tau\in\Delta$.*

Proof of Theorem 4.20. It is clear that for any singleton class $\tilde{\sigma}=\{\sigma\}$, $\bigcup\tilde{\sigma}$ satisfies the hypothesis of Lemma 4.22. Now suppose that $\tilde{\sigma}=\{\sigma,\tau,\gamma\}$ is a class in $\sim$. It is known that the function $\sigma\mapsto\dim(\sigma)$ is a rank function on P. In particular, since σ and τ cover γ, then we must have $\dim(\sigma)=\dim(\tau)=\dim(\gamma)+1$. Moreover, since $\mathcal{L}(\mathrm{P})$ is a lattice, we must have $\gamma=\sigma\cap\tau$ by Proposition 4.23. Proposition 4.13 (i) and Lemma 4.14 then show that $\bigcup\tilde{\sigma}$ is a PL ball which satisfies the hypothesis of Lemma 4.22. □

5 | ELIMINATION SYSTEMS

In order to interpolate between fine mixed subdivisions and oriented matroid covectors, we consider a generalisation of the set of forests arising from a fine mixed subdivision which we call an *elimination system*. The main result in this section is Theorem 5.12, which states that a particular poset quotient associated to an elimination system admits a factorisation into elementary quotients, as defined in Section 4.1.

5.1 | Elimination systems and their posets

For a subgraph $F\subseteq\mathrm{R}\times\mathrm{E}$ of the complete bipartite graph $K_{\mathrm{R,E}}$ and $j\in\mathrm{E}$, define the neighbourhood $F_j := \{i : (i,j)\in F\}$.

Definition 5.1. Let S be a collection of subsets of $\mathrm{R}\times\mathrm{E}$. Then S is an *elimination system* provided.

(E1) For each $F \in \mathsf{S}$ and for each $j \in \mathsf{E}$, F_j is non-empty.
(E2) If $F \subseteq G \in \mathsf{S}$ and F_j is non-empty for all $j \in \mathsf{E}$, then $F \in \mathsf{S}$.
(E3) If $F, G \in \mathsf{S}$ and $j \in \mathsf{E}$, then there exists $H \in \mathsf{S}$ such that $H_j = F_j \cup G_j$ and $H_k \in \{F_k, G_k, F_k \cup G_k\}$ for all $k \in \mathsf{E}$ with $k \neq j$.

Elimination systems are the same as generic tropical oriented matroids except without the comparability axiom; see [3, Definition 3.5].

Generalising the face poset of the polyhedral complex of Proposition 3.4 subdividing the boundary of $\bigcirc_d := \square_d + n\diamond_d$, we introduce a poset associated with an elimination system.

Definition 5.2. Given an elimination system S, we define the following poset:

$$\mathsf{P}(\mathsf{S}) := \left\{(S, F) : S \in \{-1,0,1\}^{\mathsf{R}},\ F \in \mathsf{S},\ \mathrm{supp}(S) \supseteq \mathrm{supp}_{\mathsf{R}}(F)\right\}.$$

Recall from Proposition 3.4 that $\mathrm{supp}_{\mathsf{R}}(F)$ denotes those $i \in \mathsf{R}$ such that $(i.j) \in F$ for at least one $j \in \mathsf{E}$. The ordering of the poset $\mathsf{P}(\mathsf{S})$ is given as follows: $(S, F) \leqslant (T, G)$ if and only if $S \leqslant T$ and $F \supseteq G$. Recall that here $S \leqslant T$ means that S is obtained from T by setting some entries to zero. For example, $0-0+ \leqslant +--+$; another way to see it is that the orthant labelled by S is contained in the orthant labelled by T.

5.2 | An equivalence relation of $\mathsf{P}(\mathsf{S})$

Let Π be a partition of a finite set K. We say that two sign vectors $X, Y \in \{-1,0,1\}^{\mathsf{K}}$ are *equivalent* (with respect to Π), and write $X \sim Y$, if for all $s \in \{-,+\}$ and $\pi \in \Pi$, we have $X^s \cap \pi$ is non-empty if and only if $Y^s \cap \pi$ is non-empty. For example, the following two sign vectors are equivalent with respect to the indicated partition of the coordinates:

X:	0	+	0	−	0	0	+	−	0	0	0	+	+
Y:	0	+	−	0	0	−	+	0	+	+	0	+	0

This defines an equivalence relation on $\{-1,0,1\}^{\mathsf{K}}$. We may think of each equivalence class $X/\sim$ of this equivalence relation as a sign vector in $\{0,+,-,\pm\}^{\Pi}$. For the above example, this would look like

$X/\sim\, = Y/\sim$:	0	+	−	±	+

Recall the construction of the sign matrix SA_F associated with a sign vector S and a graph F on $\mathsf{R} \sqcup \mathsf{E}$ from Definition 2.8. We introduce an equivalence relation $\sim_A$ based on the set of signs in each column of the sign matrix SA_F.

Definition 5.3. Let $\Pi := \{\mathsf{R} \times \{j\} : j \in \mathsf{E}\}$ be a partition of the edges of $K_{\mathsf{R},\mathsf{E}}$. Define the following equivalence relation $\sim_A$ on $\mathsf{P}(\mathsf{S})$: Given (S, F) and (T, G) in $\mathsf{P}(\mathsf{S})$, we say that $(S, F) \sim_A (T, G)$ if $S = T$ and $SA_F \sim SA_G$ with respect to the partition of $\mathsf{R} \times \mathsf{E}$ given by Π.

Example 5.4. Depicted below are four elements from the poset $\mathsf{P}(\mathsf{S})$ for Example 3.1. We show each element (S,F) as (S,SA_F), noting that $F = \mathrm{supp}(SA_F)$:

$$(S_1, S_1A_{F_1}) = \left(\begin{pmatrix} - \\ + \\ + \end{pmatrix}, \begin{pmatrix} + & - & + \\ + & 0 & 0 \\ - & 0 & 0 \end{pmatrix}\right), (S_2, S_2A_{F_2}) = \left(\begin{pmatrix} - \\ + \\ + \end{pmatrix}, \begin{pmatrix} 0 & - & + \\ + & 0 & 0 \\ - & - & 0 \end{pmatrix}\right),$$

$$(S_3, S_3A_{F_3}) = \left(\begin{pmatrix} - \\ + \\ + \end{pmatrix}, \begin{pmatrix} 0 & 0 & + \\ + & 0 & - \\ - & - & 0 \end{pmatrix}\right), \quad (S_4, S_4A_{F_4}) = \left(\begin{pmatrix} + \\ + \\ + \end{pmatrix}, \begin{pmatrix} 0 & 0 & - \\ + & 0 & - \\ - & - & 0 \end{pmatrix}\right).$$

Observe that these four sign vectors correspond to four full-dimensional cells in Figure 2, of which three are in the lower right orthant and the last is in the upper right orthant. They correspond to cells following the red pseudoline in Figure 3, starting from the triangle in the lower right orthant. We see right away that $(S_4, F_4) \nsim_A (S_\ell, F_\ell)$ for $\ell = 1,2,3$ as they differ in the first component. To check for the equivalence of the other three pairs, we can consider the image of the columns of $S_1A_{F_1}, S_2A_{F_2}, S_3A_{F_3}$ to $\{0,+,-,\pm\}^3$ as indicated before Definition 5.3. This yields the three vectors $(\pm,-,+), (\pm,-,+), (\pm,-,\pm)$. Hence, we get $(S_1,F_1) \sim_A (S_2,F_2) \nsim_A (S_3,F_3)$.

5.3 | Properties of the quotient $\mathsf{P}(\mathsf{S})/\!\sim_A$

We assume that we are given an elimination system S on $\mathsf{R} \times \mathsf{E}$, and a sign matrix $A \in \{-1,1\}^{\mathsf{R}\times\mathsf{E}}$. We denote the poset $\mathsf{P}(\mathsf{S})$ by P, and write $|S|$ for $|\mathrm{supp}(S)|$ of a sign vector S.

Proposition 5.5. *Suppose that (S,F) is covered by (T,G) in P. Then either $F = G$ and $|S| = |T| - 1$, or $S = T$ and $|F| = |G| + 1$.*

Proof. The fact that $(S,F) \lneq (T,G)$ means that $S \leqslant T$ and $F \supseteq G$, and either $S \lneq T$ or $F \supsetneq G$. If $S \lneq T$, then let $i \in \mathrm{supp}(T) \setminus \mathrm{supp}(S)$. Then $i \notin \mathrm{supp}(S)$, which means $i \notin \mathrm{supp}_R(F)$. Since $F \supseteq G$, this means $i \notin \mathrm{supp}_{\mathsf{R}}(G)$. If $T \setminus i$ denotes the sign vector obtained from zeroing out component i of T, then $(T \setminus i, G)$ is an element of P such that

$$(S,F) \leqslant (T \setminus i, G) \leqslant (T,G).$$

Since (S,F) is covered by (T,G), we conclude the first inequality holds with equality, and hence $F = G$ and $|S| = |T| - 1$.

Otherwise, $F \supsetneq G$. Let $(i,j) \in F \setminus G$. Then (i,j) is not the only element of F_j, since otherwise we would have $G_j = \emptyset$ which is forbidden by (E1). We therefore have $(S, F \setminus (i,j)) \in \mathsf{P}$ by (E2), and hence

$$(S,F) \leqslant (S, F \setminus (i,j)) \leqslant (T,G).$$

By covering, we conclude that the second inequality holds with equality, and hence $S = T$ and $|F| = |G| - 1$. □

Corollary 5.6. *The poset P is graded, with grading $\rho(S,F) = n + |S| - |F|$.*

Given two sign vectors $S, T \in \{-1,0,1\}^{\mathsf{R}}$, define their *intersection* $S \cap T \in \{-1,0,1\}^{\mathsf{R}}$ to be the sign vector such that $(S \cap T)^+ = S^+ \cap T^+$ and $(S \cap T)^- = S^- \cap T^-$.

Proposition 5.7. *The augmented poset* $\mathcal{L}(\mathsf{P}) := \mathsf{P} \cup \{\hat{\mathbf{0}}, \hat{\mathbf{1}}\}$ *is a lattice: if* $(S,F),(T,G) \in \mathsf{P}$ *have a common lower bound, then a greatest lower bound for both is given by* $(S \cap T, F \cup G)$.

Proof. Let (S,F) and (T,G) be elements of P with a common lower bound (L,H). Then $H \supseteq F \cup G \supseteq F, G$ which implies by (E2) that $F \cup G \in \mathsf{S}$. Similarly, we have $L \leqslant S \cap T$ and so

$$\operatorname{supp}_{\mathsf{R}}(F \cup G) \subseteq \operatorname{supp}_{\mathsf{R}}(H) \subseteq \operatorname{supp}(L) \subseteq \operatorname{supp}(S \cap T).$$

We conclude $(S \cap T, F \cup G) \in \mathsf{P}$ and is a lower bound of (S,F) and (T,G). The fact that $H \supseteq F \cup G$ and $L \leqslant S \cap T$ shows that $(S \cap T, F \cup G)$ is in fact a greatest lower bound, as (L,H) was chosen arbitrarily. □

Our next task is to generalise the equivalence relation $\sim_A$ on P from Definition 5.3, by allowing the partition Π of $\mathsf{R} \times \mathsf{E}$ to vary. We fix a partition Π of $\mathsf{R} \times \mathsf{E}$ which refines the partition $\{\mathsf{R} \times \{j\} : j \in \mathsf{E}\}$. In terms of this partition, we say $X \sim Y$ if $X^s \cap \pi$ is non-empty if and only if $Y^s \cap \pi$ is non-empty, for all $s \in \{-,+\}$ and $\pi \in \Pi$.

Definition 5.8. For $(S,F),(T,G) \in \mathsf{P}$, we say $(S,F) \sim_A (T,G)$ if and only if $S = T$ and $SA_F \sim SA_G$.

Proposition 5.9. *The equivalence relation* $\sim_A$ *on* P *is* P*-homogeneous. In particular,* $\mathsf{P}/{\sim_A}$ *is a poset.*

Proof. Let $(S,F) \leqslant (T,G)$ be two elements of P, and choose $(S,F') \sim_A (S,F)$. Our goal is to find $G' \in \mathsf{S}$ such that $(T,G') \in \mathsf{P}$ and $(S,F') \leqslant (T,G') \sim_A (T,G)$. Define

$$G' := \{(i,j) \in F' : \text{if } \pi \in \Pi \text{ contains } (i,j), \text{ then there exists } (\ell, j) \in \pi \text{ such that } (TA_G)_{\ell,j} = (SA_{F'})_{i,j}\}.$$

Thus $(S,F') \leqslant (T,G')$. The definition of G' ensures that every non-zero sign appearing in the restricted sign vector $TA_{G'}|_\pi$ also appears in $TA_G|_\pi$, for all $\pi \in \Pi$. Indeed, let $(i,j) \in \pi$ for some $\pi \in \Pi$ such that $s := (TA_{G'})_{i,j}$ is non-zero. Then there exists $(\ell, j) \in \pi$ such that $(TA_G)_{\ell,j} = (SA_{F'})_{i,j}$. Note that $(SA_{F'})_{i,j}$ is non-zero, since $(i,j) \in G' \subseteq F'$. Since $SA_{F'} \leqslant TA_{F'}$ as sign vectors, we get

$$0 \neq (TA_G)_{\ell,j} = (SA_{F'})_{i,j} \leqslant (TA_{F'})_{i,j} = (TA_{G'})_{i,j} = s,$$

and therefore $(TA_G)_{\ell,j} = s$. Conversely, if $\pi \in \Pi$ and $(TA_G)_{\ell,j}$ is non-zero for some $(\ell,j) \in \pi$, then $SA_F \sim SA_G = TA_G$ implies that there exists $(i,j) \in \pi$ such that $(TA_G)_{\ell,j} = (SA_F)_{\ell,j} = (SA_{F'})_{i,j}$, and therefore, $TA_{G'}|_\pi$ contains the sign $(TA_G)_{\ell,j}$. Note that we are using here the fact that Π refines the partition $\{\mathsf{R} \times \{j\} : j \in \mathsf{E}\}$. We conclude $TA_G \sim TA_{G'}$.

Observe that G_j is non-empty for every $j \in \mathsf{E}$ by (E1), and since $TA_G \sim TA_{G'}$ we also have G'_j is non-empty for every $j \in \mathsf{E}$. Therefore, since $G' \subseteq F'$, we have by (E2) that $G' \in \mathsf{S}$. Moreover, $\operatorname{supp}_{\mathsf{R}}(G') \subseteq \operatorname{supp}_{\mathsf{R}}(F') \subseteq \operatorname{supp}(S) \subseteq \operatorname{supp}(T)$, so that $(T,G') \in \mathsf{P}$. We conclude $(T,G') \sim_A (T,G)$. □

For a generalised sign vector $X/{\sim} \in \{0,+,-,\pm\}^{\Pi}$, let $|X/{\sim}|$ count the number of non-zero coordinates in $X/{\sim}$, with each $\pm$ counted twice. For example, if $X/{\sim} = (0,\pm,-,+,-,\pm)$ then $|X/{\sim}| = 7$. Note that if Π is the singleton partition, then $X/{\sim}$ is an ordinary sign vector and $|X/{\sim}| = |X|$.

Proposition 5.10. *The poset $\mathrm{P}/{\sim_A}$ is graded, with grading*

$$\rho((S,F)/{\sim_A}) = n + |S| - \left|SA_F/{\sim}\right|.$$

Proof. Fix $(S,F) \in \mathrm{P}$. Firstly note that (S,F) is a maximal element in the equivalence class $(S,F)/{\sim_A}$ if and only if $|(SA_F)^s \cap \pi| \leqslant 1$ for all $s \in \{-,+\}$ and all $\pi \in \Pi$. Indeed, choose any $(S,G) \sim_A (S,F)$. Then (E2) implies that we may find $(S,H) \geqslant (S,G)$ inside $(S,F)/{\sim_A}$ such that $|(SA_H)^s \cap \pi| \leqslant 1$ for all $s \in \{-,+\}$ and all $\pi \in \Pi$. In particular, this statement holds for the maximal elements of $(S,F)/{\sim_A}$.

Now, for every maximal element $(S,G) \sim_A (S,F)$, we have

$$\begin{aligned}\rho((S,G)/{\sim_A}) &= n + |S| - \left|SA_G/{\sim}\right| \\ &= n + |S| - \sum_{\pi\in\Pi} \left(\left|(SA_G)^+ \cap \pi\right| + \left|(SA_G)^- \cap \pi\right|\right) \\ &= n + |S| - |G| \\ &= \rho(S,G).\end{aligned}$$

It remains to show that ρ respects the covering relations. Suppose that $(S,F)/{\sim_A}$ is covered by $(T,G)/{\sim_A}$ in $\mathrm{P}/{\sim_A}$. By homogeneity, we may choose representatives (S,F) and (T,G) so that (S,F) is covered by (T,G) in P. Such an element (S,F) is necessarily a maximal element of the equivalence class $(S,F)/{\sim_A}$, which implies $|(SA_F)^+ \cap \pi| \leqslant 1$ and $|(SA_F)^- \cap \pi| \leqslant 1$ for all $\pi \in \Pi$. Since $(S,F) < (T,G)$, we have $TA_G = SA_G \leqslant SA_F$, and hence $|(TA_G)^+ \cap \pi| \leqslant 1$ and $|(TA_G)^- \cap \pi| \leqslant 1$ for all $\pi \in \Pi$. It follows that (T,G) is maximal in $(T,G)/{\sim_A}$. We conclude

$$\rho((S,F)/{\sim_A}) = \rho(S,F) = \rho(T,G) - 1 = \rho((T,G)/{\sim_A}) - 1. \qquad \square$$

Proposition 5.11. *The augmented poset $\mathcal{L}(\mathrm{P}/{\sim_A})$ is a lattice.*

Proof. Choose $(S,F)/{\sim_A}$ and $(T,G)/{\sim_A}$ with a common lower bound in $\mathrm{P}/{\sim_A}$. By homogeneity and Proposition 5.7, we may choose the representatives (S,F) and (T,G) so that $(S \cap T, F \cup G) \in \mathrm{P}$. By homogeneity, then, $(S \cap T, F \cup G)/{\sim_A}$ is a lower bound for both $(S,F)/{\sim_A}$ and $(T,G)/{\sim_A}$.

We show that this is a greatest lower bound. Given a lower bound $(L,H)/{\sim_A}$, we may find $(S,F') \sim_A (S,F)$ and $(T,G') \sim_A (T,G)$ such that $(L,H) \leqslant (S,F')$ and $(L,H) \leqslant (T,G')$ in P. Hence, by Proposition 5.7, $(L,H) \leqslant (S \cap T, F' \cup G') \in \mathrm{P}$. Therefore, it suffices to show

$$(S \cap T, F' \cup G') \sim_A (S \cap T, F \cup G).$$

For all $\pi \in \Pi$ and $s \in \{-,+\}$, we have

$$\begin{aligned}((S\cap T)A_{F'\cup G'})^s \cap \pi \text{ non-empty} &\iff ((SA_{F'})^s \cup (TA_{G'})^s) \cap \pi \text{ non-empty} \\ &\iff ((SA_F)^s \cup (TA_G)^s) \cap \pi \text{ non-empty} \\ &\iff ((S\cap T)A_{F\cup G})^s \cap \pi \text{ non-empty}.\end{aligned}$$

In particular, this shows $(S \cap T, F' \cup G') \sim_A (S \cap T, F \cup G)$. $\square$

We now come to the main theorem of this section.

Theorem 5.12. *The poset* $\mathtt{P}(\mathtt{S})/{\sim_A}$ *admits a factorisation* $\mathtt{P}(\mathtt{S}) = \mathtt{P}_0, \mathtt{P}_1, \dots, \mathtt{P}_k = \mathtt{P}(\mathtt{S})/{\sim_A}$ *into elementary quotients, such that the augmented poset* $\mathcal{L}(\mathtt{P}_i)$ *is a lattice for each* $i = 0, 1, \dots, k-1$.

Proof. By Proposition 5.7, $\mathcal{L}(\mathtt{P})$ is a lattice. Thus, let $\bar{\Pi}$ be a partition of $\mathsf{R} \times \mathsf{E}$ which refines the partition Π and has at least one part $\pi \in \bar{\Pi}$ such that $|\pi| \geqslant 2$. Let $e := (i,j) \in \pi$, and let $\ddot{\Pi}$ be the refinement of $\bar{\Pi}$ obtained by splitting the part π into two parts: $\{e\}$ and $\pi \setminus \{e\}$. That is,

$$\ddot{\Pi} = (\bar{\Pi} \setminus \{\pi\}) \cup \{\{e\}, \pi \setminus \{e\}\}.$$

Let $\ddot{\sim}$ and $\bar{\sim}$ denote the equivalence relations on sign vectors on $\mathsf{R} \times \mathsf{E}$ induced by $\ddot{\Pi}$ and $\bar{\Pi}$, respectively. These determine $\mathtt{P}$-homogeneous equivalence relations $\ddot{\sim}_A$ and $\bar{\sim}_A$ by Proposition 5.9. Let $\ddot{\mathtt{P}} = \mathtt{P}/\ddot{\sim}_A$. Since $\bar{\sim}_A$ is $\mathtt{P}$-homogeneous, and since $\ddot{\sim}_A$ refines $\bar{\sim}_A$, we have that $\bar{\sim}_A$ is $\ddot{\mathtt{P}}$-homogeneous. Moreover, there is a natural identification $\ddot{\mathtt{P}}/\bar{\sim}_A = \mathtt{P}/\bar{\sim}_A$. Therefore, by induction, the theorem is proved if we can show that $\ddot{\mathtt{P}}/\bar{\sim}_A$ is an elementary quotient whose augmented poset is a lattice. In fact the lattice assertion follows from Proposition 5.11.

Fix $(S,F) \in \mathtt{P}$. We would like to show that the equivalence class containing $(S,F)/\ddot{\sim}_A$ in $\ddot{\mathtt{P}}/\bar{\sim}_A$ is either a singleton, or consists of exactly three elements two of which cover a third. Note that if $SA_F|_\pi$ is the zero vector, then this equivalence class is indeed a singleton. This is because we would immediately know that $(SA_F)_e = 0$ and $SA_F|_{\pi \setminus e} = \mathbf{0}$; hence, in this case $(S,F)/\ddot{\sim}_A$ is completely determined by $(S,F)/\bar{\sim}_A$.

Otherwise, the sign vector $SA_F|_\pi$ is non-zero, and in this case there are exactly three generalised sign vectors $X_1, X_2, X_3/\ddot{\sim} \in \{0,-,+,\pm\}^{\ddot{\Pi}}$, depending on $SA_F|_\pi$ and $(SA)_e$, such that $X_1 \bar{\sim} X_2 \bar{\sim} X_3 \bar{\sim} SA_F$. The restrictions of these to π are depicted below, in all of four possible cases:

	X_1	X_2	X_3		SA_F	
e	0	$+$	$+$	$\rightarrow$	$\pm$	π
$\pi \setminus e$	$\pm$	$-$	$\pm$			

	X_1	X_2	X_3		SA_F	
e	0	$-$	$-$	$\rightarrow$	$\pm$	π
$\pi \setminus e$	$\pm$	$+$	$\pm$			

	X_1	X_2	X_3		SA_F	
e	0	$+$	$+$	$\rightarrow$	$+$	π
$\pi \setminus e$	$+$	0	$+$			

	X_1	X_2	X_3		SA_F	
e	0	$-$	$-$	$\rightarrow$	$-$	π
$\pi \setminus e$	$-$	0	$-$			

The following argument applies simultaneously to all four cases shown above. Suppose that there are at least two distinct elements $(S,F)/\ddot{\sim}_A$ and $(S,F')/\ddot{\sim}_A$ in the same equivalence class of

$\ddot{\mathsf{P}}/\tilde{\sim}_A$. Then there exists a unique $i \in \{1,2,3\}$ such that

$$\{SA_F/\ddot{\sim},\ SA_{F'}/\ddot{\sim},\ X_i/\ddot{\sim}\} = \{X_1/\ddot{\sim},\ X_2/\ddot{\sim},\ X_3/\ddot{\sim}\}.$$

for some $i \in \{1,2,3\}$. We consider the three cases separately.

– If $i = 1$ or 2, then without loss of generality assume $SA_F \ddot{\sim} X_3$.
  - If $i = 1$, then by (E2) the set $F'' = F \setminus e$ is in S, and $SA_{F''} \ddot{\sim} X_1$.
  - If $i = 2$, then by (E2) the set $F'' = F \setminus ((SA_F)^s \cap \pi)$ is in S, where s is the unique sign appearing in $X_3|_{\pi \setminus e}$ but not $X_2|_{\pi \setminus e}$, and $SA_{F''} \ddot{\sim} X_2$.

– If $i = 3$, then by (E3), we can find $F'' \in \mathsf{S}$ such that

$$SA_{F''}|_\pi = SA_{F \cup F'}|_\pi$$

$$SA_{F''}|_\tau \in \{SA_F|_\tau,\ SA_{F'}|_\tau,\ SA_{F\cup F'}|_\tau\} \text{ for all } \tau \in \bar{\Pi} \setminus \pi.$$

This shows $SA_{F''} \ddot{\sim} X_3$. We remark that this is the only time (E3) is used.

In all three cases, we therefore have found $(S,F'') \tilde{\sim}_A (S,F)$ such that $SA_{F''} \ddot{\sim} X_i$. Therefore the equivalence class of $(S,F)/\ddot{\sim}_A$ in $\ddot{\mathsf{P}}/\tilde{\sim}_A$ consists of the three distinct elements $(S,F)/\ddot{\sim}_A$, $(S,F')/\ddot{\sim}_A$, and $(S,F'')/\ddot{\sim}_A$. Their gradings in $\ddot{\mathsf{P}} = \mathsf{P}/\ddot{\sim}$ are given by, by Proposition 5.10, $n - |S| - |X_i/\ddot{\sim}|$ for $i = 1,2,3$. Inspecting the above four tables, we conclude that two of these elements cover the third in $\ddot{\mathsf{P}}$. □

6 | PROOF OF THE MAIN THEOREM

In this section we prove Theorem 3.5. We assume the fine mixed subdivision S of $n\triangle_{d-1}$, the sign matrix A and the oriented matroid $\widetilde{\mathcal{M}}$ are as defined in Section 3.2.

6.1 | Properties of the extended patchworking complex

We begin this section by establishing some technical details of the extended patchworking complex defined in Section 3.2. Note that each face of the polytope $n\triangle_{d-1}$ is in bijection with a non-empty subset $I \subseteq \mathsf{R}$. Let S_I denote the cells of S contained in the coordinate subspace $\mathbb{R}^I \times \{0\}^{\bar{I}}$ of $\mathbb{R}^d$. For $S \in \{-1,0,1\}^d \setminus \mathbf{0}$ and $\sigma \in \mathsf{S}_{\mathrm{supp}(S)}$, let $\sigma_S := \square_S + S \cdot \sigma$.

Proposition 6.1. *Define the collection of polytopes given by*

$$\hexagon_{\mathsf{S}} := \left\{ \sigma_S \,:\, S \in \{-1,0,1\}^d \setminus \mathbf{0},\ \sigma \in \mathsf{S}_{\mathrm{supp}(S)} \right\}.$$

Then the following statements hold.

(i) *$\hexagon_{\mathsf{S}}$ is a polyhedral cell complex which subdivides the boundary of $\square_d + n\diamond_d$.*
(ii) *For each $\sigma_S \in \hexagon_{\mathsf{S}}$, both S and σ can be recovered from σ_S.*
(iii) *For $\sigma_S, \tau_T \in \hexagon_{\mathsf{S}}$, we have $\sigma_S \subseteq \tau_T$ if and only if $S \geqslant T$ and $\sigma \subseteq \tau$.*

Remark 6.2. The subdivision S in the statement of Proposition 6.1 can be replaced by any polyhedral subdivision of $n\triangle_{d-1}$.

Proof. Firstly note that (2) follows from the fact that each $\sigma_S = \square_S + S\cdot\sigma$ is a Minkowski sum of two affinely independent polytopes. Therefore, projection allows us to recover both $\square_S$ and $S\cdot\sigma$, and therefore the pair (S,σ).

Recall the general fact that F is a proper face of the Minkowski sum $K+L$ of two full-dimensional polytopes K and L if and only if there exists a non-zero objective function $\mathbf{c}$ such that $F = K_{\mathbf{c}} + L_{\mathbf{c}}$, where $K_{\mathbf{c}}$ and $L_{\mathbf{c}}$ denote the faces of K and L, respectively, maximised by $\mathbf{c}$. Specialising to the case $K = \square_d$ and $L = n\Diamond_d$, we have $K_{\mathbf{c}} = \square_S$ and $L_{\mathbf{c}} = S\cdot(n\triangle_I)$, where S is the component-wise sign vector of $\mathbf{c}$, I is the set of all $i\in\mathsf{R}$ such that $|\mathbf{c}_i| = \max_{k\in\mathsf{R}}|\mathbf{c}_k|$ and $\triangle_I := \operatorname{conv}(\mathbf{e}_i : i\in I)$. It follows that the collection of proper faces of $\square_d + n\Diamond_d$ is given by

$$\left\{\square_S + S\cdot(n\triangle_I) \;:\; S\in\{-1,0,1\}^d\setminus\mathbf{0},\ \emptyset\subsetneq I\subseteq\operatorname{supp}(S)\right\}.$$

Since $\square_S + S\cdot(n\triangle_I)$ is the union of the cells $\{\sigma_S : \sigma\in\mathsf{S}_I\}$, this shows that the cells in $\bigcirc_{\mathsf{S}}$ cover the boundary of $\square_d + n\Diamond_d$. The above fact about faces of Minkowski sums can also be used to show that the faces of $\sigma_S = \square_S + S\cdot\sigma$ are given by $\{\tau_T : \tau \text{ face of } \sigma,\ T\supseteq S\}$. This establishes (3), and that $\bigcirc_{\mathsf{S}}$ is closed under taking faces.

To establish (1), it remains to show that the intersection of two intersecting cells of $\sigma_S,\tau_T\in\bigcirc_{\mathsf{S}}$ is a face of both. If $\sigma_S\cap\tau_T$ is non-empty, then S and T are conformal sign vectors since otherwise, if (say) $i\in S^+\cap T^-$, then σ_S would lie in the halfspace $x_i\geqslant 1$, while τ_T lies in the halfspace $x_i\leqslant -1$. Now, we would like to show $\sigma_S\cap\tau_T = (\sigma\cap\tau)_{S\circ T}$, where $S\circ T$ denotes sign vector composition. As argued above, $(\sigma\cap\tau)_{S\circ T}$ is a face of both σ_S and τ_T. Thus, it remains to show that $\sigma_S\cap\tau_T$ is contained in $(\sigma\cap\tau)_{S\circ T}$.

Suppose $u+s = v+t$, where $u\in\square_S$, $v\in\square_T$, $s\in S\cdot\sigma$, $t\in T\cdot\tau$. We are done if we can show $u = v$ and $s = t$. For this it suffices to show $u_i = v_i$ for all $i\in\mathsf{R}$. Since S,T are conformal, we have $u_i = v_i$ for all $i\in\operatorname{supp}(S)\cap\operatorname{supp}(T)$. For $i\in\mathsf{R}\setminus(\operatorname{supp}(S)\cup\operatorname{supp}(T))$, we have $s_i = t_i = 0$, and hence $u_i = v_i$. Thus it remains to show $u_i = v_i$ in the case when $i\in\operatorname{supp}(S)\setminus\operatorname{supp}(T)$ or $i\in\operatorname{supp}(T)\setminus\operatorname{supp}(S)$.

Suppose $i\in\operatorname{supp}(S)\setminus\operatorname{supp}(T)$. The fact $i\in\operatorname{supp}(S)$ implies $|u_i| = 1$, and the fact $i\notin\operatorname{supp}(T)$ implies $t_i = 0$. Now $u_i s_i\geqslant 0$, which implies

$$1+|s_i| = |u_i+s_i| = |v_i+t_i| = |v_i|\leqslant 1.$$

Hence $s_i = 0 = t_i$, and so $u_i = v_i$. The case $i\in\operatorname{supp}(T)\setminus\operatorname{supp}(S)$ is proven analogously. □

Recall that, by the Cayley trick, the cells in S encode the simplices in $\mathcal{T}$ for which no node in E is isolated. This directly shows that they fulfil property (E1) and (E2) of elimination systems (Definition 5.1). A proof that S satisfies (E3) can be found in [28, Proposition 4.12], and this result has been generalised to arbitrary mixed subdivisions in [19, Theorem 7.11]. Hence, we conclude the following.

Proposition 6.3. *The set of forests encoded by* S *forms an elimination system.*

Observing that $\sigma_F\in\mathsf{S}_I$ if and only if $\operatorname{supp}_{\mathsf{R}}(F)\subseteq I$, and that on the level of posets, taking the dual just amounts to reversing the ordering, we have the following.

Corollary 6.4. *The map* $(\sigma_F)^\vee_S \mapsto (S,F)$ *determines an isomorphism from the face poset of the dual complex* $\bigcirc^\vee_{\mathsf{S}}$ *to the poset* $\mathsf{P}(\mathsf{S})$ *defined in Definition 5.2.*

6.2 | The map $\varphi : \mathsf{P}(\mathsf{S})/{\sim_A} \to \mathcal{L}(\widetilde{\mathcal{M}})$

We next consider the labelling of the elements of $\mathsf{P}(\mathsf{S})/{\sim_A}$ by sign vectors. For this, we use the connection between the pairs (S,F) denoting cells of the extended patchworking complex and covectors established in Corollary 2.11.

Now, we look at the particular elimination system given by the fine mixed subdivision S. In the following proposition, let $\mathcal{L}(\widetilde{\mathcal{M}})$ denote the poset of non-zero covectors of $\widetilde{\mathcal{M}}$. Let $\mathsf{P}(\mathsf{S})/{\sim_A}$ be the poset as in Section 5.3.

Recall from Section 5.3 the system of mixed signs $\{0,+,-,\pm\}$. Using this as an intermediate step, one sees that the following map extending the map ψ_A of Definition 2.9 is well defined on its equivalence classes.

Definition 6.5. Define the map $\varphi : \mathsf{P}(\mathsf{S})/{\sim_A} \to \{-1,0,1\}^{\widetilde{\mathsf{E}}}$ by $\varphi_A(S,F) = (S, \psi_A(S,F))$.

As we fix A most of the time, we just set $\varphi(S,F) = \varphi_A(S,F)$.

Example 6.6 (Example 5.4 continued). Recall that we could identify the equivalence classes of the four sign vectors $S_\ell A_{F_\ell}$ for $\ell \in [4]$ with $(\pm,-,+), (\pm,-,+), (\pm,-,\pm)$ and $(\pm,-,-)$. This shows that the images of the three equivalence classes of (S_ℓ, F_ℓ) (as $(S_1, A_{F_1}) \sim_A (S_2, A_{F_2})$) under φ are the sign vectors

$$(-,+,+,0,-,+), (-,+,+,0,-,0), (+,+,+,0,-,-).$$

Note that a similar map was used in [19, § 6] to prove the representation theorem for tropical oriented matroids.

Since φ is constant on the equivalence classes of $/{\sim_A}$, we can just fix an element $(S,F) \in \mathsf{P}(\mathsf{S})$. With this, we associate the bipartite graph T on $\mathsf{R} \sqcup \widetilde{\mathsf{E}}$ having the edges of F and edges between the nodes of R and its copy $\widetilde{\mathsf{R}}$ within $\widetilde{\mathsf{E}}$ for each element R in the support of S. The next claim thus follows from Proposition 2.10.

Corollary 6.7. *For all* $(S,F) \in \mathsf{P}(\mathsf{S})/{\sim_A}$, *we have* $\varphi(S,F) \in \mathcal{L}(\widetilde{\mathcal{M}})$.

Proposition 6.8. *The map* $\varphi : \mathsf{P}(\mathsf{S})/{\sim_A} \to \mathcal{L}(\widetilde{\mathcal{M}})$ *is a poset map.*

Proof. Suppose $(S,F) \leqslant (S',F')$ in P. Let $(S,X) = \varphi(S,F)$, and let $(S',X') = \varphi(S',F')$. Then $S \leqslant S'$, and because $F \supseteq F'$, the passage from SA_F to $SA_{F'}$ only decreases the number of non-zero entries in each column of SA_F. However, $SA_{F'}$ still has at least one non-zero entry in each column. From this we conclude $X \leqslant X'$, and therefore φ respects order. □

Recall that a poset P is a *sphere* if its order complex is a sphere; see Definition 4.9.

Theorem 6.9 (Borsuk–Ulam). *Let* $\mathtt{P}, \mathtt{Q}$ *be posets such that both are homeomorphic to* S^{d-1} *and both are equipped with a fixed-point free involutive automorphism* $x \mapsto -x$. *Let* $\varphi : \mathtt{P} \to \mathtt{Q}$ *be a poset map satisfying* $\varphi(-x) = -\varphi(x)$ *for all* $x \in \mathtt{P}$. *Then* φ *is surjective.*

Corollary 6.10. *Assuming that* $\mathtt{P}(\mathtt{S})/{\sim_A}$ *is a* $(d-1)$*-sphere, the map* $\varphi : \mathtt{P}(\mathtt{S})/{\sim_A} \to \mathcal{L}(\widetilde{\mathcal{M}})$ *is an isomorphism.*

Remark 6.11. We show that $\mathtt{P}(\mathtt{S})/{\sim_A}$ is indeed a $(d-1)$-sphere in Section 6.4.

Proof. For $(S, F) \in \mathtt{P}(\mathtt{S})/{\sim_A}$, the interpretation of $SA_F/{\sim}$ as a generalised sign vector in $\{0, -, +, \pm\}^{\Pi}$ shows that φ is injective; here we are using the fact that every column of SA_F has at least one non-zero entry. To see that φ is surjective, we simply note that $(S, F) \mapsto -(S, F) := (-S, F)$ is a fixed-point free involutive automorphism of $\mathtt{P}(\mathtt{S})$ which descends to $\mathtt{P}(\mathtt{S})/{\sim_A}$, while $X \mapsto -X$ is one of $\mathcal{L}(\widetilde{\mathcal{M}})$. Furthermore, by definition of φ, we have $\varphi(-S, F) = (-S, -X) = -(S, X) = -\varphi(S, F)$. As $\widetilde{\mathcal{M}}$ has rank d, the poset $\mathcal{L}(\widetilde{\mathcal{M}})$ is a $(d-1)$-sphere by Theorem 6.12, and hence, the conclusion follows from the Borsuk–Ulam theorem. □

6.3 | Pseudosphere arrangements from regular cell complexes

The following result shows how to get pseudosphere arrangements from regular cell complexes.

Theorem 6.12 [6, Theorem 4.3.3, Proposition 4.3.6]. *Let* $\mathcal{M}$ *be an oriented matroid of rank* d *on the ground set* E. *Let* Δ *be a regular cell complex with face poset* $\mathtt{P}$, *such that there is a poset isomorphism* $\mathtt{P} \simeq \mathcal{L}(\mathcal{M})$. *Thus each cell* σ_X *of* Δ *is labelled by some non-zero covector* X *of* $\mathcal{M}$. *For each* $k \in E$, *define the subcomplex*

$$\Delta_k := \{\sigma_X \in \Delta : X_k = 0\}.$$

Then $\|\Delta\|$ *is a* $(d-1)$*-sphere, and the spaces* $\|\Delta_k\|$ *ranging over all* $e \in E$ *form an arrangement of pseudospheres within* $\|\Delta\|$ *representing* $\mathcal{M}$.

Remark 6.13. This theorem is really the uniqueness assertion of [6, Theorem *4.3.3*], whose proof can be traced back to [6, Proposition *4.7.23*].

6.4 | Putting it all together

With all the pieces now in place, we are ready to prove Theorem 3.5, which asserts that our patchworking procedure yields a pseudosphere representation of $\widetilde{\mathcal{M}}$.

Proof of Theorem 3.5. As shown for Proposition 6.3, a fine mixed subdivision $\mathtt{S}$ of $n\triangle_{d-1}$ gives rise to an elimination system as in Definition 5.1. Abusing notation, we denote this elimination system also by $\mathtt{S}$. We let $\mathtt{P}(\mathtt{S})$ be the poset of $\mathtt{S}$ obtained by introducing signs as in Definition 5.2.

Let $\mathtt{P}(\Delta)$ denote the face poset of $\Delta = \bigcirc^{\vee}_{\mathtt{S}}$. By Corollary 6.4, we have $\mathtt{P}(\Delta) \simeq \mathtt{P}(\mathtt{S})$. Hence the poset quotient $\mathtt{P}(\mathtt{S})/{\sim_A}$ induces a quotient $\mathtt{P}(\Delta)/{\sim_A}$. By Theorem 5.12, then, $\mathtt{P}(\Delta)/{\sim_A}$ admits a factorisation $\mathrm{P}(\Delta) = \mathrm{P}_0, \mathrm{P}_1, \dots, \mathrm{P}_k = \mathrm{P}(\Delta)/{\sim_A}$ into elementary quotients, such that the augmented poset $\mathcal{L}(\mathrm{P}_i)$ is a lattice for each $i = 0, 1, \dots, k-1$.

As a polyhedral complex on the boundary of a d-dimensional polytope, $\bigcirc_{\mathrm{S}}$ is a PL $(d-1)$-sphere. Hence, by Proposition 4.17, $\Delta = \bigcirc_{\mathrm{S}}^{\vee}$ is also a PL $(d-1)$-sphere. In particular, by Proposition 4.16, each cell $\sigma^{\vee}$ in Δ is a PL ball. It follows, by Corollary 4.21, that $\Delta/{\sim_A}$ is a regular cell complex with face poset $\mathrm{P}(\Delta)/{\sim_A}$. In particular, $\mathrm{P}(\Delta)/{\sim_A}$ is a $(d-1)$-sphere.

By Corollary 6.10, we have isomorphisms $\mathrm{P}(\Delta)/{\sim_A} \simeq \mathrm{P}(\mathrm{S})/{\sim_A} \simeq \mathcal{L}(\widetilde{\mathcal{M}})$. For $k \in \widetilde{\mathrm{E}}$, define the subcomplex

$$(\Delta/{\sim_A})_k := \left\{ \bigcup (\sigma^{\vee}_{(S,F)}/{\sim_A}) \in \Delta/{\sim_A} \ : \ \varphi(S,F)_k = 0 \right\}$$

of $\Delta/{\sim_A}$. Now, Theorem 6.12 implies that the spaces $\|(\Delta/{\sim_A})_k\|$ ranging over all $k \in \widetilde{\mathrm{E}}$ form an arrangement of pseudospheres within $\|\Delta/{\sim_A}\| = \|\Delta\|$ representing $\widetilde{\mathcal{M}}$.

It remains to show that $\|(\Delta/{\sim_A})_k\| = \|\Delta_k\|$ for all $k \in \widetilde{\mathrm{E}}$, where Δ_k is a subcomplex of Δ defined in (3.1) and (3.2). For this note that the closed cells of $\Delta/{\sim_A}$, and hence all subcomplexes of $\Delta/{\sim_A}$, each consist of a union of members of Δ. Hence, it suffices to show that for all $\sigma^{\vee} \in \Delta$ and $k \in \widetilde{\mathrm{E}}$, we have $\sigma^{\vee} \subseteq \|(\Delta/{\sim_A})_k\|$ if and only if $\sigma^{\vee} \in \Delta_k$.

For $\sigma^{\vee} \in \Delta$ and $k \in \widetilde{E}$, we have $\sigma^{\vee} \subseteq \|(\Delta/{\sim_A})_k\|$ if and only if there exists $(S,F) \in \mathrm{P}(\mathrm{S})$ such that $\varphi(S,F)_k = 0$ and

$$\sigma^{\vee} \subseteq \bigcup_{(S,G) \sim_A (S,F)} \sigma^{\vee}_{(S,G)}.$$

As Δ is a regular cell complex, the interiors of the balls in Δ are disjoint, and so the above containment holds true if and only if $\sigma^{\vee} = \sigma^{\vee}_{(S,F)}$ for some $(S,F) \in \mathrm{P}(\mathrm{S})$ such that $\varphi(S,F)_k = 0$. If $k \in \mathrm{R}$, then $\varphi(S,F)_k = 0$ if and only if $S_k = 0$. If $k \in \mathrm{E}$, we have $\varphi(S,F)_k = 0$ if and only if there exist $(i,k),(\ell,k) \in F$ such that $S_i A_{i,k} = -S_{\ell} A_{\ell,k} \neq 0$. In either case, we conclude $\sigma^{\vee} \subseteq \|(\Delta/{\sim_A})_k\|$ if and only if $\sigma^{\vee} \in \Delta_k$. □

ACKNOWLEDGEMENT

The first author was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy — The Berlin Mathematics Research Center MATH+(EXC-2046/1, project ID: 390685689). He is grateful to Josephine Yu for helpful discussions on patchworking, in particular pointing out the reference [33]. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement n° ScaleOpt-757481). The second author also thanks Xavier Allamigeon, Michael Joswig and Mateusz Skomra for inspiring discussions and Ben Smith for helpful comments. The third author was supported by Trond Mohn Foundation project 'Algebraic and topological cycles in tropical and complex geometry', Croucher Fellowship for Postdoctoral Research, and Netherlands Organisation for Scientific Research Vici grant 639.033.514 during his affiliation to University of Oslo, Brown University and University of Bern, respectively. He also thanks Cheuk Yu Mak for pointing out [31]. The second and third author both thank the *Tropical Geometry, Amoebas, and Polyhedra* semester program at Institut Mittag-Leffler for the introduction that started the project. The authors thank Laura Anderson, Jesus De Loera and Kris Shaw for comments on our work, as well as the anonymous referees for their helpful comments and suggestions.

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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