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PRELIMINARIES ON ALMOST-CONTROLLED INVARIANCE FOR SINGLE-INPUT NONLINEAR SYSTEMS

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1. Introduction

An important new development in nonlinear system theory in the last decade has been, without any doubt, the introduction of controlled and conditioned invariant distributions, see [1,2]. These concepts have proven to form suitable generalizations of controlled and conditioned invariant subspaces in linear system theory, cf. [3] and, as it is true for linear systems, turn out to be essential tools in the solvability requirements for a number of convincing control system synthesis problems as for example the disturbance decoupling problem, the model following problem, the synthesis of noninteracting controllers, see e.g. [14] and references therein. Recently, in the frame of geometric linear system theory, Willems has introduced in [5,6] the concepts of almost controlled invariant and almost conditioned invariant subspaces, which turn out to be essential in the solution of "almost" control synthesis problems where it is required that the standard synthesis problems are solvable in a suitable approximate sense, see also [7,8] for further investigations using almost concepts in linear systems. The purpose of this note is to introduce an analogous concept of almost controlled invariant distributions for (single-input) nonlinear systems. Remarkable as it may be, the definition we propose here is in fact one of the equivalent characterizations of linear almost controlled invariance, see in particular [5, appendix] and [7]. In [9] an approach for solving the almost disturbance decoupling problem for single-input single-output nonlinear systems is given. As we will indicate part of this method is essentially based on the use of the notion of almost controlled invariance. We therefore hope that our approach is connected with the idea of singular perturbations methods, [10]. An expanded version of this note will appear elsewhere, [11].

2. Local almost controlled invariance

Throughout this note we consider smooth ($= C^\infty$ or analytic) single input nonlinear systems that can be described as

$$\dot{x} = f(x,u) \tag{1}$$

In (1) the state x belongs to an open part X of an Euclidean space $\mathbb{R}^n$ and the inputs $u(\cdot)$ belong to the set of admissible controls $\mathcal{U}$. The set $\mathcal{U}$ is given as follows. There exists an integer m such that

$$\mathcal{U} = \{u:\mathbb{R} \to \mathbb{R} \mid u \text{ is smooth and } u^{(i)} \in U^i,\ i = 0,\dots,m \text{ for open intervals } U^i \in \mathbb{R}\} \tag{2}$$

So an admissible input is a smooth bounded function u with bounded derivatives $u^{(1)},\dots,u^{(m)}$. Let D be a regular involutive distribution on X of dimension k, which by the local Frobenius theorem, cf. [12], can be written in suitably chosen local coordinates $(x_1,\dots,x_n)$ around x^0 in X as $\text{span}\{\frac{\partial}{\partial x_{n-k}},\dots,\frac{\partial}{\partial x_n}\}$. Locally we may consider the quotient space $X(\text{mod } D) = \bar{X}$ which has as local coordinates $(x_1,\dots,x_{n-k}) = \bar{x}$. For $u(\cdot) \in \mathcal{U}$ and $x^0 \in X$ we denote $x(t,x^0,u)$ as the solution at time t of (1) starting at t = 0 in x^0. Moreover let $\bar{x}(t,x^0,u)$ denote $x(t,x^0,u)(\text{mod } D)$, i.e. the first n-k coordinates of $x(t,x^0,u)$. Our basic definition of a (locally) almost controlled invariant distribution for the system (1) is as follows.

<u>Definition 1</u>. Consider the smooth system (1) with admissible controls $\mathcal{U}$ given in (2) and let D be a smooth regular distribution. We call D a (<u>local</u>) <u>almost controlled invariant distribution</u> for system (1,2) if locally there exists a smooth system of the form (1) on $\bar{X} = X(\text{mod } D)$, say

$$\dot{\bar{x}} = \bar{f}(\bar{x},w) \tag{3}$$

with an input space $\mathcal{W}$ of the form (2) and which is such that the set of all modulo D trajectories $\bar{x}(t,x^0,u)$ of (1) coincides with the set of trajectories $\bar{x}(t,\bar{x}^0,w)$.

The above definition amounts to saying that for $x^0 \in X$ and $u(\cdot) \in \mathcal{U}$ there exists a neighborhoor $\mathcal{O}(x^0)$ in X and an input $w(\cdot) \in \mathcal{W}$ such that the modulo D trajectory $\bar{x}(t,\bar{x}^0,u)$ coincides with $\bar{x}(t,\bar{x}^0,w)$ as long as $x(t,\bar{x}^0,u)$ belongs to $\mathcal{O}(x_0)$. Note that this characterization indeed generalizes the linear one, cf. [5,7]. Also observe that this definition is feedback invariant: if $u = \alpha(x,v)$ is a regular feedback for (1), so $\alpha(x,\cdot)$ is a diffeomorphism for all x in X, then D being locally almost controlled invariant for (1) is equivalent to the fact that D is almost controlled invariant for $\dot{x} = f(x,\alpha(x,v))$ with v belonging to the transformed input set. It is a straightforward excercise to see that a local controlled invariant distribution D for (1), see e.g. [13], is also locally almost controlled invariant; therefore the notion of almost controlled invariance generalizes that of controlled invariance. Next we discuss two typical nonlinear examples.

<u>Example 2</u>. Consider a smooth scalar input affine system on $\mathbb{R}^n$

$$\dot{x} = f(x) + g(x)u \tag{4}$$

and let $\mathcal{U} = \{u\colon \mathbb{R} \to \mathbb{R} \mid u \text{ is smooth}\}$. Assume the vector field g is nonvanishing everywhere and consider the 1-dimensional (and thus involutive) distribution $D = \text{span}\{g\}$. Let $x^0 \in \mathbb{R}^n$, then by the local Frobenius theorem we find local coordinates $(x = (x_1,\dots,x_n)$, $-\epsilon < x_i < \epsilon$ for a certain $\epsilon > 0$, $i = 1,\dots,n$, such that locally around x_0, $D = \text{span}\{\frac{\partial}{\partial x_n}\}$. In these coordinates (4) can be written as

$$\begin{cases} \dot{x}_i = f_i(x_1,\dots,x_n),\ i = 1,\dots,n-1 \\ \dot{x}_n = f_n(x_1,\dots,x_n) + u \end{cases} \tag{5}$$

The first n-1 equations of (5) describe the dynamics on $\mathbb{R}^n(\text{mod } D)$ whereas the last equation in (5) represents the dynamics along the leaves of D. One clearly sees that the dynamics on $\mathbb{R}^n(\text{mod } D)$ is "driven" by the variable $w = x_n$, which is the state variable of the dynamics along the leaves of D. With the above choice of admissable inputs $\mathcal{U}$ we see that the state variable x_n is allowed to vary arbitrarily along the interval $(-\epsilon,\epsilon)$ and this shows that we locally end up with a smooth system on $\mathbb{R}^n(\text{mod } D)$

$$\dot{x}_i = f_i(x_1,\dots,x_{n-1},w),\ i = 1,\dots,n-1 \tag{6}$$

with inputs w belonging to $\mathcal{W} = \{w\colon \mathbb{R} \to \mathbb{R} \mid w \text{ is smooth and } w^{(0)} \in (-\epsilon,\epsilon)\}$. Therefore $D = \text{span}\{g\}$ is locally almost controlled invariant. □

Notice that example 2 is the nonlinear analogue of the fact that for a linear system $\dot{x} = Ax + bu$ the subspace Im b is an almost controlled invariant subspace for that system, cf. [5]. The next example will show a similar idea for an involutive distribution generated by the vector fields g and [f,g].

Example 3. Consider on $\mathbb{R}^5$ the smooth affine system

$$\begin{cases} \dot{x}_1 = f_1(x_1,x_2,x_3) \\ \dot{x}_2 = f_2(x_1,x_2,x_3) \\ \dot{x}_3 = f_3(x_1,x_2,x_3,x_5) + \exp(2x_4) \\ \dot{x}_4 = u \\ \dot{x}_5 = f_5(x_1,x_2,x_3,x_5) + \exp(x_4) \end{cases} \tag{7}$$

with input space $\mathcal{U} = \{u: \mathbb{R} \to \mathbb{R} \mid u \text{ is smooth}\}$. Let D be the distribution given by $D = \text{span}\{\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}\}$. In fact, if we write (7) shortly as $\dot{x} = f(x) = g(x)u$ this distribution equals the involutive closure of span{g, [f,g]}, which by itself is not involutive. The first two equations of (7) form the dynamics on $\mathbb{R}^5$(mod D), "driven" by the variable $w = x_3$ and the last three equations of (7) constitute the dynamics along the leaves of D. Note that $w = x_3$ satisfies $\dot{w} = f_3(x_1,x_2,x_3,x_5) + \exp(2x_4)$ and $\ddot{w} = \varphi(x_1,x_2,x_3,x_4,x_5) + 2\exp(2x_4)u$, for some function φ. Interpreting w as an output for (7) we can bring (7) locally in a more tractable form by applying the feedback $\varphi(x_1,x_2,x_3,x_4,x_5) + 2\exp(2x_4)u = \tilde{u}$ and the (local) state transformation $(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4,\tilde{x}_5) = (x_1,x_2,x_3,f_3(x_1,x_2,x_3,x_5) + \exp(2x_4),x_5)$:

$$\begin{cases} \dot{\tilde{x}}_1 = f_1(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3) \\ \dot{\tilde{x}}_2 = f_2(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3) \\ \dot{\tilde{x}}_3 = \tilde{x}_4 \quad ,\dot{\tilde{x}}_4 = \tilde{u}, \\ \dot{\tilde{x}}_5 = f_5(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4,\tilde{x}_5) \end{cases} \tag{8}$$

Clearly for this system the variable $w = \tilde{x}_3$ can be arbirarily chosen from $\mathcal{W} = \{w: \mathbb{R} \to \mathbb{R} \mid w \text{ is smooth and } w^{(0)} \in U^0(\tilde{x}_{30}),\ w^{(1)} \in U^1(\tilde{x}_{40})\}$, for some neighborhood U^0 and U^1 of $\tilde{x}_{30}$ resp. $\tilde{x}_{40}$. Therefore the system on $\mathbb{R}^5$(mod D) is described as

$$\dot{\tilde{x}}_1 = f_1(\tilde{x}_1,\tilde{x}_2,w),\ \dot{\tilde{x}}_2 = f_2(\tilde{x}_1,\tilde{x}_2,w) \tag{9}$$

with as input space $\mathcal{W}$. So D is locally almost controlled invariant. Note that this conclusion would not be true if for instance both x_3 and x_4 were present in both the first two equations. □

The above examples are an illustration of a far more general result, which will be given for affine systems only. The result for general systems (1) follows in fact by using Example 2.

Theorem 4. Consider an affine nonlinear system (4) with inputspace $\mathcal{U}$ as in (2). Let D_k be a regular involutive distribution given as

$$D_k = \text{involutive closure of } \{g, ad_f g, \ldots, ad_f^k g\} \tag{10}$$

Then D_k is a locally almost controlled invariant distribution if

$$\text{rank } [f,D_k] \text{ mod } D_k \le 1 \tag{11}$$

For the proof of this theorem we refer to [11]. Observe that for a linear system the subspace D_k defined via (10) automatically satisfies the condition (11). In the nonlinear setting (11) is assential condition. It enables us to define the "driven" variable as in the Examples 2 and 3 and it guarantees that the dynamics on X(mod D) only depends on one such variable and not on two (or more) depending variables like for instance w_1 and w_2 with $\dot{w}_2 = w_1$. Of course theorem 4 does not describe all almost controlled invariant distributions, but it does if we exclude (local) controlled invariant distributions, cf. [11].

Next we identify a set of local almost controlled invariant distributions for a system with outputs. Consider a smooth affine single-input single-output nonlinear system

$$\dot{x} = f(x) + g(x)u \quad , \quad y = h(x) \tag{12}$$

where the input u belongs to a set $\mathcal{U}$ of the form (2). Let ρ be the characteristic number of (12), i.e. ρ is the minimal integer for which $L_g L_f^\rho h \neq 0$. Then the supremal local controlled invariant distribution $\mathcal{V}^*$ of (12) in ker dh is given as $\bigcap_{j=0}^{\rho-1} \ker dL_f^j h$ and after feedback the system may be put into the form (e.g. [4])

$$\begin{cases} \dot{x}_i = x_{i+1},\ i = 0,\ldots,\rho-1 \\ \dot{x}_\rho = \tilde{u} \\ \dot{\bar{x}} = \tilde{f}(x,\bar{x}),\ y = x_1 \end{cases} \tag{13}$$

Recall the definition of D_k, see (10). Then it is straightforward to show that for each k the distribution $\mathcal{V}^* + D_k$ is locally almost controlled invariant for (12). Moreover, the dynamics on X mod($\mathcal{V}^* + D_k$) is given as

$$\begin{cases} \dot{x}_1 = x_2 \\ \quad \vdots \\ \dot{x}_{\rho-k} = w \quad , y = x_1 \end{cases} \tag{14}$$

In particular it follows that the distribution ker dh is almost controlled invariant. Note that the reduction procedure as sketched above is essentially one part of the procedure used in [9] for solving the almost disturbance decoupling problem for single-input single-output nonlinear systems and in fact thus generalizes the linear approach, cf [5].

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