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Dynamic Disturbance Decoupling for Nonlinear Discrete-Time Systems

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Abstract

In this paper we study the dynamic disturbance decoupling problem for nonlinear discrete-time systems in a neighbourhood of a reference trajectory. Furthermore the connection between the solvability of this problem and the solvability of the corresponding problem for the time-varying linear discrete-time system obtained by linearizing the original system along the given reference trajectory is investigated.

1. Introduction

Disturbance decoupling by means of *static* (DDP) or *dynamic state feedback* (DDDP) has received a lot of attention in the past. Complete solutions have been obtained for time-invariant linear systems. In the nonlinear continuous-time setting, local results are available for special classes of systems provided certain regularity conditions are met. In contrast with linear systems, however, the application of dynamic state feedback proved to be a stronger means in achieving disturbance decoupling. Nonlinear disturbance decoupling by static state feedback relies on the concept of *controlled invariant distributions.* In dynamic disturbance decoupling, a crucial role is played by the so-called *Singh Algorithm (SA)* and the resulting *Singh Compensator (SC).*

The purpose of this note is threefold. In Section 2 we give results concerning the disturbance decoupling for time-varying linear discrete-time systems. Section 3 deals with the solution of DDDP for nonlinear discrete- time systems given in a neighbourhood of a reference trajectory and Section 4 connects the results of the previous sections.

2. Time-Varying Linear Systems

Consider the time-varying linear discrete-time system:

$$\Sigma : \begin{cases} x(k+1) &= A(k)x(k) + B(k)u(k) + E(k)d(k) \\ y(k) &= C(k)x(k) \end{cases}$$

where $d(k) \in \mathbb{R}^q$, $k \in \mathbb{N}$, represents a sequence of unknown disturbances. Furthermore, $x(\cdot) \in \mathbb{R}^n$, $u(\cdot) \in \mathbb{R}^m$ and $y(\cdot) \in \mathbb{R}^p$. $A(\cdot)$, $B(\cdot)$, $E(\cdot)$ and $C(\cdot)$ are matrices of appropriate sizes.

Definition 2.1 *The disturbance decoupling problem by static state feedback (DDP) consists in finding a static feedback law $u(k) = F(k)x(k)$ such that in the closed loop system Σ_{BF}, for every $k_0 \in \mathbb{N}$ and arbitrary disturbance sequence $\{d(k)\}$, $k \geq k_0$, the output only depends on the initial state $x(k_0)$.*

We introduce the concept of a *controlled invariant sequence* of subspaces for time-varying linear systems in the following way:

Definition 2.2 *A sequence $\{\mathcal{V}(k)\}$, $k \in \mathbb{N}$, of subspaces of the state space $\mathbb{R}^n$ is said to be controlled invariant if for every $k_0 \in \mathbb{N}$ and arbitrary $x_0 \in \mathcal{V}(k_0)$, there exists an input $u_0 \in \mathbb{R}^m$ such that $x_{u_0}(k_0+1) \in \mathcal{V}(k_0+1)$ where $x_{u_0}(k_0+1) := A(k_0)x_0 + B(k_0)u_0$.*

The always existing largest (w.r.t. inclusion) such sequence can be computed algorithmically.

The following lemma tells us when DDP can be solved.

Lemma 2.3 *DDP is solvable iff there exists a controlled-invariant sequence $\{\mathcal{V}(k)\}$ of subspaces of $\mathbb{R}^n$ such that*

$$im E(k-1) \subseteq \mathcal{V}(k) \subseteq ker C(k) \ , \quad k \in \mathbb{N}\backslash\{0\} \ .$$

In [3] a continuous-time counterpart can be found. Furthermore one can show that DDP is solvable iff DDDP is solvable (cf. [1]).

3. DDDP for Nonlinear Systems

Consider the following nonlinear discrete-time system:

$$\Sigma : \begin{cases} x(k+1) &= f(x(k), u(k), q(k)), \quad x_0 = x(0) \\ y(k) &= h(x(k)) \end{cases}$$

where x, u, q, and y take their values in some open parts of $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^r$, and $\mathbb{R}^p$, respectively. f and h are supposed to be analytic. Let us assume furthermore that their exists a reference trajectory for Σ, that is, a set of time functions $(\overline{x}(k), \overline{u}(k), \overline{q}(k), \overline{y}(k))$ that satisfies the system equations. This way, the application of local methods become possible.

Definition 3.1 *The regular dynamic disturbance decoupling problem (DDDP) consists in finding a regular dynamic compensator (i.e. the relation between $u(\cdot)$ and $v(\cdot)$ given by C is invertible)*

$$C : \begin{cases} z(k+1) &= \psi(z(k), x(k), v(k)), \quad z_0 = z(0) \\ u(k) &= \phi_k(z(k), x(k), v(k)) \end{cases} \tag{1}$$

with $z(\cdot) \in \mathbb{R}^\nu$ and $v(\cdot) \in \mathbb{R}^m$, defined locally around a set of time functions $(\overline{z}(k), \overline{x}(k), \overline{v}(k), \overline{u}(k))$ satisfying the compensator equations such that in the compensated system $\Sigma \circ C$ the disturbances do not influence the outputs for $0 \leq k \leq k_F$ and for a certain $k_F > 0$, no matter what $v(k)$ is.

Instrumental in the solution of the formulated problem is the so-called *Singh Algorithm* and the resulting *Singh Compensator.* We only sketch the algorithm. Details can be found in e.g. [4],[1]. We consider the system $\Sigma_{\overline{q}}$ obtained from Σ by keeping the disturbances fixed to $\overline{q}(k)$. Perform SA around every point $(\overline{x}(k), \overline{u}(k), \overline{q}(k))$, $k = 0, 1, ..., k_F$. 'For every k' is to be understood accordingly in the sequel. Moreover, for $j < k$ let $i \in \mathcal{I}_{jk}$ be short hand for $j \leq i \leq k$.

Calculate $y(k+1) = h[f(x(k), u(k), \overline{q}(k))]$ and define

$$\rho^1(k) = rank\ D_u y(k+1)\Big|_{\overline{x}(k), \overline{u}(k), \overline{q}(k)}$$

where D_u denotes partial derivatives w.r.t. u. Assume that $\rho^1(k) =: \rho^1$ is constant for every k and that this rank remains constant in some neighbourhood $\mathcal{O}^1(k)$ of $(\overline{x}(k), \overline{u}(k), \overline{q}(k))$. Moreover, assume that the independent rows of $D_u y(k+1)$ are the same for every k. Permute the components of the output in such a way that the first ρ^1 rows of the matrix $D_u y(k+1)$ are linearly independent. Decompose $y(k+1)$ accordingly

$$y(k+1) = [\tilde{y}^{1^T}(k+1), \hat{y}^{1^T}(k+1)]^T$$

where $\tilde{y}^1(k+1)$ consists of the first ρ^1 (independent) components of $y(k+1)$. Since the last $p - \rho^1$ rows of the matrix $D_u y(k+1)$ are linearly dependent on the first ρ^1 rows, the corresponding components of y, viewed as functions of u and with parameters x (and $\overline{q}(k)$), are functionally dependent on the first ρ^1 components. Hence we can write

$$\begin{aligned} \tilde{y}^1(k+1) &= \tilde{a}^1(x(k), u(k), \overline{q}(k)) \\ \hat{y}^1(k+1) &= \psi^1(x(k), \overline{q}(k), \tilde{y}^1(k+1)) \ . \end{aligned}$$

Shift $\hat{y}(k+1)$ to obtain $\hat{y}(k+2)$ and calculate

$$\rho^2(k) = rank\ D_u[\tilde{y}^{1^T}(k+1), \hat{y}^{1^T}(k+2)]^T\Big|_{\overline{x}(k), \overline{u}(k), \overline{q}(k), \overline{y}(k)}$$

and proceed as above as long as the imposed assumptions are satisfied or SA terminates. Observe that the application of SA is not unique. We therefore define

Definition 3.2 *The reference trajectory is said to be regular if there is an application of SA such that all assumptions made in performing the algorithm are satisfied. It is said to be strongly regular if this holds true for an arbitrary application of the algorithm.*

It can be shown (cf. [1],[4]) that around a regular reference trajectory SA terminates in at most n steps and that the integers $\rho^1, \rho^2, \ldots$ are independent of the particular application of SA around a strongly regular reference trajectory. Let $\rho^* = \max\{\rho^l, l \geq 1\}$ and define α as the smallest $l \in \mathbb{N}$ such that $\rho^l = \rho^*$. Assume that we are working around a strongly regular reference trajectory. We then obtain a uniquely defined sequence of integers $0 \leq \rho^1 \leq \ldots \leq \rho^* \leq min(m,p)$. Applying SA around $(\overline{x}(k), \overline{u}(k), \overline{q}(k), \overline{y}(k))$ yields at the αth step:

$$\begin{aligned} \tilde{y}^1(k+1) &= \tilde{a}^1(\cdot) \\ &\vdots \\ \tilde{y}^\alpha(k+\alpha) &= \tilde{a}^\alpha(\cdot, \tilde{y}^i(k+j) : i \in \mathcal{I}_{1\alpha-1}, j \in \mathcal{I}_{i+1\alpha}) \end{aligned} \quad (2)$$

$$\hat{y}^\alpha(k+\alpha) = \psi^\alpha(\cdot, \tilde{y}^i(k+j) : i \in \mathcal{I}_{1\alpha}, j \in \mathcal{I}_{i\alpha}) \quad (3)$$

where points stand for dependence on x and u which is omitted for reasons of space. The matrix $D_u[\tilde{y}^{1^T}(k+1), \tilde{y}^{2^T}(k+2), \ldots, \tilde{y}^{\alpha^T}(k+\alpha)]^T$ has full row rank ρ^* in a neighbourhood $\mathcal{O}^\alpha(k)$ of $(\overline{x}(k), \overline{u}(k), \overline{q}(k), \overline{y}(k))$. For $i = 1, 2, \ldots, \rho^*$, let $k + \gamma_i$ be the lowest time instant and $k + \delta_i$ the highest time instant at which $\tilde{y}_i$ appears in (2). Possibly after permutation of the inputs we may assume that the Jacobian matrix of (2) with respect to $u^1 := (u_1, \ldots, u_{\rho^*})^T$ has full row rank ρ^* around the points $p(k) := (\overline{x}(k), \overline{u}(k), \{\overline{q}(k+i) : i \in \mathcal{I}_{0\alpha}\}, \{\bar{\tilde{y}}_i(k+j) : i \in \mathcal{I}_{1\rho^*}, j \in \mathcal{I}_{\gamma_i+1\delta_i}\})$. Therefore, (2) can be uniquely solved for $u^1(k)$ around $p(k)$ as a function of $x(k)$, $u^2(k) := (u_{\rho^*+1}(k), \ldots, u_m(k))$ and $\tilde{y}_i(k+j)$ by applying the Implicit Function Theorem, i.e.,

$$u^1(k) = \phi_k(x(k), \{\tilde{y}_i(k+j) : i \in \mathcal{I}_{1\rho^{\alpha-1}}, j \in \mathcal{I}_{\gamma_i+1\delta_i}\}, u^2(k)) \; .$$

Notice that, no matter how the initial state x_0 and inputs $u(\cdot)$ are chosen, the resulting trajectories may in general drift away from the reference trajectory. Therefore the solvability of (2) for u^1 can only be guaranteed up to a finite time k_F.

A SC is constructed in the following way. Let $z_i = (z_{i,1}, \ldots, z_{i,\delta_i - \gamma_i})^T$, $i \in \mathcal{I}_{1\rho^*}$, be $(\delta_i - \gamma_i)$-dimensional vectors, v^2 a vector of dimension $(m - \rho^*)$ and consider the system (4) with inputs $v^1 = (v_1, \ldots, v_{\rho^*})^T$, v^2 and outputs (u^1, u^2)

$$\begin{aligned} z_{i,1}(k+1) &= z_{i,2}(k) \\ &\vdots \qquad\qquad , \; i = 1, \ldots, \rho^* \\ z_{i,\delta_i-\gamma_i-1}(k+1) &= z_{i,\delta_i-\gamma_i}(k) \\ z_{i,\delta_i-\gamma_i}(k+1) &= v_i(k) \end{aligned} \quad (4)$$

$$\begin{aligned} u^1(k) &= \phi_k(\cdot, \{z_{i,j}(k) : j \in \mathcal{I}_{1\delta_i-\gamma_i}, v_i(k) : i \in \mathcal{I}_{1\rho^*}\}, v^2(k)) \\ u^2(k) &= v^2(k) \; . \end{aligned}$$

A trajectory about which (4) is regular is given by $\overline{z}(k), \overline{x}(k), \overline{v}(k), \overline{u}(k)$ where for $i \in \mathcal{I}_{1\rho^*}$, $j \in \mathcal{I}_{1\delta_i-\gamma_i}$

$$\overline{z} = (\overline{z}_{1,1}, \ldots, \overline{z}_{1,\delta_1-\gamma_1}, \ldots, \overline{z}_{\rho^*,\delta_{\rho^*}-\gamma_{\rho^*}})^T \; , \; \overline{v}^2(k) = \overline{u}^2(k)$$

$\overline{z}_{i,j}(k) = \bar{\tilde{y}}_i(k + \gamma_i + j - 1)$, $\overline{v}_i(k) = \bar{\tilde{y}}_i(k + \delta_i)$, $0 \leq k \leq k_F$.

Applying (4) with arbitrary initial state to $\Sigma_{\overline{\mathbf{q}}}$ yields locally for $i = 1, \ldots, \rho^*$

$$\begin{aligned} \tilde{y}_i(k + \gamma_i + j - 1) &= z_{i,j}(k), \; j = 1, \ldots, \delta_i - \gamma_i \\ \tilde{y}_i(k + \delta_i) &= v_i(k), \; 0 \leq k \leq k_F \; . \end{aligned}$$

Moreover, inspection of SA reveals that for the compensated system the outputs $\tilde{y}_i(0), \ldots, \tilde{y}_i(\gamma_i - 1)$, $i = 1, \ldots, \rho^*$, only depend on the initial conditions x_0, z_0, and $\overline{\mathbf{q}}$.

Performing SA gives in each step l a function ψ^l representing the functionally dependent part of $\hat{y}(k+l)$. Considering the effects of applying the constructed compensator to the output components $\tilde{y}_i$, it will come as no surprise that the ψ^l's will play a crucial role in solving the formulated problem. Before stating the main result of this section, let us define $\psi^0(x) := h(x)$.

Theorem 3.3 *Consider system Σ in a neighbourhood of a strongly regular reference trajectory. Apply SA to $\Sigma_{\overline{\mathbf{q}}}$. Then DDDP is finite time solvable for Σ around the given reference trajectory if and only if*

$$D_q\psi^l(f(x,u,q), \{\tilde{y}^i(k+j+1) : i \in \mathcal{I}_{1l}, j \in \mathcal{I}_{il}\}) = 0 \quad (5)$$

for $0 \leq l \leq (n-1)$ and for all (x,u,q) in a neighbourhood of $(\overline{x}(k), \overline{u}(k), \overline{q}(k))$. Moreover, DDDP can then be solved by means of the SC (4).

Solutions for nonlinear continuous-time systems can be found in [2].

4. Nonlinear DDDP and Linearization

In addition to system Σ as defined in Section 3, we consider its Jacobian linearization Σ_{li} along a strongly regular reference trajectory $(\overline{x}(k), \overline{u}(k), \overline{q}(k), \overline{y}(k))$.

$$\Sigma_{li} : \left\{ \begin{aligned} x_{li}(k+1) &= F(k)x_{li}(k) + G(k)u_{li}(k) + E(k)q_{li}(k) \\ y_{li}(k) &= H(k)x_{li}(k) \end{aligned} \right.$$

There is a close connection between SCs for Σ and Σ_{li} which is expressed in the next theorem.

Theorem 4.1 *Consider system Σ defined in a neighbourhood of a strongly regular reference trajectory and let Σ_{li} be its linearization along this trajectory. Then*

(i) The linearization of a SC for Σ is a SC for Σ_{li}.

(ii) Conversely, every SC for Σ_{li} is a first order approximation of a SC for Σ.

Suppose now that DDDP is solvable for Σ. By Theorem 3.3 it follows that for all applications of SA to Σ, we have for $0 \leq l \leq (n-1)$

$$D_q\psi^l(f(x,u,q), \{\tilde{y}^i(k+j+1) : i \in \mathcal{I}_{1l}, j \in \mathcal{I}_{il}\}) = 0 \; .$$

In [1] it has been proved that to every application of SA to Σ, there corresponds an application of SA to Σ_{li} which can be obtained by linearization. It is immediately clear that the conditions necessary for the solvability of DDDP w.r.t. Σ_{li} will also be satisfied. Unfortunately, one cannot prove the other direction without additional assumptions. The problem one faces is caused by the fact that in case DDDP is solvable for Σ_{li}, the necessary assumptions for the solvability of DDDP for Σ can only be assured using Σ_{li} for the points of the reference trajectory for Σ. We therefore have to impose additional conditions to overcome this problem. It is clear that these conditions should only be given in terms of the original system Σ. On the other hand, they should be such that their verification does not require performing SA for Σ. To obtain such conditions, we associate a number μ_i to each output component y_i of Σ as the relative degree of this component w.r.t. the disturbances. Consider the by performing SA for Σ_{li} induced permutation of the outputs y_i of Σ. Conditions (5) w.r.t. Σ are satisfied in a neighbourhood of the points of the reference trajectory (provided DDDP is solvable for Σ_{li}) if the following assumptions hold (cf. [1]):

Assumption (A) *The numbers μ_i, $i = 1, \ldots, p$, satisfy*

(i) $\mu_i > 1$.

(ii) $\mu_i > l + 1$ for μ_i's belonging to components $\tilde{y}_i^l$, $l \in \mathcal{I}_{1(\alpha-1)}$.

(iii) $\mu_i = \infty$ for μ_i's belonging to components $\hat{y}_i^l, l \in \mathcal{I}_{\alpha(n-1)}$.

One finally obtains

Theorem 4.2 *Consider system Σ together with its linearization Σ_{li} along $(\overline{x}(k), \overline{u}(k), \overline{q}(k), \overline{y}(k))$. Perform SA to Σ_{li} and let Assumption (A) be satisfied. Then DDDP is solvable for Σ if and only if (D)DDP is solvable for Σ_{li}.*

5. References

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