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THE DIFFERENCE OF TWO RENEWAL PROCESSES
LEVEL CROSSING AND THE INFIMUM

by

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ABSTRACT

We consider the difference process N of two independent renewal (counting) processes. Second-order approximations to the distribution function of the level crossing time are given. Direct application of the second-order approximation is complicated by the occurrence of an (in general) unknown term $E\tilde{M}$, which denotes the expected minimum of the stationary version of N. However, this number is obtained for a wide class of processes N, using matrix-geometric techniques. Numerical experiments have been carried out, in which the new approximations were compared to simulation, first-order and/or exact results. These results confirm that the second-order approximations are considerably better than the (known) first-order ones.

1. Introduction

Although renewal processes are among the most basic and important processes in queuing and reliability theory, some areas in renewal theory seem to have been not fully explored. In Kroese & Kallenberg [8], for example, the *difference* process $N = N_1 - N_2$ of two independent renewal processes N_1 and N_2 was studied. In applications, one is often interested in the distribution of the first time that N crosses some level n, τ_n say. For example, in a reliability model τ_n may indicate the time of a system failure. Only in very special cases it is possible to give a tractable formula for $P(\tau_n \le x)$. It is however possible to expand $P(\tau_n \le x)$ for large n. A first-order approximation to $P(\tau_n \le x)$, based on asymptotic normality, can be easily established using Cox [4] p.73. To obtain greater accuracy, a second-order approximation was obtained in [8], which reduces the approximation error from $o(1)$ to $o(n^{-1/2})$, as $n \to \infty$.

The motivation for this paper is two-fold. Firstly, we address the difficulty that direct application of the second-order approximation is complicated by the occurrence of the term $E\tilde{M} - E \inf_{t \ge 0} \tilde{N}(t)$, where $\tilde{N}$ is the stationary version of N. This number is in general intractable. However, for a large class of processes N it is possible to calculate $E\tilde{M}$ explicitly or numerically. In Sections 3-5 the distribution of the infimum of N is derived for a number of cases in which N_1 and/or N_2 has some kind of Markov structure. Moreover, the distribution function of τ_n is given for a few cases of interest (Sections 3-4). The main new results on the distribution of the infimum of N are given in Section 5, where N_1 is a Phase-type renewal process and N_2 a general delayed renewal process. This shows that the second-order approximation may be used for a wide class of N's.

Secondly, we need to verify that the second-order approximations are indeed useful for finite (small) n. Numerical experiments have been carried out to compare first- and second-order approximations to simulation or exact results. It appears (cf. Section 6) that the second-order approximations indeed give considerable improvement on the known first-order results. Numerical results show good agreement with the theoretical error structure as described in [8]. Even for small n quite satisfactory approximations are obtained. It is therefore worthwhile to use the new second-order approximation instead of the known first-order one. In these numerical examples we do not restrict ourselves to *stationary* renewal processes, although formally the expansion from which the second-order approximation is derived, has yet only been proved for the stationary case (cf. [8]). We adopt here a more pragmatic point of view and state an approximation for arbitrary delayed renewal processes. In the next section we give the definitions and model assumptions for the rest of the paper, and state the second-order approximation to $P(\tau_n \le x)$.

2. Global definitions / Approximation

The definitions and assumptions given in this section hold throughout the paper, unless otherwise specified.

Let $N_1 = (N_1(t))_{t\ge 0}$ and $N_2 = (N_2(t))_{t\ge 0}$ denote two independent delayed renewal processes with renewal sequences $(X_k^{(1)})_{k\ge 1}$ and $(X_k^{(2)})_{k\ge 1}$, respectively (cf. [8] for definitions). In other definitions N_1 and N_2 may be called renewal *counting* processes. Denote the renewal times of N_i by $S_n^{(i)} = X_1^{(i)} + .. + X_n^{(i)}$, $i=1,2$, $n \in \mathbb{N}_+$. Let F_i be the distribution function (d.f.) of $X_1^{(i)}$ and let G_i be the d.f. of $X_2^{(i)}$, $i=1,2$. We assume that the expectation μ_i, the variance σ_i^2 and the third central moment μ_{3i} of $X_2^{(i)}$ finitely exist and that G_i is non-lattice, $i=1,2$. We denote $EX_1^{(i)}$ by η_i, $i=1,2$, which we assume to be finite as well. The Laplace-Stieltjes transforms of F_i and G_i are denoted by A_i and B_i, respectively, $i=1,2$. We assume that $\mu_1 < \mu_2$, in which case N has an upward drift α^{-1},

$$\alpha^{-1} = \mu_1^{-1} - \mu_2^{-1}.$$

The *difference process* $N = (N(t))_{t\ge 0}$ of N_1 and N_2 is defined by $N(t) = N_1(t) - N_2(t)$, $t \ge 0$. Let τ_n denote the first time that N

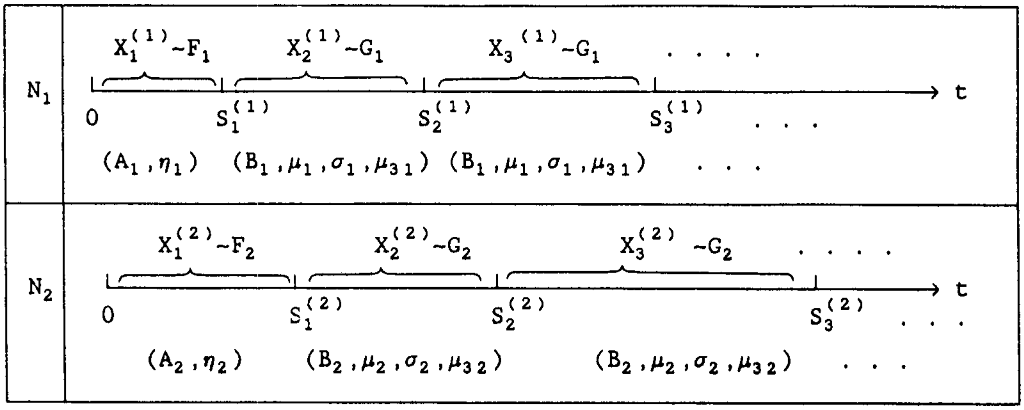

Fig. 1. *Definitions concerning the delayed renewal processes N_1 and N_2.*

crosses level $n \in \mathbb{N}_+$, that is $\tau_n = \inf\{t \geq 0 : N(t) \geq n\}$. We denote the random variable $\inf_{t \geq 0} N(t)$ by M. An overview of the definitions is given in fig. 1.

2.1 **REMARK** The *stationary version* $\tilde{N}$ of N is defined as the difference process of two stationary independent renewal processes $\tilde{N}_1$ and $\tilde{N}_2$, with corresponding inter-arrival d.f.'s $\tilde{F}_1$, $\tilde{G}_1$, $\tilde{F}_2$ and $\tilde{G}_2$, such that $\tilde{G}_i = G_i$, and (hence) $\tilde{F}_i(x) = {}_0\!\int^x (1-G_i(u))du/\mu_i$, $i=1,2$. If N is itself stationary (i.e. $F_i = \tilde{F}_i$, $i=1,2$), then $\eta_i = (\mu_i + \sigma_i^2/\mu_i)/2$, $i=1,2$ (and hence $\tilde{e} = 0$ in the next approximation).

In Kroese [7] the following second-order approximation to the d.f. of τ_n was stated. For a proof of the stationary case see [8]. In Kroese & Kallenberg [6] a similar result on the sum process of k independent delayed renewal processes was proved.

2.2 **APPROXIMATION** Let N_1 and N_2 be independent delayed renewal processes that satisfy the conditions above. Let $E\tilde{M} = E \inf_{t \geq 0} \tilde{N}(t)$ be the expected infimum of the stationary version of N (cf. Remark 2.1), then $|E\tilde{M}| < \infty$ and moreover,

$$(2.3) \qquad P(\tau_n \le x) \approx \Phi(y_n) + \frac{\varphi(y_n)}{\sqrt{n}} \left\{ p(1-y_n^2) + q \right\},$$

where Φ denotes the standard normal d.f., and φ its density function, and

$$y_n = \frac{x - n\alpha}{\gamma\alpha^{3/2}\sqrt{n}}, \qquad p = \frac{(c_1-d_1)(\gamma a_1)^{-3} - (c_2-d_2)(\gamma a_2)^{-3}}{\sqrt{\alpha}} + \frac{\gamma\sqrt{\alpha}}{2},$$

$$q = \frac{1}{2\gamma\sqrt{\alpha}} - \frac{\gamma\sqrt{\alpha}}{2} - \frac{\tilde{e}}{\sqrt{\alpha}} - \frac{1}{\gamma\sqrt{\alpha}} E\tilde{M}, \qquad \gamma = \sqrt{\sigma_1^2\mu_1^{-3} + \sigma_2^2\mu_2^{-3}},$$

$$a_i = \mu_i^{3/2}\sigma_i^{-1}, \qquad c_i = \frac{1}{6}\mu_{3i}\sigma_i^{-3}\mu_i^{1/2}, \qquad d_i = \frac{1}{2}\sigma_i\mu_i^{-1/2},$$

$$e_i = -d_i - a_i/2 + \mu_i^{1/2}\sigma_i^{-1}\eta_i, \quad i=1,2 \quad \text{and} \quad \tilde{e} = e_1(\gamma a_1)^{-1} - e_2(\gamma a_2)^{-1}.$$

2.4 **REMARK** A first-order approximation is obtained from (2.3) by suppressing the second term in the right-hand side of (2.3). The approximation error is in this case of order $o(1)$, as $n\to\infty$, whereas the second-order approximation (presumably) yields an error of order $o(n^{-1/2})$, as $n\to\infty$. For a proof in the stationary case, cf. [8].

Note that $E\tilde{M}$ is not specified further because it is in general intractable. This complicates the direct application of the approximation. However, we can derive this quantity for a number of important cases for which the exact d.f. of τ_n is very difficult to compute (or cannot be computed at all) and hence an accurate approximation is welcome. In Sections 3-5 we consider typical cases for which $E\tilde{M}$ can be computed. In each case at least one of the renewal processes has a Markov structure (phase-type or Poisson).

A similar approximation can be found for the difference process of two processes K_1 and K_2, which themselves are sum processes of a finite number of independent renewal processes, cf. [7].

3. N_1 Delayed, N_2 Poisson

In this section N_1 is a delayed renewal process and N_2 a Poisson process. We first give an explicit expression for the d.f. of τ_n in terms of μ_2, F_1 and G_1. This expression is in general too complicated to be handled numerically. However a simple expression for EM, in terms of the first few moments of F_1 and G_1 is found (along with the generating function of M) which makes the second-order approximation very simple to evaluate. A numerical example is given in section 6.2.

3.1 **PROPOSITION** Let N_1 be an <u>ordinary</u> renewal process ($F_1 = G_1$). Denote the n-fold convolution of an arbitrary function F by F^{n*}. The d.f. of τ_n is given by

(3.2) $$P(\tau_n \le t) = H^{n*}(t),\ t \ge 0,$$

where H is defined by

(3.3) $$H(t) = \sum_{k=1}^{\infty} \int_0^t \frac{(u/\mu_2)^{k-1}}{k!} e^{-u/\mu_2}\, dG_1^{k*}(u),\ t \ge 0.$$

<u>Proof</u> Let T_i be the first entrance time of N into level i, starting from level $i-1$, $i=1,..,n$. Note that since N_2 is a Poisson process, $T_1, T_2,..., T_n$ are i.i.d. random variables. Moreover $\tau_n = T_1 + T_2 + .. + T_n$. Since we can regard T_1 as the busy period of an M/G/1 queue, we have by Cohen [3], p.250, that $P(T_1 \le t) = H(t)$ and hence (3.2) follows. □

Next we consider the distribution of M and derive an expression for its expectation. We do not restrict ourselves to ordinary renewal processes N_1 this time. It is seen in (3.6) that the expression for EM is easily computed and depends only on the first two moments of G_1,

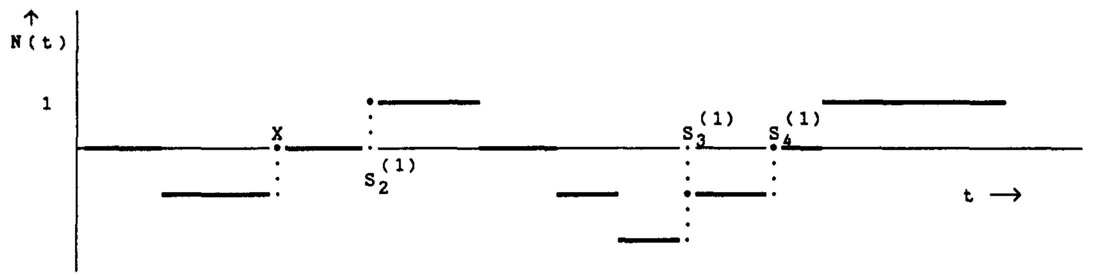

Fig. 2. *Difference of renewal processes. N_1 delayed, N_2 Poisson.*

the first moment of F_1 and μ_2. For all $x \in \mathbb{R}$, let $x^- = min(0,x)$ and $x^+ = max(0,x)$.

3.4 THEOREM For all $r \in \mathbb{C}$, $|r|\leq 1$ we have

$$(3.5) \qquad Er^{-M} = \frac{A_1((1-r)/\mu_2)(r-1)(1-\mu_1/\mu_2)}{r - B_1((1-r)/\mu_2)},$$

and moreover,

$$(3.6) \qquad EM = -\eta_1/\mu_2 - \frac{(\sigma_1^2+\mu_1^2)/\mu_2^2}{2(1-\mu_1/\mu_2)}.$$

<u>Proof</u> First let N_1 be an <u>ordinary</u> renewal process. Let M_n be the minimum of N until time $S_n^{(1)}$, that is

$$(3.7) \qquad M_n = min\ (\ N(S_1^{(1)}),\ldots,N(S_n^{(1)})\ ) - 1\ ,\ n \geq 1.$$

Define

$$\tilde{M}_n = min\ (\ N(S_2^{(1)}),\ldots,N(S_{n+1}^{(1)})\ ) - N(X_1^{(1)}) - 1\ ,\ n \geq 1.$$

Note that $M = \inf_{n\geq 1} M_n$. Denote $S_1^{(1)} - X_1^{(1)}$ by X. It is easy to see that M_n and $\tilde{M}_n$ have the same distribution. Moreover $\tilde{M}_n$ and $N(X)$ are independent and the following recurrence relation holds (see fig. 2 for an illustration)

(3.8) $$M_{n+1} = [\tilde{M}_n+1]^- + N(X) - 1,\ n \geq 1.$$

There are several ways to obtain the generating function of M. Note for example that M is the minimum of the random walk $(N(S_n^{(1)}) - 1;\ n \geq 1)$. We adopt however another point of view here. A short reflection will show that $-M_n$ has the same distribution as the number of customers U_n in an M/G/1 system, just after the departure of the nth customer. For, if we denote the number of customers that arrive during the service of the nth customer by Z_n, we have for all $n\geq 1$, $U_n = [U_{n-1}-1]^+ + Z_n$, $U_0 = 0$, $Z_n \sim -N(X) + 1$ and the random variables U_{n-1} and Z_n are independent. Hence by (3.8), $M_n \sim -U_n$, for all $n\geq 1$. Therefore the generating function of M is by Cohen [3], Ch.4 §2 (4.17) equal to

(3.9) $$Er^{-M} = \frac{B_1((1-r)/\mu_2)(r-1)(1-\mu_1/\mu_2)}{r - B_1((1-r)/\mu_2)}.$$

Next let N_1 be a <u>delayed</u> renewal processes (F_1 is not necessarily equal to G_1).
Similarly to (3.8) we have

(3.10) $$M \sim [M^*+1]^- + N(X) - 1,$$

where M^* and X are independent random variables. M^* is distributed as the random variable M in (3.9) and, for $|r|\leq 1$, $r \in \mathbb{C}$,

(3.11) $$Er^{-N(X)+1} = E\, E_X r^{N_2(X)} = Ee^{-(1-r)X/\mu_2} = A_1((1-r)/\mu_2).$$

Hence,

(3.12) $$Er^{-M} = Er^{-[M^*+1]^-} A_1((1-r)/\mu_2).$$

Moreover,

(3.13) $$\begin{aligned} Er^{-[M^*+1]^-} &= (1-r^{-1})P(M^*=0) + r^{-1}Er^{-M^*} \\ &= (1-r^{-1})(1-\mu_1/\mu_2) + r^{-1}Er^{-M^*}. \end{aligned}$$

so that (3.5) follows from (3.9), (3.12) and (3.13). (3.6) follows directly from (3.5) after some calculations. N.B.: In (3.11) E_X denotes conditional expectation w.r.t. X. □

3.14 **REMARK** The d.f. of τ_n in (3.3) is difficult to evaluate. Matters become even more complicated (numerically) in the case that N_2 is an arbitrary phase-type renewal process. However, the interpretation of τ_n as a busy period of a PH/GI/1-queue (starting with n customers) is still valid, and a generalization of Proposition 3.1 can be given for this case. Suppose that N_2 is a phase-type renewal process with m phases. Let V_r be the number of transitions involved in crossing level $r\geq 1$ (the crossing time is τ_r), and let J_r denote the phase of N_2 at time τ_r. Denote by P_i the probability measure under which N_2 starts at phase i, $i \in \{1,...,m\}$. By Neuts [10], Chapter 2, we can interpret τ_r as the first passage time from level r to level 0 of a Markov renewal process of M/G/1 type. Hence, if we write $G_{ij}^{(r)}(k;x)$ for $P_i(\tau_r\leq x,\ J_r=j,\ V_r=k)$, $r\geq 1$, $x\geq 0$, $k\geq 0$, $1\leq i,j\leq m$, then, formally, a generalization of Proposition 3.1 is given by Lemma 2.2.1 and Theorem 2.2.1 of Neuts [10]. Is seems not very likely that an easy to compute analogue of equations (3.5) or (3.6) can be obtained for this "Delayed, Phase-type" case.

4. N_1 Poisson, N_2 Poisson

In this section N_1 and N_2 are Poisson processes with intensities $\lambda = \mu_1^{-1}$ and $\nu = \mu_2^{-1}$, respectively. Of course this model is a special case of the one considered in the previous section. The reason for including this particular case is that the d.f. of τ_n can actually be evaluated numerically. This gives us the opportunity to compare the exact d.f. of τ_n with the first- and second-order approximation. This is done for several values of $\lambda - \nu$ in Section 6.3.

4.1 **PROPOSITION** Let I_n denote the modified Bessel function of order n, then

$$P(\tau_n\leq t) = \int_0^t e^{-(\lambda+\nu)u}\, n\lambda^{n/2}\nu^{-n/2}\, u^{-1}\, I_n(2u\sqrt{\lambda\nu})du.$$

Proof Let f denote the probability density of the distribution of τ_n and denote its Laplace transform by d. Let a be the Laplace transform of T_1, where T_1 is defined in the previous section. Define for $0 \le s \le 1$ the functions $g_s : \mathbb{C} \to \mathbb{C}$ by

$$g_s(z) = z - \frac{\lambda}{s + \nu(1-z) + \lambda}, \quad z \in \mathbb{C}.$$

By Cohen [3], p.250, $a(s)$ is the only zero of g_s inside the unit circle, thus

$$a(s) = \frac{\lambda+\nu+s - \sqrt{(\lambda+\nu+s)^2-4\lambda\nu}}{2\nu}$$

and hence, in view of (3.2),

$$d(s) = \left[\frac{\lambda+\nu+s - \sqrt{(\lambda+\nu+s)^2-4\lambda\nu}}{2\nu} \right]^n$$

After some manipulation with inverse Laplace transforms (cf. for example [12]), we find $f(t) = e^{-(\lambda+\nu)t} n \lambda^{n/2} \nu^{-n/2} I_n(2t\sqrt{\lambda\nu})/t$. □

5. N_1 Phase-Type, N_2 Delayed

In this section N_1 is a Phase-type renewal process (PH-renewal process) and N_2 a delayed renewal process. This is the most general case (together with the "Delayed, Phase-type" case) for which one could expect to obtain explicit, numerically feasible expressions for the distribution of M. Since phase-type distributions are widely used in applications, and since any lifetime distribution G_1 of renewal process N_1 can be approximated by phase-type distributions, this case is also of practical relevance. In contrast to the two previous sections it is not possible to write τ_n as the sum of n i.i.d. random variables here. We can however still derive the distribution of M. An essential difference with the model in Section 3 is that there EM only depended on the first two moments of F_1 and G_1, but here EM possibly depends on all moments of F_2 and G_2. Numerical results are given in Sections 6.4-6.5.

Let $p \in \mathbb{N}_+$ and let γ denote a probability distribution on $\{1,2,..,p\}$, we use the same letter γ to denote the corresponding row vector. We introduce a family of probability measures (P^{γ}) on our probability space $(\Omega,\mathcal{H})$ such that under P^{γ} N_1 is a PH-renewal process with p phases, characteristics (β,S) and with initial phase distribution γ (cf. Neuts [9] for definitions on PH-renewal processes). Moreover, under these probability measures we let N_2 be a delayed renewal process, independent of N_1. Let P_i denote the probability measure under which N_1 starts at phase i, $i \in \{1,..,p\}$, in other words $P_i = P^{\delta_i}$, where δ_i is the Dirac measure at i on $\{1,..,p\}$. Define the $p \times p$ matrices $P(n,t)$, $n \in \mathbb{N}$, $t \geq 0$, by $(P(n,t))_{ij} = P_i(N_1(t)=n,\ Z_t=j)$, $i,j \in \{1,..,p\}$, where Z_t denotes the phase of N_1 at time t. Define $p \times p$ matrices $A(k)$ and $\tilde{A}(k)$, $k \in \mathbb{N}$ by

$$A(k) = \int_0^{\infty} P(k,t)dG_2(t) \quad and \quad \tilde{A}(k) = \int_0^{\infty} P(k,t)dF_2(t).$$

Not surprisingly, these matrices are the same as the ones that frequently arise in PH/G/1- or GI/PH/1-queuing systems, see for example [9], [10] and [11].

In the next theorem we give the distribution of M. Note that M is the limit of the decreasing sequence of random variables (M_n), $M_n = \min\{0,\ N(S_1^{(2)}),..,N(S_n^{(2)})\}$, $n \geq 0$. We denote $X_1^{(2)}$ and $N(X_1^{(2)})$ by X and Y, respectively. And Z denotes the phase of N_1 at time X. Let $1 = [1,..,1]^T$.

5.1 THEOREM Let G be the unique minimal non-negative solution to the matrix equation

$$X = \sum_{k=0}^{\infty} A(k)X^k, \tag{5.2}$$

then for $m \in \mathbb{N}_+$,

$$P^{\gamma}(M = -m) = \gamma\, H(I-G)G^{m-1}\, 1, \tag{5.3}$$

and hence

$$P^{\gamma}(M = 0) = \gamma\, (I-H)\, 1,$$

where

$$H = \sum_{k=0}^{\infty} \tilde{A}(k)G^k. \tag{5.4}$$

As a consequence we have,

$$E^{\gamma}M = -\gamma H(I-G)^{-1}1. \tag{5.5}$$

Proof First suppose N_2 is an ordinary renewal process ($F_2 = G_2$). Denote the negative ladder epochs of $\{N(S_n^{(2)})\}$ by T_1, T_2,.. put $T_0=0$ and, for $n\geq 1$, let J_n denote the phase of N_1 at time T_n if $T_n < +\infty$, and $J_n = \partial$ else, for some cemetery state ∂. Let G be the matrix (g_{ij}), where $g_{ij} = P_i(J_1=j)$, $i,j \in \{1,..,p\}$. We first show that if G is sub-stochastic then

$$\mathbb{P}(M=-m) = (I-G)G^m 1, \quad \text{for all } m \in \mathbb{N}, \tag{5.6}$$

where $\mathbb{P}(M=m) = [P_1(M=m),..,P_p(M=m)]^T$. The proof of (5.6) goes as follows: Obviously (J_n) is a Markov chain taking values in $\{1,..,p\}\cup\{\partial\}$, with transition matrix

$$Q = \begin{bmatrix} G & (I-G)1 \\ 0 & 1 \end{bmatrix}.$$

Because $M = \min\{0, N(S_1^{(2)}), N(S_2^{(2)}),...\}$, we have for all $m \in \mathbb{N}_+$

$$P_i(M=-m) = P_i(J_{m+1} = \partial,\ J_1,..,J_m \neq \partial) \tag{5.7}$$

$$= \sum_{j=1}^{p} P_i(J_{m+1}=\partial,\ J_2,..,J_m \neq \partial \mid J_1=j)P_i(J_1=j) = \sum_{j=1}^{p} P_j(M=-m+1)P_i(J_1=j),$$

which leads to (5.6), provided that G is sub-stochastic (which we will show later). Next we show that G satisfies matrix equation (5.2). This follows immediately from Neuts [10], Chapter 2. Namely, matrix G is exactly the same as matrix G defined on Page 81 of [10]. Hence by Theorem 2.2.2 and (2.215) of [10] we deduce that G is the minimal non-negative solution to (5.2).

It remains to show that G is sub-stochastic and hence that (5.6) holds. This follows from an argument from PH/G/1-queues. Matrix G (the *rate matrix*) is stochastic if and only if the mean service time σ is smaller than or equal to the mean inter-arrival time λ, cf. Neuts [9] p. 122. In our case $\sigma = \mu_2 > \mu_1 = \lambda$ by assumption, which implies that G is sub-stochastic.

Finally consider a <u>delayed</u> renewal process N_2. Let $M = \inf_{t\geq 0} N(t)$ and let M^* denote a random variable with the same distribution as the M if N_2 were an <u>ordinary</u> renewal process (in other words, the distribution of M^* is given by (5.7)). Since now $P_i(Y=k-1,\ Z=j) = \tilde{A}(k)_{ij}$, we have for $i \in \{1,..,p\}$ and $m \in \mathbb{N}_+$,

$$(5.8) \qquad P_i(M=-m) = \sum_{k=0}^{\infty} \sum_{j=1}^{p} P_i(M=-m \mid Y=k-1,\ Z=j) P_i(Y=k-1,\ Z=j)$$
$$= \sum_{k=0}^{\infty} \sum_{z=1}^{p} P_j(M^*=-m-k+1)\tilde{A}(k)_{ij},$$

which in matrix notation is

$$(5.9) \qquad \mathbb{P}(M=-m) = \sum_{k=0}^{\infty} \tilde{A}(k)(I-G)G^{m+k-1}1 = H(I-G)G^{m-1}1, \text{ for } m \in \mathbb{N}_+,$$

so that $\mathbb{P}(M=0) = (I-H)1$. This concludes the proof, since $P^{\gamma}(M=-m) = \gamma\,\mathbb{P}(M=-m)$, for all $m \in \mathbb{N}$. □

5.10 COROLLARY If N_1 is a Poisson process with intensity $\lambda = 1/\mu_1$, then the generating function of M is given by

$$(5.11) \qquad Er^{-M} = 1 + \frac{(r-1)A_2(\lambda(1-g))}{1 - rg}, \text{ for all } r \in \mathbb{C},\ |r|\leq 1,$$

and hence

$$(5.12) \qquad EM = -\frac{A_2(\lambda(1-g))}{1 - g},$$

where g is the only zero of $f(y) = y - B_2(\lambda(1-y))$ inside the unit circle. (Remember that A_2 and B_2 are the Laplace-Stieltjes-transforms of F_2 and G_2, respectively.)

<u>Proof</u> Let $0 \leq g < 1$ denote the minimal non-negative solution to (5.2), which reduces in this case to

$$(5.13) \qquad x = \sum_{k=0}^{\infty} \int_0^{\infty} x^k \frac{(\lambda t)^k}{k!} e^{-\lambda t} dG_2(t) = B_2(\lambda(1-x)).$$

By the proof of Theorem 5.1, we know that $0\leq g<1$. By Rouché's theorem the function $f:\mathbb{C}\to\mathbb{C}$, defined by $f(y) = y - B_2(\lambda(1-y))$, $y \in \mathbb{C}$ has only one solution inside the unit circle, which must therefore be equal to g. Let $h = H$ as defined in (5.4), then similarly to (5.13) we have $h = A_2(\lambda(1-g))$, so that (5.11) and (5.12) follow easily from equations (5.3) and (5.5). □

5.14 REMARK In Neuts [10] is shown that G can be computed by means of succesive substitutions in (5.2). Direct application of this procedure requires however a lot storage and is numerically not very simple. An efficient algorithm for solving the non-linear matrix equation (5.2) is given in Lucantoni & Ramaswami [11], where it is shown that G also satisfies the following system of equations

$$(5.15) \qquad X = \sum_{n=0}^{\infty} \gamma_n L_n$$

$$(5.16) \qquad \begin{cases} L_0 = I \\ L_{n+1} = PL_n + P^0 \beta L_n X \ , \ n \geq 0 \end{cases} ,$$

where $\theta = \max_{1\leq i\leq m} -S_{ii}$, $\gamma_n = \int_0^\infty e^{-\theta t}\frac{(\theta t)^n}{n!} dG_2(t)$, $P = \theta^{-1}S + I$, $P^0 = 1 - P1$.

Remember that (β,S) are the characteristics of N_1. The matrix G can be computed by iterating equations (5.15) and (5.16) by starting with the initial iterate $X(0) = 0$. Moreover, a short reflection will show that matrix H is given by

$$(5.17) \qquad H = \sum_{n=0}^{\infty} \tilde{\gamma}_n L_n, \quad \text{where} \quad \tilde{\gamma}_n = \int_0^\infty e^{-\theta t} \frac{(\theta t)^n}{n!} dF_2(t), \ n \in \mathbb{N}.$$

5.18 REMARK Other closely related results on first-passage times and busy periods may be found in Hsu & He [5] and Asmussen [1], Chapter X.

6. Numerical results

To verify how well the asymptotic theory applies for finite n, several experiments were carried out. Exact calculations were performed for the case of Section 4. FORTRAN routines from the NAG library were used to compute modified Bessel functions and to perform numerical integration. In all other cases the true d.f. of τ_n was estimated through Monte-Carlo simulation of size 10000. This simulation size reduces the error in the estimated d.f. to less than 0.01 with confidence 0.95.

For each example (except 6.3) a table is presented with the (estimated) true d.f of τ_n, the (normal) first-order approximation and the second-order approximation which is given in (2.3). The first-order approximation is obtained from (2.3) by suppressing the second term on the right-hand side. Along with these tables come pictures of the approximation errors for the first- and second-order approximation. In the picture we made use of the unrounded numerical results.

6.1 REMARK Note that in the second-order approximations we need to calculate $E\tilde{M}$ through the procedures described in Sections 3-5. Now it is important to remember that although N_1 and N_2 can be arbitrary delayed renewal processes characterized by F_1, F_2, G_1 and G_2, the number $E\tilde{M}$ is the expected infimum of $\tilde{N}$, the stationary version of N (see Remark 2.1).

6.2 N_1 WEIBULL, N_2 POISSON

In our first example N_2 is a Poisson process with intensity ν, N_1 is stationary and G_1 is a Weibull d.f. with shape parameter β and scale parameter c, i.e.

$$G_1(x) = 1 - exp((-cx)^{\beta}),\ x\geq 0.$$

For the parameter values we take $\beta = 3$, $c = 0.3$, $\nu = 1/\mu_2 = 0.2$ and $n = 20$. Note that since G_1 is a Weibull d.f.,

$$\mu_1 = \Gamma(1+1/\beta)/c \quad and \quad \sigma_1^2 = \Gamma(1+2/\beta)/c^2 - \left(\Gamma(1+1/\beta)/c\right)^2$$

(cf. p.31 of Beichelt & Franken [2]). Here Γ is the Gamma function. Since N_1 is stationary, we have $\eta_1 = \left(\mu_1 + \sigma_1^2/\mu_1\right)/2$, where in the

Table 1. *Estimated d.f. of τ_n together with first- and second-order approximation. Here N_1 is a stationary renewal process such that G_1 has a Weibull d.f. with parameters β=3.0, c=0.3. N_2 is a Poisson process with intensity 0.2 . Level n=20.*

X	ESTIMATED D.F.	NORMAL APPR.	2ND.ORDER APPR.
58.90	0.00	0.02	0.00
67.73	0.00	0.04	0.01
76.55	0.02	0.05	0.03
85.37	0.05	0.08	0.07
94.19	0.09	0.12	0.11
103.01	0.16	0.16	0.18
111.83	0.25	0.21	0.25
120.65	0.33	0.27	0.33
129.47	0.43	0.34	0.42
138.29	0.52	0.42	0.51
147.11	0.60	0.50	0.60
155.93	0.67	0.58	0.67
164.75	0.74	0.66	0.73
173.57	0.79	0.73	0.79
182.39	0.83	0.79	0.83
191.21	0.86	0.84	0.86
200.03	0.89	0.88	0.88
208.85	0.92	0.92	0.91
217.67	0.94	0.95	0.92
226.49	0.95	0.96	0.94

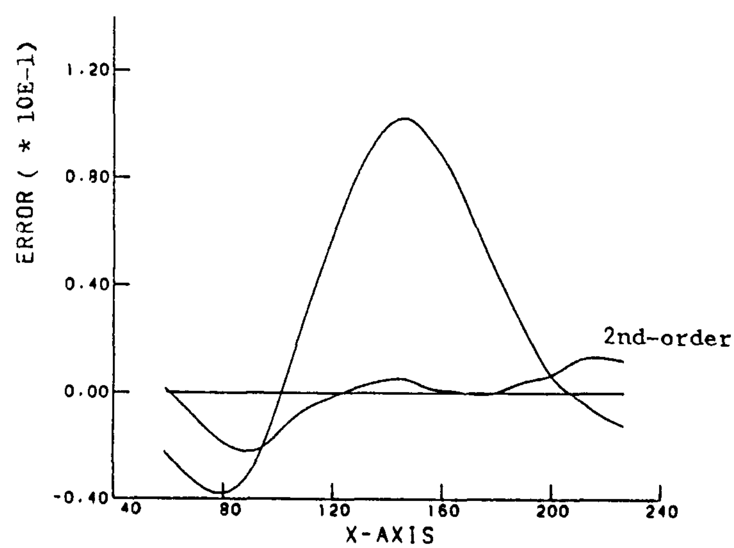

Fig. 3. *Error structure for the first- and second-order approximation of table 1.*

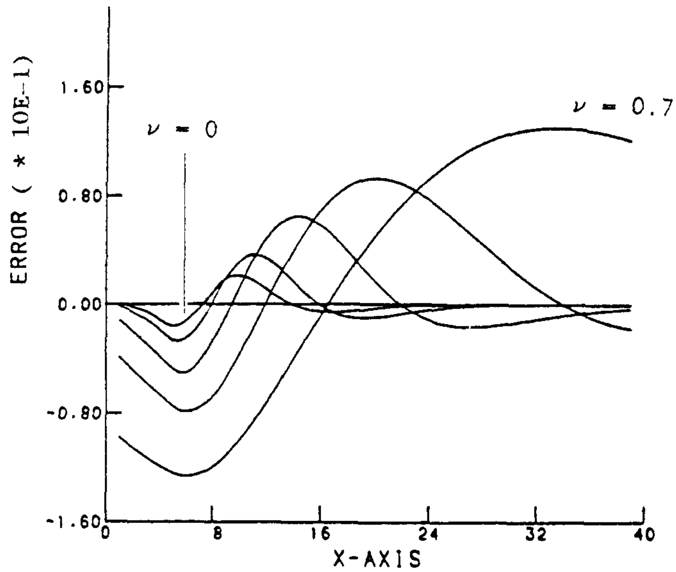

Fig. 4. *Error structure for second-order approximation to the d.f. of* τ_n *in the case that* N_1 *and* N_2 *are Poisson processes with* $\lambda=1$. *Here* $n=10$ *and* ν *takes the values 0.0, 0.1, 0.3, 0.5 and 0.7.*

present case we have $\mu_1 = 2.976598342$ and $\sigma_1 = 1.0818341148$, so that $\eta_1 = 1.6848935579$. Hence, by (3.6) we find $EM = E\tilde{M} = -0.83270345$. Note that, since we deal here with stationary renewal processes, $\tilde{e} = 0$ in Approximation 2.2 (cf. Remark 2.1).

6.3 N_1 POISSON, N_2 POISSON

We can expect poor results of approximation 2.2 when the drift term α^{-1} is to close to 0. In order to study the influence of α^{-1} on the approximation we compare the second-order approximation for the doubly Poisson case of Section 4 with the exact d.f. of τ_n, given in Proposition 4.1. This is done for several values of α^{-1}. Specifically, we fix the intensity λ of N_1 to 1 and let the intensity ν of N_2 vary from $\nu=0$, 0.1, 0.3, 0.5 to 0.7 and we let $n=10$.

6.4 N_1 POISSON, N_2 HYPER-EXPONENTIAL

In this example N_1 is a Poisson process and N_2 a ordinary renewal process with hyper-exponential d.f. G_2, that is

$$G_2(x) = p(1-\exp(-\nu_1 x)) + (1-p)(1-\exp(-\nu_2 x)),$$

for some $0 \le p \le 1$ and $\nu_i \ge 0$, $i=1,2$. Here we take $\nu_1 = 1$, $\nu_2 = 2$, $p = 1/3$ and $\lambda = 2$. Moreover, we take $n = 15$. It is easy to see that

$$B_2(s) = p\,\frac{\nu_1}{\nu_1+s} + (1-p)\,\frac{\nu_2}{\nu_2+s}.$$

We need to calculate $E\tilde{M}$, which is obtained by applying Corollary 5.10 to the stationary version $\tilde{N}$ of N. Let $\tilde{F}_2$ denote the d.f. of the first inter-arrival epoch of $\tilde{N}_2$, as in Remark 2.1, and let $\tilde{A}_2$ denote the corresponding Laplace-Stieltjes-transform. Then it is not difficult to see that

$$\tilde{A}_2(s) = q\,\frac{\nu_1}{\nu_1+s} + (1-q)\,\frac{\nu_2}{\nu_2+s},$$

where

$$q = \frac{p\nu_2}{p\nu_2 + (1-p)\nu_1}.$$

In order to apply (5.12), we need to find g, the only zero inside the unit circle of function f, given by

$$f(y) = y - \frac{1/3}{1+2(1-y)} - \frac{4/3}{2+2(1-y)}, \quad |y|<1,$$

which is

$$g = (15 - \sqrt{33})/12 = 0.771286446.$$

And with $\tilde{A}_2(\lambda(1-g)) = 3/4$, we find $E\tilde{M} = -3.279210992$.

6.5 N_1 Erlang2, N_2 Erlang2

In the last numerical example we apply the theory developed in Section 5. We use the same notation as in Section 5. Let N_1 be an ordinary

Table 2. *Estimated d.f. of τ_n together with first- and second-order approximation. Here N_1 is a Poisson process with intensity 2 and N_2 is a ordinary renewal process such that G_2 has a hyper-exponential d.f. with parameters $\nu_1 = 1$, $\nu_2 = 2$, $p = 1/3$. Level $n=15$.*

X	ESTIMATED D.F.	NORMAL APPR.	2ND.ORDER APPR.
0.00	0.00	0.08	0.02
3.66	0.00	0.11	0.07
7.31	0.03	0.15	0.13
10.97	0.11	0.19	0.20
14.63	0.22	0.24	0.28
18.28	0.33	0.29	0.36
21.94	0.45	0.35	0.45
25.60	0.54	0.42	0.53
29.25	0.62	0.49	0.61
32.91	0.68	0.55	0.67
36.57	0.73	0.62	0.72
40.22	0.78	0.68	0.76
43.88	0.81	0.74	0.79
47.54	0.84	0.79	0.81
51.19	0.87	0.84	0.83
54.85	0.89	0.88	0.84
58.51	0.91	0.91	0.85
62.16	0.92	0.93	0.87
65.82	0.93	0.95	0.88
69.48	0.94	0.97	0.90

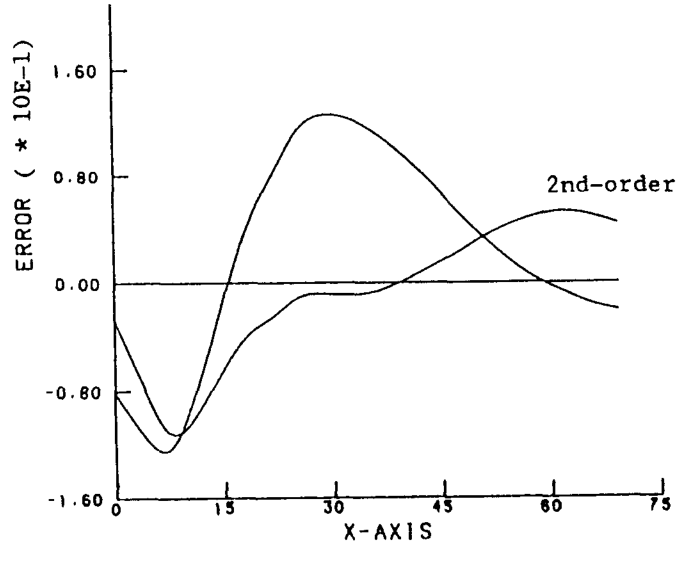

Fig. 5. *Error structure for the first- and second-order approximation of table 2.*

Table 3. *Estimated d.f. of τ_n together with first- and second-order approximation. Here N_1 and N_2 are ordinary Erlang2-renewal processes and level n=10.*

X	ESTIMATED DIST.FNC	NORMAL APPR.	2ND.ORDER APPR.
0.00	0.00	0.03	-0.01
3.03	0.00	0.04	0.00
6.07	0.00	0.06	0.02
9.10	0.02	0.09	0.06
12.13	0.06	0.12	0.11
15.16	0.14	0.17	0.18
18.20	0.24	0.22	0.27
21.23	0.35	0.28	0.36
24.26	0.45	0.35	0.45
27.30	0.55	0.43	0.55
30.33	0.63	0.51	0.63
33.36	0.71	0.59	0.70
36.40	0.76	0.66	0.76
39.43	0.81	0.73	0.80
42.46	0.85	0.79	0.83
45.49	0.88	0.84	0.85
48.53	0.90	0.89	0.87
51.56	0.92	0.92	0.89
54.59	0.94	0.95	0.91
57.63	0.95	0.96	0.92

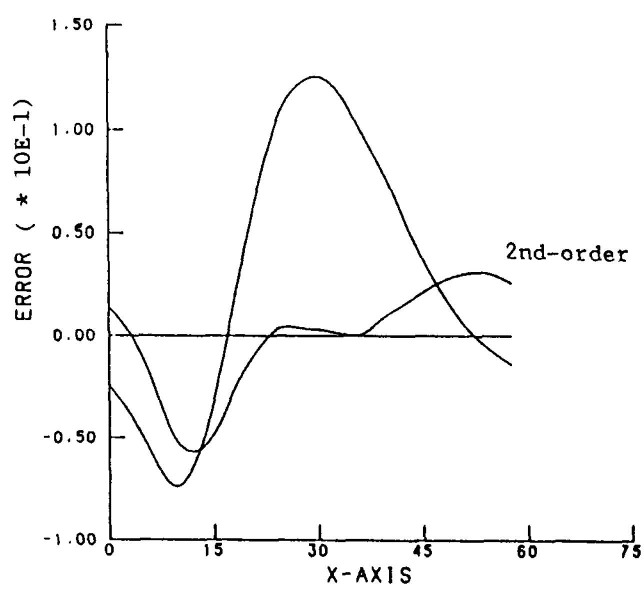

Fig. 6. *Error structure for the first- and second-order approximation of table 3.*

PH-process with characteristics (β, S) (under P^{β}), given by

$$S = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, \text{ and } \beta = [1,0],$$

and let N_2 be an ordinary renewal process with (generalized) Erlang2 inter-arrival d.f. G_2, given by

$$G_2(x) = 1 + \frac{\nu_1\nu_2}{\nu_2-\nu_1}\left(\frac{e^{-\nu_2 x}}{\nu_2} - \frac{e^{-\nu_1 x}}{\nu_1}\right), \quad x\geq 0,$$

where we take $\nu_1=1$ and $\nu_2=2$. Note that G_1 is an Erlang2 d.f. as well. Moreover, N_1 is an ordinary renewal process only under P^{β}. Under P^{γ}, with $\gamma = [1/2, 1/2]$, N_1 is stationary. We take $n = 10$.

$E\tilde{M}$ is obtained by application of Theorem 5.1 to the stationary version $\tilde{N}$ of N. Using the iteration scheme of Remark 5.14, we find that the matrices G and H are equal to

$$G = \begin{bmatrix} 0.2312992 & 0.3382979 \\ 0.078248025 & 0.3457446 \end{bmatrix} \text{ and } H = \begin{bmatrix} 0.3333333 & 0.3333333 \\ 0.077099734 & 0.4460993 \end{bmatrix}.$$

Application of (5.5) with $\gamma = [1/2, 1/2]$, yields $E\tilde{M} = -1.12027$.

6.6 CONCLUSIONS

Several other experiments have been carried out, all giving similar results. The second-order approximations indeed give considerable improvement on the first-order approximations, even for quite small n. The results show good agreement with the theoretical error structure as described in [8]. Compared to the first-order approximation, the new one has additional corrections for systematic shift and skewness, corresponding to the terms $q\varphi(y_n)n^{-1/2}$ and $p(1-y_n^{-1/2})\varphi(y_n)n^{-1/2}$, respectively, in (2.3). In [6] similar results for the sum process of k independent renewal processes were found. Approximation 2.2 tends to give the best results for t near the expectation of τ_n. In the various examples we see that for values of t that are relatively far away from this expectation the second-order approximation is only slightly better than the first-order one. This is partly due to the relative smallness of q in these cases, which means that there is only an additional correction for skewness in (2.3), when compared to the first-order approximation. But perhaps these kind of approximations are not very suited for such small or large t, and one should be looking for other approximations in this range. When α^{-1} is close to zero, application of approximation 2.2 is not very appropriate, as indicated by figure 4. In the case that $\alpha^{-1} = 0$, τ_n has, when properly normalized, for large n approximately a stable distribution of order 1/2, and not a normal distribution.

REFERENCES

[1] Asmussen, S. (1987). *Applied Probability and Queues*. Wiley & Sons, New York.

[2] Beichelt, F. & Franken, P. (1983). *Zuverlässigkeit und Instandhaltung: Mathematische Methoden*. VEB Verlag, Berlin.

[3] Cohen, J.W. (1982). *The Single Server Queue*. North-Holland, Amsterdam.

[4] Cox, D.R. (1962). *Renewal Theory*. Methuen, London.

[5] Hsu & He (1991). The distribution of the First Passage Time for the Markov processes of GI/M/1 type. To appear in *Stochastic Models* 7 (3) 397-417.

[6] Kroese, D.P. & Kallenberg, W.C.M. (1989). Approximations to the lifetime distribution of k-out-of-n systems with cold standby. *Math. Oper. Res.* 14 (3) 485-501.

[7] Kroese, D.P. (1990). The difference of two renewal processes: Level crossing and the infimum. Memorandum No. 845, University of Twente.

[8] Kroese, D.P. & Kallenberg, W.C.M. (1991). Second-order Asymptotics in Level Crossing for Differences of Renewal Processes. To appear in *Stochastic Processes*.

[9] Neuts, M.F. (1981). *Matrix-Geometric Solutions in Stochastic Models : An Algorithmic Approach*. The Johns Hopkins University Press, Baltimore.

[10] Neuts, M.F. (1989). *Structured Stochastic Matrices of M/G/1 Type and Their Applications*. Marcel Dekker, Inc., New York.

[11] Lucantoni, D.M. & Ramaswami, V. (1985). Efficient algorithms for solving the non-linear matrix equations arising in phase type queues. *Stochastic Models* 1 29-51.

[12] Spiegel, M.R. (1965). *Schaum's outline of Theory and Problems of Laplace Transforms*. McGraw-Hill, New York.

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