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On a conjecture of Nikiforov involving a spectral radius condition for a graph to contain all trees ☆

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ABSTRACT

We partly confirm a Brualdi-Solheid-Turán type conjecture due to Nikiforov, which is a spectral radius analogue of the well-known Erdős-Sós Conjecture that any tree of order t is contained in a graph of average degree greater than $t-2$. We confirm Nikiforov's Conjecture for all brooms and for a larger class of spiders. For our proofs we also obtain a new Turán type result which might turn out to be of independent interest.

1. Introduction

A central problem in extremal graph theory is the following Turán-type problem: for a given graph H, what is the maximum number of edges in an H-free graph with a given order? In the past decades, much attention has been paid to a spectral version of this question, that is, what is the maximum spectral radius of an H-free graph with a given order? The latter type of problem is called a Brualdi-Solheid-Turán type problem in [16] by Nikiforov. Examples of such problems are numerous since every Turán type problem gives rise to a corresponding Brualdi-Solheid-Turán type problem. As argued in [16], "the study of Brualdi-Solheid-Turán type problems is an important topic in spectral graph theory". Several groups of researchers have studied the relationship between the spectral radius and forbidden subgraphs such as cliques, paths, cycles and complete bipartite subgraphs. We refer to [1,9,13,14,16,18–20] for more information.

Motivated by these problems and earlier works, we study a conjecture due to Nikiforov [16], which is a spectral radius analogue of the well-known Erdős-Sós Conjecture that a graph of average degree greater than $t-2$ contains any tree of order t. Before we give more details concerning our work, we start by giving some essential definitions and introducing some useful notation.

Let $G=(V(G), E(G))$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. We use $|G|:=|V(G)|$ and $e(G):=|E(G)|$ to denote the order and size of G, respectively. Let $\mu(G)$ be the largest eigenvalue of the adjacency matrix $A(G)$ of G. We call $\mu(G)$ the *spectral radius* of G. A graph G is said to be H*-free* if H is not a subgraph of G. Note that we do not require H to be an induced subgraph. The *Turán number* of H is the maximum number of edges in an H-free graph of order n, and denoted by $\mathrm{ex}(n,H)$. Given two disjoint graphs G and H, the *disjoint union* of G and H, denoted by $G\cup H$, is the graph with vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)$. We use mG to denote the disjoint union of m copies

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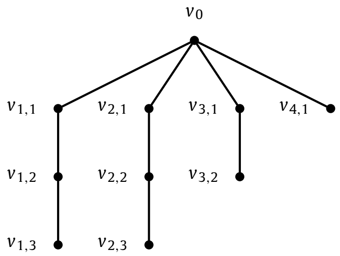

Fig. 1. The spider $S(3,3,2,1)$.

of G. The *join* of G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by adding edges joining every vertex of G to every vertex of H.

Let P_k denote the path of order k. Adopting the notation of [16], let $S_{n,k}$ denote the graph obtained by joining every vertex of a complete graph K_k to every vertex of an independent set of order $n-k$, that is, $S_{n,k} = K_k \vee \overline{K_{n-k}}$. Let $S^+_{n,k}$ be the graph obtained from $S_{n,k}$ by adding a single edge joining two vertices of the independent set of $S_{n,k}$. In addition, from [16] we know

$$\mu(S_{n,k}) = \frac{k-1}{2} + \sqrt{kn - \frac{3k^2+2k-1}{4}}, \tag{1}$$

and

$$\mu(S_{n,k}) < \mu(S^+_{n,k}) < \mu(S_{n,k}) + \frac{1}{n-k-2\sqrt{(n-k)/k}}. \tag{2}$$

Based on the Erdős-Sós Conjecture, in 2010 Nikiforov proposed the following Brualdi-Solheid-Turán type conjecture concerning trees.

Conjecture 1.1. ([16]) *Let $k \geq 2$ and let G be a graph of sufficiently large order n.*

(a) *If $\mu(G) \geq \mu(S_{n,k})$, then G contains all trees of order $2k+2$, unless $G = S_{n,k}$.*
(b) *If $\mu(G) \geq \mu(S^+_{n,k})$, then G contains all trees of order $2k+3$, unless $G = S^+_{n,k}$.*

In [16], Nikiforov proved that Conjecture 1.1 holds for paths.

Theorem 1.2. ([16]) *Let $k \geq 2$ and let G be a graph of sufficiently large order n.*

(a) *If $\mu(G) \geq \mu(S_{n,k})$, then G contains a P_{2k+2}, unless $G = S_{n,k}$.*
(b) *If $\mu(G) \geq \mu(S^+_{n,k})$, then G contains a P_{2k+3}, unless $G = S^+_{n,k}$.*

Recently, Hou, Liu, Wang, Gao and Lv [10] proved that Conjecture 1.1 (a) holds for all trees of diameter at most four. Liu, Broersma and Wang [12] also made progress on Conjecture 1.1. Let $\mathcal{T}$ be the family of all trees of diameter at most four on $2k+3$ vertices with the exception of the tree obtained from a star on $k+2$ vertices by subdividing each of its $k+1$ edges once. They proved in [12] that for $k \geq 8$ and large enough n, if G is an n-vertex graph with $\mu(G) \geq \mu(S_{n,k})$, then G contains all trees in $\mathcal{T}$, unless $G = S_{n,k}$.

In this paper, we study Conjecture 1.1 for spiders. A *spider* is a tree with at most one vertex of degree more than 2. The vertex of degree more than 2 is called the *center* of the spider (if all vertices have degree 1 or 2, the spider is a path and any vertex can be taken to be the center). A *leg* of a spider is a path from the center to a vertex of degree 1. The *length* of a leg is the number of edges of the leg. We use $S(t_1, t_2, \ldots, t_m)$ to denote a spider consisting of one designated center and m legs with lengths $t_1, t_2, \ldots, t_m$; see Fig. 1 for an example. Thus $S(t_1, t_2, \ldots, t_m)$ has $1 + \sum_{i=1}^m t_i$ vertices and $\sum_{i=1}^m t_i$ edges.

The aforementioned Erdős-Sós Conjecture has been confirmed for several classes of spiders in a series of papers; see [6–8]. Our first contribution is the following spectral radius result on the existence of a class of spiders. By imposing two restrictions on the number of odd legs, i.e., legs of odd length, we can prove the following theorem involving spiders.

Theorem 1.3. *Let $k \geq 2$ and let S be a spider of order $2k+3$ with r odd legs and s legs of length 1. If $r \geq 3$, $2s - r \geq 2$ and n is sufficiently large, then every graph G of order n with $\mu(G) \geq \mu(S_{n,k})$ contains S as a subgraph.*

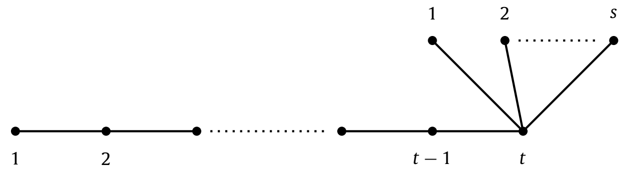

Fig. 2. The broom $B_{s,t}$.

We postpone all proofs to later sections. Theorem 1.3 confirms Conjecture 1.1 (b) for all spiders satisfying the condition in the statement of the theorem. Since $\mu(S_{n,k}) < \mu(S^+_{n,k})$, Theorem 1.3 is in fact a stronger result. Let S be a spider of order $2k+2$ with r odd legs and s legs of length 1 such that $r \geq 2$ and $2s - r \geq 1$. Let S' be the graph obtained from S by adding an extra pendant edge at the center of S. Then S' is a spider of order $2k+3$ with $r+1$ odd legs and $s+1$ legs of length 1 such that $r+1 \geq 3$ and $2(s+1) - (r+1) \geq 2$. Applying Theorem 1.3, we immediately derive the following result which confirms Conjecture 1.1 (a) for a class of spiders.

Corollary 1.4. *Let $k \geq 2$ and let S be a spider of order $2k+2$ with r odd legs and s legs of length 1. If $r \geq 2$, $2s - r \geq 1$ and n is sufficiently large, then every graph G of order n with $\mu(G) \geq \mu(S_{n,k})$ contains S as a subgraph.*

Our next result deals with the special subclass of spiders called brooms. For $s, t \geq 1$, a *broom* $B_{s,t}$ is a tree on $s+t$ vertices obtained by identifying the center of a star $K_{1,s}$ and an end-vertex of a path P_t; see Fig. 2. Note that the broom $B_{s,t}$ can be viewed as a spider $S(t_1, t_2, \ldots, t_{s+1})$, where $t_1 = \cdots = t_s = 1$ and $t_{s+1} = t - 1$. Moreover, $B_{1,t} = P_{t+1}$ and $B_{s,1} = K_{1,s}$. It is easy to check that if n is large enough, then $S_{n,k}$ contains all brooms of order $2k+3$ except for $B_{1,2k+2}$ and $B_{2,2k+1}$, and $S^+_{n,k}$ contains all brooms of order $2k+3$ except for $B_{1,2k+2}$.

The next theorem confirms Conjecture 1.1 for all brooms.

Theorem 1.5. *Let $k \geq 2$ and let G be a graph of sufficiently large order n.*

(a) *If $\mu(G) \geq \mu(S_{n,k})$, then G contains all brooms of order $2k+2$, unless $G = S_{n,k}$.*
(b) *If $\mu(G) \geq \mu(S^+_{n,k})$, then G contains all brooms of order $2k+3$, unless $G = S^+_{n,k}$.*

In order to prove Theorem 1.5, we shall use Theorems 1.2, 1.3 and the following Turán type result for connected graphs involving the broom $B_{2,2k+1}$. Theorem 1.6 below might be of independent interest.

Theorem 1.6. *For $k \geq 2$ and n sufficiently large, let G be a connected graph of order n. If $e(G) \geq e(S^+_{n,k}) = kn - \frac{k(k+1)}{2} + 1$, then G contains $B_{2,2k+1}$ as a subgraph.*

The remainder of this paper is organized as follows. In the next section, we provide some auxiliary results that will be used in our proofs. In Section 3, we prove Theorem 1.3. Section 4 is devoted to our proof of Theorem 1.5. In Section 5, we prove Theorem 1.6. Finally, we conclude this paper with some remarks on generalized brooms and by presenting some open problems in Section 6.

2. Preliminaries

In this section, we provide some additional terminology and lemmas that we will use. Let G be a connected graph. For any vertex $u \in V(G)$, let $N^d(u) := \{v \in V(G) \colon d_G(v,u) = d\}$, where $d_G(v,u)$ is the distance between u and v in G. Let $d_G(u)$ be the degree of u in G and let $\delta(G)$ be the minimum degree of G. For a non-empty subset $U \subseteq V(G)$, let $G[U]$ be the subgraph of G induced by U, $E(U)$ be the edge set of $G[U]$, and $e(U) := |E(U)|$. For two disjoint vertex sets $U, V \subseteq V(G)$, let $E(U,V)$ be the set of edges in G with one end-vertex in U and one end-vertex in V, and let $e(U,V) := |E(U,V)|$. Given a path $P = v_1 v_2 \cdots v_t$, we denote the sub-path $v_i v_{i+1} \cdots v_j$ by $v_i P v_j$. All logarithms in this paper are to the base 2. We use the standard Bachmann-Landau notation to indicate asymptotic growth rates of functions.

We will use the following known lemma on matrices in the set-up of the proof of Theorem 1.3. Given an $n \times n$ matrix A, let A_{ij} be the (i,j)-th entry of A for $1 \leq i, j \leq n$.

Lemma 2.1. ([9]) *Given $a, b \in \mathbb{Z}^+$ and an $n \times n$ nonnegative symmetric irreducible matrix A, let μ be the largest eigenvalue of A and μ' be the largest root of $f(x) = x^2 - ax - b$. Define $B = f(A) = A^2 - aA - bI$ and let $B_j = \sum_{i=1}^{n} B_{ij}$ $(1 \leq j \leq n)$. If $B_j \leq 0$ for all $j \in \{1, 2, \ldots, n\}$, then $\mu \leq \mu'$, with equality holding if and only if $B_j = 0$ for all $j \in \{1, 2, \ldots, n\}$.*

A *linear forest* is a forest all whose components are paths. We shall use the following known results on the Turán numbers of paths and linear forests in the proof of Theorem 1.3.

Lemma 2.2. ([5]) *For any positive integers $t > 1$ and $n > 1$, we have* $\mathrm{ex}(n, P_t) \le \frac{t-2}{2}n$.

Lemma 2.3. (see [4, Theorem 2.2]) *For any integer $\ell \ge 2$ and sufficiently large n, we have* $\mathrm{ex}(n, \ell P_3) < \left(\ell - \frac{1}{2}\right)n$.

Lemma 2.4. (see [11, Theorem 2]) *For any integer $\ell \ge 2$, let $F = \bigcup_{1\le i\le \ell} P_{a_i}$ be a linear forest with $a_i \ge 2$ for all $i \in [\ell]$. If at least one a_i is not 3, then for n sufficiently large,*

$$\mathrm{ex}(n, F) \le \left(\sum_{1\le i\le \ell} \left\lfloor \frac{a_i}{2} \right\rfloor - 1\right)n.$$

Next we introduce two lemmas that will be used in the proof of Theorem 1.6.

Lemma 2.5. ([2]) *Let G be a graph and for each vertex $v \in V(G)$, let p_v be the length of a longest path in G starting at v. Then* $e(G) \le \sum_{v\in V(G)} \frac{p_v}{2}$.

Lemma 2.6. ([3]) *For $k \ge 2$ and n sufficiently large, let G be a connected graph of order n with $G \ne S^+_{n,k}$. If $e(G) \ge e(S^+_{n,k}) = kn - \frac{k(k+1)}{2} + 1$, then G contains P_{2k+3} as a subgraph.*

We shall also apply the following partial solution of the Erdős-Sós Conjecture.

Lemma 2.7. ([8]) *If G is a graph on n vertices with $e(G) > \frac{(t-2)n}{2}$, then G contains every t-vertex spider with three legs.*

We end this section with some known results about $\mu(G)$.

Lemma 2.8. ([15]) *If G is a graph with n vertices, m edges and $\delta(G) = \delta$, then*

$$\mu(G) \le \frac{\delta - 1}{2} + \sqrt{2m - \delta n + \frac{(\delta+1)^2}{4}}.$$

Lemma 2.9. ([17]) *If G is a graph with m edges, then*

$$\mu(G) \le -\frac{1}{2} + \sqrt{2m + \frac{1}{4}}.$$

Lemma 2.10. ([16]) *Let the numbers $c \ge 0$, $k \ge 2$ and $n \ge 2^{4k}$, and let G be a graph of order n. If $\delta(G) \le k-1$ and $\mu(G) \ge \frac{k-1}{2} + \sqrt{kn - k^2 + c}$, then there exists a subgraph H of G satisfying one of the following conditions:*

(i) $\mu(H) > \sqrt{(2k+1)|H|}$;
(ii) $|H| \ge \sqrt{n}$, $\delta(H) \ge k$ *and* $\mu(H) > \frac{k-1}{2} + \sqrt{k|H| - k^2 + c + \frac{1}{2}}$.

3. Proof of Theorem 1.3

Let $r \ge 3$ and S be a spider with order $2k+3$, where S has r odd legs, s legs of length 1, and $2s - r \ge 2$. Suppose n is sufficiently large and G is an n-vertex graph with $\mu(G) \ge \mu(S_{n,k})$. For a contradiction, suppose that G contains no copy of S. Without loss of generality, we may assume that $S = S(t_1, t_2, \ldots, t_m)$, where $t_i = 1$ for all $1 \le i \le s$, $t_i \ge 3$ is odd for all $s+1 \le i \le r$, and $t_i \ge 2$ is even for all $r+1 \le i \le m$.

Let $f(x) = x^2 - (k-1)x - k(n-k)$. Note that $\mu(S_{n,k})$ is the largest root of $f(x)$ (see Section 2.1 of [16]). Let $B = f(A(G)) = A(G)^2 - (k-1)A(G) - k(n-k)I$ and $B_v = \sum_{i=1}^{n} B_{iv}$ for any $v \in V(G)$. By Lemma 2.1 and since $\mu(G) \ge \mu(S_{n,k})$, there exists a vertex $u \in V(G)$ with $B_u \ge 0$. Let L_u be the graph with vertex set $N^1(u) \cup N^2(u)$ and edge set $E(N^1(u)) \cup E(N^1(u), N^2(u))$. Since the (i,u)-entry of $A(G)^2$ is the number of walks of length 2 between i and u, we have

$$B_u = \sum_{x\in N^1(u)} d_{L_u}(x) - (k-2)d_G(u) - k(n-k). \tag{3}$$

We complete the proof by first proving the following claim and then distinguishing two cases based on the degree of u.

Claim 3.1. $d_G(u) \geq k+1$.

Proof. Since $d_{L_u}(x) \leq n-2$ for all $x \in N^1(u)$, using (3) we have

$$0 \leq B_u \leq d_G(u)(n-2) - (k-2)d_G(u) - k(n-k) = (d_G(u)-k)(n-k). \tag{4}$$

Thus $d_G(u) \geq k$. If $d_G(u) \geq k+1$, then we are done. If $d_G(u) = k$, then inequality (4) implies $0 = B_u = d_G(u)(n-2) - (k-2)d_G(u) - k(n-k)$. This implies that $d_{L_u}(x) = n-2$ for all $x \in N^1(u)$. Then G contains $S_{n,k}$ as a subgraph.

Since S is a bipartite graph, we may assume that (V', V'') is a bipartition of $V(S)$, say with $|V'| \geq |V''|$. Then $|V'| - |V''| = r-1 \geq 2$. Moreover, since $|V(S)| = 2k+3$ is odd, we have that $|V'| - |V''|$ is odd, so $r-1 \geq 3$. Then $|V''| \leq k$. Thus S is a subgraph of $S_{n,k}$. This contradicts the assumption that G contains no copy of S. □

We divide the rest of the proof into two cases.

Case 1. $k+1 \leq d_G(u) \leq \log n$.
In this case, since $B_u \geq 0$ and by equality (3), we have

$$\begin{aligned} e(N^1(u), N^2(u)) &= \sum_{x \in N^1(u)} d_{L_u}(x) - 2e(N^1(u)) \\ &> (k-2)d_G(u) + k(n-k) - d_G^2(u) = kn - o(n). \end{aligned}$$

Thus $|N^2(u)| > \frac{e(N^1(u),N^2(u))}{|N^1(u)|} > (1-o(1))\frac{kn}{\log n}$.

We claim that there are at least $\frac{n}{2\log n}$ vertices in $N^2(u)$ each of which has at least k neighbors in $N^1(u)$. Otherwise, we have $e(N^1(u), N^2(u)) < \frac{n}{2\log n}|N^1(u)| + (|N^2(u)| - \frac{n}{2\log n})(k-1) < \frac{n}{2} + (k-1)n \ll kn - o(n)$, a contradiction.

Therefore, there exist $\frac{n}{2\log n}\frac{1}{\binom{|N^1(u)|}{k}} > \frac{n}{\log^{k+1} n}$ vertices in $N^2(u)$ which have k common neighbors in $N^1(u)$. Without loss of generality, let $X \subseteq N^1(u)$ and $Y \subseteq N^2(u)$ with $|X| = k$ and $|Y| = \frac{n}{\log^{k+1} n} \gg 2k+2$ such that X is completely joined to Y. Similarly as in the last paragraph of the proof of Claim 3.1, we derive that $G[X \cup Y]$ contains a copy of S, a contradiction.

Case 2. $d_G(u) > \log n$.
We consider two subcases based on the number of edges between $N^1(u)$ and $N^2(u)$.

Subcase 2.1. $e(N^1(u), N^2(u)) > kd_G(u) + (2k-2)|N^2(u)| - k(n-k)$.
By equality (3) and since $B_u \geq 0$, we have $\sum_{x \in N^1(u)} d_{L_u}(x) \geq (k-2)d_G(u) + k(n-k)$. Then

$$\begin{aligned} e(L_u) &= \frac{1}{2}\left(\sum_{x \in N^1(u)} d_{L_u}(x) + e(N^1(u), N^2(u))\right) \\ &> \frac{1}{2}((k-2)d_G(u) + k(n-k) + kd_G(u) + (2k-2)|N^2(u)| - k(n-k)) \\ &= (k-1)(d_G(u) + |N^2(u)|) = (k-1)|V(L_u)|. \end{aligned} \tag{5}$$

We next claim that L_u contains the following linear forest.

Claim 3.2. *There is a $\bigcup_{s+1 \leq i \leq m} P_{t_i}$ in L_u such that each P_{t_i} has an end-vertex in $N^1(u)$.*

Proof. If $t_i = 2$ for all $s+1 \leq i \leq m$, then $s = r \geq 3$. Since $2k+3$ is odd, we have that r is even. Hence, we further have $s = r \geq 4$. Thus $m - s = (2k+3-1-s)/2 \leq k-1$. By Lemmas 2.2 and 2.4, for sufficiently large N, we have $\mathrm{ex}(N, (m-s)P_2) \leq (k-2)N$. By inequality (5), there is an $(m-s)P_2$ in L_u. By the definition of L_u, each P_2 has an end-vertex in $N^1(u)$.

Next, we assume that at least one t_i $(s+1 \leq i \leq m)$ is not 2. We now show that there is a $\bigcup_{s+1 \leq i \leq m} P_{t_i+1}$ in L_u.

If $r = s$ and $m - s = m - r = 1$, then $t_m + 1 \leq 2k+3-1-r+1 \leq 2k$ since $r \geq 3$. By Lemma 2.2 and inequality (5), there is a P_{t_m+1} in L_u.

If $m = r$ and $m - s = r - s = 1$, then $s \geq 3$ since $2 \leq 2s - r = 2s - (s+1)$. In this case, we also have $t_m + 1 \leq 2k+3-1-s+1 \leq 2k$. Thus there is a P_{t_m+1} in L_u by Lemma 2.2 and inequality (5).

If $m - s \geq 2$, then by Lemma 2.4 and since $2s - r \geq 2$, for sufficiently large N, we have

$$\mathrm{ex}\left(N, \bigcup_{s+1 \leq i \leq m} P_{t_i+1}\right) \leq \left(\sum_{s+1 \leq i \leq m} \left\lfloor \frac{t_i+1}{2} \right\rfloor - 1\right) N$$

$$
\begin{aligned}
&= \left(\sum_{s+1\le i\le r} \frac{t_i+1}{2} + \sum_{r+1\le i\le m} \frac{t_i}{2} - 1 \right) N = \left(\sum_{s+1\le i\le m} \frac{t_i}{2} + \sum_{s+1\le i\le r} \frac{1}{2} - 1 \right) N \\
&= \left(\frac{\sum_{s+1\le i\le m} t_i + \sum_{s+1\le i\le r} 1}{2} - 1 \right) N = \left(\frac{2k+3-1-s+r-s}{2} - 1 \right) N \\
&= \frac{2k+r-2s}{2} N \le \frac{2k-2}{2} N = (k-1)N.
\end{aligned}
$$

By inequality (5), there is a $\bigcup_{s+1\le i\le m} P_{t_i+1}$ in L_u. For each $s+1\le i\le m$, let v_i' and v_i'' be the first and second vertex along the path P_{t_i+1}, respectively. By the definition of L_u, at least one of v_i' and v_i'' is contained in $N^1(u)$. Hence, there is a $\bigcup_{s+1\le i\le m} P_{t_i}$ in L_u such that each P_{t_i} has an end-vertex in $N^1(u)$. □

By Claim 3.2, there is a $\bigcup_{s+1\le i\le m} P_{t_i}$ in L_u such that each P_{t_i} has an end-vertex in $N^1(u)$. Together with vertex u and s additional vertices in $N^1(u)$, this $\bigcup_{s+1\le i\le m} P_{t_i}$ forms a copy of S in G, a contradiction.

Subcase 2.2. $e(N^1(u), N^2(u)) \le kd_G(u) + (2k-2)|N^2(u)| - k(n-k)$.

By equality (3) and since $B_u \ge 0$, we have $\sum_{x\in N^1(u)} d_{L_u}(x) \ge (k-2)d_G(u) + k(n-k)$. Since $d_G(u) > \log n$ and n is sufficiently large, we have

$$
\begin{aligned}
e(N^1(u)) &= \frac{1}{2}\left(\sum_{x\in N^1(u)} d_{L_u}(x) - e(N^1(u), N^2(u)) \right) \\
&\ge \frac{1}{2}((k-2)d_G(u) + k(n-k) - kd_G(u) - (2k-2)|N^2(u)| + k(n-k)) \\
&= -d_G(u) + k(n - |N^2(u)|) - k^2 + |N^2(u)| \\
&\ge -d_G(u) + k(d_G(u)+1) - k^2 + |N^2(u)| \\
&= (k-1)d_G(u) + k - k^2 + |N^2(u)| \\
&> \frac{2k-3}{2} d_G(u) + \frac{1}{2} d_G(u) - k^2 \\
&> \frac{2k-3}{2}|N^1(u)| + \frac{1}{2}\log n - k^2 > \frac{2k-3}{2}|N^1(u)|. \qquad (6)
\end{aligned}
$$

We next claim that $G[N^1(u)]$ contains the following linear forest.

Claim 3.3. *There is a* $\bigcup_{s+1\le i\le m} P_{t_i}$ *in* $G[N^1(u)]$.

Proof. If $t_i = 3$ for all $s+1\le i\le m$, then $m = r$. Since $2s - r \ge 2$, we have $s \ge r - s + 2 = m - s + 2$. We first show that $m - s \le k/2$. Indeed, if $m - s > k/2$, then $s \ge m - s + 2 > (k+4)/2$. On the other hand, $s = 2k+3-1-3(m-s) < 2k+2-3k/2 = (k+4)/2$, a contradiction. Hence, $m - s \le k/2$. By Lemmas 2.2 and 2.3, we have $\mathrm{ex}(N, (m-s)P_3) \le (k/2 - 1/2)N \le (2k-3)N/2$ for sufficiently large N. By inequality (6), there is an $(m-s)P_3$ in $G[N^1(u)]$.

Next, we assume that at least one t_i $(s+1\le i\le m)$ is not 3. If $r = s$ and $m-s = m-r = 1$, then $t_m \le 2k+3-1-r \le 2k-1$ since $r \ge 3$. By Lemma 2.2 and inequality (6), there is a P_{t_m} in $G[N^1(u)]$. If $m = r$ and $m - s = r - s = 1$, then $s \ge 3$ since $2 \le 2s - r = 2s - (s+1)$. In this case, we also have $t_m \le 2k+3-1-s \le 2k-1$. Thus there is a P_{t_m} in $G[N^1(u)]$ by Lemma 2.2 and inequality (6). If $m - s \ge 2$, then by Lemma 2.4 and since $r \ge 3$, for sufficiently large N, we have

$$
\begin{aligned}
&\mathrm{ex}\left(N, \bigcup_{s+1\le i\le m} P_{t_i}\right) \le \left(\sum_{s+1\le i\le m} \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) N \\
&= \left(\sum_{s+1\le i\le r} \frac{t_i-1}{2} + \sum_{r+1\le i\le m} \frac{t_i}{2} - 1 \right) N = \left(\sum_{s+1\le i\le m} \frac{t_i}{2} - \sum_{s+1\le i\le r} \frac{1}{2} - 1 \right) N \\
&= \left(\frac{\sum_{s+1\le i\le m} t_i - \sum_{s+1\le i\le r} 1}{2} - 1 \right) N = \left(\frac{2k+3-1-s-(r-s)}{2} - 1 \right) N \\
&= \frac{2k-r}{2} N \le \frac{2k-3}{2} N.
\end{aligned}
$$

By inequality (6), there is a $\bigcup_{s+1\le i\le m} P_{t_i}$ in $G[N^1(u)]$. □

By Claim 3.3, there is a $\bigcup_{s+1\le i\le m} P_{t_i}$ in $G[N^1(u)]$. Together with vertex u and s additional vertices in $N^1(u)$ (recall that $d_G(u) > \log n$), this $\bigcup_{s+1\le i\le m} P_{t_i}$ forms a copy of S in G, a contradiction. This contradiction completes our proof of Theorem 1.3.

4. Proof of Theorem 1.5

In this section, we give our proof of Theorem 1.5, which confirms Conjecture 1.1 for all brooms. We first prove that the result holds for $B_{2,2k+1}$.

Theorem 4.1. *For integers $k \ge 2$ and n sufficiently large, every graph G of order n with $\mu(G) \ge \mu(S^+_{n,k})$ contains $B_{2,2k+1}$ as a subgraph.*

Proof. We first consider the case that G is connected. If $e(G) \ge kn - k(k+1)/2 + 1$, then G contains $B_{2,2k+1}$ as a subgraph by Theorem 1.6. Next we assume that $e(G) \le kn - k(k+1)/2$. If $\delta(G) \ge k$, then by Lemma 2.8 and equality (1), we have

$$\begin{aligned}\mu(G) &\le \frac{\delta(G)-1}{2} + \sqrt{2e(G) - \delta(G)n + \frac{(\delta(G)+1)^2}{4}}\\ &\le \frac{k-1}{2} + \sqrt{2e(G) - kn + \frac{(k+1)^2}{4}}\\ &\le \frac{k-1}{2} + \sqrt{2\left(kn - \frac{k(k+1)}{2}\right) - kn + \frac{(k+1)^2}{4}}\\ &= \mu(S_{n,k}) < \mu(S^+_{n,k}),\end{aligned}$$

which is a contradiction. Thus $\delta(G) \le k-1$. We briefly remark that the second inequality in this computation follows from the fact that $f(\delta) = \frac{\delta-1}{2} + \sqrt{2e(G) - \delta n + \frac{(\delta+1)^2}{4}}$ is a decreasing function when n is sufficiently large (also see [15]).

Let $c = (k-1)^2/4$. By Lemma 2.10, there is a subgraph H of G satisfying either (i) $\mu(H) > \sqrt{(2k+1)|H|}$ or (ii) $|H| \ge \sqrt{n}$, $\delta(H) \ge k$ and $\mu(H) > (k-1)/2 + \sqrt{k|H| - k^2 + c + 1/2}$.

If (i) holds, then by Lemma 2.9, we have $\sqrt{(2k+1)|H|} < \mu(H) \le \sqrt{2e(H)}$. So $(2k+1)|H| < 2e(H)$. Thus G contains $B_{2,2k+1}$ as a subgraph by Lemma 2.7.

If (ii) holds, then H contains a component H' with $\delta(H') \ge k$ and $\mu(H') = \mu(H) > (k-1)/2 + \sqrt{k|H| - k^2 + c + 1/2} \ge (k-1)/2 + \sqrt{k|H'| - k^2 + c + 1/2}$. Moreover, since $|H| \ge \sqrt{n}$ and $\mu(H') = \mu(H) > (k-1)/2 + \sqrt{k|H| - k^2 + c + 1/2}$, the order of H' is large when n is large. Next we will prove $\mu(H') > (k-1)/2 + \sqrt{k|H'| - (3k^2 + 2k - 3)/4} > \mu\big(S^+_{|H'|,k}\big)$. For a contradiction, suppose that $(k-1)/2 + \sqrt{k|H'| - (3k^2 + 2k - 3)/4} \le \mu\big(S^+_{|H'|,k}\big)$. By inequality (2), we have

$$\begin{aligned}\mu\big(S^+_{|H'|,k}\big) &< \mu\big(S_{|H'|,k}\big) + \frac{1}{|H'| - k - 2\sqrt{(|H'|-k)/k}}\\ &= \frac{k-1}{2} + \sqrt{k|H'| - \frac{3k^2+2k-1}{4}} + \frac{1}{|H'| - k - 2\sqrt{(|H'|-k)/k}}.\end{aligned}$$

Hence,

$$\sqrt{k|H'| - \frac{3k^2+2k-3}{4}} - \sqrt{k|H'| - \frac{3k^2+2k-1}{4}} < \frac{1}{|H'| - k - 2\sqrt{(|H'|-k)/k}},$$

so

$$\frac{1/2}{\sqrt{k|H'| - (3k^2+2k-3)/4} + \sqrt{k|H'| - (3k^2+2k-1)/4}} < \frac{1}{|H'| - k - 2\sqrt{(|H'|-k)/k}}.$$

Since $|H'|$ is sufficiently large, the above inequality is impossible. Hence, $\mu(H') > \mu\big(S^+_{|H'|,k}\big)$. By an analogous argument as in the first paragraph of the proof (recall that $|H'|$ is large and $\delta(H') \ge k$), we can derive a contradiction. This contradiction completes the proof for connected G.

If G is disconnected, then G contains a component G' with $\mu(G') = \mu(G)$. Then $\mu(G') \ge \mu(S^+_{n,k}) \ge \mu\big(S^+_{|G'|,k}\big)$. Thus we can apply the above arguments to G' and complete the proof. □

Now we have all ingredients to present our proof of Theorem 1.5.

Proof of Theorem 1.5. For positive integers s and t, let $B_{s,t}$ be a broom. Note that $B_{1,t}$ is in fact a path. In this case, the result follows from Theorem 1.2. Hence, we may assume that $s \geq 2$ in the following arguments.

The broom $B_{s,t}$ can be viewed as a spider $S(t_1, t_2, \ldots, t_{s+1})$, where $t_1 = \cdots = t_s = 1$ and $t_{s+1} = t - 1$. Such a spider has at least s legs of length 1. Let r be the number of odd legs. So $s \leq r \leq s+1$. We first prove Theorem 1.5 (a). Since $s \geq 2$, we have $2s - r \geq 2s - (s+1) \geq 1$. By Corollary 1.4, the result holds. We next prove Theorem 1.5 (b). The case for $B_{2,2k+1}$ follows from Theorem 4.1. For the case $s \geq 3$, we have $2s - r \geq 2s - (s+1) \geq 2$. By Theorem 1.3, the result holds. □

5. Proof of Theorem 1.6

Let G be a connected graph of order n with $e(G) \geq e(S^+_{n,k}) = kn - k(k+1)/2 + 1$. We first consider the case $G = S^+_{n,k}$. Let $v_1, v_2, \ldots, v_k$ denote the k vertices with degree $n-1$, $u_1, u_2, \ldots, u_{n-k-2}$ the $n-k-2$ vertices with degree k, and w_1, w_2 the two vertices with degree $k+1$. Then $w_1w_2v_1u_1v_2u_2 \cdots v_ku_k$ is a path of order $2k+2$. Together with the edge v_ku_{k+1}, this path forms a $B_{2,2k+1}$ in G.

Next, we consider the case $G \neq S^+_{n,k}$. For a contradiction, suppose that G contains no $B_{2,2k+1}$. Let $P = v_1v_2 \cdots v_\ell$ be a longest path in G. By Lemma 2.6, we have $\ell \geq 2k+3$. Let $X = V(P)$ and $Y = V(G) \setminus V(P)$. We next state and prove two claims.

Claim 5.1. $2k+3 \leq \ell \leq 4k-1$.

Proof. We first suppose $4k \leq \ell \leq n-1$. Since P is a longest path, we have $uv_1 \notin E(G)$ and $uv_\ell \notin E(G)$ for any vertex $u \in Y$. Since G is connected, there is an $i \in \{2, 3, \ldots, \ell-1\}$ and a vertex $w \in Y$ such that $v_iw \in E(G)$. Since $\ell \geq 4k$, at least one of the paths $v_1 \cdots v_i$ and $v_i \cdots v_\ell$ has order at least $2k+1$, say $v_1 \cdots v_i$. Together with the edge v_iw, the path $v_1 \cdots v_iv_{i+1}$ forms a graph that contains $B_{2,2k+1}$ as a subgraph, a contradiction.

We next suppose $\ell = n$. Since $e(G) \geq kn - k(k+1)/2 + 1 > n$, there exist two vertices v_i and v_j with $i+2 \leq j$ and $\{i, j\} \neq \{1, n\}$ such that $v_iv_j \in E(G)$. Since n is sufficiently large, at least one of the paths $v_1 \cdots v_i$, $v_i \cdots v_j$ and $v_j \cdots v_n$ has order greater than $2k+1$. It is easy to check that there is a $B_{2,2k+1}$ in G, a contradiction. □

Claim 5.2. *Let v be any vertex in Y. Then*

(i) *there is no edge between v and $\{v_1, v_2, \ldots, v_{\ell-2k}\} \cup \{v_{2k+1}, v_{2k+2}, \ldots, v_\ell\}$; and*
(ii) *v has at most $\left\lceil \frac{4k-\ell}{2} \right\rceil$ neighbors in X and cannot be adjacent to two consecutive vertices on the path $v_{\ell-2k+1} \cdots v_{2k}$.*

Proof. Since P is a longest path, there is no edge between v and $\{v_1, v_\ell\}$. Since G contains no $B_{2,2k+1}$, there is no edge between v and $\{v_2, \ldots, v_{\ell-2k}\} \cup \{v_{2k+1}, \ldots, v_{\ell-1}\}$. Thus (i) holds. For any $1 \leq i \leq \ell-1$, at most one of vv_i and vv_{i+1} is an edge of G; otherwise the path $v_1 \cdots v_ivv_{i+1} \cdots v_\ell$ is longer than P. Combining with (i), the vertex v has at most $\lceil(\ell - 2(\ell-2k))/2\rceil = \lceil(4k-\ell)/2\rceil$ neighbors in X. Thus (ii) holds. □

If $e(X, V(H)) + e(H) \leq (k-1/2)|H|$ for each component H in $G - X$, then $e(G) \leq (k-1/2)(n-\ell) + \binom{\ell}{2} < kn - k(k+1)/2 + 1$ when n is sufficiently large, a contradiction. Thus there exists a component H in $G - X$ with $e(X, V(H)) + e(H) > (k - 1/2)|H|$. In the remainder of this proof, let H denote such a component. Our aim is to show that $e(X, V(H)) + e(H) \leq (k-1/2)|H|$, contradicting the above. For any vertex $v \in V(H)$, let s_v be the number of neighbors of v in X, and let p_v be the length of a longest path in H starting at v (so this longest path has $p_v + 1$ vertices). We next state and prove three claims.

Claim 5.3. *For any vertex $v \in V(H)$, we have $vv_{\ell-2k+1} \notin E(G)$ and $vv_{2k} \notin E(G)$.*

Proof. By symmetry, we only prove $vv_{\ell-2k+1} \notin E(G)$ for any $v \in V(H)$. For a contradiction, suppose that there is a vertex $u \in V(H)$ with $uv_{\ell-2k+1} \in E(G)$. Note that $|H| \geq 2$; otherwise if $|H| = 1$, then $e(X, V(H)) + e(H) = s_u \leq k-1 \leq (k-1/2)|H|$ by Claims 5.1 and 5.2 (ii), a contradiction. Before we derive at a contradiction, we first make three observations.

Observation 1: For any vertex $v \in V(H)$, we have $p_v \leq \ell - 2k - 1$. Indeed, if $v \in V(H)$ and $vv_{\ell-2k+1} \in E(G)$, then to avoid a $P_{\ell+1}$ we must have $p_v \leq \ell - 2k - 1$. If $v \in V(H)$, $vv_{\ell-2k+1} \notin E(G)$ and $p_v \geq \ell - 2k$, then we may assume that $vw_1w_2 \cdots w_{\ell-2k}$ is a path in H. In order to avoid a $P_{\ell+1}$ or a $B_{2,2k+1}$, we have $u \notin \{v, w_1, w_2, \ldots, w_{\ell-2k}\}$, and there is no path in H connecting u and the path $vw_1w_2 \cdots w_{\ell-2k}$. This contradicts the fact that H is connected and thus we have verified Observation 1.

Observation 2: When $4k - \ell$ is odd, there is at most one vertex in H which is adjacent to $\lceil(4k-\ell)/2\rceil = (4k-\ell+1)/2$ vertices in X. Otherwise, suppose that there are two vertices $v', v'' \in V(H)$ with $s_{v'} = s_{v''} = (4k-\ell+1)/2$. This implies that v' (resp., v'') is either adjacent to both $v_{\ell-2k+1}$ and $v_{\ell-2k+3}$ or adjacent to both $v_{\ell-2k+2}$ and $v_{\ell-2k+4}$. Since H is connected, there is a path Q in H connecting v' and v''. It is easy to check that $G[X \cup V(Q)]$ contains a path longer than P, a contradiction. Thus we have verified Observation 2.

Observation 3: When $4k-\ell$ is even, there are at most two vertices in H which are adjacent to $\lceil(4k-\ell)/2\rceil = (4k-\ell)/2$ vertices in X. Otherwise, suppose that there are three vertices $v', v'', v''' \in V(H)$ with $s_{v'} = s_{v''} = s_{v'''} = (4k-\ell)/2$. This implies that v' (resp., v'', v''') is adjacent to both $v_{\ell-2k+1}$ and $v_{\ell-2k+3}$, both $v_{\ell-2k+2}$ and $v_{\ell-2k+4}$, or both $v_{\ell-2k+1}$ and $v_{\ell-2k+4}$. Since H is connected, there is a path Q of length at least two in H connecting two of the vertices v', v'' and v'''. It is easy to check that $G[X \cup V(Q)]$ contains a path longer than P, a contradiction. Thus we have verified Observation 3.

Therefore, by Lemma 2.5, Claim 5.2 (ii) and recalling that $|H| \geq 2$, we have

$$\begin{aligned} &e(X, V(H)) + e(H) \leq \sum_{v \in V(H)} \left(s_v + \frac{p_v}{2}\right) \\ &\leq \left(\frac{4k-\ell+1}{2} + \frac{\ell-2k-1}{2}\right) + (|H|-1)\left(\frac{4k-\ell+1}{2} - 1 + \frac{\ell-2k-1}{2}\right) \leq \left(k - \frac{1}{2}\right)|H| \end{aligned}$$

when $4k-\ell$ is odd, and

$$\begin{aligned} &e(X, V(H)) + e(H) \leq \sum_{v \in V(H)} \left(s_v + \frac{p_v}{2}\right) \\ &\leq 2\left(\frac{4k-\ell}{2} + \frac{\ell-2k-1}{2}\right) + (|H|-2)\left(\frac{4k-\ell}{2} - 1 + \frac{\ell-2k-1}{2}\right) \leq \left(k - \frac{1}{2}\right)|H| \end{aligned}$$

when $4k-\ell$ is even, a contradiction. □

Claim 5.4. *For any vertex $v \in V(H)$, if $s_v \neq 0$, then $s_v + \frac{p_v}{2} \leq \min\left\{k - \frac{1}{2}, \frac{\ell-1}{2} - \frac{p_v}{2}\right\}$.*

Proof. Let v be a vertex in H with $s_v \neq 0$. By Claims 5.2 and 5.3, we have $s_v \leq \lceil(4k-\ell-2)/2\rceil$. Let $v_{i_1}, \ldots, v_{i_{s_v}}$ be all the neighbors of v in X, where $i_1 < \cdots < i_{s_v}$. If $i_{s_v} \geq \ell - p_v$, then choosing a path $w_1 w_2 \cdots w_{p_v} v$ in H, we can combine this path with $v v_{i_{s_v}} v_{i_{s_v - 1}} \cdots v_1$ to form a $P_{\ell+1}$, a contradiction. Thus $i_{s_v} \leq \ell - (p_v + 1)$, and a similar argument shows that $i_1 \geq p_v + 2$. Therefore,

$$i_{s_v} - i_1 \leq \ell - (p_v + 1) - (p_v + 2) = \ell - 2p_v - 3.$$

Combining this inequality with the observation that $\lceil(i_{s_v} - i_1 + 1)/2\rceil \geq s_v$ gives the inequality $2(s_v + p_v) \leq \ell - 1$ or equivalently, $s_v + p_v/2 \leq (\ell-1)/2 - p_v/2$. Now when $p_v \geq \ell - 2k + 1$, we have $(\ell-1)/2 - p_v/2 < k - 1/2$ so that

$$s_v + p_v/2 \leq \min\{k - 1/2, (\ell-1)/2 - p_v/2\}. \tag{7}$$

When $p_v \leq \ell - 2k$,

$$s_v + p_v/2 \leq \lceil(4k-\ell-2)/2\rceil + (\ell-2k)/2 \leq k - 1/2.$$

Note that $p_v \leq \ell - 2k$ also implies $k - 1/2 \leq (\ell-1)/2 - p_v/2$, so (7) holds in this case as well. □

Claim 5.5. *If $\ell = 2k+3$, then for any vertex $v \in V(H)$, we have $vv_{k+2} \notin E(G)$.*

Proof. The case $k = 2$ follows from Claim 5.3. We next consider the case $k = 3$. In this case, for any vertex $v \in V(H)$, v_5 is the only possible neighbor of v in X by Claims 5.2 and 5.3. Let $V' := \{v \in V(H) : vv_5 \in E(G)\}$ and $V'' := V(H) \setminus V'$. In order to avoid a $B_{2,7}$, each vertex $v \in V'$ (resp., $v \in V''$) has at most two neighbors in V' (resp., $V(H)$). Thus

$$\begin{aligned} e(X, V(H)) + e(H) &\leq e(X, V') + e(V') + e(V', V'') + e(V'') \\ &\leq |V'| + \frac{1}{2}\sum_{v \in V'} 2 + \sum_{v \in V''} 2 = 2|H| < (k - 1/2)|H|, \end{aligned}$$

a contradiction. In the following arguments, we may assume that $k \geq 4$.

For a contradiction, suppose that there is a vertex $u \in V(H)$ with $uv_{k+2} \in E(G)$. Let $\mathcal{H}$ be the set of all components of $H - u$ and let $H' \in \mathcal{H}$. For any vertex $v \in V(H')$, let p'_v be the length of a longest path in H' starting at v. Then $p'_v \leq p_v$. In the following, we show that $s_v + p'_v/2 + \mathbb{1}_{(vu \in E(G))} \leq k - 1/2$ for any vertex $v \in V(H')$, where $\mathbb{1}_{(vu \in E(G))} := 1$ if $vu \in E(G)$, and $\mathbb{1}_{(vu \in E(G))} := 0$ if $vu \notin E(G)$.

We first consider a vertex $v \in V(H')$ with $s_v \neq 0$. By Claim 5.4, we have $s_v + p'_v/2 \leq s_v + p_v/2 \leq \min\{k - 1/2, k + 1 - p_v/2\}$. If $vu \notin E(G)$ (resp., $p_v \geq 5$), then $s_v + p'_v/2 + \mathbb{1}_{(vu \in E(G))} \leq s_v + p_v/2 \leq k - 1/2$ (resp., $s_v + p'_v/2 + \mathbb{1}_{(vu \in E(G))} \leq k + 1 - 5/2 + 1 = k - 1/2$). Now we consider the case $vu \in E(G)$ and $p_v \leq 4$. In order to avoid a P_{2k+4}, there is no edge between v and $\{v_k, v_{k+1}, v_{k+3}, v_{k+4}\}$. Combining with Claim 5.2 (i) and Claim 5.3, we have $s_v \leq 2\lceil(k-5)/2\rceil + 1 \leq k - 3$. Moreover, if we further have $p_v = 4$, then $vv_5, vv_{2k-1} \notin E(G)$ for avoiding a P_{2k+4}, so $s_v \le \max\{2\lceil (k-6)/2\rceil + 1, 1\} \le \max\{k-4, 1\}$ in this case. Now, if $k \ge 5$, then

$$s_v + p'_v/2 + 1 \le s_v + p_v/2 + 1 \le \begin{cases} k-3+3/2+1, & \text{if } p_v \le 3, \\ k-4+4/2+1, & \text{if } p_v = 4, \end{cases}$$

so $s_v + p'_v/2 + 1 \le k - 1/2$. If $k = 4$, then $p_u \le 4$ for avoiding a P_{2k+4}, and thus $p'_v \le 3$. Hence, if $k = 4$, then $s_v + p'_v/2 + 1 \le 1 + 3/2 + 1 = k - 1/2$.

We next consider a vertex $v \in V(H')$ with $s_v = 0$. We now show that $p'_v \le 2k-2$. Otherwise, suppose that Q is a path in H' of length $2k-1$ starting at v. Let w be the other end-vertex of Q. Since H is connected and H' is a component of $H-u$, there is a path Q' connecting u and Q. We choose such a path Q' with the minimum length, that is, $|V(Q) \cap V(Q')| = 1$, say $V(Q) \cap V(Q') = \{x\}$. Then one of the paths vQx and xQw has length at least k, say vQx. Then $v_1 \cdots v_{k+2}uQ'xQv$ is a path of length at least $2k+3$. This contradiction implies $p'_v \le 2k-2$. In order to avoid a P_{2k+4}, if $p'_v \ge k$, then $vu \notin E(G)$. Thus $s_v + p'_v/2 + \mathbb{1}_{(vu \in E(G))} \le 0 + \max\{(2k-2)/2, (k-1)/2 + 1\} \le k - 1/2$.

Since $\ell = 2k+3$, we have $s_u \le k-2$ by Claims 5.2 and 5.3. From the above arguments and by Lemma 2.5, we have

$$\begin{aligned} e(X, V(H)) + e(H) = &\ e(\{u\}, X) + \sum_{H' \in \mathcal{H}} \big(e(V(H'), X) + e(H') + e(V(H'), \{u\})\big) \\ \le &\ s_u + \sum_{H' \in \mathcal{H}} \sum_{v \in V(H')} \big(s_v + p'_v/2 + \mathbb{1}_{(vu \in E(G))}\big) \\ \le &\ k-2 + \sum_{H' \in \mathcal{H}} (k-1/2)|V(H')| < \ (k-1/2)|H|, \end{aligned}$$

a contradiction. □

Let $A := \{v \in V(H) \colon s_v \ne 0\}$, $B := \{v \in V(H) \colon s_v = 0 \text{ and } p_v \le 2k-1\}$ and $C := \{v \in V(H) \colon s_v = 0 \text{ and } p_v \ge 2k\}$. Then for any vertex $v \in A \cup B$, we have $s_v + p_v/2 \le k - 1/2$ by Claim 5.4. The next claim deals with vertices in C.

Claim 5.6. *If $C \ne \emptyset$, then for any vertex $v \in C$, the degree of v in H is at most $k-1$.*

Proof. Let v be an arbitrary vertex in C. By the definition of C, we may assume that $w_0w_1w_2 \cdots w_{2k}$ is a path in H, where $w_0 := v$. Let $W = \{w_1, w_2, \ldots, w_{2k}\}$. Since G is connected and $s_v = 0$, there is a path Q connecting a vertex of X and a vertex of W. We choose such a path with minimal length, that is, Q has exactly one common vertex with X and exactly one common vertex with W. We may assume that $Q = u_0u_1 \cdots u_t$ for some $t \ge 1$, where $u_0 \in X$ and $u_t = w_i$ for some $1 \le i \le 2k$.

We first show that $\ell \ge 2k+4$. Otherwise if $\ell = 2k+3$, then $vv_{k+2} \notin E(G)$ by Claim 5.5. Thus one of the paths v_1Pu_0 and u_0Pv_ℓ has order at least $k+3$. In order to avoid a P_{2k+4}, we have $k+3+t+i \le 2k+3$ and $k+3+t+2k-i \le 2k+3$. Thus $k+t \le i \le k-t$, which is impossible. Hence, $\ell \ge 2k+4$.

Note that one of the paths v_1Pu_0 and u_0Pv_ℓ has order at least $\lceil(\ell+1)/2\rceil$, say v_1Pu_0. In order to avoid a $P_{\ell+1}$, we have $\lceil(\ell+1)/2\rceil + t + i \le \ell$ and $\lceil(\ell+1)/2\rceil + t + 2k - i \le \ell$. Thus $2k + t - \lfloor(\ell-1)/2\rfloor \le i \le \lfloor(\ell-1)/2\rfloor - t$. Moreover, for any $0 \le j \le j_0 := \lceil(\ell+1)/2\rceil + t + i - 2k - 1$, the path $v_1Pu_0Qw_iw_{i-1}\cdots w_j$ has order at least $2k+1$. Note that

$$j_0 = \lceil(\ell+1)/2\rceil + t + i - 2k - 1 \ge \lceil(\ell+1)/2\rceil + t + 2k + t - \lfloor(\ell-1)/2\rfloor - 2k - 1 \ge 2.$$

In order to avoid a $B_{2,2k+1}$, the vertex v has no neighbor in $\{w_2, \ldots, w_{j_0}\}$ and has at most one neighbor in $\{w_{i+1}, \ldots, w_{2k}\} \cup (V(H) \setminus (W \cup V(Q)))$. Since $\ell \ge 2k+4$, the number of neighbors of v in H is at most $|\{w_1\}| + |\{w_{j_0+1}, \ldots, w_i\}| + t - 1 + 1 = i - j_0 + t + 1 = 2k + 2 - \lceil(\ell+1)/2\rceil \le k-1$. □

Recall that for any vertex $v \in A \cup B$, we have $s_v + p_v/2 \le k - 1/2$. By Lemma 2.5 and Claim 5.6, we have

$$\begin{aligned} e(X, V(H)) + e(H) &= e(X, A) + e(A \cup B) + e(C) + e(C, A \cup B) \\ &\le \sum_{v \in A \cup B} \left(s_v + \frac{p_v}{2}\right) + \sum_{v \in C} d_H(v) \\ &\le |A \cup B| \left(k - \frac{1}{2}\right) + |C|(k-1) \le \ \left(k - \frac{1}{2}\right)|H|. \end{aligned}$$

This contradiction completes the proof of Theorem 1.6.

6. Concluding remarks

In this paper, we proved that a Brualdi-Solheid-Turán type conjecture due to Nikiforov (Conjecture 1.1) holds for a class of spiders. In particular, we also confirmed Conjecture 1.1 for all brooms. For integers $s \geq 1$ and $t \geq \ell \geq 1$, a *generalized broom* $B^{\ell}_{s,t}$ is a tree on $s+t$ vertices obtained from a path P_t by attaching s pendant edges at the ℓ-th vertex along the path. Note that $B^{\ell}_{s,t} = B^{t+1-\ell}_{s,t}$ and $B^{1}_{s,t} = B^{t}_{s,t} = B_{s,t}$. Using Theorem 1.3, it is easy to derive the following result for generalized brooms.

Corollary 6.1. *For integers $k \geq 2$, let $\mathcal{T}$ (resp., $\mathcal{T}'$) be the set of all generalized brooms $B^{\ell}_{s,t}$ of order $2k+3$ with $s \geq 3$ (resp., of order $2k+2$ with $s \geq 2$). Then every graph G of sufficiently large order n with $\mu(G) \geq \mu(S_{n,k})$ contains all graphs in $\mathcal{T}$ and $\mathcal{T}'$.*

Proof. For $s \geq 3$, the generalized broom $B^{\ell}_{s,t}$ can be viewed as a spider $S(t_1, t_2, \ldots, t_{s+2})$, where $t_1 = \cdots = t_s = 1$, $t_{s+1} = \ell - 1$ and $t_{s+2} = t - \ell$. Such a spider has at least s legs of length 1. Let r be the number of odd legs. So $s \leq r \leq s+2$. Since $s \geq 3$, we have $r \geq 3$. If $s \geq 4$, then $2s - r \geq 2s - (s+2) \geq 2$. If $s = 3$, then $r = s+1$ since $2k+3$ is odd, so we also have $2s - r \geq 2$. By Theorem 1.3, G contains all graphs in $\mathcal{T}$. Since every graph $B^{\ell}_{s,2k+2-s}$ $(s \geq 2)$ is a subgraph of $B^{\ell}_{s+1,2k+3-(s+1)}$, we can further deduce that G contains all graphs in $\mathcal{T}'$. $\square$

An interesting and natural question is to study Conjecture 1.1 for generalized brooms $B^{\ell}_{s,t}$ of order $2k+3$ with $s \leq 2$ (resp., of order $2k+2$ with $s = 1$) and $2 \leq \ell \leq t-1$. Another direction is to study Conjecture 1.1 for other classes of spiders. Hopefully this will also lead to new ideas and approaches for resolving Conjecture 1.1 for general trees.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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