Received: 27 August 2021 | Accepted: 22 February 2022

DOI: 10.1002/jgt.22822

ARTICLE

WILEY

The Erdős–Gyárfás function with respect to Gallai-colorings

Xihe Li[1,2] | **Hajo Broersma**[2] | **Ligong Wang**[1]

[1]School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an, Shaanxi, China

[2]Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, Enschede, The Netherlands

Correspondence
Hajo Broersma, Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands.
Email: h.j.broersma@utwente.nl

Funding information
National Natural Science Foundation of China, Grant/Award Number: 11871398; China Scholarship Council, Grant/Award Number: 201906290174

Abstract

For fixed p and q, an edge-coloring of the complete graph K_n is said to be a (p, q)-coloring if every K_p receives at least q distinct colors. The function $f(n, p, q)$ is the minimum number of colors needed for K_n to have a (p, q)-coloring. This function was introduced about 45 years ago, but was studied systematically by Erdős and Gyárfás in 1997, and is now known as the Erdős–Gyárfás function. In this paper, we study $f(n, p, q)$ with respect to Gallai-colorings, where a Gallai-coloring is an edge-coloring of K_n without rainbow triangles. Combining the two concepts, we consider the function $g(n, p, q)$ that is the minimum number of colors needed for a Gallai-(p, q)-coloring of K_n. Using the anti-Ramsey number for K_3, we have that $g(n, p, q)$ is nontrivial only for $2 \le q \le p - 1$. We give a general lower bound for this function and we study how this function falls off from being equal to $n - 1$ when $q = p - 1$ and $p \ge 4$ to being $\Theta(\log n)$ when $q = 2$. In particular, for appropriate p and n, we prove that $g = n - c$ when $q = p - c$ and $c \in \{1, 2\}$, g is at most a fractional power of n when $q = \lfloor \sqrt{p-1} \rfloor$, and g is logarithmic in n when $2 \le q \le \lfloor \log_2(p-1) \rfloor + 1$.

KEYWORDS
Erdős–Gyárfás function, Gallai-coloring, Ramsey theory

MATHEMATICAL SUBJECT CLASSIFICATION
05C55, 05D10

1 | INTRODUCTION

Let p and q be positive integers with $2 \le q \le \binom{p}{2}$. An edge-coloring of the complete graph K_n is said to be a (p, q)-*coloring* if every K_p receives at least q distinct colors. The function $f(n, p, q)$ is defined to be the minimum number of colors that are needed for K_n to have a (p, q)-coloring. This function was first introduced by Erdős and Shelah [9,10], but Erdős and Gyárfás [11] were the first to study it in depth; it is now known as the Erdős–Gyárfás function. This function generalizes the multicolored Ramsey number, since determining $f(n, p, 2)$ is equivalent to determining the Ramsey number of K_p.

In [11], Erdős and Gyárfás determined various thresholds for $f(n, p, q)$. In particular, they proved that $q = \binom{p}{2} - p + 3$ is the smallest value of q such that $f(n, p, q)$ is linear in n, and $q = \binom{p}{2} - \left\lfloor \frac{p}{2} \right\rfloor + 2$ is the smallest value of q such that $f(n, p, q)$ is quadratic in n. In [7], Conlon et al. proved that $q = p$ is the smallest value of q such that $f(n, p, q)$ is polynomial in n.

The exact value of the Erdős–Gyárfás function is very difficult to determine, even for some small values of p and q. For example, the best known lower bound for $f(n, 4, 3)$ is $O(\log n)$ [17], while the best until now upper bound is $e^{O(\sqrt{\log n})}$ [26]. There is clearly a large gap between the lower and upper bound. On the other hand, some special cases of this function are closely related to other interesting problems. For example, $f(n, 9, 34)$ relates to a Turán-type hypergraph problem posed by Brown, Erdős, and Sós [4,11], $f(n, 5, 9)$ relates to sets containing no 3-term arithmetic progression [1], and $f(n, 3, 3)$ and $f(n, 5, 9)$ relate to some problems on properly colored complete graphs [11,28]. For more information on this function, we refer to [2,6,7,15,27,29,30] and Section 3.5.1 of [8].

A graph with an edge-coloring is called *rainbow* if all its edges are colored differently. A *Gallai-k-coloring* is a k-edge-coloring of a complete graph K_n without rainbow triangles (i.e., every triangle receives at most two colors). In this paper, we investigate the Erdős–Gyárfás function within the framework of Gallai-colorings. A Gallai-coloring of the complete graph K_n is said to be a *Gallai-*(p, q)*-coloring* if every K_p receives at least q distinct colors. We define $g(n, p, q)$ to be the minimum number of colors that are needed for K_n to have a Gallai-(p, q)-coloring. Clearly, we have $f(n, p, q) \le g(n, p, q)$ if both functions are defined for these values of n, p, and q.

For studying $g(n, p, q)$ it is convenient to introduce the following function. For $1 \le q \le \binom{p}{2}$, let $g_q^k(p)$ be the smallest positive integer n such that every Gallai-k-coloring of K_n contains a copy of K_p receiving at most q distinct colors. Restated, $g_q^k(p) - 1$ is the largest positive integer n such that there is a Gallai-k-coloring of K_n in which every K_p receives at least $q + 1$ distinct colors, that is, such that $g(n, p, q + 1) \le k$. Throughout the remainder of the paper, we concentrate on the function $g_q^k(p)$ and derive upper and lower bounds and some exact values for this function. We reflect on what these results on $g_q^k(p)$ imply for the function $g(n,p,q)$ in Section 8. It is worth noting that Erdős introduced an analogue of the function $g_q^k(p)$ when he posed the problem on $f(n,p,q)$ in his original paper [9].

We first point out that $g_q^k(p)$ is nontrivial only for $1 \le q \le p-2$ (equivalently, $g(n,p,q)$ is nontrivial only for $2 \le q \le p-1$). When $q \ge p-1$, we can deduce $g_q^k(p)$ using the following anti-Ramsey result.

Theorem 1.1 (Erdős et al. [14] and Gyárfás and Simonyi [21]). *At most $p-1$ colors can be used in any Gallai-coloring of K_p.*

Corollary 1.2. *For integers $k \ge 1, p \ge 3$, and $q \ge p-1$, there is no Gallai-k-coloring of K_n in which every K_p receives at least $q+1$ distinct colors. Thus $g_q^k(p) = p$ for $q \ge p-1$.*

Moreover, if $k < q$, then it is obvious that $g_q^k(p) = p$. In the sequel, we will always assume that $k \ge q$ and $1 \le q \le p-2$ when we consider $g_q^k(p)$. Note that we have the following inequalities:

$$g_q^k(p) \le g_q^{k+1}(p), \quad g_{q+1}^k(p) \le g_q^k(p), \quad \text{and} \quad g_{q+1}^{k+1}(p) \le g_q^k(p),$$

as we now explain. The first two inequalities hold by the definition of $g_q^k(p)$. For the third inequality, let $n_0 = g_{q+1}^{k+1}(p) - 1$. Then there exists a Gallai-$(k+1)$-coloring G of K_{n_0} in which every K_p receives at least $q+2$ colors. Let G' be an edge-coloring of K_{n_0} obtained from G by unifying colors k and $k+1$. Clearly G' is a Gallai-k-coloring in which every K_p receives at least $q+1$ colors. Thus $g_q^k(p) \ge n_0 + 1 = g_{q+1}^{k+1}(p)$.

In [16], Fox, Grinshpun, and Pach proved the following asymptotic result. Note that for $k = 3$ and $q = 2$, this result is a special case of the multicolor generalization of the well-known Erdős–Hajnal conjecture.

Theorem 1.3 (Fox et al. [16]). *Let k and q be fixed positive integers with $q \le k$. Every Gallai-k-coloring of K_n contains a set of order $\Omega\left(n^{\binom{q}{2}/\binom{k}{2}} \log_2^{c_{k,q}} n\right)$ which uses at most q colors, where $c_{k,q}$ is only depending on k and q. Moreover, this bound is tight apart from the constant factor.*

It is worth noticing that the problem studied by Fox, Grinshpun, and Pach is to find the largest subgraph K_p using at most q colors in every Gallai-k-coloring of K_n, for fixed k and q, when n is sufficiently large. But in this paper, we mainly focus on the problem to determine the smallest n such that there is a K_p using at most q colors in every Gallai-k-coloring of K_n, for fixed p and q, when $k \in [1, +\infty)$ (or $k \to \infty$). Therefore, the above theorem cannot give us much support, since it requires that n is sufficiently large, in fact,

$$n \ge n_0 = 2^{2^{2^{2^{8k^2}}}}.$$

But we can prove an upper bound of $2^{\frac{2k(p-2)}{q}+1}$ on $g_q^k(p)$ (see Theorem 1.4). If $2^{\frac{2k(p-2)}{q}+1} \geq n_0$, then $k = o(p)$, which implies that for fixed p and q, only $o(p) g_q^k(p)$'s can be bounded using the above theorem.

Theorem 1.4. *For integers p, q, k with $p \geq 3, 1 \leq q \leq p-2$, and $k \geq q$, we have $g_q^k(p) \leq 2^{\frac{2k(p-2)}{q}+1}$.*

We postpone all proofs of our results to later sections. Note that Theorem 1.4 implies that $g(n, p, q) > \frac{q-1}{2(p-2)}(\log_2 n - 1)$, where $p \geq 3$, $2 \leq q \leq p-1$, and $n \geq 2^{2p-3}$. In [11], Erdős and Gyárfás obtained an upper bound for $f(n, p, q)$ using the Lovász Local Lemma. However, it seems difficult to determine a nontrivial general upper bound for $g(n, p, q)$ (or, equivalently, lower bound for $g_q^k(p)$). Although we can prove some nontrivial results (see, e.g., Proposition 1.5) using the Lovász Local Lemma, it cannot help us much in determining an upper bound for $g(n, p, q)$. A graph with an edge-coloring is called *q-colored* if its edges are colored with at most q distinct colors.

Proposition 1.5. *For fixed integers s, q, k, and appropriately large integer p with $s \geq 4$ and $k \geq \max\left\{\binom{s}{2}, 2q+1\right\}$, there exists a k-edge-coloring of K_n with*

$$n = \left(\frac{(s-2)pL^{1/\left(1-\binom{s}{2}\right)}}{(c+o(1))\left(\binom{s}{2}-2.1\right)\ln\left(pL^{1/\left(1-\binom{s}{2}\right)}\right)}\right)^{\left(\binom{s}{2}-2.1\right)/(s-2)}$$

such that there is neither a rainbow K_s nor a q-colored K_p, where c is a constant and $L = \binom{s}{2}(k-1)^{2-\binom{s}{2}}(k-2)\cdots\left(k-\binom{s}{2}+1\right)$.

When $q = 1$, $g_1^k(p)$ is the smallest positive integer n such that every Gallai-k-coloring of K_n contains a monochromatic copy of K_p. Fox, Grinshpun, and Pach [16] posed the following conjecture, which was verified independently by Chung and Graham [5] and Gyárfás et al. [20] for $p = 3$, and by Liu et al. [24] for $p = 4$, using the language of Gallai–Ramsey numbers.

Conjecture 1.6 (Fox et al. [16]). *For integers $k \geq 3$ and $p \geq 3$,*

$$g_1^k(p) = \begin{cases} (R_2(K_p)-1)^{k/2}+1 & \text{if } k \text{ is even,} \\ (p-1)\cdot(R_2(K_p)-1)^{(k-1)/2}+1 & \text{if } k \text{ is odd,} \end{cases}$$

where $R_2(K_p)$ is the 2-colored Ramsey number for K_p.

We can slightly improve Theorem 1.4 for $q = 1$ by proving the following upper bound on $g_1^k(p)$.

Proposition 1.7. *For integers* $k \geq 3$ *and* $p \geq 5$, *we have* $g_1^k(p) < 2^{2k(p-2)-3}$.

When $q = p - 2$, we can prove the following result, thereby improving some results obtained in [3].

Theorem 1.8. *For integers* $p \geq 4$ *and* $k \geq p - 2$, *we have* $g_{p-2}^k(p) = k + 2$.

The above result is equivalent to $g(n, p, p - 1) = n - 1$, where $n \geq p \geq 4$. Using Theorem 1.8, we can show that $g_q^k(p)$ is at least quadratic in k for $q = \lfloor\sqrt{p-1}\rfloor - 1$.

Theorem 1.9. *For integers* $p \geq 17$ *and* $k \geq \lfloor\sqrt{p-1}\rfloor - 1$, *we have* $g_{\lfloor\sqrt{p-1}\rfloor-1}^k(p) \geq k^2 + 2k + 2$.

Note that Theorem 1.9 implies that $g(n, p, \lfloor\sqrt{p-1}\rfloor) \leq \lceil\sqrt{n}\rceil - 1$ for $p \geq 17$ and $n \geq (\lfloor\sqrt{p-1}\rfloor + 1)^2$. When $q = p - 3$, we can prove the following result, which is equivalent to $g(n, p, p - 2) = n - 2$ for $n \geq p \geq 8$.

Theorem 1.10. *For integers* $p \geq 8$ *and* $k \geq p - 3$, *we have* $g_{p-3}^k(p) = k + 3$.

Furthermore, we can determine the exact value of $g_2^k(5)$. Using this result, we can show that $g_q^k(p)$ is exponential in k for all $1 \leq q \leq \lfloor\log_2(p-1)\rfloor$.

Theorem 1.11. *For integers* $k \geq 2$, *we have* $g_2^k(5) = 2^k + 1$.

Theorem 1.12. *For integers* $p \geq 5$ *and* $k \geq \lfloor\log_2(p-1)\rfloor$, *we have* $g_{\lfloor\log_2(p-1)\rfloor}^k(p) \geq 2^k + 1$.

Note that Theorem 1.11 is equivalent to $g(n, 5, 3) = \lceil\log_2 n\rceil$, where $n \geq 5$. Theorem 1.12 implies that $g(n, p, \lfloor\log_2(p-1)\rfloor + 1) \leq \lceil\log_2 n\rceil$, where $p \geq 5$ and $n \geq 2(p-1)$.

Finally, motivated by the problem introduced by Erdős, Hajnal, and Rado (see Section 18 of [12]) to find the minimum integer n such that for any k-coloring of K_n there is a $(k-1)$-colored K_m, we study $g_{k-1}^k(p)$ for $k \leq p - 1$. If p is sufficiently larger than k, then $g_{k-1}^k(p) = O((p/\log_2^c p)^{k/(k-2)})$ by Theorem 1.3. So we will focus on the case $k/p \to 1$. By Theorems 1.8 and 1.10, we have $g_{k-1}^k(p) = p + 1$ for $k \in \{p-1, p-2\}$ and large enough p. A natural question is whether $g_{k-1}^k(p) = p + 1$ for $k = p - c$, where c is a constant and p is large enough. The following theorem answers this question.

Theorem 1.13. *For integers* c, p *and* k *with* $c \geq 1, p \geq 2(8+c)^{c+1} - 1$, *and* $k = p - c$, *we have* $g_{k-1}^k(p) = p + 1$.

The remainder of this paper is organized as follows. In Section 2, we provide some useful results and additional terminology. In Section 3, we prove Theorem 1.4 and Propositions 1.5 and 1.7. In Section 4, we give our proof of Theorem 1.8, and we prove Theorem 1.9 in a more general form. In Section 5, we present our proof of Theorem 1.10. In Section 6, we prove Theorems 1.11 and 1.12. Section 7 is devoted to our proof of Theorem 1.13. Finally, we will conclude the paper with some reflections on what our results for $g_q^k(p)$ imply for the function $g(n, p, q)$ in Section 8. There we also present a conjecture and some open problems.

2 | PRELIMINARIES

We begin with some terminology and notation. Given a graph G, let c: $E(G) \to [k]$ be a k-edge-coloring of G, where $[k] := \{1, 2, ..., k\}$. For an edge $e \in E(G)$, let $c(e)$ be the color used on edge e. For nonempty subsets $U, V \subset V(G)$ with $U \cap V = \varnothing$, let $E(U, V) = \{uv \in E(G): u \in U, v \in V\}$ and $C(U, V) = \{c(e): e \in E(U, V)\}$. If $|C(U, V)| = 1$, then we use $c(U, V)$ to denote the unique color in $C(U, V)$. The subgraph of G induced by U is denoted by $G[U]$, and $G - U$ is shorthand for $G[V(G)\setminus U]$. If U consists of a single vertex u, then we simply write $E(\{u\}, V)$, $C(\{u\}, V)$, $c(\{u\}, V)$, and $G - \{u\}$ as $E(u, V)$, $C(u, V)$, $c(u, V)$, and $G - u$, respectively. Let $C(G)$, $C(G[U])$, and $C(G - U)$ denote the set of colors used on $E(G)$, $E(G[U])$, and $E(G - U)$, respectively. We also use the abbreviation $C(U)$ for $C(G[U])$. For a color i, the *subgraph induced by color* i is the subgraph that contains all the edges with color i and the vertices that are incident with at least one edge of color i.

The following structural result on Gallai-colorings was first obtained by Gallai [18], using the terminology of transitive orientations, and restated by Gyárfás and Simonyi [21] in the language of graph theory.

Theorem 2.1 (Gallai [18] and Gyárfás and Simonyi [21]). *In any Gallai-coloring of a complete graph, the vertex set can be partitioned into at least two nonempty parts such that there is only one color on the edges between every pair of parts, and there are at most two colors between the parts in total.*

We call a vertex partition as given by Theorem 2.1 a *Gallai-partition*. Since every 2-edge-coloring of K_n contains a connected monochromatic spanning subgraph, we have the following corollary.

Corollary 2.2. *In any Gallai-coloring of a complete graph, there is a connected monochromatic spanning subgraph.*

We shall also use the following simple result in our proofs.

Lemma 2.3. *Let G be a Gallai-coloring of a complete graph, $V \subset V(G)$ and $v \in V(G)\setminus V$. Then there is at most one color on the edges between v and V that is not used on any edge within V (i.e., $|C(v, V)\setminus C(V)| \le 1$).*

Proof. Suppose that $c(vu) = 1, c(vw) = 2$ and $1, 2 \notin C(V)$, where $u, w \in V$. Then we may further assume that $c(uw) = 3$. Now $\{u, v, w\}$ forms a rainbow triangle, a contradiction. □

Finally, we introduce the Lovász Local Lemma. Let $(\Omega, \mathcal{F}, \Pr)$ be a probability space and let $A_1, A_2, \ldots, A_n$ be events. A graph D with $V(D) = \{v_1, v_2, \ldots, v_n\}$ is called a *dependency graph* for events $A_1, A_2, \ldots A_n$ if for every i, the event A_i is mutually independent of all A_j with $v_i v_j \notin E(D)$ and $i \neq j$, that is, A_i is independent of any Boolean function of the events in $\{A_j : v_i v_j \notin E(D), i \neq j\}$. We shall use the following form of the Local Lemma due to Spencer.

Lemma 2.4 (Lovász Local Lemma [13,31]). *Let $A_1, A_2, \ldots, A_n$ be events in a probability space $(\Omega, \mathcal{F}, \Pr)$ with dependency graph D. If there exist positive real numbers $y_1, y_2, \ldots, y_n$ such that for each i, $y_i \Pr(A_i) < 1$ and $\ln y_i > \sum_{v_i v_j \in E(D)} y_j \Pr(A_j)$, then $\Pr(\wedge_{i=1}^n \overline{A}_i) > 0$.*

3 | GENERAL UPPER AND LOWER BOUNDS

Before proving Theorem 1.4, we first prove two lemmas. The proof ideas of Lemmas 3.1 and 3.2 are from [17]. For an edge-colored K_n, a vertex $v \in V(K_n)$ and a color i, let $d_i(v)$ be the number of edges in color i incident with v.

Lemma 3.1. *If an edge-coloring of K_n with $n \geq 4$ satisfies $d_i(v) \leq \frac{n}{4}$ for each $v \in V(K_n)$ and each color i, then there exists a rainbow copy of K_3.*

Proof. It suffices to show that the number of nonrainbow K_3's is less than $\binom{n}{3}$. Note that for any vertex v and any color i, there are at most $\binom{d_i(v)}{2}$ nonrainbow K_3's with two edges in color i incident with vertex v. Thus the number of nonrainbow K_3's is at most

$$\sum_{v \in V(K_n)} \sum_i \frac{d_i(v)(d_i(v) - 1)}{2} \leq 4n \frac{(n/4)(n/4 - 1)}{2} < \binom{n}{3},$$

where the first inequality holds since $\sum_i \frac{d_i(v)(d_i(v)-1)}{2} \leq 4 \frac{(n/4)(n/4-1)}{2}$ (using $0 \leq d_i(v) \leq \frac{n}{4}$, $\sum_i d_i(v) = n - 1$, and noting that the function $f(x) = \frac{x(x-1)}{2}$ is convex with $f(x) \geq f(1) = 0$ for any $x \geq 1$). □

Let $[k] = \{1, 2, \ldots, k\}$ be a set of colors and $t_q = \sum_{i=1}^q \binom{k}{i}$. Let $\mathcal{I} = \{I \subseteq [k] : 1 \leq |I| \leq q\} = \{I_1, I_2, \ldots, I_{t_q}\}$. Then we define $g_q^k(p_1, p_2, \ldots, p_{t_q})$ to be the smallest positive integer n such that every Gallai-k-coloring of K_n contains a copy of K_{p_i} all edges of which have colors from one set I_i for some i.

Lemma 3.2. *We have*

$$g_q^k(p_1, p_2, \ldots, p_{t_q}) \leq 4 \cdot \max_{1 \leq i \leq k} g_q^k\left(p_1^{(i)}, p_2^{(i)}, \ldots, p_{t_q}^{(i)}\right),$$

where $p_j^{(i)} = p_j - 1$ if $i \in I_j$, and $p_j^{(i)} = p_j$ otherwise.

Proof. Let $n \geq 4 \cdot \max_{1\leq i\leq k} g_q^k\left(p_1^{(i)}, p_2^{(i)}, \ldots, p_{t_q}^{(i)}\right)$. By Lemma 3.1, for every Gallai-coloring of K_n, there exists a vertex v and a color ℓ with $d_\ell(v) > \frac{n}{4}$. Let $N_\ell(v) = \{u : c(uv) = \ell\}$. Then $|N_\ell(v)| > g_q^k\left(p_1^{(\ell)}, p_2^{(\ell)}, \ldots, p_{t_q}^{(\ell)}\right)$. In this case there is a copy of K_{p_i} all edges of which have colors from one set I_i for some i. This proves the statement of the lemma. □

Now we have all ingredients to present our proofs of Theorem 1.4 and Proposition 1.7.

Proof of Theorem 1.4. Note that $g_q^k(p) = g_q^k(p, p, \ldots, p)$. We can repeatedly apply Lemma 3.2 until in some step we get $g_q^k(p_1, p_2, \ldots, p_{t_q}) \leq 2$. In each step, we have $g_q^k(p_1, p_2, \ldots, p_{t_q}) \leq 4 \cdot g_q^k\left(p_1^{(i)}, p_2^{(i)}, \ldots, p_{t_q}^{(i)}\right)$ for some i. For each $i \in [k]$, let $\alpha(i)$ be the number of steps in which we apply Lemma 3.2 for color i. By the definition of $g_q^k(p_1, p_2, \ldots, p_{t_q})$, we have $g_q^k(p_1, p_2, \ldots, p_{t_q}) = 1 < 2$ if $p_j = 1$ for some $j \in [t_q]$. We also have $g_q^k(p_1, p_2, \ldots, p_{t_q}) = 2$ if $p_j = 2$ for all $j \in [t_q]$ with $|I_j| = q$. Thus $\sum_{I\in\mathcal{I},|I|=q}\sum_{i\in I}\alpha(i) \leq (p-2)\binom{k}{q}$. Then $\sum_{i=1}^k \alpha(i) = \frac{1}{\binom{k-1}{q-1}}\sum_{I\in\mathcal{I},|I|=q}\sum_{i\in I}\alpha(i) \leq \frac{1}{\binom{k-1}{q-1}}\binom{k}{q}(p-2) = \frac{k(p-2)}{q}$. We conclude that $g_q^k(p) \leq 4^{\frac{k(p-2)}{q}} \cdot 2 = 2^{\frac{2k(p-2)}{q}+1}$, completing the proof of Theorem 1.4. □

Proof of Proposition 1.7. The proof is similar to the proof of Theorem 1.4. The only difference is that we repeatedly apply Lemma 3.2 until in some step we get $g_1^k(p_1, p_2, \ldots, p_{t_q}) < 32$. Note that we have $g_1^k(2) = 2 \leq 32, g_1^k(2, \ldots, 2, 6) = 6 < 32, g_1^k(2, \ldots, 2, 3, 5) = R(K_3, K_5) = 14 < 32$ [19], $g_1^k(2, \ldots, 2, 4, 4) = R_2(K_4) = 18 < 32$ [19], $g_1^k(2, \ldots, 2, 3, 3, 4) = g_1^3(3, 3, 4) = 17 < 32$ [24], and $g_1^k(2, \ldots, 2, 3, 3, 3, 3) = g_1^4(3) = 26 < 32$ [5,20]. Thus we have $\sum_{i=1}^k \alpha(i) \leq k(p-2) - 4$ in this case, so $g_1^k(p) < 4^{k(p-2)-4} \cdot 32 = 2^{2k(p-2)-3}$. □

In the rest of this section, we prove Proposition 1.5, using a similar method to that used in [32].

Proof of Proposition 1.5. Consider a k-edge-coloring G of K_n, where each edge receives color i $(1 \leq i \leq k-1)$ with probability $\frac{r}{k-1}$ and color k with probability $1 - r$ (for small r, to be determined shortly), and these probabilities are mutually independent. For each set S of s vertices, let A_S be the event that $G[S]$ is a rainbow K_s. For each set T of p vertices, let B_T be the event that $G[T]$ is a q-colored K_p. We shall show that $\Pr((\wedge_S \overline{A}_S) \wedge (\wedge_T \overline{B}_T)) > 0$.

Define a graph D with a vertex set corresponding to all possible A_S and B_T such that (the vertex corresponding to) A_S is adjacent to (the vertex corresponding to) B_T if and only if $|S \cap T| \geq 2$, and A_S (resp., B_T) is adjacent to $A_{S'}$ (resp., $B_{T'}$) if and only if $|S \cap S'| \geq 2$ (resp., $|T \cap T'| \geq 2$). Then D is a dependency graph. We define N_{AA}, N_{AB}, N_{BA}, and N_{BB} such that N_{XY} is the number of vertices in D of type Y (so corresponding either to a number of A_S

vertices or a number of B_T vertices) adjacent to a fixed vertex of type X (so either one A_S vertex or one B_T vertex). To be able to apply Lemma 2.4, for each S, let the positive real number $y_i = y$ correspond to event A_S, and for each T, let $y_i = z$ correspond to event B_T. By Lemma 2.4, to show that $\Pr((\wedge_S \overline{A}_S) \wedge (\wedge_T \overline{B}_T)) > 0$, it suffices to show that there exist positive real numbers r, y, z such that

$$r < 1, \quad y\Pr(A_s) < 1, \quad z\Pr(B_T) < 1, \tag{1}$$

$$\ln y > y\Pr(A_s)N_{AA} + z\Pr(B_T)N_{AB}, \tag{2}$$

$$\ln z > y\Pr(A_s)N_{BA} + z\Pr(B_T)N_{BB}. \tag{3}$$

Note that for r small, we have

$$\begin{aligned}
\Pr(A_S) &\leq \binom{k-1}{\binom{s}{2}}\binom{s}{2}!\left(\frac{r}{k-1}\right)^{\binom{s}{2}} + \binom{k-1}{\binom{s}{2}-1}\left(\binom{s}{2}-1\right)!\binom{s}{2}(1-r)\left(\frac{r}{k-1}\right)^{\binom{s}{2}-1} \\
&= \binom{s}{2}(k-1)(k-2)\cdots\left(k-\binom{s}{2}+1\right)\left(\frac{r}{k-1}\right)^{\binom{s}{2}-1}\left(\frac{k-\binom{s}{2}}{\binom{s}{2}}\cdot\frac{r}{k-1}+1-r\right) \\
&\leq Lr^{\binom{s}{2}-1}
\end{aligned}$$

and

$$\begin{aligned}
\Pr(B_T) &\leq \binom{k-1}{q}\left(\frac{qr}{k-1}\right)^{\binom{p}{2}} + \binom{k-1}{q-1}\left(1-r+\frac{(q-1)r}{k-1}\right)^{\binom{p}{2}} \\
&\leq \binom{k-1}{q}\left(\frac{r}{2}\right)^{\binom{p}{2}} + \binom{k-1}{q-1}\left(1-\frac{r}{2}\right)^{\binom{p}{2}} \\
&\leq \left(\binom{k-1}{q}+\binom{k-1}{q-1}\right)\left(1-\frac{r}{2}\right)^{\binom{p}{2}} \\
&\leq \binom{k}{q}\exp\left(-\frac{r}{2}\binom{p}{2}\right) = \exp\left(-\frac{rp^2}{4}+\frac{rp}{4}+\ln\binom{k}{q}\right).
\end{aligned}$$

We bound N_{AA}, N_{AB}, N_{BA}, and N_{BB} as follows:

$$N_{AA} \leq \binom{s}{2}\binom{n-2}{s-2} \leq s^2 n^{s-2}, \quad N_{AB} \leq \binom{s}{2}\binom{n-2}{p-2} \leq s^2 n^{p-2},$$

$$N_{BA} \leq \binom{p}{2}\binom{n-2}{s-2} \leq p^2 n^{s-2}, \quad N_{BB} \leq \binom{p}{2}\binom{n-2}{p-2} \leq p^2 n^{p-2}.$$

Let $\alpha = (s-2)/\left(\binom{s}{2}-2.1\right)$ and $\beta = 1/\left(\binom{s}{2}-1\right)$. We set

$$r = c_1 n^{-\alpha} L^{-\beta}, \quad p = c_2 n^{\alpha}(\ln n)L^{\beta}, \quad y = 1+\epsilon, \quad z = \exp(c_3 n^{\alpha}(\ln n)^2 L^{\beta}),$$

where $\epsilon \ll 1, c_1, c_2, c_3$ are appropriately chosen and n tends to infinity. Then we have

$$y\Pr(A_S)N_{AA} \le (1+\epsilon)Lr^{\binom{s}{2}-1}s^2 n^{s-2} = (1+\epsilon)s^2 c_1^{\binom{s}{2}-1} n^{\frac{-1.1(s-2)}{\binom{s}{2}-2.1}},$$

$$y\Pr(A_S)N_{BA} \le (1+\epsilon)Lr^{\binom{s}{2}-1}p^2 n^{s-2} = (1+\epsilon)c_1^{\binom{s}{2}-1}c_2^2 L^{2\beta} n^{\alpha-\frac{0.1(s-2)}{\binom{s}{2}-2.1}}(\ln n)^2,$$

$$\begin{aligned} z\Pr(B_T)N_{AB} &\le \exp\left(c_3 n^{\alpha}(\ln n)^2 L^{\beta} - \frac{rp^2}{4} + \frac{rp}{4} + \ln\binom{k}{q} + 2\ln s + (p-2)\ln n\right) \\ &\le \exp\left(c_3 n^{\alpha}(\ln n)^2 L^{\beta} - \frac{c_1 c_2^2}{4} n^{\alpha}(\ln n)^2 L^{\beta} + o(n^{\alpha}(\ln n)^2) + c_2 \right. \\ &\qquad \left. n^{\alpha}(\ln n)^2 L^{\beta}\right) \\ &\le \exp\left(\left(c_3 - \frac{c_1 c_2^2}{4} + c_2 + o(1)\right)n^{\alpha}(\ln n)^2 L^{\beta}\right), \end{aligned}$$

and

$$\begin{aligned} z\Pr(B_T)N_{BB} &\le \exp\left(c_3 n^{\alpha}(\ln n)^2 L^{\beta} - \frac{rp^2}{4} + \frac{rp}{4} + \ln\binom{k}{q} + 2\ln p + (p-2)\ln n\right) \\ &\le \exp\left(\left(c_3 - \frac{c_1 c_2^2}{4} + c_2 + o(1)\right)n^{\alpha}(\ln n)^2 L^{\beta}\right). \end{aligned}$$

If we choose c_1, c_2, c_3 such that $c_3 - \frac{c_1 c_2^2}{4} + c_2 + o(1) < 0$, then in Equations (1)–(3) hold. Setting $c = c_2$ in the above expression for p, and expressing n in terms of p, we have

$$n \ge \left(\frac{(s-2)pL^{1/\left(1-\binom{s}{2}\right)}}{(c+o(1))\left(\binom{s}{2}-2.1\right)\ln\left(pL^{1/\left(1-\binom{s}{2}\right)}\right)}\right)^{\left(\binom{s}{2}-2.1\right)/(s-2)}.$$

□

4 | PROOFS OF THEOREMS 1.8 AND 1.9

We first present our proof of Theorem 1.8.

Proof of Theorem 1.8. We first show that there is a Gallai-k-coloring of K_{k+1}, in which there is no K_p receiving at most $p-2$ distinct colors. The case $k = p-2$ is trivial since K_{p-1} contains no K_p. For $k \ge p-1$, let $V(K_{k+1}) = \{v_1, v_2, ..., v_{k+1}\}$. For every

$1 \leq i < j \leq k+1$, we color the edge $v_i v_j$ using color i. Note that for any three vertices v_i, v_j, v_k with $i < j < k$, we have $c(v_i v_j) = c(v_i v_k)$, so there are no rainbow triangles. For any p vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ with $i_1 < i_2 < \cdots < i_p$, we have $C(\{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\}) = \{i_1, i_2, \ldots, i_{p-1}\}$, so every K_p receives $p-1$ distinct colors.

Next, we show that $g_{p-2}^k(p) \leq k+2$ by induction on k. For the base case, if $k = p-2$, then it is trivial that $g_{p-2}^k(p) = p$. Now assume that it holds for every $p-2 \leq k' \leq k-1$, and we will prove it for k.

For a contradiction, suppose that G is a Gallai-k-coloring of K_{k+2} without a $(p-2)$-colored K_p. Using Theorem 2.1, let $V_1, V_2, \ldots, V_m$ $(m \geq 2)$ be a Gallai-partition of $V(G)$. Note that $m \leq p-1$ since $p-2 \geq 2$. If $m \geq 4$, then we can choose nonempty subsets $V_i' \subseteq V_i$ $(1 \leq i \leq m)$ such that $\sum_{i=1}^m |V_i'| = p$. Since G is a Gallai-coloring, we have $|C(V_i')| \leq |V_i'| - 1$ $(1 \leq i \leq m)$ by Theorem 1.1. Then $|C(\bigcup_{i=1}^m V_i')| \leq 2 + \sum_{i=1}^m (|V_i'| - 1) = 2 + p - m \leq p-2$. Thus there is a $(p-2)$-colored K_p in G, a contradiction. Hence, we have $m \leq 3$. Note that if G contains a Gallai-partition with exactly three parts, then G also contains a Gallai-partition with exactly two parts. Thus we may assume that $m = 2$ and $c(V_1, V_2) = 1$. □

Claim 4.1. $1 \notin C(V_1)$ and $1 \notin C(V_2)$.

Proof. By symmetry, we only prove $1 \notin C(V_1)$. If $1 \in C(V_1)$, then we may choose $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$ such that $1 \in C(V_1')$ and $|V_1'| + |V_2'| = p$. Thus $|C(V_1' \cup V_2')| \leq |C(V_1')| + |C(V_2')| \leq |V_1'| - 1 + |V_2'| - 1 = p-2$, a contradiction. □

Claim 4.2. $|V_1| = |C(V_1)| + 1$ and $|V_2| = |C(V_2)| + 1$.

Proof. By symmetry, we only prove it for V_1. By Theorem 1.1, we have $|V_1| \geq |C(V_1)| + 1$, so it suffices to prove $|V_1| \leq |C(V_1)| + 1$. Suppose for a contradiction that $|V_1| \geq |C(V_1)| + 2$. If $|C(V_1)| \leq p-3$, then $|V_1| \leq p-1$ to avoid a $(p-2)$-colored K_p. Thus we can choose $V_2' \subseteq V_2$ with $|V_1| + |V_2'| = p$. Since $|C(V_2')| \leq |V_2'| - 1$, we have $|C(V_1 \cup V_2')| \leq 1 + |C(V_1)| + |V_2'| - 1 \leq |C(V_1)| + p - |V_1| \leq |C(V_1)| + p - (|C(V_1)| + 2) = p-2$, a contradiction. Thus $|C(V_1)| \geq p-2$, and then we have $|V_1| \leq |C(V_1)| + 1$ by Claim 4.1 and the induction hypothesis. □

We now show that $C(V_1) \cap C(V_2) = \varnothing$. Otherwise, suppose $2 \in C(V_1) \cap C(V_2)$. We choose $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$ such that $2 \in C(V_1'), 2 \in C(V_2')$ and $|V_1'| + |V_2'| = p$. Then $|C(V_1' \cup V_2')| \leq 1 + |C(V_1')| + |C(V_2')| - 1 \leq |V_1'| - 1 + |V_2'| - 1 = p-2$, a contradiction. Finally, by Claims 4.1 and 4.2, we have $k+2 = |V(G)| = |V_1| + |V_2| = |C(V_1)| + 1 + |C(V_2)| + 1 \leq k - 1 + 2 = k+1$, a contradiction.

In the following, instead of proving Theorem 1.9, we will prove the following more general result.

Theorem 4.3. *For integers $p \gg m \geq 2$ and $k \geq \lfloor \sqrt[m]{p-1} \rfloor - 1$, we have $g^k_{\lfloor \sqrt[m]{p-1} \rfloor - 1}(p) \geq (k+1)^m + 1$.*

Proof. Let $q = \lfloor \sqrt[m]{p-1} \rfloor - 1$. By Theorem 1.8, we have $g_q^k(q+2) > k+1$. Let G_0 be a Gallai-k-coloring of K_{k+1} in which the largest q-colored complete subgraph has order at most $q+1$, and let $G_1 = G_0$. Suppose that for some $1 \le i < m$ we have constructed a k-edge-coloring G_i of $K_{(k+1)^i}$. Then we construct G_{i+1} by substituting $k+1$ copies of G_i into vertices of G_0. Finally, we obtain a k-edge-coloring G_m of $K_{(k+1)^m}$. It is easy to check that G_m is a Gallai-coloring and that the largest q-colored complete subgraph has order at most $(q+1)^m \le p-1$. Thus we have $g_q^k(p) \ge (k+1)^m + 1$. □

5 | PROOF OF THEOREM 1.10

For the lower bound, we will construct a Gallai-k-coloring of K_{k+2} without a $(p-3)$-colored K_p. The case $k = p-3$ is trivial since K_{p-1} contains no K_p. For $k \ge p-2$, let $V(K_{k+2}) = \{v_1, v_2, ..., v_{k+2}\}$. For every $1 \le i \le k$ and $i < j \le k+2$, we color the edge $v_i v_j$ using color i, and we color the edge $v_{k+1}v_{k+2}$ with color k. Then we obtain a desired edge-coloring.

For the upper bound, we will use induction on k. For the base case, if $k = p-3$, then it is trivial that $g_{p-3}^k(p) \le k+3$. Now assume that it holds for every $p-3 \le k' \le k-1$, and we will prove it for k. For a contradiction, suppose that G is a Gallai-k-coloring of K_{k+3} without a $(p-3)$-colored K_p. By the induction hypothesis, we may assume that all the k colors appear in G (i.e., $C(G) = [k]$). Using Theorem 2.1, let $V_1, V_2, ..., V_m$ $(m \ge 2)$ be a Gallai-partition of $V(G)$. We choose it such that m is minimum.

Case 1. $m \ge 4$.

In this case, by the minimality of m, there are exactly two colors used between the parts, say colors 1 and 2. If $m \ge 5$, then we can choose one vertex v_i from each V_i $(1 \le i \le 5)$ to form a 2-colored K_5. Then we choose another $p-5$ vertices $v_6, v_7, ..., v_p$ one by one arbitrarily. Note that for each $6 \le i \le p$, when we add vertex v_i to $G_{i-1} = G[\{v_1, v_2, ..., v_{i-1}\}]$, we add at most one new color that is not used in G_{i-1}, by Lemma 2.3. Thus we obtain a $(p-3)$-colored K_p, a contradiction. Hence, we have $m = 4$.

Claim 5.1. For any $i \in [4]$, we have $1, 2 \notin C(V_i)$. For any $1 \le i < j \le 4$, we have $C(V_i) \cap C(V_j) = \emptyset$.

Proof. If $C(V_i) \cap \{1,2\} \ne \emptyset$ for some $i \in [4]$, then we can choose nonempty subsets $V_l' \subseteq V_l$ $(1 \le l \le 4)$ such that $\sum_{l=1}^4 |V_l'| = p$ and $C(V_i') \cap \{1,2\} \ne \emptyset$. Since G is a Gallai-coloring, we have $|C(V_l')| \le |V_l'| - 1$ $(1 \le l \le 4)$ by Theorem 1.1. Then $|C(\bigcup_{l=1}^4 V_l')| \le 2 + (\sum_{l=1}^4 |C(V_l')|) - 1 \le 2 + (\sum_{l=1}^4 (|V_l'| - 1)) - 1 = 2 + p - 4 - 1 = p - 3$. Thus there is a $(p-3)$-colored K_p in G, a contradiction. If $C(V_i) \cap C(V_j) \ne \emptyset$ for some $1 \le i < j \le 4$, say $c_0 \in C(V_i) \cap C(V_j)$, then we can choose nonempty subsets $V_l' \subseteq V_l$ $(1 \le l \le 4)$ such that $\sum_{l=1}^4 |V_l'| = p$, $c_0 \in C(V_i')$ and $c_0 \in C(V_j')$. Then $|C(\bigcup_{l=1}^4 V_l')| \le 2 + (\sum_{l=1}^4 |C(V_l')|)$ $-1 \le 2 + (\sum_{l=1}^4 (|V_l'| - 1)) - 1 = p - 3$. Thus there is a $(p-3)$-colored K_p in G, a contradiction. □

Claim 5.2. For any $i \in [4]$, we have $|V_i| \le |C(V_i)| + 1$.

Proof. Suppose for a contradiction that $|V_i| \geq |C(V_i)| + 2$ for some $i \in [4]$, say $i = 1$. If $|C(V_1)| \leq p - 5$, then $|V_1| \leq p - 4$ to avoid a $(p-3)$-colored K_p. Thus we can choose nonempty subsets $V'_j \subseteq V_j$ $(2 \leq j \leq 4)$ such that $|V_1| + \sum_{j=2}^{4}|V'_j| = p$. Then $|C(V_1 \cup (\bigcup_{j=2}^{4} V'_j))| \leq 2 + |C(V_1)| + \sum_{j=2}^{4}(|V'_j| - 1) \leq 2 + |C(V_1)| + (p - |V_1|) - 3 \leq 2 + |C(V_1)| + p - (|C(V_1)| + 2) - 3 = p - 3$, a contradiction.

If $|C(V_1)| \geq p - 3$, then $|V_1| \leq |C(V_1)| + 2$ by Claim 5.1 and the induction hypothesis. If $|C(V_1)| = p - 4$, then $|V_1| \leq p - 2 = |C(V_1)| + 2$ to avoid a $(p-3)$-colored K_p. Thus $|V_1| = |C(V_1)| + 2$ whenever $|C(V_1)| \geq p - 4$. By Theorem 1.8, there is a $(p_1 - 2)$-colored K_{p_1} in $G[V_1]$ for every $4 \leq p_1 \leq |V_1|$. Let H be a copy of a $(p-5)$-colored K_{p-3} in $G[V_1]$. Then we can choose one vertex from each V_j $(2 \leq j \leq 4)$ such that these vertices together with H form a $(p-3)$-colored K_p, a contradiction. □

By Claims 5.1 and 5.2, we have $k + 3 = |V(G)| = \sum_{i=1}^{4}|V_i| \leq \sum_{i=1}^{4}(|C(V_i)| + 1) \leq k - 2 + 4 = k + 2$, a contradiction.

Case 2. $2 \leq m \leq 3$.

By the minimality of m, we may assume that $m = 2$ and $c(V_1, V_2) = 1$.

Claim 5.3. At most one of V_1 and V_2 contains an edge with color 1.

Proof. If $1 \in C(V_1)$ and $1 \in C(V_2)$, then we can choose $V'_1 \subseteq V_1$ and $V'_2 \subseteq V_2$ such that $|V'_1| + |V'_2| = p$, $1 \in C(V'_1)$ and $1 \in C(V'_2)$. Then $|C(V'_1 \cup V'_2)| \leq |C(V'_1)| + |C(V'_2)| - 1 \leq |V'_1| - 1 + |V'_2| - 1 - 1 = p - 3$, a contradiction. □

Claim 5.4. We have $|V_i| = |C(V_i)| + 1$ and $|V_{3-i}| = |C(V_{3-i})| + 2$ for some $i \in [2]$.

Proof. Recall that $|V_i| \geq |C(V_i)| + 1$ for each $i \in [2]$ by Theorem 1.1. First suppose that $|V_i| \geq |C(V_i)| + 2$ for all $i \in [2]$. Note that for each $i \in [2]$, since $|V_i| \geq 2$, we have $|C(V_i)| \geq 1$ and thus $|V_i| \geq 3$. Moreover, if $|C(V_1)| = 1$ (resp., $|C(V_2)| = 1$), then $G[V_1]$ (resp., $G[V_2]$) is a monochromatic complete subgraph of order at least 3, and if $|C(V_1)| \geq 2$ (resp., $|C(V_2)| \geq 2$), then $G[V_1]$ (resp., $G[V_2]$) contains a $(p' - 2)$-colored $K_{p'}$ for every $4 \leq p' \leq |V_1|$ (resp., $4 \leq p' \leq |V_2|$) by Theorem 1.8. Thus we can choose a $(p_i - 2)$-colored K_{p_i} in $G[V_i]$ for each $i \in [2]$ such that $3 \leq p_i \leq |V_i|$ and $p_1 + p_2 = p$, so there is a $(p-3)$-colored K_p in G, a contradiction. Hence, we may assume that $|V_1| = |C(V_1)| + 1$ without loss of generality.

If $|V_2| = |C(V_2)| + 1$, then $k + 3 = |V_1| + |V_2| = |C(V_1)| + |C(V_2)| + 2$, so $|C(V_1)| + |C(V_2)| = k + 1$. Then $C(V_1) \cap C(V_2) \neq \varnothing$. Let $C' = C(V_1) \cap C(V_2)$, and we have $1 \notin C'$ by Claim 5.3. If $1 \notin C(V_1)$ and $1 \notin C(V_2)$, then $|C'| \geq 2$ (otherwise we have $|C(V_1)| + |C(V_2)| \leq k$). Then we can choose $V'_1 \subseteq V_1$ and $V'_2 \subseteq V_2$ with $|V'_1 \cup V'_2| = p$ such that $|C(V'_1) \cap C(V'_2)| \geq 2$. Then $|C(V'_1 \cup V'_2)| \leq 1 + |C(V'_1)| + |C(V'_2)| - 2 \leq 1 + |V'_1| - 1 + |V'_2| - 1 - 2 = p - 3$, a contradiction. Hence, without loss of generality, we may assume that $1 \in C(V_1)$, $c_0 \in C'$ and $c_0 \neq 1$. Then we can choose $V'_1 \subseteq V_1$ and $V'_2 \subseteq V_2$ with $|V'_1 \cup V'_2| = p$ such that $\{1, c_0\} \subseteq C(V'_1)$ and $c_0 \in C(V'_2)$. Then $|C(V'_1 \cup V'_2)| \leq |C(V'_1)| + |C(V'_2)| - 1 \leq |V'_1| - 1 + |V'_2| - 1 - 1 = p - 3$, a contradiction. Therefore, we have $|V_2| \geq |C(V_2)| + 2$.

If $1 \leq |C(V_2)| \leq p-4$, then $|V_2| \leq p-2$ to avoid a $(p-3)$-colored K_p. Let $V_1' \subseteq V_1$ such that $|V_1' \cup V_2| = p$. Then $|V_2| = p - |V_1'| \leq p - (|C(V_1')| + 1) \leq p - (|C(V_1' \cup V_2)| - |C(V_2)|) \leq p - (p - 2 - |C(V_2)|) = |C(V_2)| + 2$, where the second inequality is by $|C(V_1' \cup V_2)| \leq 1 + |C(V_1')| + |C(V_2)|$, and the last inequality follows from the assumption that G contains no $(p-3)$-colored K_p. If $p-3 \leq |C(V_2)| \leq k-1$, then by the induction hypothesis we have $|V_2| \leq |C(V_2)| + 2$. If $|C(V_2)| = k$, then $|V_2| = k + 3 - |V_1| \leq k + 2 = |C(V_2)| + 2$. Therefore, we have $|V_2| = |C(V_2)| + 2$. □

By Claim 5.4, we may assume that $|V_1| = |C(V_1)| + 1$ and $|V_2| = |C(V_2)| + 2$ without loss of generality.

Claim 5.5. $C(V_1) \cap C(V_2) = \varnothing$.

Proof. For a contradiction, suppose that $C' = C(V_1) \cap C(V_2) \neq \varnothing$. Similar to the second paragraph in the proof of Claim 5.4, we have $1 \notin C(V_1)$, $1 \notin C(V_2)$, and $|C'| = 1$, say $C' = \{c_0\}$.

If $|V_2| \leq p-2$, then we can choose $V_1' \subseteq V_1$ such that $|V_1' \cup V_2| = p$ and $c_0 \in C(V_1')$. Now we have $|C(V_1' \cup V_2)| \leq 1 + |C(V_1')| + |C(V_2)| - 1 \leq 1 + (|V_1'| - 1) + (|V_2| - 2) - 1 = p - 3$, a contradiction. Thus $|V_2| \geq p-1$ and $|C(V_2)| = |V_2| - 2 \geq p-3$. Let uv be an edge within V_1 with $c(uv) = c_0$. Then $G[V_2 \cup \{u, v\}]$ is a $(|C(V_2)| + 1)$-colored $K_{|V_2|+2}$. If $|C(V_1)| \geq 2$, then $|C(V_2 \cup \{u, v\})| \leq k-1$, and thus we can derive a contradiction by the induction hypothesis. Thus we have $C(V_1) = \{c_0\}$ and $|V_1| = 2$.

By Theorem 1.8, we may assume that H is a copy of a $(p-5)$-colored K_{p-3} in $G[V_2]$. If $c_0 \in C(H)$, then $G[V(H) \cup V_1]$ is a $(p-4)$-colored K_{p-1}. For any vertex $w \in V_2 \backslash V(H)$, we have $|C(w, V(H) \cup V_1) \backslash C(V(H) \cup V_1)| \leq 1$ by Lemma 2.3, which implies a $(p-3)$-colored K_p, a contradiction. If $c_0 \notin C(H)$ and there is an edge xy with color c_0 such that $x \in V(H)$ and $y \in V_2 \backslash V(H)$, then $C(y, V(H)) \backslash C(H) = \{c_0\}$ by Lemma 2.3. Then $G[V(H) \cup V_1 \cup \{y\}]$ is a $(p-3)$-colored K_p, a contradiction. Hence, $G[V_2]$ contains no edge in color 1 which has an end-vertex in $V(H)$. Thus we may assume that xy is an edge with color c_0 such that $x, y \in V_2 \backslash V(H)$. By Theorem 1.8, we may further assume that H' is a copy of a $(p-6)$-colored K_{p-4} in H. By Lemma 2.3, we have $|C(x, V(H')) \backslash C(H')| \leq 1$ and $C(y, V(H') \cup \{x\}) \backslash C(V(H') \cup \{x\}) = \{c_0\}$. Then $G[V(H') \cup V_1 \cup \{x, y\}]$ is a $(p-3)$-colored K_p, a contradiction. □

By Claim 5.4, we have $|C(V_1)| + |C(V_2)| = |V_1| + |V_2| - 3 = k$. Then we have either $1 \in C(V_1)$ or $1 \in C(V_2)$ by Claims 5.3 and 5.5. We first consider the case $1 \in C(V_1)$ and $1 \notin C(V_2)$. We define a subset $V_2' \subseteq V_2$ as follows. If $|V_2| \leq p-3$, then $V_2' = V_2$. If $|V_2| \geq p-2$, then we choose V_2' such that $G[V_2']$ is a $(p-4)$-colored K_{p-2} (using Theorem 1.8). Then let $V_1' \subset V_1$ such that $|V_1'| = p - |V_2'|$ and $1 \in C(V_1')$. Since $|C(V_1' \cup V_2')| \leq |C(V_1')| + |C(V_2')| \leq |V_1'| - 1 + |V_2'| - 2 = p - 3$, we derive a contradiction. Next, we consider the case $1 \notin C(V_1)$ and $1 \in C(V_2)$. In this case, we have $|V_2| \geq p$, since otherwise if $|V_2| \leq p-1$, then we can choose $V_1' \subset V_1$ with $|V_1' \cup V_2| = p$ such that $|C(V_1' \cup V_2)| \leq |C(V_1')| + |C(V_2)| \leq |V_1'| - 1 + |V_2| - 2 = p - 3$, a contradiction.

By Theorem 1.8, we may assume that H is a copy of a $(p-4)$-colored K_{p-2} in $G[V_2]$. Let u be any vertex in V_1. If $1 \in C(H)$, then $G[V(H) \cup \{u\}]$ is a $(p-4)$-colored K_{p-1}. For any vertex

$v \in V_2 \backslash V(H)$, we have $|C(v, V(H) \cup \{u\}) \backslash C(V(H) \cup \{u\})| \le 1$ by Lemma 2.3, which implies a $(p-3)$-colored K_p, a contradiction. If $1 \notin C(H)$ and there is an edge xy with color 1 such that $x \in V(H)$ and $y \in V_2 \backslash V(H)$, then $C(y, V(H)) \backslash C(H) = \{1\}$ by Lemma 2.3. Then $G[V(H) \cup \{u, y\}]$ is a $(p-3)$-colored K_p, a contradiction. If $1 \notin C(H)$ and $G[V_2]$ contains no edge with color 1 incident with a vertex of H, then we may assume that xy is an edge with color 1 such that $x, y \in V_2 \backslash V(H)$. By Theorem 1.8, we may further assume that H' is a copy of a $(p-5)$-colored K_{p-3} in H. By Lemma 2.3, we have $|C(x, V(H')) \backslash C(H')| \le 1$ and $C(y, V(H') \cup \{x\}) \backslash C(V(H') \cup \{x\}) = \{1\}$. Then $G[V(H') \cup \{x, y, u\}]$ is a $(p-3)$-colored K_p. This contradiction completes the proof of Theorem 1.10.

Remark 5.6. The bound $p \ge 8$ in Theorem 1.10 is the best possible. Indeed, if $p = 7$, then we can show that $g_4^5(7) > 8$ by the following counterexample. Let G_1 (resp., G_2) be a K_4 using colors 1 and 2 (resp., colors 3 and 4) such that colors 1 and 2 (resp., colors 3 and 4) induce two monochromatic copies of a P_4. Let G be a 5-colored K_8 obtained by joining G_1 and G_2 using edges that all get color 5. It is easy to check that G contains neither a rainbow K_3 nor a 4-colored K_7. For the case that $p = 6$, we can prove that $g_3^4(6) = 8$ and $g_3^5(6) = 10$ (the full proofs can be found in an arXiv version of this paper; see [arXiv:2011.01592]). When $p = 5$, the function $g_2^k(5)$ is exponential in k by Theorem 1.11.

6 | PROOFS OF THEOREMS 1.11 AND 1.12

We first present our proof of Theorem 1.11.

Proof of Theorem 1.11. We first show that there is a Gallai-k-coloring of K_{2^k}, in which there is no K_5 receiving at most two distinct colors. For $k = 2$, let G_2 be an edge-coloring of K_4 with colors 1 and 2 such that color 1 induces a perfect matching and color 2 induces a C_4. It is easy to check that there is neither a rainbow K_3 nor a monochromatic K_3 in G_2, and G_2 contains no 2-colored K_5 clearly. Suppose for some $2 \le i \le k-1$ we have constructed a Gallai-i-coloring G_i of K_{2^i} in which there is neither a monochromatic K_3 nor a 2-colored K_5. Then we construct an $(i+1)$-edge-coloring G_{i+1} of $K_{2^{i+1}}$ by joining two copies of G_i with edges that all get color $i+1$. Since G_i contains no rainbow K_3, there is no rainbow K_3 in G_{i+1}. Since G_i contains neither a monochromatic K_3 nor a 2-colored K_5, there is no 2-colored K_5 in G_{i+1}. By repeating this process, we finally obtain a Gallai-k-coloring G_k of K_{2^k} without a 2-colored K_5.

We now prove that $g_2^k(5) \le 2^k + 1$ by induction on k. For the base case, it is trivial that $g_2^2(5) = 5$. Now assume that it holds for every $2 \le k' \le k-1$, and we will prove it for $k \ge 3$.

For a contradiction, suppose that G is a Gallai-k-coloring of K_{2^k+1} without a 2-colored K_5. Using Theorem 2.1, let $V_1, V_2, \ldots, V_m$ $(m \ge 2)$ be a Gallai-partition of $V(G)$. We choose it such that m is minimum. Since there is no 2-colored K_5, we have $m \le 4$. If $m = 4$, then by the minimality of m, there are exactly two colors used between the parts, say colors 1 and 2. To avoid a 2-colored K_5, there is no edge with color 1 or 2 within each part. If $k = 3$, then there is only color 3 within these parts. Note that $\max_{1 \le i < j \le 4} |V_i \cup V_j| \ge 5$, so there is a 2-colored K_5. Thus $k \ge 4$. By the induction hypothesis, we have $|V(G)| = |V_1| + |V_2| + |V_3| + |V_4| \le 4 \cdot 2^{k-2} = 2^k$, a contradiction.

Thus $2 \leq m \leq 3$, and by the minimality of m we may assume $m = 2$ and $c(V_1, V_2) = 1$. If $1 \notin C(V_1)$ and $1 \notin C(V_2)$, then by the induction hypothesis, we have $|V(G)| = |V_1| + |V_2| \leq 2^{k-1} + 2^{k-1} = 2^k$, a contradiction. If $1 \in C(V_1)$ and $1 \in C(V_2)$, then G contains a monochromatic K_4. By Lemma 2.3, G contains a 2-colored K_5, a contradiction. Thus we may assume that $1 \in C(V_1)$ and $1 \notin C(V_2)$ without loss of generality. □

Claim 6.1. Color 1 induces a bipartite graph within V_1.

Proof. We first show that $G[V_1]$ contains no monochromatic K_3 with color 1. Otherwise, suppose $\{u, v, w\}$ forms a monochromatic K_3 with color 1 within V_1. Then for any vertex $x \in V_2$, we have that $\{u, v, w, x\}$ forms a monochromatic K_4. By Lemma 2.3, there is a 2-colored K_5 in G, a contradiction.

We next show that $G[V_1]$ contains no C_4 with exactly three edges in color 1. Otherwise, if $G[V_1]$ contains such C_4, say $c(uv) = c(vw) = c(wz) = 1$ and $c(zu) = 2$. To avoid a rainbow K_3, we have $c(uw) \in \{1, 2\}$ and $c(vz) \in \{1, 2\}$. Then for any vertex $x \in V_2$, we have that $\{u, v, w, z, x\}$ forms a 2-colored K_5, a contradiction.

Finally, we show that $G[V_1]$ contains no monochromatic odd cycle in color 1 (thus color 1 induces a bipartite graph within V_1). Suppose that $C_{2t+1} = a_1a_2 \cdots a_{2t+1}a_1$ $(t \geq 2)$ is a monochromatic cycle using color 1 in $G[V_1]$. Since there is no C_4 with exactly three edges in color 1, we have $c(a_1a_4) = 1$, so $c(a_1a_6) = 1, c(a_1a_8) = 1, ..., c(a_1a_{2t}) = 1$. Then $\{a_1, a_{2t}, a_{2t+1}\}$ forms a monochromatic K_3 in color 1, a contradiction. □

Let E_1 be the set of edges with color 1 in $G[V_1]$, and let $V_1' \subseteq V_1$ be the set of vertices incident with some edge of E_1. By Claim 6.1, we may partition V_1' into two parts A and B such that $1 \notin C(A)$ and $1 \notin C(B)$. Since $1 \in C(V_1)$, we have $A \neq \emptyset$ and $B \neq \emptyset$. Let $V_1'' = V_1 \setminus V_1'$ (it is possible that $V_1'' = \emptyset$).

Claim 6.2. The following statements hold:

(1) for any color $i \in C(V_1')$, we have $i \notin C(V_2)$;
(2) $|V_2| \leq 2^{|C(V_2)|}$;
(3) $|A| \leq 2^{|C(A)|}$ and $|B| \leq 2^{|C(B)|}$.

Proof.

(1) If $i = 1$, then it holds clearly. If $i \neq 1$, then we may assume that $c(uv) = i$ for some $u, v \in V_1'$. Since $u \in V_1'$, there exists a vertex $w \in V_1' \setminus \{u, v\}$ with $c(uw) = 1$. To avoid a rainbow K_3, we have $c(vw) \in \{1, i\}$. If $i \in C(V_2)$, then there is a 2-colored K_5 using colors 1 and i in G, a contradiction.
(2) Let $|C(V_2)| = j$ $(0 \leq j \leq k - 1)$. If $j = 0$, then $|V_2| = 1 = 2^0$. If $j = 1$, then $G[V_2]$ is a monochromatic complete subgraph. Suppose $|V_2| \geq 3$. Then G contains a 2-colored K_5 since $1 \in C(V_1)$, a contradiction. Thus $|V_2| = 2 = 2^1$. If $2 \leq j \leq k - 1$, then $|V_2| \leq 2^j$ by the induction hypothesis.
(3) By symmetry, we only prove it for A. If $|C(A)| \neq 1$, then $|A| \leq 2^{|C(A)|}$ by the same argument as in (2). If $|C(A)| = 1$, then $G[A]$ is a monochromatic complete subgraph. Suppose $|A| \geq 3$, say $u, v, w \in A$. Recall that $1 \notin C(A)$. By the definition of A and B, there exists a vertex $x \in B$ such that $c(ux) = 1$. To

avoid a rainbow K_3, we have $C(x, \{v, w\}) \subseteq C(A) \cup \{1\}$. Note that for any vertex $y \in V_2$, we have $c(y, \{u, v, w, x\}) = 1$. Thus $\{u, v, w, x, y\}$ forms a 2-colored K_5, a contradiction. □

Claim 6.3. $|A| \geq 2$ and $|B| \geq 2$.

Proof. By symmetry, we only prove $|A| \geq 2$. If $|A| = 1$, say $A = \{u\}$, then we have $c(u, B) = 1$ by the definition of B. By Claim 6.2(1), we have $C(B) \cap C(V_2) = \varnothing$, so $|C(B)| + |C(V_2)| \leq k - 1$. If $V_1'' = \varnothing$, then by Claim 6.2(2) and (3) we have $|V(G)| = |A| + |B| + |V_2| \leq 1 + 2^{|C(B)|} + 2^{|C(V_2)|} \leq 1 + (2^{|C(B)|+|C(V_2)|} + 1) \leq 1 + 2^{k-1}+1 < 2^k + 1$, a contradiction. Thus $V_1'' \neq \varnothing$, say $v \in V_1''$. Note that v is not incident with any edge in color 1. Thus we may further assume that $c(uv) = 2$. To avoid a rainbow K_3, we have $c(v, B) = 2$. Then $2 \notin C(B)$ and $2 \notin C(V_2)$ to avoid a 2-colored K_5, so $|C(B)| + |C(V_2)| \leq k - 2$. Then $|V(G)| = |A| + |B| + |V_2| + |V_1''| \leq 1 + 2^{|C(B)|} + 2^{|C(V_2)|} + 2^{k-1} \leq 1 + (2^{|C(B)|+|C(V_2)|} + 1) + 2^{k-1} \leq 1 + 2^{k-2} + 1 + 2^{k-1} < 2^k + 1$, a contradiction. □

Claim 6.4. $C(B) \backslash C(A) \neq \varnothing$ and $C(A) \backslash C(B) \neq \varnothing$.

Proof. By symmetry, we only prove $C(B) \backslash C(A) \neq \varnothing$. Recall that $1 \notin C(A)$ and $1 \notin C(B)$. By Claim 6.3, we have $|C(B)| \geq 1$. Since $G[B]$ is a Gallai-coloring, there exists a color, say color 2, inducing a connected spanning subgraph of $G[B]$ by Corollary 2.2. We will show that $2 \notin C(A)$. For a contradiction, suppose that there are two vertices $u, v \in A$ with $c(uv) = 2$. We may further assume that $c(uw) = 1$ and $c(wx) = 2$, where $w, x \in B$. Then $c(vw) \in \{1, 2\}$ and $c(ux) \in \{1, 2\}$. Thus $c(vx) \notin \{1, 2\}$, since otherwise $\{u, v, w, x\}$ together with a vertex in V_2 forms a 2-colored K_5. Then $c(vw) = c(ux) = 2$ to avoid a rainbow K_3. Since v is incident with some edge in color 1, we may assume that $c(vy) = 1$ for some $y \in B \backslash \{w, x\}$. To avoid a rainbow K_3, we have $C(y, \{u, w\}) \subseteq \{1, 2\}$. Then $\{u, v, w, y\}$ together with a vertex in V_2 forms a 2-colored K_5, a contradiction. □

By Claim 6.2(1), we have $C(A) \cap C(V_2) = \varnothing$ and $C(B) \cap C(V_2) = \varnothing$. Recall that $1 \notin C(A)$ and $1 \notin C(B \cup V_1'')$. By Claim 6.4, we further have $|C(A)| \leq k - |\{1\}| - |C(B) \backslash C(A)| - |C(V_2)| \leq k - 2 - |C(V_2)|$. Thus $|V(G)| = |A| + |V_2| + |B \cup V_1''| \leq 2^{k-2-|C(V_2)|} + 2^{|C(V_2)|} + 2^{k-1} \leq 2^{k-2} + 1 + 2^{k-1} < 2^k + 1$, a contradiction. This completes the proof of Theorem 1.11.

Next, we present our proof of Theorem 1.12.

Proof of Theorem 1.12. Let $q = \lfloor \log_2(p-1) \rfloor$. By Theorem 1.11, we have $g_2^q(5) = 2^q + 1 \leq p$. Thus every Gallai-$q$-colored K_p contains a 2-colored K_5. Let $g = g_q^k(p)$. Then every Gallai-k-coloring of K_g contains a Gallai-q-colored K_p, and thus a 2-colored K_5. Hence, $g_q^k(p) \geq g_2^k(5) = 2^k + 1$. □

In fact, we can generalize Theorem 1.12 as follows.

Theorem 6.5. *For integers p, q, k with $q \leq \log_2(p-1)$ and $k \geq q$, we have* $g_q^k(p) \geq \lfloor (p-1)^{1/q} \rfloor^k + 1$.

Proof. Let $m = \lfloor (p-1)^{1/q} \rfloor$. We show that there is a Gallai-k-coloring of K_{m^k}, in which there is no K_p receiving at most q distinct colors. Let G_1 be a monochromatic copy of K_m with color 1. Suppose for some $1 \le i \le k-1$ we have constructed a Gallai-i-coloring G_i of K_{m^i}. Then we construct an $(i+1)$-edge-coloring G_{i+1} of $K_{m^{i+1}}$ by joining m copies of G_i using edges that all get color $i+1$. Finally, we obtain a Gallai-k-coloring G_k of K_{m^k}. For any q distinct colors $c_1, c_2, \ldots, c_q$, the largest complete subgraph in G_k using only these q colors has order at most $m^q \le p-1$. Thus $g_q^k(p) \ge m^k + 1$. □

By Theorems 1.4 and 6.5, we have $\lfloor (p-1)^{1/q} \rfloor^k + 1 \le g_q^k(p) \le 2^{\frac{2k(p-2)}{q}+1}$ for $2 \le q \le \log_2(p-1)$. In the case $q = 1$, we have $2^{\left(\frac{1}{4}+o(1)\right)kp} \le g_1^k(p) \le 2^{(2-o(1))kp}$, where the lower bound follows from a construction given in [16] and the upper bound follows from Proposition 1.7. In this case, when p is large, the gap between the lower and upper bounds is much smaller than the gap between the abovementioned lower and upper bounds. In the case $p = 5$ and $q = 2$, our Theorem 1.11 shows that $g_2^k(5) = 2^k + 1$ for $k \ge 2$, which closes the gap in this case.

7 | PROOF OF THEOREM 1.13

We first introduce two additional definitions and prove some useful lemmas. An *exact Gallai-k-coloring* is a Gallai-k-coloring in which all the k colors are used. A *star* of order $t+1 \ge 2$ is a connected graph with a vertex of degree t having t neighbors of degree 1, and is usually denoted as $K_{1,t}$.

Lemma 7.1. *For any $n \ge 2$, there are at least $\left\lceil \frac{n}{2} \right\rceil$ colors each inducing a star in every exact Gallai-$(n-1)$-coloring of K_n.*

Proof. We prove the statement by induction on n. For the base case, if $2 \le n \le 3$, the statement clearly holds. Now assume that it holds for every $2 \le n' \le n-1$, and we will prove it for n. Let G be an exact Gallai-$(n-1)$-coloring of K_n. Let $V_1, V_2, \ldots, V_m$ be a Gallai-partition of $V(G)$ such that m is minimum. If $m \ge 4$, then by Theorem 1.1 we have $|C(G)| \le 2 + \sum_{i=1}^m |C(V_i)| \le 2 + \sum_{i=1}^m (|V_i| - 1) \le 2 + n - m \le n - 2$, a contradiction. Thus $2 \le m \le 3$, so $m = 2$ by the minimality of m. Without loss of generality, we may assume that $c(V_1, V_2) = 1$ and $|V_1| \ge |V_2|$. We claim that $1 \notin C(V_1) \cup C(V_2)$ and $C(V_1) \cap C(V_2) = \varnothing$, since otherwise $|C(G)| \le 1 + |C(V_1)| + |C(V_2)| - 1 \le |V_1| - 1 + |V_2| - 1 \le n - 2$. If $|V_2| = 1$, then $G[V_1]$ is an exact Gallai-$(n-2)$-coloring of K_{n-1}. By the induction hypothesis, the number of colors each inducing a star is at least $1 + \lceil (n-1)/2 \rceil \ge \lceil n/2 \rceil$. If $|V_1| \ge |V_2| \ge 2$, then by the induction hypothesis, the number of colors each inducing a star is at least $\lceil |V_1|/2 \rceil + \lceil |V_2|/2 \rceil \ge \lceil n/2 \rceil$. □

Lemma 7.2. *For integers c, n, N with $c \ge 2$, $n \ge 2(7+c)^c$ and $N \ge n - \frac{3n}{(7+c)^c}$, we have* $\frac{N}{(2+c)(6+c)^{c-1}} - 2 \ge \frac{n}{(7+c)^c}$.

Proof. Let $a = 6 + c$. Then $a \geq 8$. Since

$$\begin{aligned}
&(2+c)(6+c)^{c-1}\Big(\tfrac{N}{(2+c)(6+c)^{c-1}} - 2 - \tfrac{n}{(7+c)^c}\Big)\\
&\geq \Big(n - \tfrac{3n}{(7+c)^c}\Big) - 2(2+c)(6+c)^{c-1} - \tfrac{(2+c)(6+c)^{c-1}n}{(7+c)^c}\\
&= \Big(1 - \tfrac{3}{(7+c)^c} - \tfrac{(2+c)(6+c)^{c-1}}{(7+c)^c}\Big)n - 2(2+c)(6+c)^{c-1}\\
&\geq \Big(1 - \tfrac{3}{(7+c)^c} - \tfrac{(2+c)(6+c)^{c-1}}{(7+c)^c}\Big)2(7+c)^c - 2(2+c)(6+c)^{c-1}\\
&= 2(7+c)^c - 4(2+c)(6+c)^{c-1} - 6\\
&= 2((a+1)^{a-6} - 2(a-4)a^{a-7}) - 6\\
&= 2\Big(\textstyle\sum_{i=0}^{a-6}\binom{a-6}{i}a^i - 2a^{a-6} + 8a^{a-7}\Big) - 6\\
&= 2\Big(\textstyle\sum_{i=0}^{a-8}\binom{a-6}{i}a^i + (a-6)a^{a-7} + a^{a-6} - 2a^{a-6} + 8a^{a-7}\Big) - 6\\
&= 2\textstyle\sum_{i=0}^{a-8}\binom{a-6}{i}a^i + 4a^{a-7} - 6 \geq 0,
\end{aligned}$$

we have $\frac{N}{(2+c)(6+c)^{c-1}} - 2 \geq \frac{n}{(7+c)^c}$. □

Lemma 7.3. *For any $c \geq 1$ and $n \geq 2(7+c)^c$, there are at least $\frac{n}{(7+c)^c}$ colors each inducing a star in every exact Gallai-$(n-c)$-coloring of K_n.*

Proof. We prove the statement by induction on c. For the base case, if $c = 1$, the statement holds by Lemma 7.1. Now assume that it holds for every $1 \leq c' \leq c-1$, and we will prove it for c with $c \geq 2$. Let G be an exact Gallai-$(n-c)$-coloring of K_n using colors $1, 2, \ldots, n-c$. For a contradiction, suppose that the number of colors each inducing a star in G is less than $\frac{n}{(7+c)^c}$. □

Claim 7.4. Let N be an integer satisfying $N \leq n$ and $\frac{N}{(2+c)(6+c)^{c-1}} - 2 \geq \frac{n}{(7+c)^c}$. For any $V' \subseteq V(G)$ with $|V'| = N$, $C(V') \cap C(V', V(G)\backslash V') = \varnothing$, and $C(V') \cap C(V(G)\backslash V') = \varnothing$, let $G' = G[V']$. If G' is an exact Gallai-$(N-c)$-coloring of K_N, then $V(G')$ has a Gallai-partition consisting of exactly two parts V_1' and V_2', such that $|C(V_1')| = |V_1'| - c$, $|C(V_2')| = |V_2'| - 1$, $c(V_1', V_2') \notin C(V_1') \cup C(V_2')$, and $C(V_1') \cap C(V_2') = \varnothing$.

Proof. Note that the integer N and subset V' satisfying the above conditions exist since we can choose $N = n$, $V' = V(G)$, and $G' = G$. Without loss of generality, let $C(G') = [N-c]$. First, we assume that there exists some color $\ell \in [N-c]$ such that the subgraph of G' induced by color ℓ has at least two nontrivial components. Then we recolor all the edges of color ℓ in one of its nontrivial components with color $N-c+1$. Let G'' be the resulting coloring of K_N. It is easy to check that G'' is an exact Gallai-$(N-(c-1))$-coloring of K_N. Since $\frac{N}{(2+c)(6+c)^{c-1}} - 2 \geq \frac{n}{(7+c)^c}$, we have

$N \geq 2(7+(c-1))^{c-1}$. By the induction hypothesis, there are at least $\frac{N}{(7+(c-1))^{c-1}}$ colors each inducing a star in G''. Recall that $C(V') \cap C(V', V(G)\backslash V') = \varnothing$ and $C(V') \cap C(V(G)\backslash V') = \varnothing$. There are at least $\frac{N}{(7+(c-1))^{c-1}} - 2 \geq \frac{n}{(7+c)^c}$ colors each inducing a star in G, a contradiction.

Next, we may assume that every color induces a subgraph with exactly one nontrivial component in G'. Let $V_1', V_2', \ldots, V_m'$ be a Gallai-partition of $V(G')$ such that m is minimum and $|V_1'| = \max_{1 \leq i \leq m}\{|V_i'|\}$, and let S be the set of colors used between these parts. Then $1 \leq |S| \leq 2$ and $(C(V_i') \cap C(V_j'))\backslash S = \varnothing$ for every $1 \leq i < j \leq m$.

If $m \geq 4$, then $N - c = |C(G')| \leq |S| + \sum_{i=1}^{m}|C(V_i')| \leq 2 + \sum_{i=1}^{m}(|V_i'| - 1) \leq N + 2 - m$, so $m \leq 2 + c$. Thus $|V_1'| \geq N/(2+c) \geq 2(6+c)^{c-1} = 2(7+(c-1))^{c-1}$. Moreover, $|C(V_1')| \geq N - c - |S| - \sum_{i=2}^{m}|C(V_i')| \geq N - c - 2 - \sum_{i=2}^{m}(|V_i'| - 1) = |V_1'| - c + m - 3 \geq |V_1'| - c + 1$. Let $C(V_1') = \{c_i : i = 1, 2, \ldots, |C(V_1')|\}$. For all $|V_1'| - c + 2 \leq j \leq |C(V_1')|$ (if $|C(V_1')| > |V_1'| - c + 1$), we recolor all the edges of color c_j with color c_1 in $G'[V_1']$, so we obtain an exact Gallai-$(|V_1'| - (c-1))$-coloring of $K_{|V_1'|}$. By the induction hypothesis, there are at least $\frac{|V_1'|}{(7+(c-1))^{c-1}}$ colors each inducing a star in $G'[V_1']$. Thus the number of colors each inducing a star in G is at least $\frac{|V_1'|}{(7+(c-1))^{c-1}} - |S| \geq \frac{N}{(2+c)(7+(c-1))^{c-1}} - 2 \geq \frac{n}{(7+c)^c}$, a contradiction.

Thus $2 \leq m \leq 3$, so $m = 2$ by the minimality of m. Then $|V_1'| \geq N/2 \geq 2(7+(c-1))^{c-1}$. Note that $|C(V_1')| \geq N - c - |C(V_2')| - |S| \geq N - c - |V_2'| + 1 - 1 = |V_1'| - c$. If $|C(V_1')| \geq |V_1'| - c + 1$, then we can derive a contradiction by a similar argument as above. Thus we have $|C(V_1')| = |V_1'| - c$, so $|C(V_2')| = |V_2'| - 1$, $S \cap (C(V_1') \cup C(V_2')) = \varnothing$, and $C(V_1') \cap C(V_2') = \varnothing$. □

We will use an algorithm to find $\lceil \frac{n}{(7+c)^c} \rceil$ colors each inducing a star in G. Let $V_1^{(0)} := V(G)$, $V_2^{(0)} := \varnothing$, $G^{(0)} := G$, $t := 1$, $A := \varnothing$, and $B := \varnothing$. The algorithm at time $i \geq 1$ consists of two steps.

Step 1. By applying Claim 7.4 to $N = |V_1^{(i-1)}|$, $V' = V_1^{(i-1)}$, and $G' = G[V_1^{(i-1)}]$, we obtain a Gallai-partition $V_1^{(i)}, V_2^{(i)}$ of $V(G')$ such that $|C(V_1^{(i)})| = |V_1^{(i)}| - c$, $|C(V_2^{(i)})| = |V_2^{(i)}| - 1$, $c(V_1^{(i)}, V_2^{(i)}) \notin C(V_1^{(i)}) \cup C(V_2^{(i)})$, and $C(V_1^{(i)}) \cap C(V_2^{(i)}) = \varnothing$.

Step 2. If $|V_2^{(i)}| = 1$, then let $c_t = c(V_1^{(i)}, V_2^{(i)})$, $t := t + 1$, and $A := A \cup V_2^{(i)}$; otherwise if $|V_2^{(i)}| \geq 2$, then let $B := B \cup V_2^{(i)}$.

We repeat the above steps until $t \geq \frac{n}{(7+c)^c} + 1$. Finally, we obtain $t - 1$ distinct colors $c_1, c_2, \ldots, c_{t-1}$ each inducing a star in G. It remains to show that the above algorithm is valid. Since for any $j \leq i - 1$ with $V_2^{(j)} \subseteq B$ we have $|V_2^{(j)}| \geq 2$ and $|C(V_2^{(j)})| = |V_2^{(j)}| - 1$, the number of colors each inducing a star in $G[V_2^{(j)}]$ is at least $\lceil |V_2^{(j)}|/2 \rceil$ by Lemma 7.1. Recall that $c(V_1^{(j)}, V_2^{(j)}) \notin C(V_1^{(j)}) \cup C(V_2^{(j)})$ and $C(V_1^{(j)}) \cap C(V_2^{(j)}) = \varnothing$ for every $j \leq i - 1$. Thus

$|B| < \frac{2n}{(7+c)^c}$; otherwise the number of colors each inducing a star in G is at least $\frac{n}{(7+c)^c}$, a contradiction. Thus $|V_1^{(i-1)}| = n - |B| - |A| > n - \frac{3n}{(7+c)^c}$. By Lemma 7.2, $|V_1^{(i-1)}|$ satisfies the condition of N in Claim 7.4. Moreover, $V_1^{(i-1)}$ (resp., $G^{(i-1)}$) satisfies the condition of V' (resp., G') in Claim 7.4. Thus we can apply Claim 7.4 in Step 1, so the algorithm is valid.

Now we have all ingredients to present our proof of Theorem 1.13.

Proof of Theorem 1.13. The cases $c \in \{1, 2\}$ follow from Theorems 1.8 and 1.10, so we may assume that $c \geq 3$. The lower bound $g_{k-1}^k(p) > p$ is trivial. For the upper bound, let G be a Gallai-k-coloring of K_{p+1}. We may assume that G is an exact Gallai-k-coloring, where $k = p - c = p + 1 - (c + 1)$. By Lemma 7.3, the number of colors each inducing a star in G is at least $\frac{p+1}{(7+(c+1))^{c+1}} \geq 2$. Let i be a color that induces a star in G, and let v be a vertex with a maximum degree in this star. Then $G - v$ is a copy of K_p using at most $k - 1$ colors. The result follows. □

8 | CONCLUDING REMARKS

In this paper, we studied the behavior of $g(n, p, q)$, which is the minimum number of colors that are needed for K_n to have a Gallai-coloring in which every K_p receives at least q distinct colors. For this purpose it was convenient to consider the closely related function $g_q^k(p)$. We now recapitulate what the above results on $g_q^k(p)$ imply for the function $g(n, p, q)$.

Corollary 1.2 implies that $g(n, p, q)$ makes sense only for $2 \leq q \leq p - 1$. Theorem 1.4 implies that $g(n, p, q) > \frac{q-1}{2(p-2)}(\log_2 n - 1)$. For appropriate p and n, Theorems 1.8, 1.10, 1.11, 1.9, and 1.12 imply that $g(n, p, p - 1) = n - 1$, $g(n, p, p - 2) = n - 2$, $g(n, 5, 3) = \lceil \log_2 n \rceil$, $g(n, p, \lfloor \sqrt{p-1} \rfloor) \leq \lceil \sqrt{n} \rceil - 1$, and $g(n, p, \lfloor \log_2(p - 1) \rfloor + 1) \leq \lceil \log_2 n \rceil$, respectively.

We remark that the behavior of $g(n, p, q)$ is very different from $f(n, p, q)$, as may be seen by noting that in the case $q = p - 1$, Conlon et al. [7] proved that $f(n, p, p - 1)$ is subpolynomial in n, but here we show that $g(n, p, p - 1) = n - 1$. A natural problem is to find the threshold for linear $g(n, p, q)$, that is, the smallest q such that $g(n, p, q)$ is linear in n. We were not able to solve this problem, but in light of Theorems 1.8, 1.10, and 1.13, we conjecture the following.

Conjecture 8.1. *For any constant $c \geq 2$, there exists a p_0 such that for all integers $p \geq p_0$ and $k \geq p - c$, we have $g_{p-c}^k(p) = k + c$. (Equivalently, for any constant $c' \geq 1$, there exists a p_0 such that for all integers $p \geq p_0$ and $n \geq p$, we have $g(n, p, p - c') = n - c'$).*

The following construction shows that $g_{p-c}^k(p) \geq k + c$. Let G be a copy of K_{k+c-1} with vertex set $\{v_1, v_2, ..., v_{k+c-1}\}$. For every $1 \leq i \leq k$ and $i < j \leq k + c - 1$, we color the edge $v_i v_j$ using color i, and we color all the remaining edges with color k. In the case $c \in \{2, 3\}$, Theorems 1.8 and 1.10 confirm Conjecture 8.1. For $c \geq 4$, Theorem 1.13 shows that $g_{p-c}^k(p) = k + c$ for $k = p - c + 1$ and sufficiently large p.

Theorems 1.9 and 1.12 imply that $g(n, p, \lfloor \sqrt{p-1} \rfloor)$ and $g(n, p, \lfloor \log_2(p - 1) \rfloor + 1)$ are at most $O(n^{1/2})$ and $O(\log n)$, respectively. We know that $g(n, p, p - 1)$ is linear in n, $g(n, p, 2)$ is logarithmic in n, and $g(n, p, q) \geq g(n, p, q - 1)$. Thus for any fixed p (where p is large enough), there exists a value q such that $g(n, p, q)$ is polynomial in n and $g(n, p, q - 1)$ is subpolynomial in n. Another natural problem is to find the smallest q such that $g(n, p, q) = \Theta(n^c)$ for some constant $0 < c < 1$, and the largest q such that $g(n, p, q) = \Theta(\log n)$.

Recently, Krueger [22] studied the minimum number of colors in an edge-coloring of K_n such that every P_m receives at least q colors. For a general fixed graph H, one can also study the minimum number of colors in a Gallai-coloring of K_n such that every H receives at least q colors. This problem generalizes the concept of Gallai–Ramsey numbers of graphs. For recent results on Gallai–Ramsey theory, we refer the interested reader to [23–25].

ACKNOWLEDGMENTS

The authors are grateful to the anonymous referees for valuable comments, suggestions, and corrections which improved the presentation of this paper. Supported by the National Natural Science Foundation of China (No. 11871398) and China Scholarship Council (No. 201906290174).

ORCID

Hajo Broersma http://orcid.org/0000-0002-4678-3210
Ligong Wang http://orcid.org/0000-0002-6160-1761

REFERENCES

1. M. Axenovich, *A generalized Ramsey problem*, Discrete Math. **222** (2000), 247–249.
2. M. Axenovich, Z. Füredi, and D. Mubayi, *On generalized Ramsey theory: The bipartite case*, J. Combin. Theory Ser. B. **79** (2000), 66–86.
3. G. Beam and M. Budden, *Weakened Gallai–Ramsey numbers*, Surv. Math. Appl. **13** (2018), 131–145.
4. W. G. Brown, P. Erdős, and V. T. Sós, *Some extremal problems on r-graphs*, New Directions in the Theory of Graphs (F. Harary, ed.), Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971, Academic Press, New York, 1973, pp. 53–63.
5. F. R. K. Chung and R. L. Graham, *Edge-colored complete graphs with precisely colored subgraphs*, Combinatorica. **3** (1983), 315–324.
6. D. Conlon, J. Fox, C. Lee, and B. Sudakov, *On the grid Ramsey problem and related questions*, Int. Math. Res. Not. **2015** (2015), 8052–8084.
7. D. Conlon, J. Fox, C. Lee, and B. Sudakov, *The Erdős–Gyárfás problem on generalized Ramsey numbers*, Proc. Lond. Math. Soc. **110** (2015), 1–18.
8. D. Conlon, J. Fox, and B. Sudakov, *Recent developments in graph Ramsey theory*, Surveys in Combinatorics 2015, (A. Czumaj, A. Georgakopoulos, D. Král, V. Lozin, and O. Pikhurko, eds.), Cambridge University Press, Cambridge, 2015, pp. 49–118.
9. P. Erdős, *Problems and results on finite and infinite graphs*, Recent Advances in Graph Theory (M. Fiedler, ed.), Proc. Second Czechoslovak Sympos., Prague, 1974, Academia, Prague, 1975, pp. 183–192.
10. P. Erdős, *Solved and unsolved problems in combinatorics and combinatorial number theory*, Congr. Numer. **32** (1981), 49–62.
11. P. Erdős and A. Gyárfás, *A variant of the classical Ramsey problem*, Combinatorica. **17** (1997), 459–467.
12. P. Erdős, A. Hajnal, and R. Rado, *Partition relations for cardinal numbers*, Acta Math. Acad. Sci. Hungar. **16** (1965), 93–196.
13. P. Erdős and L. Lovász, *Problems and results on 3-chromatic hypergraphs and some related questions*, Infinite and Finite Sets (A. Hajnal, R. Rado, and V. T. Sós, eds.), Colloq. Math. Soc. J. Bolyai, North-Holland, Amsterdam, 1975, pp. 609–627.
14. P. Erdős, A. Simonovits, and V. T. Sós, *Anti-Ramsey theorems*, Infinite and Finite Sets (A. Hajnal, R. Rado, and V. T. Sós, eds.), Colloq. Math. Soc. J. Bolyai, North-Holland, Amsterdam, 1975, pp. 633–643.
15. S. Fish, C. Pohoata, and A. Sheffer, *Local properties via color energy graphs and forbidden configurations*, SIAM J. Discrete Math. **34** (2020), 177–187.

16. J. Fox, A. Grinshpun, and J. Pach, *The Erdős–Hajnal conjecture for rainbow triangles*, J. Combin. Theory Ser. B **111** (2015), 75–125.
17. J. Fox and B. Sudakov, *Ramsey-type problem for an almost monochromatic K_4*, SIAM J. Discrete Math. **23** (2009), 155–162.
18. T. Gallai, *Transitiv orientierbare Graphen*, Acta Math. Acad. Sci. Hungar. **18** (1967), 25–66.
19. R. E. Greenwood and A. M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math. **7** (1955), 1–7.
20. A. Gyárfás, G. N. Sárközy, A. Sebő, and S. Selkow, *Ramsey-type results for Gallai colorings*, J. Graph Theory. **64** (2010), 233–243.
21. A. Gyárfás and G. Simonyi, *Edge colorings of complete graphs without tricolored triangles*, J. Graph Theory. **46** (2004), 211–216.
22. R. A. Krueger, *Generalized Ramsey numbers: Forbidding paths with few colors*, Electron. J. Combin. **27** (2020), #P1.44.
23. H. Lei, Y. T. Shi, Z.-X. Song, and J. M. Zhang, *Gallai–Ramsey numbers of C_{10} and C_{12}*, Australas. J. Combin. **79** (2021), 380–400.
24. H. Liu, C. Magnant, A. Saito, I. Schiermeyer, and Y. T. Shi, *Gallai–Ramsey number for K_4*, J. Graph Theory. **94** (2020), 192–205.
25. Y. C. Liu and Y. J. Chen, *Gallai and ℓ-uniform Ramsey numbers of complete bipartite graphs*, Discrete Appl. Math. **301** (2021), 131–139.
26. D. Mubayi, *Edge-coloring cliques with three colors on all 4-cliques*, Combinatorica. **18** (1998), 293–296.
27. C. Pohoata and A. Sheffer, *Local properties in colored graphs, distinct distances, and difference sets*, Combinatorica. **39** (2019), 705–714.
28. V. Rosta, *Note on Gy. Elekes's conjectures concerning unavoidable patterns in proper colorings*, Electron. J. Combin. **7** (2000), #N3.
29. G. N. Sárközy and S. Selkow, *On edge colorings with at least q colors in every subset of p vertices*, Electron. J. Combin. **8** (2001), #R9.
30. G. N. Sárközy and S. Selkow, *An application of the regularity lemma in generalized Ramsey theory*, J. Graph Theory. **44** (2003), 39–49.
31. J. Spencer, *Asymptotic lower bounds for Ramsey functions*, Discrete Math. **20** (1977), 69–76.
32. Z. Wang, Y. P. Mao, H. Z. Li, and S. P. Cui, Asymptotic lower bounds for Gallai–Ramsey functions and numbers, arXiv:2007.04895v1, 2020. https://doi.org/10.48550/arXiv.2007.04895

How to cite this article: X. Li, H. Broersma, and L. Wang, *The Erdős–Gyárfás function with respect to Gallai-colorings*, J. Graph Theory. 2022;101:242–264. https://doi.org/10.1002/jgt.22822