1 Introduction

Flexure joints are rapidly gaining ground in precision applications [1–5]. Flexure joints allow excellent predictable motion as they do not suffer from friction and backlash and have low hysteresis, in contrast to other bearings. However, their range of motion is limited due to a stress limitation and a loss of support stiffness in deformed configurations. The support stiffness can be significantly increased by using leafsprings of which the width and thickness vary over the length of the leafspring. This paper presents formulations for two beam elements with a varying cross section that can be used for the efficient modeling of these types of leafsprings. One of these beam-formulations includes the modeling of the warping due to torsion, which is shown to be essential for accurate modeling. The 90% accuracy in stiffness results and 80% accuracy in stress results, in comparison with results of finite element analyses, are sufficient for the evaluation of concept-designs. Optimizations show that the support stiffness of two typical flexure joints can be increased by a factor of up to 4.0 keeping the same range of motion, by allowing the cross section to vary over the length of the leafspring. In these two flexure joints, 98% of this improvement can already be obtained by only varying the thickness, and keeping a constant width. [DOI: 10.1115/1.4056482]

Fig. 1 The free-flex pivot includes leafsprings with varying width. Left: full pivot, right: pivot with gap to show inside.
noted that this modeling approach results in a small error as shown by Boley [29]. Therefore more accurate modeling methods are proposed [30–32], but these formulations are valid only in two dimensions, and the resulting beam elements do not account for constraint torsional warping, making the formulations less suitable for the modeling of leafsprings. For optimization purposes, the simplified approach is sufficiently accurate as the error introduced by this approach is small for small variations of the cross section. However, this approach has never been formulated in the generalized strain formulation that is used in SPACAR, hence it cannot be used for the numerically efficient design optimizations.

This paper presents two beam formulations with varying cross section in the generalized strain formulation, for beams with isotropic material properties. One of the formulations includes warping due to torsion. Both formulations are presented in Sec. 2 and include second-order effects in the deformation. In Sec. 3 the accuracies of the formulations are verified and it is shown that the support stiffness of several flexure joints can be increased by varying the cross section of leavesprings. The paper ends with the most important conclusions.

2 Beam Formulations

Section 2.1 presents the relations between nodal forces and displacements for a beam with varying cross section (hereinafter VC-beam) in local coordinates. These relations are used in Sec. 2.2 to obtain the stiffness in terms of deformation modes, so as to make the formulation applicable in the generalized strain formulation. The corresponding deformation modes are derived in Sec. 2.3. In Sec. 2.4 warping due to torsion is included in this beam formulation (the resulting element will be referred to as the variational cross section warping (VCW)-beam). Second-order terms in the deformation are derived in Sec. 2.5 and the mass matrix of the beam is derived in Sec. 2.6.

2.1 Relations Between Nodal Forces and Nodal Displacements. In this section, the relations between forces and displacements are derived by integrating the elasticity coefficients over the length of the beam. The resulting integrals will not be evaluated analytically, although these integrals can be evaluated for many standard variations in the cross section [22,23,25]. Evaluating the integrals numerically allows for more freedom in the variation of the cross section and little computation time is required for this numerical integration.

2.1.1 Axial Deformation. Figure 2(a) shows a beam on which axial force is applied. The resulting axial displacement can be computed from

\[ u(\xi) = u^0 + F x L_0 \int_0^\xi \frac{1}{E A(s)} ds \]  

(2.1)

where \( \xi \) is the natural x-coordinate of the undeformed configuration (\( \xi \equiv x/L_0 \)), \( u^0 \) is the displacement of the left node, \( F x \) is the axial force, \( L_0 \) is the undeformed length of the beam, \( E \) is the elasticity, \( A \) is the cross-sectional area which depends on the x-coordinate and \( s \) is the integration variable. The integral-term will be denoted by \( p_1(\xi) \)

\[ p_1(\xi) \equiv \int_0^\xi \frac{1}{E A(s)} ds \]  

(2.2)

The relation between the axial displacement of the left and right node and the forces on the nodes can therefore be expressed as

\[ u^0 - u^p = F x L_0 p_1(1) \]  

(2.3)

where \( u^p \) is the displacement of the right node.

In the remainder of the derivation, more integrals will be defined and denoted by \( p_i \). The numbers in the subscript are related to the type of deformation, \( p_2 \) is related to torsion, deformation in the xz-plane is indicated by a subscript that starts with “3” and deformation in the xy-plane is indicated by a subscript that starts with “5.” This numbering is consistent with the numbering of the deformation modes in SPACAR, which will be defined in Sec. 2.2. Hereinafter we write an integral, \( p_i \), evaluated at \( \xi = 1 \) with capital \( P_i \), so \( P_1 \equiv p_1(1) \).

2.1.2 Torsion. Torsion is shown in Fig. 2(b) and can be computed by

\[ \psi_1(\xi) = \phi_0^x + M x L_0 \int_0^\xi \frac{1}{G I_1(s)} ds \]  

(2.4)

where \( \phi_0^x \) is the rotation around the local x-axis of the left node, \( M x \) is the applied moment, \( G \) is the shear modulus, and \( I_1 \) is the Saint-Venant’s torsion constant. By introducing

\[ p_2(\xi) \equiv \int_0^\xi \frac{1}{G I_1(s)} ds \]  

(2.5)

we can write

\[ \phi_1^x - \phi_0^x = M x L_0 p_2 \]  

(2.6)

where \( \phi_0^x \) is the rotation of the right node.

2.1.3 Bending and Shear in the xz-Plane. Figure 2(c) shows the deformation in the xz-plane. The internal bending moment about the y-axis found by equilibrium

\[ M y(\xi) = M y^0 - F z L_0 \xi \]  

(2.7)

The rotation around the y-axis, \( \psi_y \), in the beam is

\[ \psi_y(\xi) = \phi_0^y + M y^0 \int_0^\xi \frac{1}{E I_2(s)} ds \]  

(2.8)
we can express the displacement in the z-direction as
\[ w(\xi) = w^p - \phi_y^p L_0 - M_y^d L_0^2 p_{31}(\xi) + F_z L_0^2 p_{34}(\xi) \] (2.13)

Equations (2.9) and (2.13) can be evaluated at the right node of the beam, i.e., at \( \xi = 1 \), and combined to
\[
\begin{bmatrix}
\phi_y^q - \phi_y^p \\
\psi^q - \psi^p \phi_y^p L_0
\end{bmatrix}
= 
\begin{bmatrix}
L_0 p_{31} & -L_0^2 p_{32} \\
-L_0^2 p_{33} & L_0^2 p_{34}
\end{bmatrix}
\begin{bmatrix}
\phi_y^p \\
F_z
\end{bmatrix}
\]
(2.14)

This is a relation between the nodal displacements and nodal forces in the xy-plane in terms of four integrals. However, the four integrals are not independent. A relation exists between three of these integrals which means that only three integrals have to be evaluated to obtain the relation between nodal forces and displacements. To show this relation, the rule of partial integration can be used, which implies that for two arbitrary functions \( f(\xi) \) and \( g(\xi) \)
\[
\int_0^1 f(\xi)g'(\xi)d\xi = \left[ f(\xi)g(\xi) \right]_0^1 - \int_0^1 f'(\xi)g(\xi)d\xi
\]
By substituting \( f(\xi) = \xi \) and \( g(\xi) = \xi^1 1/El_f(s)ds \) we obtain
\[
\int_0^1 \frac{1}{El_f(s)}d\xi = \left[ \frac{\xi}{El_f(s)} \right]_0^1 - \int_0^1 \frac{1}{El_f(s)}dsd\xi
\]
(2.16)
which is equivalent to \( P_{32} = P_{31} - P_{33} \).

2.1.4 Bending and Shear in the xy-Plane. The relation between forces and displacements in the xy-plane can be obtained in a similar way to the derivation for the z-plane, resulting in
\[
\begin{bmatrix}
\phi_y^q - \phi_y^p \\
\psi^q - \psi^p \phi_y^p L_0
\end{bmatrix}
= 
\begin{bmatrix}
L_0 p_{31} & L_0^2 p_{32} \\
L_0^2 p_{33} & L_0^2 p_{34}
\end{bmatrix}
\begin{bmatrix}
\phi_y^p \\
F_y
\end{bmatrix}
\]
(2.17)
where \( P_{31} \) to \( P_{34} \) are similar to \( P_{31} \) to \( P_{34} \), except that \( I_y \) and \( k_y \) are replaced by \( I_z \) and \( k_z \), respectively.

This section shows the relation between nodal forces and nodal displacements in terms of 10 integrals that depend only on the distribution of the elasticity coefficients over the length of the beam-element.

2.2 Stiffness in Terms of Deformation Modes. This section will define deformation modes and use the relations from Sec. 2.1 to derive the stiffness matrix in terms of these deformation modes. The generalized coordinates of these deformation modes are called generalized deformations (in other literature also referred to as ‘generalized strains’ although they are related to displacements instead of strain). The generalized deformations are denoted by \( e \) and are directly related to the global nodal coordinates \( x \)
\[
\epsilon = D(x)
\]
(2.18)
The global coordinates are the global positions and orientations of both nodes; note that these coordinates are different from the local coordinates used in Sec. 2.1. In this paper, the deformation modes will be chosen in such a way that this function \( D \) is equivalent to that of Jonker and Meijer [17]. Therefore this relation is not further detailed in this paper. The relations between the generalized deformations and the local nodal displacements are called boundary conditions and are listed in Fig. 3.

The generalized forces of the modes are called generalized stresses (although they are related to forces and moments instead of stress) and are denoted by \( \sigma \). According to the principle of virtual work, the element is in state of equilibrium if
\[
\delta e^T \sigma = \delta u^T F \quad \forall \delta e
\]
(2.19)
where \( u \) is the vector with the 12 local nodal displacements (the three translations, \( u, v, w \) and the three rotations \( \phi_x, \phi_y, \phi_z \)) and \( F \) is the vector with forces in the corresponding directions (the three force components \( F_x, F_y, \) and \( F_z \)) and three moment components \( M_x, M_y, \) and \( M_z \) for both nodes of the element). Equation (2.19) can be used to define relations between the stress resultants and nodal forces. The twelve boundary conditions can be rewritten to \( u = \Phi e \). By substituting this into the equation we obtain
\[
\delta e^T \sigma = \delta u^T F = \delta e^T \Phi^T F \quad \forall \delta e \Rightarrow \sigma = \Phi^T F
\]
(2.20)
These are six relations, which are given in Fig. 3 and referred to as stress relations.

The stiffness relation between the generalized deformations and stress resultants can be expressed for a beam with a double symmetric cross section by a stiffness matrix as

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix}
= 
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\
S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\
S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\
S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\
S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\
S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\epsilon_4 \\
\epsilon_5 \\
\epsilon_6
\end{bmatrix}
\]
(2.21)
The coefficients \( S_{ij} \) can be obtained by substituting the boundary conditions and stress relations in the equations of Sec. 2.1, as explained below. These coefficients are given in Fig. 3. For the axial deformation, \( S_{11} \) can be obtained by substituting the boundary conditions and stress relations of Fig. 3(a) in Eq. (2.3). The stiffness coefficient for torsion, \( S_{22} \), can be obtained similarly, by substituting the relations in Fig. 3(b) into Eq. (2.6). For bending and shear in the xz-plane, the boundary conditions, and stress relations can be substituted in Eq. (2.14), and the result can be rewritten to
\[
\begin{bmatrix}
\epsilon_3 \\
\epsilon_4
\end{bmatrix}
= 
\begin{bmatrix}
L_0^4 & -L_0^4 & -L_0^4 & -L_0^4 & -L_0^4 & -L_0^4
\end{bmatrix}
\begin{bmatrix}
P_{31} - P_{32} - P_{34} & P_{34} \\
P_{34} & P_{32} - P_{34}
\end{bmatrix}
\begin{bmatrix}
\sigma_3 \\
\sigma_4
\end{bmatrix}
\]
(2.22)
where we used the relation \( P_{32} = P_{31} - P_{33} \) to substitute \( P_{33} \).

Inverting this matrix results in the stiffness coefficients that are given in Fig. 3(c).

The stiffness coefficients for the \( xy \)-plane are obtained similarly.

### 2.3 Mode Shapes

The local displacements can be expressed in terms of the generalized deformations using the mode shapes derived in this section. These mode shapes are visualized in Fig. 3 and will be used in Secs. 2.4–2.6.

The axial force \( F_x \) can be expressed in terms of the displacements using Eq. (2.3). Substituting the result in Eq. (2.1) and using the boundary conditions from Fig. 3(a) gives the axial mode shape \( N_1(\xi) \)

\[
\begin{align*}
\nu(\xi) &= N_1(\xi) e_1, \quad N_1(\xi) = \frac{p_1(\xi)}{P_1} \\
\phi_1(\xi) &= \frac{1}{L_0} N_2(\xi) e_2, \quad N_2(\xi) = \frac{p_2(\xi)}{P_2}
\end{align*}
\] (2.23)

The torsional mode shape can be found similarly

\[
\phi_1(\xi) = \frac{1}{L_0} N_2(\xi) e_2, \quad N_2(\xi) = \frac{p_2(\xi)}{P_2}
\] (2.24)

To obtain the displacement \( w \), the force \( F_y \) and moment \( M_yz \) can be expressed in terms of the nodal displacements by Eq. (2.14) and then substituted in Eq. (2.13). This gives an expression for the displacements in terms of the nodal displacements. After substituting the boundary conditions from Fig. 3(c), the displacement is expressed in terms of \( e_3 \) and \( e_4 \)

\[
w(\xi) = N_3(\xi) e_3 + N_4(\xi) e_4 \] (2.25)

where

\[
\begin{align*}
N_3(\xi) &= \frac{1}{L_0} \left[ -P_{32} + P_{34} - P_{31} + P_{32} + P_{34} \right] \\
N_4(\xi) &= \frac{1}{L_0} \left[ -P_{32} - P_{34} - P_{31} + P_{32} + P_{34} \right]
\end{align*}
\] (2.26)

\(D_{24}\) is defined in Fig. 3(c). The rotation can also be specified using mode shapes

\[
\phi_3(\xi) = \frac{1}{L_0} \left( N_3(\xi) e_3 + N_4(\xi) e_4 \right)
\] (2.27)
where

\[ N_{31}(\xi) = -1 + \frac{(P_{34} - P_{32})p_{31}(\xi) + P_{32}p_{35}(\xi)}{D_{34}} \]
\[ N_{41}(\xi) = \frac{P_{34}p_{31}(\xi) + (P_{32} - P_{31})p_{35}(\xi)}{D_{34}} \]

(2.28)

If the shear deformation is disregarded, \( \phi_1(\xi) = -w'(\xi)/L_0 \) and then the mode shapes are also related: \( N_{31}(\xi) = -N_{51}(\xi), \)
\( N_{41}(\xi) = -N_{61}(\xi). \)

The mode shapes for deformation in the xy-plane can be obtained similarly

\[ v(\xi) = N_5(\xi)e_5 + N_6(\xi)e_6, \quad \phi_1(\xi) = \frac{1}{L_0}(N_{51}(\xi)e_5 + N_{61}(\xi)e_6) \]

(2.29)

where

\[ N_5(\xi) = -\xi - \frac{(P_{32} - P_{34})p_{31}(\xi) - P_{32}p_{34}(\xi)}{D_{34}} \]
\[ N_6(\xi) = -\xi - \frac{P_{34}p_{31}(\xi) + (P_{31} - P_{32})p_{34}(\xi)}{D_{34}} \]
\[ N_{51}(\xi) = -1 + \frac{(P_{34} - P_{32})p_{35}(\xi) + P_{32}p_{35}(\xi)}{D_{34}} \]
\[ N_{61}(\xi) = \frac{P_{34}p_{35}(\xi) + (P_{32} - P_{31})p_{35}(\xi)}{D_{34}} \]

(2.30)

2.4 Warping Due to Torsion. This section explains how the effect of warping can be taken into account for a beam with double symmetric cross section, the resulting element will be referred to as a VCW-beam. According to the Saint-Venant torsion theory, torsion generally causes warping of the cross section in axial direction, which can be obtained by

\[ u_w = \omega(y, z)x(x) \]

(2.31)

where the warping coordinate \( x(x) \) is the derivative of the torsion: \( x(x) = \phi'(x) \). The warping shape \( \omega(y, z) \) depends on the shape of the cross section. The resistance against this warping is modeled by a variable called the bimoment

\[ B(x) = -E\omega(x)\frac{d^2x}{dx^2} = -E\omega(x)\frac{d^2\phi}{dx^2}, \quad I_w(x) = \int_A \omega^2(y, z)dA \]

(2.32)

For thin rectangular cross sections, with the local y-axis in the thickness direction, the warping shape equals \( \omega(y, z) = yz \), such that the warping rigidity becomes \( I_w = w^3L/144 \). According to Vlasov’s torsion theory [36] the total torsional moment, \( M_t \), is composed of the Saint-Venant torsion \( T_t \) and the derivative of the bimoment

\[ M_t(x) = T_t(x) + dB \quad T_t(x) = GL(x)\frac{d\phi}{dx} \]

(2.33)

In Secs. 2.1 and 2.2, exact relations for stiffness were found and in Sec. 2.3, exact mode shapes were found. This is possible because the distribution of the internal forces can be expressed as a function of the forces on the nodes, based on equilibrium.

The distribution of the Saint-Venant torsional moment and bimoment depends on the variation of the cross-sectional dimensions over the beam axis, and cannot be found easily. However, if the extra stiffness because of warping is small (which is usually the case far from the clamped areas), the mode shape is given by \( p_2(\xi)/P_2 \) as defined in Eq. (2.24). In order to include the effect of warping, this mode shape is split into three mode shapes, visualized in Fig. 4

\[ N_2(\xi) = p_2(\xi)/P_2 \cdot ((1 + x^2) - (1 + x^2)) \]
\[ N_7(\xi) = p_2(\xi)/P_2 \cdot (1 - 2x^2 + x^4) \]
\[ N_8(\xi) = p_2(\xi)/P_2 \cdot (x^2 + x^4) \]

Note that the sum of these mode shapes equals the old mode shape \( p_2(\xi)/P_2 \).

The following boundary conditions follow from this mode shapes:

\[ N_{278}(\xi) = N_2(\xi) + N_7(\xi) + N_8(\xi) = p_2(\xi)/P_2 \]

In Fig. 4 Mode shapes of VCW beam with \( p_2(\xi) = 0.6z^2 + 0.4z^2 \)
\[ \sigma_2 = M/L_0, \quad \sigma_7 = z^2 B_1^2/L_0^2, \quad \sigma_8 = z^2 B_2^2/L_0^2 \]  
(2.39)

The stiffness matrix of the deformation modes for the VCW-beam is

\[
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6 \\
\sigma_7 \\
\sigma_8 \\
\end{pmatrix} =
\begin{bmatrix}
S_{11} & S_{22} & S_{27} & S_{28} \\
S_{22} & S_{33} & S_{34} & \vdots \\
S_{33} & S_{34} & \ddots & \vdots \\
S_{43} & S_{44} & \ddots & \vdots \\
S_{55} & S_{56} & \ddots & \vdots \\
S_{65} & S_{66} & \ddots & \vdots \\
S_{72} & S_{77} & \ddots & \vdots \\
S_{82} & S_{87} & \ddots & S_{88} \\
\end{bmatrix}
\begin{pmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\epsilon_4 \\
\epsilon_5 \\
\epsilon_6 \\
\epsilon_7 \\
\epsilon_8 \\
\end{pmatrix}
\]

(2.40)

All the terms that are not related to torsion or warping are not changed and have the values given in Fig. 3. The terms related to torsion (deformation modes 2, 7, and 8) can be found from the energy potential of these terms

\[
E_{pot}^{tot} = \frac{1}{2} \int_{0}^{1} \left( GL_1 \phi_{s,xxx} + EI_{L_0} \phi_{s,xx}^2 \right) d\xi
\]

(2.41)

The derivatives of \( \phi_s \) can be expressed in terms of \( \epsilon_{278} \) using the mode shapes in Eq. (2.36). By substituting these mode shapes, the potential energy of torsion can be written as

\[
E_{pot}^{tot} = \frac{1}{2} \int_{0}^{1} \left( GL_1 \phi_{s,xxx} + EI_{L_0} \phi_{s,xx}^2 \right) d\xi
\]

(2.42)

where the stiffness matrix is

\[
S_{278} = \frac{1}{L_0} \left[ N_{278}^T(\xi) \right]^T GL_1 \left[ N_{278}(\xi) \right] d\xi
\]

(2.43)

(2.44)

\[
\begin{array}{c}
\dot{\epsilon}_1 = \int_{0}^{1} \left[ \frac{du}{d\xi} + \frac{1}{2} \left( \frac{du}{d\xi} \right)^2 + \frac{1}{2} \left( \frac{dv}{d\xi} \right)^2 + \frac{1}{2} I_{L_0} \left( \frac{d\phi}{d\xi} \right)^2 \right] d\xi \\
\end{array}
\]

The values inside the integral are obtained by numerical integration of these mode shapes.

2.5 Second-Order Expression. The axial elongation is influenced by bending; this effect is called foreshortening. The trapeze effect (also Wagner term) [37,38] couples the axial elongation to the twist. A better approximation for the axial deformation as defined in the first generalized deformation is, therefore [39,40]

\[
\dot{\epsilon}_1 = \frac{1}{2} \left[ H_{11}^{(1)} \epsilon_1^2 + 2H_{12}^{(1)} \epsilon_1 \epsilon_2 + H_{22}^{(1)} \epsilon_2^2 + 2H_{34}^{(1)} \epsilon_3 \epsilon_4 + H_{44}^{(1)} \epsilon_4^2 + 2H_{55}^{(1)} \epsilon_5^2 + 2H_{66}^{(1)} \epsilon_6^2 + H_{77}^{(1)} \epsilon_7^2 + 2H_{88}^{(1)} \epsilon_8^2 + H_{88}^{(1)} \epsilon_8^2 \right]
\]

(2.45)

where

\[
H_{ij}^{(1)} = \int_{0}^{1} N_i^{(1)}(\xi) N_j^{(1)}(\xi) d\xi
\]

(2.46)

This is the expression for the VCW-beam. For the VC-beam the terms that depend on \( \epsilon_7 \) and \( \epsilon_8 \) do not exist, such that only the term \( H_{35}^{(1)} \epsilon_5^2 \) remains in the trapeze effect.

Another nonlinear effect is caused by the fact that a rotation matrix is not linear in the local rotations \( \phi_1, \phi_2, \) and \( \phi_3 \). This couples the torsion and bending deformations. Appendix A shows how this effect can be included up to the second order. All second-order generalized deformations are expressed in terms of the first-order generalized deformations, and can, therefore, be included by modifying the relation between the absolute nodal coordinates and the generalized deformations

\[
\dot{\epsilon} = \dot{\mathcal{D}}(x) = f(\mathcal{D}(x))
\]

(2.47)

Section 3.4 evaluates the significance of these second-order effects.

2.6 Mass Matrix. The mass matrix can be derived based on the kinetic energy. The kinetic energy of a beam element can be expressed as

\[
E_{kin} = \frac{1}{2} \int_{V} \left[ \dot{u}(\xi) \right]^T \left[ \dot{u}(\xi) \right] dV
\]

(2.48)

where \( \dot{u} \) and \( \dot{\phi} \) are the local velocity and rotational velocity of the beam, respectively, and \( Q(\xi) \) contains the inertia properties of the cross section.

\[
Q(\xi) = \rho \cdot \text{diag} \left( \left[ A(\xi) \quad A(\xi) \quad A(\xi) \quad I_p(\xi) \quad I_p(\xi) \quad I_s(\xi) \quad I_s(\xi) \quad I_s(\xi) \right] \right)
\]

(2.49)

where \( \rho \) is the density of the material and \( I_p(\xi) \) is the polar moment of area. The terms related to \( I_s(\xi) \) and \( I_s(\xi) \) are usually smaller than the terms related to \( A(\xi) \) and \( A(\xi) \) and can therefore be ignored.

The local (rotational) velocities in the beam can be expressed in terms of the (rotational) velocities of both nodes using the Craig–Bampton boundary modes, \( \Psi(\xi) \)

\[
\left\{ \begin{array}{c}
\dot{u}(\xi) \\
\dot{L}_0 \phi(\xi) \\
L_0^2 \phi(\xi) \\
\dot{L}_0 x(\xi) \\
\end{array} \right\} = \Psi(\xi) U, \quad U \equiv \left\{ \begin{array}{c}
\dot{u} \\
L_0 \phi \\
L_0^2 \phi \\
\dot{L}_0 x \\
\end{array} \right\}
\]

(2.50)

The matrix with Craig–Bampton boundary modes are closely related to the deformation modes defined in Secs. 2.3 and 2.4 and can be written as

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where

\[
\begin{align*}
\psi_1 &= 1 - N_3(\xi) \\
\psi_2 &= N_1(\xi) \\
\psi_3 &= 1 - \xi - N_3(\xi) + N_6(\xi) \\
\psi_4 &= -N_3(\xi) \\
\psi_5 &= N_6(\xi) \\
\psi_6 &= 1 - \xi + N_3(\xi) - N_4(\xi) \\
\psi_7 &= -N_3(\xi) \\
\psi_8 &= N_6(\xi) \\
\psi_9 &= N_4(\xi) \\
\psi_{10} &= -N_3(\xi) \\
\psi_{11} &= 1 - N_2(\xi) \\
\psi_{12} &= N_2(\xi) \\
\psi_{13} &= 1 + N_3(\xi) - N_4(\xi) \\
\psi_{14} &= -N_3(\xi) \\
\psi_{15} &= -1 - N_3(\xi) + N_4(\xi) \\
\psi_{16} &= N_4(\xi) \\
\psi_{17} &= -1 - N_3(\xi) + N_6(\xi) \\
\psi_{18} &= -N_3(\xi) \\
\psi_{19} &= 1 + N_3(\xi) - N_6(\xi) \\
\psi_{20} &= N_6(\xi) \\
\psi_{21} &= N_3(\xi) \\
\psi_{22} &= N_6(\xi) \\
\psi_{23} &= -N_2(\xi) \\
\psi_{24} &= N_3(\xi) \\
\psi_{25} &= N_2(\xi) \\
\psi_{26} &= N_6(\xi)
\end{align*}
\]  

These are the mode shapes for the VCW-beam. For the VC-beam we can ignore the row corresponding to \(Z(\xi)\) and the columns corresponding to \(x^0\) and \(x^1\), and use \(N_2(\xi)\) as defined in Fig. 3(b). By substituting the mode shapes into Eq. (2.48) the kinetic energy can be expressed in terms of the local mass matrix

\[
\begin{align*}
E_{\text{kin}} &= \frac{1}{2} U^T M U, \\
M &= L_o \int_0^L \bar{\Psi}^T(\xi) \bar{Q}(\xi) \bar{\Psi}(\xi) d\xi
\end{align*}
\]  

The global mass matrix can be obtained by rotation

\[
M = B M_0 B^T, \\
B &= \begin{bmatrix} R^o & L_0 R^o & L_0^2 & L_0 R^o & L_0^2 \\ L_0 & L_0^2 & L_0 & L_0^2 & L_0 \end{bmatrix}
\]  

where rotation matrices \(R^0\) and \(R^1\) define the orientations of both nodes.

3 Validation

In Sec. 3.1, both beam formulations are validated by the analysis of a single leafspring. In Secs. 3.2 and 3.3 two flexure joints are optimized and the optimization results are analyzed. The [41] covariance matrix adaptation evolution strategy (CMA-ES) algorithm is used for the optimizations, with a population size of ten. All optimizations are run two times. These optimization pairs always converge to the same design, suggesting that global optima were found. The computation of the maximum Von Mises stress in these two sections is explained in Appendix B. In all the results, the second-order terms as derived in Sec. 2.5 were included. These second-order terms are evaluated in Sec. 3.4.

3.1 Analysis of a Leafspring in Bending

The accuracy of the stiffness computed by the beam element is analyzed by a single leafspring, shown in Fig. 5. The leafspring is clamped at the base. The tip is displaced out of plane (in the \(y\)-direction) and in this case study, the rotation about the \(x\)-axis and \(z\)-axis are prescribed to be zero. The leafspring is modeled by ten serial connected VC-beams or VCW-beams. A model in ANSYS is used as a reference, where the leafsprings were modeled by about 15,000 solid-shell elements (SOLSH190), with three layers of elements in the thickness direction. The length of the leafspring is 100 mm and is made of steel with Young’s modulus of 200 GPa and a Poisson ratio of 0.3. Three different designs are considered, one design with a constant cross section and two designs where the width or thickness is varied linearly from the base to the tip:

- C: width: 40 mm, thickness: 0.5 mm;
- W: width: from 60 to 40 mm, thickness: 0.5 mm;
- T: width: 40 mm, thickness: from 1.0 to 0.4 mm.
Figure 6 shows the axial stiffness and in-plane stiffness during the tip-displacement. The axial stiffness is computed more than 93% accurately by both beam elements. For the in-plane direction, the results of the VC-beam become less accurate for increasing displacements, but the resulting errors of the VCW-beam in comparison with ANSYS are small: The stiffness at zero displacement is more than 99% accurate for the C-design and T-design, but it deviates 4% for the W-design. This 4% deviation is mainly because the variation of the width is significant with respect to the length of the leafspring which is known to result in a small error, as shown by Boley [29]. This is because the assumption that the cross section does not deform, which was used in the derivation of the Timoshenko beam theory, is not valid for highly varying cross sections. Another way to view this is that the Timoshenko beam theory implies that the normal stress in case of bending and extension is exclusively in axial direction. However, the boundary conditions at the top and bottom surface imply a different stress, see Fig. 7. According to Refs. [30] and [32], the resulting axial stiffness of a cross section is therefore a factor of \( w^2 / 10 \) too high.

Table 1 shows the driving stiffness in undeformed configuration. This stiffness is not affected by the warping from torsion, so the model with the VC-beam and that with the VCW-beam give the same result. There is a difference of about 5% between the beam elements and ANSYS. This difference is caused almost completely by the constraint anticlastic bending effect at both sides of the leafspring, which is not modeled by the beam elements. This was verified by computations with a Poisson ratio of zero, which give almost identical results.

For a deflection of 8% of the beam length, the stiffness can generally be modeled with 90% accuracy with the new beam elements with respect to finite elements, an exception is the in-plane stiffness of the W-design, which is about 85% accurate. Using beam elements the computation time to compute this deflection is more than 100 times shorter.

### 3.2 Optimization of a Parallel Flexure Guidance.

A parallel flexure guidance is optimized for support stiffness to study the value of varying the cross section of leafsprings. Figure 8 shows the dimensions of the mechanism. The elasticity of the leafsprings is 200 GPa and the Poisson ratio 0.3. Each leafspring is modeled using six flexible VCW-beam elements. The thick parts are assumed to be infinitely stiff. The mechanism is specified to be able to move 20 mm, without exceeding the stress limit of 600 MPa. The thickness of the leafsprings is specified to be at least 0.3 mm. The support stiffness is evaluated in the deformed configuration, in the initial center of compliance, where the displacement in the x-direction was constrained and the motions in all other directions were free to move.

Some initial design optimizations were run, in which leafsprings without reinforcements were considered the varying thickness of which was optimized. The leafsprings in the resulting optimized designs were, however, typically reinforced leafsprings, i.e., the middle parts of the leafsprings became very thick. This is because deflection of the inner parts typically does not contribute significantly to the motion, but their thickness has a significant influence on the support stiffness. Therefore six designs are considered in more detail of which five designs have reinforced leafsprings. For the first four designs the width of the leafsprings is constant: 50 mm. The thickness is always a function of design parameters:

- C: leafsprings without reinforcement, with a constant cross section with thickness \( t_1 \).
- R: reinforced leafsprings of which the reinforced part has length \( d_r \), the slender parts have a constant thickness, \( t_1 \).
- TLR: reinforced leafsprings of which the thickness of the slender parts varies linearly over the length.

| Table 1 Driving stiffness of the leafspring in undeformed configuration |
|-------------------|---|---|---|
|                  | C  | W  | T  |
| VCW-beam (N/mm)  | 1.00 | 1.24 | 2.43 |
| ANSYS (N/mm)     | 1.06 | 1.32 | 2.57 |
| Error with respect to ANSYS (%) | 5.7 | 6.5 | 5.4 |

A leafspring, with a highlighted cross section. The stress boundary condition indicates that the normal stress is not exclusively in the axial direction.
determined by thickness $t_1$ (at the base) and $t_3$ (at the reinforced part).

**TQR:**
reinforced leafsprings of which the thickness of the slender parts varies quadratic over the length, determined by thickness $t_1$, $t_1$, and $t_2$ (at the center of the slender parts).

**WLR:**
reinforced leafsprings of which the width of the slender parts varies linearly over the length, determined by the width at the base (which is 50 mm) and $w_3$ (at the reinforced part).

**TQWLR:**
reinforced leafsprings of which the thickness of the slender parts varies quadratic over the length and the width varies linear.

The support stiffness in three directions has been optimized: for the $y$-direction, for the $z$-direction, and for rotation around the $x$-axis. Figure 9 shows the optimized support stiffnesses and Table 2 shows the corresponding design variables. The resulting designs for all three directions of the support stiffness were similar, except for the WLR-design as indicated in the table.

To understand the results, it should be noted that thicker leafsprings will result in extra support stiffness, but also in a higher stress in the deformed configuration. Therefore, in the designs with a varying thickness or width, the stress over the whole length of the deforming parts in deformed configuration is close to the maximum allowed value.

The support stiffness with respect to the reinforced design (R-design) can be increased by a factor of 1.3 by allowing a varying thickness of the slender part of the leafsprings. Using a quadratic thickness variation only marginally increases the support stiffness with respect to a linear thickness variation. This is because the bending moment varies approximately linearly over the beams length, such that a linear thickness function can already result in almost maximum stress over the whole deforming range.

The WLR-design shows that reducing the width at the reinforced part results in a higher support stiffness. This is because this results in more deformation near the reinforced part, reducing the stress near the base and the guided body, allowing additional thickness to increase the overall stiffness. In the TQWLR-design the width becomes maximal and the result is the same as the TQR-design.

The results are validated in ANSYS, where the leafsprings were modeled by about 15,000 solid-shell elements (SOLSH190) in total, with three layers of elements in the thickness direction. Table 3 shows the stiffness in the rotational stiffness around the $x$-axis computed in the undeformed configuration with the beam elements and with ANSYS. The results show a significant error in the stiffness only for the case where the width is varied; this error is about 10%.

3.3 Optimization of a Spherical Joint With Folded Leafsprings.
Naves et al. [42] proposed multiple configurations for a spherical flexure joint with folded leafsprings. This section shows how to optimize these joint configurations for a spherical joint.

### Table 2 Design parameters of the parallel flexure guidance, optimized for support stiffness

<table>
<thead>
<tr>
<th>Design</th>
<th>$t_1$ (mm)</th>
<th>$d_1$ (mm)</th>
<th>$t_3$ (mm)</th>
<th>$t_2$ (mm)</th>
<th>$w_3$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0.49</td>
<td>73.3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td>0.30</td>
<td>75.1</td>
<td>0.30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TLR</td>
<td>0.35</td>
<td>75.1</td>
<td>0.30</td>
<td>0.32</td>
<td></td>
</tr>
<tr>
<td>TQR</td>
<td>0.35</td>
<td>77.2</td>
<td>0.30</td>
<td></td>
<td>38.5</td>
</tr>
<tr>
<td>WLR</td>
<td>0.3</td>
<td>74.03</td>
<td>0.30</td>
<td>0.32</td>
<td>47.8</td>
</tr>
<tr>
<td>TQWLR</td>
<td>0.35</td>
<td>75.1</td>
<td>0.30</td>
<td>0.32</td>
<td>50.0</td>
</tr>
</tbody>
</table>

### Table 3 Support stiffness in undeformed configuration and stress results

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>R</th>
<th>TLR</th>
<th>WLR</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{h,y}$—VCW-beam (kN/m)</td>
<td>20.5</td>
<td>46.8</td>
<td>54.1</td>
<td>41.5</td>
</tr>
<tr>
<td>$k_{h,z}$—ANSYS (kN/m)</td>
<td>20.8</td>
<td>47.7</td>
<td>54.9</td>
<td>45.8</td>
</tr>
<tr>
<td>$k_{h,z}$—error with respect to ANSYS (%)</td>
<td>1.3</td>
<td>1.8</td>
<td>1.6</td>
<td>9.3</td>
</tr>
</tbody>
</table>

The WLR-design is the design that is optimized for rotational stiffness around the $x$-axis.
the extent to which variation of the cross section of leafsprings can improve the performance for two of these configurations, shown in Fig.11:

FL: The single spherical joint, consisting of three folded leafsprings. These folded leafsprings are placed in such a way that lines through the folds coincide in the center of the joint. In this way, the deformation of the leafsprings allows a large rotation of the end-effector around all three axis through this center point. This is the most simple design constraining three translations and having three rotational degrees of freedom shown in Ref. [42].

SFL: The serial stacked spherical joint, consisting of two FL-joints. The six folded leafsprings are placed in such a way that lines through the folds coincide in the center of the joint. This is the best performing design in Ref. [42].

The support stiffness in the $z$-direction is optimized when considering a range of motion of 30 deg tip-tilt angle (any rotation angle perpendicular to the $z$-axis). The maximum allowable stress in the material due to deformation is 600 MPa, the elasticity of the material is 200 GPa and the Poisson ratio is 0.3. The build space for the mechanism is limited to a cylinder aligned with the $z$-axis with a radius of 75 mm. The algorithm presented in Ref. [43] was used to detect collision of the leafsprings. At least five design parameters were used, which are shown in Fig. 11 (the length $L$ is computed based on the build space and the other design parameters). Four different designs for each of the two mentioned configurations are considered:

- **C**: Leafsprings with constant cross section (so exactly the same design as in Ref. [42]).
- **W**: The thickness of the leafsprings is kept constant, while the width is varied quadratically. The width is determined by the width at the base ($w_1$), the width at the center ($w_2$), and the width at the fold ($w_3$).
- **T**: A constant width and a quadratically varying thickness, which is determined by $t_1$, $t_2$, and $t_3$, defined at the same positions as in the W-design.
- **WT**: The width and the thickness are both varied quadratically.

The flexure joints are optimized with VC-beams and with VCW-beams, with four elements per side of each folded leafspring. Table 4 shows the resulting dimensions for the latter case. Figure 12 shows the optimized support stiffnesses. The results of the VC-beam are up to a factor of five worse than the results of the VC-beam. This is mainly because of the extra stress due to the constraint warping. This indicates the importance of the modeling of warping in beam elements that are used to model leafsprings.

The results of the VCW-beam show that the support stiffness of the SFL-joint can be increased by a factor of 4.0. The reason for this large factor is that the extra stress due to the constraint warping is decreased by making the leafsprings thicker around the clamped interfaces. The T-design in this case results in about 98% of the support stiffness of the WT-design. This indicates that a large part of the improvement can already be achieved by only varying one dimension, which may simplify manufacturing.

Figure 13 shows the resulting support stiffness over the full range of motion. It indicates that the error of the VCW-beam compared to the finite element simulation is less than 10%.

### 3.4 Evaluation of the Second-Order Terms

The second-order terms derived in Sec. 2.5 increase the accuracy, but including these terms increases the computation time, especially the time to obtain the generalized deformations as a function of the absolute nodal coordinates, i.e., $\mathbf{D}(x)$. Figure 14 shows results for...
one design of the parallel flexure guidance and one design of the folded leafspring based spherical joint, both modeled using VCW-beams. Both cases show a significant improvement in the accuracy of the support stiffness. The foreshortening effect is the only relevant second-order term for the parallel flexure guidance. The torsion-bending coupling is the most significant term for the spherical joint. The simulation time increases by about 30%, but the increase in accuracy is generally more than 30%. This indicates that it is beneficial to include second-order terms.

### Conclusions

Two beam elements with varying cross section have been formulated in the generalized strain formulation. The formulation is

<table>
<thead>
<tr>
<th>Case</th>
<th>Design</th>
<th>Configuration</th>
<th>L (mm)</th>
<th>r (mm)</th>
<th>$\psi$ (°)</th>
<th>$\theta_1$ (°)</th>
<th>$w_1$ (mm)</th>
<th>$w_2$ (mm)</th>
<th>$w_3$ (mm)</th>
<th>$t_1$ (mm)</th>
<th>$t_2$ (mm)</th>
<th>$t_3$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FL</td>
<td>C</td>
<td>FL</td>
<td>72.3</td>
<td>27.3</td>
<td>83</td>
<td>45</td>
<td>23.5</td>
<td></td>
<td></td>
<td>0.30</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>W</td>
<td>FL</td>
<td>67.7</td>
<td>26.0</td>
<td>76</td>
<td>40</td>
<td>39.9</td>
<td>21.5</td>
<td>25.1</td>
<td>0.42</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>FL</td>
<td>70.3</td>
<td>30.7</td>
<td>82</td>
<td>45</td>
<td>29.8</td>
<td></td>
<td></td>
<td>0.62</td>
<td>0.42</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>WT</td>
<td>FL</td>
<td>70.9</td>
<td>30.6</td>
<td>83</td>
<td>45</td>
<td>27.5</td>
<td>30.8</td>
<td>29.4</td>
<td>0.66</td>
<td>0.42</td>
<td>0.48</td>
</tr>
<tr>
<td>SFL</td>
<td>C</td>
<td>SFL</td>
<td>69.2</td>
<td>29.4</td>
<td>97</td>
<td>39</td>
<td>28.1</td>
<td></td>
<td></td>
<td>0.56</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>W</td>
<td>SFL</td>
<td>62.8</td>
<td>30.7</td>
<td>87</td>
<td>36</td>
<td>50.2</td>
<td>25.2</td>
<td>28.3</td>
<td>0.77</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>SFL</td>
<td>66.2</td>
<td>35.3</td>
<td>92</td>
<td>39</td>
<td>32.9</td>
<td></td>
<td></td>
<td>1.20</td>
<td>0.76</td>
<td>0.81</td>
</tr>
<tr>
<td></td>
<td>WT</td>
<td>SFL</td>
<td>65.3</td>
<td>35.1</td>
<td>92</td>
<td>39</td>
<td>36.5</td>
<td>32.4</td>
<td>32.9</td>
<td>1.11</td>
<td>0.75</td>
<td>0.80</td>
</tr>
</tbody>
</table>

Fig. 12 Optimized support stiffness in z-direction of the spherical joint in deformed configuration

Fig. 13 Spherical joint, vertical support stiffness, $k_z$, of the SFL-joint. The design that was obtained using the VC-beams is evaluated, the stiffness is obtained using VCW-beam elements. The tip-tilt rotation axis is initial the x-axis and this axis rotates by angle $\phi$ around the z-axis (so for $\phi = 90$ deg it rotates around the y-axis).
based on the Timoshenko beam equations in which the variation of the stiffness coefficients is taken into account. In one of the elements, the effect of warping due to torsion has been included by exploiting Vlasov’s warping theory. The beam elements are applied to model leafsprings, which have a thin rectangular cross section. If the variation of the cross section is small compared to the length (which is always the case for thickness variations) the errors in stiffness are typically below 10% and errors in stress below 20%.

The new beam elements are used to optimize the support stiffness of several flexure joints that should allow a certain large motion without exceeding stress limits. The maximum stress computed by the VC-beam (in which the constrained torsional warping is not modeled), is typically much lower than the stress computed by the VCW-beam. Therefore the optimized support stiffness of a spherical flexure joint was about a factor of five lower by modeling the constraint warping. This highlights the importance of modeling constraint warping.

The results indicate that allowing a variation of the width and the thickness of the leafsprings a parallel flexure guidance increases the support stiffness up to a factor of 1.3. The support stiffness of a spherical joint can be increased by a factor of about 4.0. This large improvement factor is because the warping causes extra stresses at the clamp which can be reduced by locally increasing the cross section near the clamp. In all the cases that were modeled with the VCW-beam, more than 98% of the support stiffness could already be obtained by only varying the thickness of the leafsprings, keeping the width constant.

**Funding Data**

- Netherlands Organization for Scientific Research (NWO) (HTSM 2017 with project number 16210).

**Data Availability Statement**

The datasets generated and supporting the findings of this article are obtainable from the corresponding author upon reasonable request.

**Appendix A: Second-Order Term—Torsion-Bending Coupling**

This appendix derives the second-order term in the deformation functions that couples torsion and bending modes. This effect occurs because the local rotation matrix is not a linear function of the local rotations. The effect has been derived in Refs. [17] and [40] for beams with a constant cross section. The second-order expression of the rotation matrix is (see Eq. 45 of Ref. [40])

$$R(\xi) = \begin{bmatrix} 1 - \phi_{10}^2/2 - \phi_2^2/2 & -\phi_2 + \phi_3\phi_1 & \phi_1 + \phi_3\phi_2 \\ \phi_2 & 1 - \phi_1^2/2 - \phi_3^2/2 & -\phi_1 + \phi_3\phi_2 \\ -\phi_3 & \phi_1 & 1 - \phi_1^2/2 - \phi_2^2/2 \end{bmatrix} \quad (A.1)$$

The relation between the second-order generalized deformations and the global coordinates, \(\tilde{\xi} = D(x)\) is based on the global rotation matrices of both nodes. This relation can also be expressed in terms of the local coordinates using Eq. (A.1). The second-order expression is (see Eq. (54) of Ref. [40])

$$\begin{align*}
\varepsilon_2 &= L_0\phi_1^0 + \frac{1}{2L_0}(\epsilon_3 - \epsilon_4)(\epsilon_5 + \epsilon_6) \\
\varepsilon_3 &= -L_0\phi_2^0 \\
\varepsilon_4 &= L_0\phi_2^0 + \frac{1}{L_0}\epsilon_2\epsilon_6 \\
\varepsilon_5 &= -L_0\phi_3^0 \\
\varepsilon_6 &= L_0\phi_3^0 - \frac{1}{L_0}\epsilon_2\epsilon_4 \\
\end{align*} \quad (A.2)$$

The local displacements can be expressed in terms of the local rotations of the nodes using the mode shapes defined in Secs. 2.3 and 2.4

$$\begin{align*}
v(\xi) &= -N_3(\xi) L_0\phi_1^0 + N_4(\xi) L_0\phi_2^0 \\
w(\xi) &= -N_3(\xi) L_0\phi_2^0 + N_4(\xi) L_0\phi_3^0 \\
\phi_1(\xi) &= N_2(\xi)\phi_1^0 + N_3(\xi)\epsilon_1/L_0 + N_4(\xi)\epsilon_2/L_0 \\
\phi_2(\xi) &= -N_3(\xi)\phi_2^0 + N_4(\xi)\phi_3^0 \\
\phi_3(\xi) &= -N_5(\xi)\phi_2^0 + N_6(\xi)\phi_3^0 \quad (A.3)\end{align*}$$

A second-order expression for the torsional curvature can be obtained based on the second-order rotation matrix in Eq. (A.1). The second-order definition of the second generalized deformation is the integration of this curvature over the beam (see also Eqs. (47) and (63) of Ref. [40])

$$\tilde{\varepsilon}_2 = L_0\int_0^L \kappa_1(s) \, ds = L_0\int_0^L \left( \phi_1'(\xi) - \phi_2(\xi)\phi_1'(\xi) \right) d\xi \quad (A.4)$$

By substituting the local displacements of Eq. (A.3) and using Eq. (A.2) to express this in terms of the old generalized deformations we obtain
\[ \dot{\varepsilon}_2 = \varepsilon_2 + \frac{1}{L_0} \left( \left( H_{35}^{(2)} - \frac{1}{2} \right) \varepsilon_3 \varepsilon_5 + \left( H_{56}^{(2)} - \frac{1}{2} \right) \varepsilon_5 \varepsilon_6 \right) + \left( H_{45}^{(2)} + \frac{1}{2} \right) \varepsilon_4 \varepsilon_5 + \left( H_{46}^{(2)} + \frac{1}{2} \right) \varepsilon_4 \varepsilon_6 \] (A.5)

with

\[ H_{ik}^{(2)} = \int_0^1 N_i^k(\zeta) N_j^k(\zeta) d\zeta \] (A.6)

The second-order expressions for the bending modes can be obtained similarly (see also Eq. (64) of Ref. [40])

\[ \dot{\varepsilon}_3 = \varepsilon_3 + \frac{1}{L_0} \sum_{n=1}^{2,7,8} 2 \bar{I}_n (H_{35}^{(4)} \varepsilon_3 \varepsilon_5 + H_{56}^{(4)} \varepsilon_5 \varepsilon_6) \]

\[ \dot{\varepsilon}_4 = \varepsilon_4 + \frac{1}{L_0} \sum_{n=1}^{2,7,8} 2 \bar{I}_n (H_{45}^{(4)} \varepsilon_4 \varepsilon_5 + H_{46}^{(4)} \varepsilon_4 \varepsilon_6) \]

\[ \dot{\varepsilon}_5 = \varepsilon_5 + \frac{1}{L_0} \sum_{n=1}^{2,7,8} 2 \bar{I}_n (H_{55}^{(4)} \varepsilon_5 \varepsilon_5 + H_{46}^{(5)} \varepsilon_4 \varepsilon_6) \]

\[ \dot{\varepsilon}_6 = \varepsilon_6 + \frac{1}{L_0} \sum_{n=1}^{2,7,8} 2 \bar{I}_n (H_{66}^{(4)} \varepsilon_6 \varepsilon_6 + H_{46}^{(6)} \varepsilon_4 \varepsilon_6) + \varepsilon_2 \varepsilon_6 \] (A.7)

with

\[ H_{ik}^{(4)} = \int_0^1 (1 - \zeta) N_i(\zeta) N_j^k(\zeta) d\zeta, \quad s = 3, 5 \]

\[ H_{ik}^{(6)} = \int_0^1 \zeta N_i(\zeta) N_j^k(\zeta) d\zeta, \quad s = 4, 6 \] (A.8)

This defines the torsion-bending coupling for VCW-beams. The result for the VC-beam is very similar. The only difference is that the terms that depend on \( \varepsilon_5 \) or \( \varepsilon_6 \) in Eq. (A.8) do not exist, so instead of the summation-term we can use \( i = 2 \).

**Appendix B: Stress Computation**

This appendix explains how the maximum Von Mises stress has been computed in the results of Secs. 3.2 and 3.3. The default method in classical beam theory is used. After computing the static deformed equilibrium configuration, the deformation and the reaction forces at both nodes of each beam element are known, see Fig. 15. From this, the internal forces and internal moments at a cross section are computed based on equilibrium considerations.

For the VCW-beam, the bimoment is linearly interpolated between its values at both nodes. The Saint-Venant-torsion in the VCW-beam is derived based on the local torsional rotation, which is obtained based on the mode-shapes

\[ T_x(\chi) = GI_x^i \frac{d\phi_x}{dx} \] (B.1)

For the VC-beam we use \( T_x = M_z \) and \( B = 0 \). The axial stress is computed by

\[ \sigma_{xz}(x, y, z) = \frac{F_z}{A} + \frac{M_z y}{I_y} + \frac{M_y z}{I_y} + \frac{B \omega(y, z)}{I_y} \] (B.2)

The shear stress in the xz-direction is caused by the shear force in \( \chi \)-direction and Saint-Venant torsion. The stress due to the torsion is obtained using Prandtl’s membrane analogy [44], resulting in an infinite series of which only the first terms have to be computed for reasonable accuracy

\[ \tau_{\chi z} = \frac{F_z}{I_y} \left( \frac{z^2}{8} - \frac{z^2}{2} \right) - \frac{87 \omega w}{\pi^2 I_y} \sum_{n=1,3,5,\ldots} \frac{(-1)^{n+1}}{n^2} \sinh \left( \frac{n \pi x}{L_x} \right) \cos \left( \frac{n \pi y}{L_y} \right) \cos \left( \frac{n \pi z}{L_z} \right) \] (B.3)

The shear stress in the yz-direction is caused by the shear force in \( \gamma \)-direction and torsion

\[ \tau_{\gamma z} = \frac{F_z}{I_y} \left( \frac{y^2}{8} - \frac{y^2}{2} \right) + \frac{87 \omega w}{\pi^2 I_y} \sum_{n=1,3,5,\ldots} \frac{(-1)^{n+1}}{n^2} \sinh \left( \frac{n \pi y}{L_y} \right) \cos \left( \frac{n \pi z}{L_z} \right) \] (B.4)

The other stress-components are zero, so the Von Mises stress can be obtained by

\[ \sigma_{\text{Von Mises}} = \sqrt{\sigma_{xx}^2 + 3\sigma_{xy}^2 + 3\sigma_{xz}^2} \] (B.5)

**References**


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[38] Hodges, D. H., 2006, Nonlinear Composite Beam Theory (Progress in Aeronautics and Aeronautics), American Institute of Aeronautics and Astronautics, Reston, VA.


