NETWORKING: THEORY AND TEACHING PRACTICE USING LESSON STUDY

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This paper describes the search of six Dutch teachers for the integration of theories to make sense of mathematics. The mathematics teachers investigated their teaching practices using lesson study. Two successive research lessons about the introduction of the derivative were jointly planned, implemented, and live observed. The teachers revised and re-teached the research lessons based on collaborative discussions at school and reflections at the university. The results of the study show that making sense of the derivative starts with encouraging students to communicate intuitively using own words. This is followed by the iconic development of visualizations and finally results in the use of symbols: operations with numbers and reasoning about operations with numbers.

INTRODUCTION

This study focusses on teachers' collaborative investigation to integrate theories of teaching and learning to make sense of mathematics. In 2008 the Dutch government recognized a stagnated progress in numeracy at scientific studies as a consequence of a lack of students' mastering of mathematical skills. This resulted in an increased attention for algorithms and correct calculations at secondary schools. The balance moved from a focus on Skemp's (1976) relational understanding to instrumental understanding of mathematical concepts.

Research at the University of Twente focusses on the effects of teacher design teams. In this context a number of mathematics teachers collaborated with the intention to improve mathematics education. The researcher (first author) invited six mathematics teachers to start a lesson study team. Teacher selection was based on good experiences the researcher had with the teachers during teacher trainee supervision. Lesson study is a professional development strategy in which teachers collaboratively investigate teaching and learning practices by means of live classroom observations and post-lesson discussions (Stepanek, Appel, Leong, Mangan, & Mitchell, 2007). We used lesson study to collaboratively investigate the integration of theories to make sense of mathematics.

Making sense of mathematics arises initially through coherent perception and action, and develops through coherent use of operations in arithmetic and algebra (Tall, 2012). The lesson study team's investigation of making sense of mathematics builds on Skemp's (1976) types of understanding and Tall's (2008) framework of long-term mathematical thinking in relation with Bruner's (1966) representations.

PROBLEM DEFINITION AND RESEARCH QUESTION

Lesson study is in Japan widely used and deeply rooted for over a century. Lesson study makes teaching approaches more practical and understandable to teachers through a deeper understanding of content and student thinking (Murata, 2011). In 2009 a four-year lesson study project was initiated at the University of Twente. The first project year focussed on the effects of lesson study on teachers' professional development. The results showed complexities with regard to culture differences with Japan (Verhoef & Tall, 2011). The second project year showed a positive effect of the use of GeoGebra in the context of the introduction of the derivative. This paper reports the third project year in which the search for the integration of theories to make sense of mathematics was central. Our research question is:

How to make sense of mathematics integrating theories in the context of the introduction of the derivative?

THEORETICAL FRAMEWORK

(a) A sensible approach to mathematics

A sensible approach to mathematics takes account of the structures of mathematics and of the increasing levels of sophistication as learning progresses from sense through perception, then through the relationships of operation and a developing sense of reason (Chin &Tall, 2012). This approach relates to Bruner's (1966) successive modes of representation. Bruner distinguished: (a) action based enactive representation, (b) image based iconic representation, and (c) symbolic representation including not only written and spoken language but also the symbolism of arithmetic and the language of logic. Enactive means gesture, movement of the body, and physical showing of ideas. Iconic, incorporating enactive, is visual and includes all forms of sensory recognition, touch and smell etc. In his framework Tall (2008) puts enactive and iconic together as conceptual embodiment. The enactive and iconic modes of human perception and action develop into the mental world of perceptual and mental thought experiment. Operational symbolism develops from embodied actions, such as counting and measuring, and encapsulates as symbols in arithmetic. The higher level of logic specified by Bruner is seen as a distinct level based on set-theoretic definitions and formal proof.

We suggest that, to make sense of mathematical thinking, the teacher should be aware of the changing needs of the student in new situations, to build on previous success and to realize that what worked before will need a new approach to make sense of the new situation. To do this we consider how the learner makes sense through perception based on fundamental conceptual embodiment and thought experiment, then through the coherent relationships in operational symbolism, and later in terms of reasoning based on definition and deduction. In school mathematics, reasoning develops in various forms, like the transition from the practical tangent $(f(x+\Delta x)-f(x))/\Delta x$ measuring the slope from x to $x+\Delta x$ to the theoretical tangent the ratio of the component of a tangent vector. In this paper we typify sense making of mathematical thinking in terms of *perception*, *operation* and *reasoning.* We distinguish the practical enactive and iconic representations, and the theoretical symbolic representation of the derivative. Perception means the dynamical look along the curve to *see* the changing gradient as the changing curves direction. Operation applies the changing practical slope and the relationship between the visual changing slope and symbolic computation of the slope that stabilizes on the derivative function. Reasoning develops practically (perception and operation) based on experiment, and theoretically based on definition and deduction.

(b) Lesson study as a strategy for professional teacher development

Lesson study can be typified as a *live* research lesson. The live research lesson creates a unique learning opportunity for teaching. Lewis, Perry and Murata (2006) describe three specific areas that develop through the lesson study process: (1) teachers' knowledge, (2) teachers' commitment, and (3) community and learning resources. While teaching is considered an independent and often isolated practice in many countries, lesson study brings teachers together to share goals, discuss ideas, and work collaboratively.

Murata (2011) reports the following five attention points. Firstly, lesson study is centred around teachers' interests. Teachers should perceive lesson study goals to be important and relevant for their own classroom practice. Secondly, lesson study is student focussed. The lesson study activities should direct teachers' attention to student learning and the relation between learning and teaching. Thirdly, lesson study has a research potential. Teachers share physical observation experiences and these provide research opportunities. Fourthly, lesson study is a reflective process. Teachers have to reflect on their teaching practice and subsequent student learning in an educational community. Fifthly, in lesson study teachers work interdependently and collaboratively. Isoda (2010) characterized the lesson study cycle process as consisting of the following collaborative elements: planning (preparation), doing (observation), and seeing (discussion and reflection). He advocated the use of scientific literature as a basis for deepening teaching strategies.

RESEARCH METHOD

Participants

Six mathematics teachers from different secondary schools participated in the lesson study team during the school year 2011-2012, see Table 1. The first three teachers participated in previous years. School management facilitated the teachers by giving them half a day weekly for participating in the lesson study project.

Table 1: Description of participants

	Work experience in 2010	Education and teaching experience
A	17 years	BSc math + MSc math education; lower level to upper level high school students
B	14 years	BSc math + BSc math education; mostly upper level high school students
C	one year	BSc engineering + MSc math education; mostly upper level high school students
D	26 years	MSc math + MSc math education; mathematics teacher team leader
E	19 years	BSc math + MSc math education; lower level to upper level high school students

F	one year	MSc math + BSc math education; a mathematics PhD

Besides the teachers, the lesson study team consisted of four staff members of the University of Twente: a mathematician, a mathematics teacher trainer, a PhD-candidate (second author) and the researcher (first author). The staff members had specific roles in the lesson study team.

Research instruments

The research instruments consisted of three lesson plans, field notes of student observations and written reports of the discussions at the teachers' school, and the plenary reflections at the university. The observers were participants of the lesson study team plus interested school colleagues.

Context of the study

The teachers revised the textbook with regard to the introduction of the derivative with a focus on sense making. They intended to pick up the textbook approach, with a focus on mastering differentiation rules, after the introduction.

Based on last year's experiences with lesson study, the teachers decided to use GeoGebra for sense making of the derivative. The teachers started a process of zooming in at a fixed point on the graph being aware of the rate of change using the visualization (Bruner's enactive representation). They wanted to introduce an icon to develop a link to the use of numbers (Bruner's symbolic representation).

The teachers worked in three pairs (P1, P2 and P3). In each pair, one teacher did not have any previous experience with lesson study. The pairs started successively teaching two lessons. Firstly, P1 started with the first research lesson. P2 continued the same day at another location. Secondly, P1 continued with the second research lesson next day while P2 continued later. The lessons were planned collaboratively, observed and discussed at the teacher's school. The first four lessons (from P1 and P2) were evaluated in a plenary meeting at the university. This resulted in a revision of the research lessons, and this was used in class by P3. This lesson was collaboratively discussed at the teacher's school and plenary evaluated at a university meeting.

Data collection, processing and analysis

P1's lesson plan was summarized. The other lesson plans were described in relation with P1's lesson plan. The field notes of the student observations were classified with regard to Chin and Tall's (2012) categorizations: perception, operation and reasoning. Remarkable (discussion and reflection) report statements were coded as practical (enactive, iconic) or theoretical (symbolic) based on Bruner's (1966) framework of representations. The classifications, codes and analysis were member checked with the teachers afterwards.

RESULTS

Lesson plans

Below, first the results of the lesson plans will be reported, followed by field notes of student observations, and finally elements from the discussions and plenary reflections. P1's lesson plan emphasized student interaction. The teachers tried to make sense by activating students' communication explicitly in their lesson plan (Figure 1).

> The teacher introduces the increasing and decreasing graph in comparison with a jumping frog in the first lesson. The Power Point sheet shows the words: increasing/decreasing; monotonic; tangent; slope. The teacher gives each student pair one assignment. One student of each pair, sitting with backs against each other, receives an arbitrary graph on paper. The student describes the given graph in own words, the other student tries to draw the graph on his empty paper.
>
> The teacher continues plenary with the graph of a parabola. He has drawn arrows on the graph (first two figures below). The teacher reminds the students of the computer game Angry Birds, making sense to the graph's change in one point. The teacher shows the third figure below and asks 'Do you know the right place of these numbers'?
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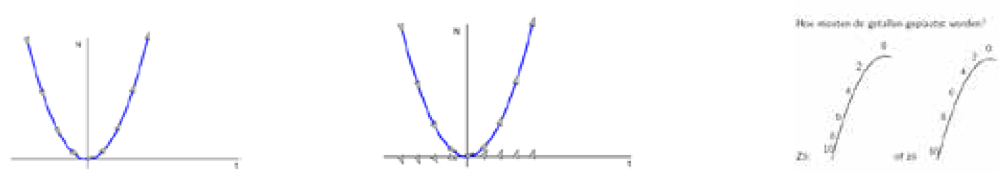
> 
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> The teacher continues the second lesson using numbers (slopes) illustrating a change each. He uses squares on his board and puts line segments in here with the comparable slopes. The students get an arbitrary graph on paper each. The teacher asks to put numbers – illustrating a change each - at some fixed points on the delivered graph.
> The teacher ends plenary with the calculation of the slope of a straight line through two closed points on the graph, suggesting this gives one answer exactly: the change in one point on the graph.

Figure 1: Lesson plan of the first pair

P2's lesson plan emphasizes operations with symbols. The teachers replace student interaction with worksheets. They want to reveal students thinking on paper as much as possible. The worksheet focusses on reasoning about numbers as rates of change. Figure 2 lists the core problem.

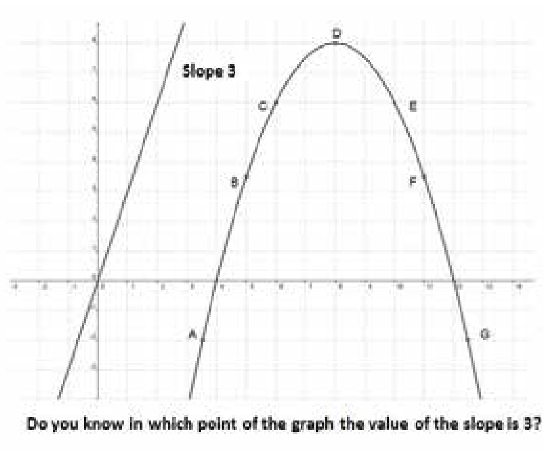

Figure 2: The slope in a point on the graph

The teachers use the problem in which students describe a graph in own words at the end of the lesson instead of in the beginning. The teachers don't use arrows. They emphasize local straightness in a point nearby the top (more curved). They introduce the zooming in process as an analogy with a view at the earth from space. The teachers continue the procedure of zooming in according to two closed points using GeoGebra suggesting that the process of zooming in gives the same result (one slope). After that they calculate the derivative in different points on the graph and establish that the numbers are elements of one straight line. They end with the ratios Δy/Δx and (f(x+Δx) - f(x)/Δx).

P3's first lesson starts again with a student describing a graph in own words, and continues with the graph of the parabola with arrows, see the first graph in Figure 1.The students solve problems on a worksheet in pairs. The teachers ask the students to add numbers to the arrows on the graph. They continue with asking numbers as rates of change, see Figure 2. They end with asking the right numbers on the right places, see Figure 3.

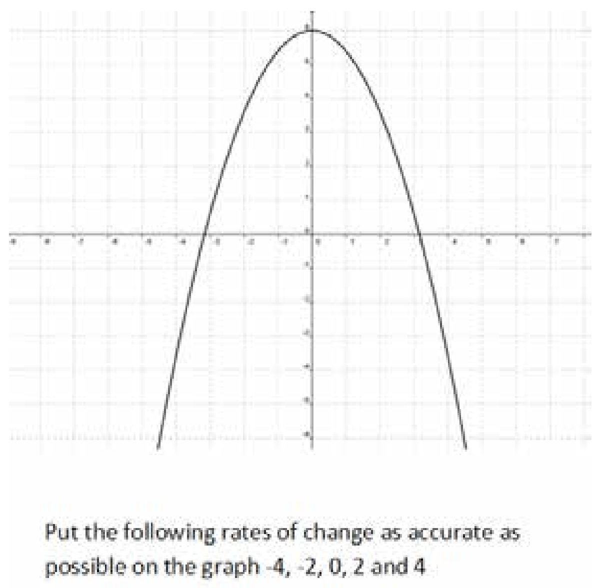

Figure 3: Right number on the right place

Field notes of student observations

Table 2 lists characteristic field notes of the student observations. The first column lists the successive pairs and the successive two lessons. The rest of the columns shows the classifications *perception*, *operation* and *reasoning*. The cells contain characteristic field notes of student observations per pair. The dotted line marks the reflective meeting at the university after P2's lessons.

Table 2: Characteristic field notes of student observations

	Perception Students see:	Operation Students:	Reasoning Students wonder whether:
P1-1	- the words 'increasing/ decreasing; monotonic; tangent; slope' standing on the board	- draw arrows (dove tails) on the graph	- does it involve the ratio on the coordinate system or the rate of increasing?
P1-2	- line segments in a square on the board indicating the change from picture to number	- fold paper and slide with it - calculate the difference quotient	- why numbers are important, if you are able to fold the change?
P2-1	- no data	- draw chords, not the tangent line in a point on the graph - try by calculating - calculate x- and y-coordinates	- make no comments
P2-2	- zooming in with GeoGebra - local straightness - a chord on a small interval - zooming out giving the tangent	- slide the lines, estimate and calculate - calculate the difference quotient	- make no comments
P3-1	- the words 'increasing/ decreasing; monotonic; tangent; slope' are given in advance	- sketch on the worksheet - slide a line with a ruler - draw chords instead of a tangent line - don't mention the word tangent	- do arrows on a straight line be the same ? - do the closer the arrows on the x-axis be, the flatter the graph? - does an arrow on the x-axis show the height of the graph?
P3-2	- zooming in with GeoGebra - local straightness - that Δx becomes smaller	- make a curve straight - slide the lines, estimate and calculate - formulate a straight line	- what happens if A coincides B? - what happens if Δx becomes zero? - does dy/dx not be defined, dx=0?

Perception: the perception moves to the visualization of symbolic representations during the lesson study process. The words 'increasing/ decreasing; monotonic; tangent; slope' in P1's first lesson stimulates students' communication. P1's students start to fold paper spontaneously when the teacher asks numbers – illustrating a change each - at some fixed points on the delivered graph (Figure 1, the last sentence of the last but one paragraph). P2 does not focus on perception in the first lesson.

Operation: the number of student activities increases during the lesson study process.

Reasoning: the intuitive reasoning becomes more important. P2 does not focus on reasoning at all. P1's students are trying to refine the icon 'arrow' (as a dove tail) to a line segment in a square. P2's students are impeded by the incorrect use of the 'tangent line method' - indicating some kind of smoothness of the graph -, learned from the physics teacher. They start with chords on a large interval (to prevent measurement errors as in physics). P3's students start to reason about the icon 'arrow'. They move to reasoning about Δx and dx without any transition. These students have to use symbols to be able to differentiate functions.

Reports from discussions and reflections

Table 3 shows the characteristic teacher comments from the discussions and the reflections based on the student observations. The first column lists the successive pairs and the successive two lessons. The rest of the columns shows the components in types of mathematical thinking: practical (enactive and iconic) and theoretical (symbolic subdivided in local or global). The cells contain characteristic comments from the reports. The dotted line marks the reflective meeting after P2's lessons.

Table 3: Teachers' reflections

	Practical		Theoretical	
	Enactive Reflection on action	Iconic Reflection on visualization	Symbolic (local) Reflection on symbols	Symbolic (global) Reflection on symbols
P1-1	- *no* words increasing/ decreasing; monotonic; tangent; slope in advance	- an arrow with direction refers to a move, preference an 'arrow' without any direction	- students not simply work with numbers after working with arrows	- differentiation globally should develops gradually
P1-2	- maximum and minimum are not a problem	- preference of an arrow without direction	- awareness of coaching to numbers	- support with calculations /difference quotient
P2-1	conflict with physics: derivative means chord	- the use of an icon is necessary	- change the context to a coordinate system	- calculations with a difference quotient
P2-2	- misconception that drawing a chord on a small interval when zooming in, gives a tangent line	- the use of an icon is necessary without giving a direction	- give equations of parabola and line and support students to reason about the slope	- calculations with a difference quotient
P3-1	- cut with a scissors, sew like a sewing machine does and emphasize two sides	- line segment with a dot halfway works best	- avoid stagnation in a chord, continue in coordinate system	- local discontinuous; connect increasing and monotonic increasing
P3-2	- zooming in on paper is not possible - calculation with small Δx takes time!	no data	- students think that a Δx of 0,0001 works exactly!	- there is a lack of numeracy

The teachers prefer to stimulate students' intuitive communication in own words, *not* giving the words 'increasing/ decreasing; monotonic; tangent; slope' in advance (enactive representation). They introduce new ideas like cutting with a scissors, and sewing like a sewing machine does and emphasizing two sides. The icon develops during the lesson study from a dove tail shaped arrow (suggesting a movement), via an arrow with a direction, to a line segment with a dot halfway – the picture of the graph magnified so highly that it looks like a straight line. For the students the transition to the use of numbers is impeded by a too large gap between perception (practical) and calculation (theoretical) by themselves. For the teachers the idea grows to introduce a coordinate system to insert both a line segment related to the slope value and a graph in relation with its equation. The teachers agree to stimulate communication with the students regarding three possibilities to calculate the slope at an interval: the interval [A-x, A+x], the interval [A, A+x] and the interval [A-x, x]. The discussions focus on a too large or a too small number. Another possibility is the use of counter examples like the relation with the graph of y=abs(x).

CONCLUSIONS AND DISCUSSION

The study shows that the use of an icon influences operational symbolism positively when the icon is chosen practically incorporating enaction and visualizing including sensory recognition, touch, smell etc. The 'dove tail icon' at the graph, seems to hide a line segment inside from the top to the bottom of the arrow, which may give rise to the idea that the concept of the derivative is inseparable from a difference quotient. Subsequently, the difference quotient gives rise to the differential quotient with which dividing by zero appears as an obstacle. The 'arrow icon' as a line segment and a v-sign on top, may give rise to the assumption that there is a continuous move because the direction is given and it resembles a vector used in physics. The 'line segment icon' with halfway a dot, gives rise to Skemp's (1976) relational understanding of the concept of a vector field as a basis for understanding differential equations in a later phase (Figure 4).

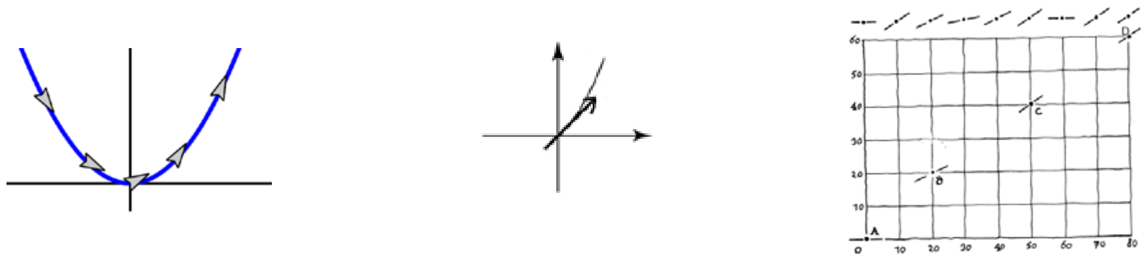

Figure 4: The development of an icon

The experiences with the development of an icon as being useful for sense making of mathematics were based on the student observation, discussion and reflection. Essentially, lesson study focused on classroom practices. The discussions after class and the plenary reflections contributed to teachers' own relational understanding of the derivative. The different theories encouraged the teachers to re-think and to refine the lessons in spite their textbook approach. Teachers' epistemological perspectives changed by the networking of theories (Oshimaa, Horinoa, Oshimab, Yamamotoc, Inagakid, Takenakae, Yamaguchif, Murayamaa, & Nakayamaf, 2006).

The Dutch textbooks are strongly influenced by Freudenthal's (1984) philosophy of mathematics as an activity, the principle of re-invention. Teachers' plenary reflections with regard to the 'line segment icon' with a dot halfway in a vector field gave rise to the derivative related to modelling of changing processes using differential equations. Being aware of this, the teachers made sense by thinking on the introduction of the derivative as a phenomenon. The teachers discussed the possibility to introduce the derivative on a curved area whereby student activities focused on describing the changing curved area. The teachers indicated students' instrumental understanding of a differential equation. They argued that students' tend to solve differential equations algorithmically and that students were unable to set up a differential equation, not being aware of the derivative as a rate of change necessarily for setting up a differential equation (Verhoef, Zwarteveen, Van Joolingen, & Pieters, 2013).

This study indicates that networking of theories - Skemp's (1976) types of understanding and Tall's (2012) framework of long-term mathematical thinking integrating by Bruner's (1966) representations – refines teaching practices. Lesson study stimulates the process of refining teaching practices.

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