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Connections between the Algebraic Riccati Equation and the Hamiltonian for Riesz-Spectral Systems*

C.R. Kuiper H.J. Zwart

Abstract

The algebraic Riccati equation (ARE) has been studied in great detail for finite-dimensional systems. For infinite-dimensional systems almost all results concentrate on the relation with the linear quadratic optimal control problem. The object of this paper is to consider solutions of the ARE for infinite-dimensional systems from a more general point of view. The relation between linear, bounded solutions of the ARE and the eigenvectors of the Hamiltonian will be studied, in the case when the Hamiltonian is a Riesz-spectral operator.

We present a general form of all possible linear, bounded solutions of the ARE in terms of the eigenvectors of the Hamiltonian. Characterizations for self-adjoint, nonnegative and stabilizing solutions are given as well. The derived results shall be applied to the heat equation.

Key words: infinite-dimensional systems, algebraic Riccati equation, Hamiltonian operator, Riesz-spectral operator

AMS Subject Classifications: 93C25, 93B28, 47A62, 49N05, 49R20, dv 93C20

1 Introduction

The algebraic Riccati equation (ARE) has been studied in great detail for finite-dimensional systems. This has led to many publications. We shall only mention those which are directly related with this paper. In Potter [22], a formula expressing the solutions of the ARE in terms of the eigenvectors of the associated Hamiltonian has been obtained. This theory has

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been extended in Mårtensson [18], e.g. in case not all the eigenvalues of the Hamiltonian are distinct. We should also mention the well-known paper of Willems [26], where a complete classification of all solutions of the ARE are given and optimal control problems with respect to quadratic performance criteria are analytically solved via the ARE. Recently, Kučera [10] published an overview of these results. He presented also some important results about existence and uniqueness of hermitian and definite solutions of the ARE, using invariant subspaces of the associated Hamiltonian.

Since the ARE is a successful tool for solving problems in finite-dimensions, it motivates extensions to infinite-dimensional settings. Many people have studied the ARE for infinite-dimensional systems, even when the input and output operators are unbounded. We shall only mention a few. Almost all results concerning the ARE concentrate on the relation with the linear quadratic optimal control problem. In the beginning of the mid-sixties, this relation was studied by Lukes and Russell [17]. A full analysis on existence and uniqueness of the stabilizing solution of the ARE was completed first by Zabczyk [29]. Sorine [23] and Louis and Wexler [16] gave relations between the stabilizing solution of the ARE and the associated Hamiltonian. Recently, Louis and Wexler [16] and Weiss [25] considered cost functions, which are not necessarily positive. The latter uses a Popov function approach to give necessary and sufficient conditions for the existence of a stabilizing solution of the ARE. He also gives connections between a general ARE, H_∞-design and robust control.

The object of this paper is to consider solutions of the ARE for infinite-dimensional systems from a general point of view. The relation between linear, bounded solutions of the ARE and the eigenvectors of the associated Hamiltonian will be studied, when the Hamiltonian is a Riesz-spectral operator. This type of operator was introduced in Curtain and Zwart [5].

Our approach does not simplify the linear quadratic optimal control problem, but makes it more clear, like in the finite-dimensional case. We present a general form of all linear, bounded solutions of the ARE in terms of the eigenvectors of the Hamiltonian. Then, characterizations for self-adjoint, nonnegative and stabilizing solutions follow in a natural way. Hence, we do not only show when we have the existence of a solution of the ARE, but we give also a method for constructing all solutions.

This paper contains generalizations to a Hilbert space of the results proved by Mårtensson [18] and Potter [22]. We also generalize some results of Willems [26], Kučera [10], and Kano and Nishimura [7]. Some preliminary results have already been published in Kuiper and Zwart [12], [13]. We remark that in the recent paper of Callier, Dumortier and Winkin [1] the nonnegative solutions of the ARE are investigated using the Hamiltonian.

Basic concepts of functional analysis, which we use, can be found in, for example, Curtain and Pritchard [2], Kreyszig [9] and Yosida [27].

In more detail, the content of this work is as follows. In Section 2 we give the definitions of Riesz basis, Riesz-spectral operator and exponentially stability of a C_0-semigroup. We introduce the Hamiltonian H as

$$H = \begin{pmatrix} A & -Q_1 \\ -Q_2 & -A^* \end{pmatrix}, \tag{1.1}$$

where

1. $A : \mathrm{D}(A) \subset Z \mapsto Z$ is the infinitesimal generator of a C_0-semigroup on the Hilbert space Z, and
2. Q_1 and Q_2 are self-adjoint, linear, bounded operators on Z.

Throughout this paper we assume that the Hamiltonian H is a Riesz-spectral operator.

The operator X is a linear, bounded solution of the ARE if $X \in \mathcal{L}(Z)$ and

$$\langle Az_1, Xz_2\rangle + \langle X^* z_1, Az_2\rangle + \langle Q_2 z_1, z_2\rangle - \langle Q_1 X^* z_1, Xz_2\rangle = 0 \tag{1.2}$$

for all $z_1, z_2 \in \mathrm{D}(A)$.

In Section 3 we give some results about Riesz bases. These are necessary for answering the question, in Section 5, of whether or not every ARE has a linear bounded solution. The main result of Section 4 gives sufficient conditions on the infinitesimal generator A such that the Hamiltonian H is a Riesz-spectral operator. These conditions are needed, because a perturbation of a Riesz-spectral operator is not necessarily Riesz-spectral. In Section 5, the main theorem gives necessary and sufficient conditions in terms of the eigenvectors of the Hamiltonian H such that the ARE has a linear, bounded solution. A method for constructing all solutions is given as well. For this we do not assume nonnegativeness of the operators Q_1 and Q_2. Also note that here the solution of the ARE need not be self-adjoint. Characterizations for self-adjoint and nonnegative solutions are studied in Section 6. In Section 7, we deal with the stabilizing solutions of the ARE. We show that if there exits a stabilizing one, then it is unique and self-adjoint. We also give a method for constructing it. Under extra assumptions on the operators Q_1 and Q_2, we give necessary and sufficient conditions for the existence and uniqueness of a stabilizing solution of the ARE. We shall present a proof without using optimal control theory.

In Section 8, we apply the derived results to the example of the heat equation. Finally, in Section 9 we end with some conclusions.

2 Preliminaries

In this section, we shall give some basic definitions, results and assumptions, which we shall need throughout this paper. We assume that all Hilbert spaces in this article are separable.

First, we give some notations:

Let Z_1 and Z_2 be Hilbert spaces and T a linear operator defined on some subspace of Z_1 with values in Z_2. Then

D(T) := domain of T,
$\rho(T)$:= resolvent set of T,
$\sigma(T)$:= spectrum of T,
$\sigma_p(T)$:= point spectrum of T (= set of all eigenvalues of T),
$\mathbf{Z}_0 := \mathbf{Z} \setminus \{0\}$,
$\mathcal{L}(Z_1, Z_2)$:= vector space of linear bounded operators from Z_1 to Z_2,
$\mathcal{L}(Z_1) := \mathcal{L}(Z_1, Z_1)$.

From Curtain and Zwart [5, section 2.4] we have the following definitions and results:

Definition 2.1 *A sequence of vectors* $\{\phi_n, n \geq 1\}$ *in a Hilbert space* Z *forms a* Riesz basis *for* Z *if*

1. $$\overline{\operatorname*{span}_{n\geq 1}\{\phi_n\}} = Z, \text{ and} \tag{2.1}$$

2. *there exist positive constants* m *and* M *such that for arbitrary* $N \in \mathbf{N}$ *and arbitrary scalars* α_n, $n = 1, \ldots, N$, *there holds*
$$m \sum_{n=1}^{N} |\alpha_n|^2 \leq \| \sum_{n=1}^{N} \alpha_n \phi_n \|^2 \leq M \sum_{n=1}^{N} |\alpha_n|^2. \tag{2.2}$$

Of course, one can define a Riesz basis for Z comprised of a sequence of vectors ϕ_n belonging to a countable subset of $\mathbf{Z} = \{n = 0, \pm 1, \pm 2, \ldots\}$; for example $\{\phi_n, n \geq 0\}$, $\{\phi_n,\ n = \pm 1, \pm 2, \ldots\}$ and $\{\phi_n, n \in \mathbf{Z}\}$ are often used in this paper. The following theory for Riesz bases is independent of the choice of the countable subset of $\mathbf{Z}$.

From the definition, it is obvious that an orthonormal basis is a Riesz basis. In the next lemma we shall show that if $\{\phi_n, n \geq 1\}$ is a Riesz basis, then every element in Z can be uniquely represented as a linear combination of the ϕ_n, even if the ϕ_n are not orthogonal, by means of the biorthogonal sequence of $\{\phi_n, n \geq 1\}$. $\{\phi_n, \psi_n\}$ forming a *biorthogonal sequence* if

$$\langle \phi_n, \psi_m \rangle = \delta_{mn}. \tag{2.3}$$

Lemma 2.2 *Suppose that the closed, linear operator* T *on the Hilbert space* Z *has simple eigenvalues* $\{\lambda_n,\ n \geq 1\}$ *and that its corresponding eigenvectors* $\{\phi_n,\ n \geq 1\}$ *form a Riesz basis for* Z. *Then*

1. *if* $\{\psi_n,\ n \geq 1\}$ *are the eigenvectors of the adjoint of* T *corresponding to the eigenvalues* $\{\overline{\lambda_n},\ n \geq 1\}$, *then the* $\{\psi_n\}$ *can be suitably scaled so that* $\{\phi_n,\ \psi_n\}$ *forms a biorthogonal sequence, where* $\overline{\lambda_n}$ *is the complex conjugate of* λ_n;

2. *every $z \in Z$ can be represented uniquely by*

$$z = \sum_{n=1}^{\infty} \langle z, \psi_n \rangle \phi_n, \tag{2.4}$$

and there exist constants m' and $M' > 0$ such that

$$m' \sum_{n=1}^{\infty} \mid \langle z, \psi_n \rangle \mid^2 \leq \|z\|^2 \leq M' \sum_{n=1}^{\infty} \mid \langle z, \psi_n \rangle \mid^2;$$

and

3. *$\{\psi_n, n \geq 1\}$ forms a Riesz basis for Z.*

Definition 2.3 *Suppose that T is a linear, closed operator on a Hilbert space Z, with simple eigenvalues $\{\lambda_n, n \geq 1\}$ and suppose that the corresponding eigenvectors $\{\phi_n, n \geq 1\}$ form a Riesz basis for Z. If the closure of $\{\lambda_n, n \geq 1\}$ is totally disconnected, then we call T a* Riesz-spectral operator.

By totally disconnected we mean that no two points $\lambda, \mu \in \overline{\{\lambda_n, n \geq 1\}}$, the closure of $\{\lambda_n, n \geq 1\}$, can be joined by a segment lying entirely in $\overline{\{\lambda_n, n \geq 1\}}$. So Definition 2.3 does cover the case that T has finitely many accumulation points.

Theorem 2.4 *Suppose that T is a Riesz-spectral operator with simple eigenvalues $\{\lambda_n, n \geq 1\}$ and corresponding eigenvectors $\{\phi_n, n \geq 1\}$. Let $\{\psi_n, n \geq 1\}$ be the eigenvectors of T^*, the adjoint of T, such that $\langle \phi_n, \psi_n \rangle = \delta_{nm}$. Then*

1. *$\rho(T) = \{\lambda \in \mathbf{C} \mid \inf_{n \geq 1} |\lambda - \lambda_n| > 0\}$ and for $\lambda \in \rho(T)$ $(\lambda I - T)^{-1}$ is given by*

$$(\lambda I - T)^{-1} = \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n} \langle \cdot, \psi_n \rangle \phi_n; \tag{2.5}$$

2. *T has the representation*

$$Tz = \sum_{n=1}^{\infty} \lambda_n \langle z, \psi_n \rangle \phi_n, \tag{2.6}$$

for $z \in \mathrm{D}(T)$, and

$$\mathrm{D}(T) = \{z \in Z \mid \sum_{n=1}^{\infty} \mid \lambda_n \mid^2 \mid \langle z, \psi_n \rangle \mid^2 < \infty\};$$

and

3. *T^* is also a Riesz-spectral operator with simple eigenvalues $\{\overline{\lambda_n}, n \geq 1\}$ and corresponding eigenvectors $\{\psi_n, n \geq 1\}$, where $\overline{\lambda_n}$ is the complex conjugate of λ_n.*

We make the following notation and assumptions:

Assumption 2.5

1. *Z is a separable complex Hilbert space;*
2. *$A : \mathrm{D}(A) \subset Z \to Z$ is the infinitesimal generator of the C_0-semigroup $T(t)$;*
3. *$Q_1 = Q_1^* \in \mathcal{L}(Z)$;*
4. *$Q_2 = Q_2^* \in \mathcal{L}(Z)$;*
5. *H denotes the* Hamiltonian *defined by*

$$H = \begin{pmatrix} A & -Q_1 \\ -Q_2 & -A^* \end{pmatrix} \tag{2.7}$$

on the Hilbert space $Z \oplus Z$ with domain $\mathrm{D}(H) = \mathrm{D}(A) \oplus \mathrm{D}(A^*)$.

Assumption 2.6 *The Hamiltonian H given by equation (2.7) is a Riesz-spectral operator with eigenvalues $\{\lambda_n, n \in \mathbf{Z}_0\}$ and eigenvectors $\{\Phi_n = \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix}, n \in \mathbf{Z}_0\}$.*

Since the Hamiltonian H, given by equation (2.7), is a Riesz-spectral operator, we shall speak about a Riesz-spectral system.

We also need the algebraic Riccati equation (ARE). By definition, $X \in \mathcal{L}(Z)$ is a solution of the ARE if

$$\langle Az_1, Xz_2 \rangle + \langle X^* z_1, Az_2 \rangle + \langle Q_2 z_1, z_2 \rangle - \langle Q_1 X^* z_1, X z_2 \rangle = 0 \tag{2.8}$$

for all $z_1, z_2 \in \mathrm{D}(A)$.

We end this subsection by giving two definitions, which we need in Sections 6 and 7.

Definition 2.7 *A C_0-semigroup, $T(t)$, on a Hilbert space Z, is* exponentially stable *if there exist positive constants M and α such that*

$$\|T(t)\| \leq M e^{-\alpha t}, t \geq 0. \tag{2.9}$$

Definition 2.8 *Suppose that A generates the C_0-semigroup $T(t)$ on the Hilbert space Z and $B \in \mathcal{L}(U, Z)$, where U is a Hilbert space. If there exists an $F \in \mathcal{L}(Z, U)$ such that $A + BF$ generates an exponentially stable C_0-semigroup, $T_F(t)$, then we say that (A, B) is* exponentially stabilizable.

3 Results about Riesz Bases

By assumption 2.6, we have that the Hamiltonian H, given by equation (2.7), is a Riesz-spectral operator. This implies that the corresponding eigenvectors $\{\Phi_n = \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix}, n \in \mathbf{Z}_0\}$ form a Riesz basis for the Hilbert space $Z \oplus Z$. Under these assumptions, we shall show that the question whether every ARE has a linear bounded solution is equivalent to the question whether there exists an index set $\mathbf{J} \subset \mathbf{Z}_0$ such that $\{\eta_n, n \in \mathbf{J}\}$ is a Riesz basis for Z (Theorem 5.6). In this section, we shall investigate the existence of such an index set $\mathbf{J}$ in general. For this study we do not need the special form of the Hamiltonian H or the fact that $\{\Phi_n = \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix}, n \in \mathbf{Z}_0\}$ are the corresponding eigenvectors. Therefore we take an arbitrary sequence of vectors in the Hilbert space $Z \oplus Z$, which forms a Riesz basis for $Z \oplus Z$. We shall prove that if $\{\Phi_n = \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix}, n \in \mathbf{Z}_0\}$ forms a Riesz basis for $Z \oplus Z$, then there exists a subset $\mathbf{J} \subset \mathbf{Z}_0$ such that $\{\eta_n, n \in \mathbf{J}\}$ is maximal in Z and every finite subset is linearly independent. However, in general this need not imply that $\{\eta_n, n \in \mathbf{J}\}$ forms a Riesz basis for Z.

Lemma 3.1 *Let $\{\Phi_n = \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix}, n \in \mathbf{Z}_0\}$ be a sequence of vectors in a Hilbert space $Z \oplus Z$. If $\overline{\operatorname{span}_{n\in\mathbf{Z}_0}\{\Phi_n\}} = Z \oplus Z$, then $\overline{\operatorname{span}_{n\in\mathbf{Z}_0}\{\eta_n\}} = Z$.*

Proof: Suppose that $\overline{\operatorname{span}_{n\in\mathbf{Z}_0}\{\eta_n\}} \neq Z$. Then there would exist a $v \neq 0 \in Z$ such that

$$\langle v, \eta_n \rangle = 0 \qquad \text{for all } n \in \mathbf{Z}_0.$$

This implies that

$$\langle \begin{pmatrix} v \\ 0 \end{pmatrix}, \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix} \rangle = 0 \qquad \text{for all } n \in \mathbf{Z}_0.$$

This is in contradiction with $\overline{\operatorname{span}_{n\in\mathbf{Z}_0}\{\Phi_n\}} = Z \oplus Z$. ■

Theorem 3.2 *Let $\{\Phi_n = \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix}, n \in \mathbf{Z}_0\}$ be a sequence of vectors in a Hilbert space $Z \oplus Z$. If $\overline{\operatorname{span}_{n\in\mathbf{Z}_0}\{\Phi_n\}} = Z \oplus Z$, then there exists an index set $\mathbf{J} \subset \mathbf{Z}_0$ such that $\overline{\operatorname{span}_{n\in\mathbf{J}}\{\eta_n\}} = Z$, and every finite combination of the η_n's is linearly independent.*

The proof follows from Zorn's Lemma; see Kuiper and Zwart [14].
So if $\{\Phi_n = \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix}, n \in \mathbf{Z}_0\}$ is a Riesz basis for $Z \oplus Z$, then there exists an index $\mathbf{J}$ such that $\{\eta_n, n \in \mathbf{J}\}$ is a basis. However, this does not imply that it is a Riesz basis, see Kuiper and Zwart [14].

In the next theorem, we shall show that under an extra assumption there exists an index set $\mathbf{J} \subset \mathbf{Z}_0$ such that $\{\eta_n,\, n \in \mathbf{J}\}$ is a Riesz basis for Z.

Theorem 3.3 *Let* $\{\Phi_n = \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix}, n \in \mathbf{Z}_0\}$ *be a Riesz basis for* $Z \oplus Z$*. If there exists an index set* $\mathbf{J} \subset \mathbf{Z}_0$ *such that*

1. $\overline{\operatorname{span}_{n\in\mathbf{J}}\{\eta_n\}} = Z$*, and*
2. *there exists a bounded* X *satisfying* $X\eta_n = \zeta_n$ *for* $n \in \mathbf{J}$*,*

then $\{\eta_n,\, n \in \mathbf{J}\}$ *is a Riesz basis for* Z*.*

Proof: Note that the index set $\mathbf{J} \subset \mathbf{Z}_0$ is isomorphic to $\mathbf{N}$. Therefore, without loss of generality, we may assume that $\mathbf{J} = \mathbf{N}$, and so

$$\overline{\operatorname{span}_{n\in\mathbf{N}}\{\eta_n\}} = Z.$$

Define $\Phi_0 = 0$. Since $\{\Phi_n, n \in \mathbf{Z}_0\}$ is a Riesz basis for $Z \oplus Z$, we can choose positive constants m and M such that for arbitrary $N \in \mathbf{N}$ and arbitrary scalars α_n $(n = -N, -N+1, \cdots, N)$, there holds

$$m \sum_{n=-N}^{N} |\alpha_n|^2 \leq \| \sum_{n=-N}^{N} \alpha_n \eta_n \|^2 + \| \sum_{n=-N}^{N} \alpha_n \zeta_n \|^2 \leq M \sum_{n=-N}^{N} |\alpha_n|^2. \quad (3.1)$$

This implies that for arbitrary $N \in \mathbf{N}$ and arbitrary scalars α_n $(n = 1, \cdots, N)$, there exists a positive constant $M' (= M)$ such that

$$\| \sum_{n=1}^{N} \alpha_n \eta_n \|^2 \leq M' \sum_{n=1}^{N} |\, \alpha_n \,|^2 \,.$$

It follows also from equation (3.1) that for arbitrary $N \in \mathbf{N}$ and arbitrary scalars α_n $(n = 1, \ldots, N)$

$$\begin{aligned} m \sum_{n=1}^{N} |\alpha_n|^2 &\leq \| \sum_{n=1}^{N} \alpha_n \eta_n \|^2 + \| \sum_{n=1}^{N} \alpha_n \zeta_n \|^2 \\ &= \| \sum_{n=1}^{N} \alpha_n \eta_n \|^2 + \| X \left(\sum_{n=1}^{N} \alpha_n \eta_n \right) \|^2 \quad \text{since } X\eta_n = \zeta_n \end{aligned}$$

$$\leq \quad (1 + \|X\|^2) \left(\| \sum_{n=1}^{N} \alpha_n \eta_n \|^2 \right)$$
$$\leq \quad (1 + K^2) \left(\| \sum_{n=1}^{N} \alpha_n \eta_n \|^2 \right) \qquad \text{since } X \text{ is bounded.}$$

Hence there exists a positive constant $m' = \frac{m}{1+K^2}$ such that for arbitrary $N \in \mathbf{N}$ and arbitrary scalars α_n $(n = 1, \cdots, N)$ there holds

$$m' \sum_{n=1}^{N} | \alpha_n |^2 \leq \| \sum_{n=1}^{N} \alpha_n \eta_n \|^2 .$$

So we conclude that $\{\eta_n, n \in \mathbf{N}\}$ is a Riesz basis for Z. ■

4 When is a System Riesz-Spectral?

Recall our assumption that the Hamiltonian H, defined by (2.7), should be a Riesz-spectral operator; i.e. our system should be Riesz-spectral. In this section, we shall give sufficient conditions on the infinitesimal generator A such that the Hamiltonian H is a Riesz-spectral operator. To derive these conditions on A, we need the theory of discrete (spectral) operators developed by Dunford and Schwartz [6]. Note that in many examples it is easier to check that H is a Riesz-spectral operator directly, for instance, if H is normal and has compact resolvent. We start with some definitions and properties of discrete (spectral) operators, which can be found in Dunford and Schwartz [6] and Zwart [32].

Definition 4.1 *A linear operator $T : Z \longrightarrow Z$ is* discrete *if there is a number λ in its resolvent set $\rho(T)$ for which the resolvent operator $R(\lambda, T) = (\lambda I - T)^{-1}$ is compact.*

Lemma 4.2 *If a linear operator $T : Z \longrightarrow Z$ is discrete, then*

1. *the resolvent operator $R(\lambda, T)$ is compact for every $\lambda \in \rho(T)$, and*
2. *the spectrum of $T, \sigma(T)$, consists of only isolated eigenvalues with finite multiplicity.*

Let the discrete operator $T : Z \longrightarrow Z$ have eigenvalues $\{\lambda_n, n \geq 1\}$. The *spectral projection* $E(\lambda_n)$ is defined by

$$E(\lambda_n)z = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - T)^{-1} z \mathrm{d}\lambda, \tag{4.1}$$

where Γ is a simple closed curve surrounding only the eigenvalue λ_n. This is possible, since all the eigenvalues λ_n are isolated. Γ is traversed once in the positive direction (counter-clockwise).

Using the spectral projection we can give the definition of a discrete spectral operator.

Definition 4.3 *A discrete operator $T : Z \longrightarrow Z$ is* spectral *if the spectral projections $E(\lambda_n)$ defined by (4.1) satisfy*

1. *the family of sums of finite collections of projections $E(\lambda_n)$ is uniformly bounded, and*
2. *no non-zero $z \in Z$ satisfies all the equations $E(\lambda_n)z = 0$, $\lambda_n \in \sigma(T)$.*

Remark 4.4 Suppose that T is a discrete Riesz-spectral operator on the Hilbert space Z with simple eigenvalues $\{\lambda_n, n \geq 1\}$ and corresponding eigenvectors $\{\phi_n, n \geq 1\}$. Let $\{\psi_n, n \geq 1\}$ be the eigenvectors of the adjoint of T such that $\langle \phi_n, \psi_n \rangle = \delta_{nm}$. Using Theorem 2.4 part 1 and the Cauchy theorem, it follows that

$$E(\lambda_n)z = \frac{1}{2\pi i} \int_\Gamma (\lambda I - T)^{-1} z \mathrm{d}\lambda = \langle z, \psi_n \rangle \phi_n.$$

Hence we have for arbitrary $N \in \mathbf{N}$

$$\begin{aligned} \| \sum_{n=1}^{N} E(\lambda_n) z \|^2 &= \| \sum_{n=1}^{N} \langle z, \psi_n \rangle \phi_n \|^2 \\ &\leq M \sum_{n=1}^{N} |\langle z, \psi_n \rangle|^2 \qquad \text{since } \{\phi_n\} \text{ is Riesz basis} \\ &\leq M \sum_{n=1}^{\infty} | \langle z, \psi_n \rangle |^2 \\ &\leq \frac{M}{m'} \|z\|^2 \qquad \text{by Lemma 2.2 part 2,} \end{aligned}$$

where m' and M are positive constants.

This gives us that the family of sums of finite collections of projections $E(\lambda_n)$ is uniformly bounded. Hence a discrete Riesz-spectral operator is a discrete spectral operator.

We are now in a position to study the opposite question: is a discrete spectral operator a Riesz-spectral operatorΓ The next lemma shows that this is indeed the case, if all the eigenvalues of the discrete spectral operator are simple. For the proof we refer to Kuiper and Zwart [14] again.

Lemma 4.5 *Suppose that T is a discrete spectral operator on the Hilbert space Z with simple eigenvalues $\{\lambda_n,\, n \geq 1\}$ and corresponding normalized eigenvectors $\{\phi_n,\, n \geq 1\}$. Let $\{\psi_n,\, n \geq 1\}$ be the eigenvectors of the adjoint of T corresponding to the eigenvalues $\{\bar{\lambda}_n,\, n \geq 1\}$. Scale ψ_n such that $\langle \phi_n, \psi_m \rangle = \delta_{nm}$ (see Lemma 2.2). Then*

1. $\sum_{n=1}^{\infty} E(\lambda_n) = I$ *(completeness property),* *where the convergence is in the strong topology.*
2. $$E(\lambda_n)z = \langle z,\, \psi_n \rangle \phi_n, \qquad \textit{and} \tag{4.2}$$
3. $\{\phi_n,\, n \geq 1\}$ *forms a Riesz basis for Z.*

Note that in Lemma 4.5 we have proven that the normalized eigenvectors of an operator form a Riesz basis. Since it is always possible to normalize the eigenvectors, it poses no extra "assumption."

Using Lemma 4.2, Remark 4.4 and Lemma 4.5 we can deduce the following corollary.

Corollary 4.6 *Suppose that T is an operator on a Hilbert space Z with simple eigenvalues $\{\lambda_n,\, n \geq 1\}$ and corresponding normalized eigenvectors $\{\phi_n,\, n \geq 1\}$. Then the following two conditions are equivalent:*

1. *T is a discrete spectral operator, and*
2. *T is a discrete Riesz-spectral operator.*

Now we can formulate the main theorem of this section. In this theorem we shall give sufficient conditions on the infinitesimal generator A such that the Hamiltonian H is a Riesz-spectral operator.

Theorem 4.7 *Let A be the infinitesimal generator of the C_0-semigroup $T(t)$ on the Hilbert space Z and let $\{\mu_n,\, n \in \mathbf{Z}_0\}$ be an enumeration of $\sigma_p(A) \cup -\overline{\sigma_p(A)}$, where the bar denotes the complex conjugate. Define d_n to be the distance from μ_n to $\{\sigma_p(A) \cup -\overline{\sigma_p(A)}\} - \{\mu_n\}$. If*

1. *A is a discrete Riesz-spectral operator,*
2. *for all $\mu_n \in \sigma_p(A)$, except a finite number of n, $-\overline{\mu_n} \notin \sigma_p(A)$,*
3. $\sum_{n=-\infty,\, n\neq 0}^{\infty} \frac{1}{d_n^2} < \infty$, *where the sum is taken over those n for which $d_n \neq 0$, and*
4. *all the eigenvalues of the Hamiltonian $H = \begin{pmatrix} A & -Q_1 \\ -Q_2 & -A^* \end{pmatrix}$ are simple, where $Q_1 = Q_1^* \in \mathcal{L}(Z)$ and $Q_2 = Q_2^* \in \mathcal{L}(Z)$,*

then

1a. $\begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$ *is a discrete spectral operator, and*

1b. $H = \begin{pmatrix} A & -Q_1 \\ -Q_2 & -A^* \end{pmatrix}$ *is a (discrete) Riesz-spectral operator, whose normalized eigenvectors form a Riesz basis.*

Proof:
1a: By Remark 4.4 we have that A is a discrete spectral operator. Using Lemma XIX 5.4 of Dunford and Schwartz [6], we obtain that $-A^*$ is also a discrete spectral operator.

To show that $A_e := \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$ is a discrete spectral operator, we shall first prove that this operator is discrete. Since A and $-A^*$ are discrete,

$$\sigma(A_e) = \sigma(A) \cup \sigma(-A^*) = \sigma_p(A) \cup -\overline{\sigma_p(A)}, \tag{4.3}$$

where the bar denotes the complex conjugate.

Since $\sigma_p(A) \cup -\overline{\sigma_p(A)}$ is countable, this implies that $\rho(A_e) \neq \emptyset$. For $\lambda \in \rho(A_e)$ it is easy to see that

$$(\lambda I - A_e)^{-1} = \begin{pmatrix} (\lambda I - A)^{-1} & 0 \\ 0 & (\lambda I + A^*)^{-1} \end{pmatrix}, \tag{4.4}$$

which imply that $\rho(A_e) = \rho(A) \cap \rho(-A^*)$, and that A_e is a discrete operator. By equation (4.3), $\sigma_p(A_e)$ is enumerated as $\{\mu_n, n \in \mathbf{Z}_0\}$. So, for $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in Z \oplus Z$ the spectral projection corresponding to the eigenvalue μ_n of the operator A_e, becomes

$$\begin{aligned} E(\mu_n)z &= \frac{1}{2\pi i} \int_\Gamma (\lambda I - A_e)^{-1} z \, \mathrm{d}\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma \begin{pmatrix} (\lambda I - A)^{-1} z_1 \\ (\lambda I + A^*)^{-1} z_2 \end{pmatrix} \mathrm{d}\lambda \qquad \text{by (4.4)} \\ &= \begin{pmatrix} E_A(\mu_n) z_1 \\ E_{-A^*}(\mu_n) z_2 \end{pmatrix}. \end{aligned}$$

Note that if μ_n is not an eigenvalue of A, then $(\lambda I - A)^{-1}$ is an analytic function inside and on the contour Γ. So, by the Cauchy's Theorem, $E_A(\mu_n)z_1 = \frac{1}{2\pi i} \int_\Gamma (\lambda I - A)^{-1} z_1 \, \mathrm{d}\lambda = 0$ (an analogous argument holds for $-A^*$). By assumption 2 and equation (4.3) we have that for all $\mu_n \in \sigma_p(A)$, except a finite number of n, $\mu_n \notin \sigma_p(-A^*)$. Hence, except for finite many n, one of the components of $\begin{pmatrix} E_A(\mu_n)z_1 \\ E_{-A^*}(\mu_n)z_2 \end{pmatrix}$ is zero.

Using the above formula for $E(\mu_n)z$ and the fact that A and $-A^*$ are discrete spectral operators, it follows immediately from Definition 4.3 that A_e is a spectral operator too.

1b. Note that the Hamiltonian H is equal to a linear bounded perturbation $\begin{pmatrix} 0 & -Q_1 \\ -Q_2 & 0 \end{pmatrix}$ of the nominal operator $\begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$. By assumption 3 we can apply Theorem XIX 2.7 and Corollary XIX 2.9 of Dunford and Schwartz [6]. This gives that the Hamiltonian H is a discrete spectral operator. Since all the eigenvalues of H are simple, we have by Corollary 4.6 that H is a (discrete) Riesz-spectral operator and that its normalized eigenvectors form a Riesz basis. ■

Theorem 4.7 is still valid if we do not assume that $Q_1 = Q_1^*$ and $Q_2 = Q_2^*$. The same yields for Theorem 4.8. However, in Sections 5, 6 and 7 we need that Q_1 and Q_2 are self-adjoint operators.

Note that for some systems assumption 3 of Theorem 4.7 is not satisfied. Therefore we shall try to weaken this assumption. We know that the Hamiltonian $H = \begin{pmatrix} A & -Q_1 \\ -Q_2 & -A^* \end{pmatrix}$ is equal to the linear bounded perturbation $\begin{pmatrix} 0 & -Q_1 \\ -Q_2 & 0 \end{pmatrix}$ of the operator $\begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$. Hence we have to study the following question: when is a linear bounded perturbation of a discrete spectral operator again discrete and spectralΓ For the general case only a constraint like assumption 3 of Theorem 4.7 is known. However, if we additionally assume that the perturbation is finite-dimensional, i.e. the rank of the perturbation is finite-dimensional, then we can weaken this assumption. Finite-dimensional perturbations are very important, since they come from control problems with finite dimensional input and output spaces, where $Q_1 = BR^{-1}B^*$ and $Q_2 = CC^*$.

For developing an analogue of Theorem 4.7 in case we have a finite-dimensional perturbation we use results from Sun Shun-Hua [24]. We shall only give the main lines of the derivation of this theorem, because the details are rather technical.

For the sake of convenience we introduce the abbreviation

$$Q = \begin{pmatrix} 0 & -Q_1 \\ -Q_2 & 0 \end{pmatrix}, \tag{4.5}$$

where Q_1, $Q_2 \in \mathcal{L}(Z)$. So we assume that this linear bounded perturbation has rank m.

In the work of Sun Shun-Hua [24] a one dimensional perturbation of the form $\langle \cdot, g \rangle b$ with g, $b \in Z$, is considered. By induction, one can show that the results still hold for perturbations of the form

$$\sum_{i=1}^{m} \langle \cdot, g_i \rangle b_i, \tag{4.6}$$

where g_i, $b_i \in Z$ for $i = 1, \ldots, m$.

Since Q has rank m we can write Q it in this form.

Theorem 4.8 *Let A be the infinitesimal generator of the C_0-semigroup $T(t)$ on the Hilbert space Z and let $\{\mu_n, n \in \mathbf{Z}_0\}$ be the enumeration of $\sigma_p(A) \cup -\overline{\sigma_p(A)}$, where the bar denotes the complex conjugate. Assume that $\begin{pmatrix} 0 & -Q_1 \\ -Q_2 & 0 \end{pmatrix}$ has rank m and is written as*

$$\begin{pmatrix} 0 & -Q_1 \\ -Q_2 & 0 \end{pmatrix} = \sum_{i=1}^{m} \langle \cdot, g_i \rangle_{Z \oplus Z} b_i,$$

where g_i, $b_i \in Z \oplus Z$ for $i = 1, \ldots, m$. If

1. *A is a discrete Riesz-spectral operator;*
2. *for all $\mu_n \in \sigma_p(A)$, $-\overline{\mu_n} \notin \sigma_p(A)$;*
3. *all the eigenvalues of*

$$\left(\begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} + \sum_{i=1}^{k} \langle \cdot, g_i \rangle_{Z \oplus Z} b_i \right)$$

   are simple for $k = 1, \ldots, m$;

4. *$\inf_{\forall n \neq j} \mid \mu_n - \mu_j \mid = \delta > 0$ with $n, j \neq 0$; and*
5. *$\sup_{-\infty < n < \infty} \sum_{j=-\infty}^{\infty} \frac{1}{|\mu_j - \mu_n|^2} = \tau < \infty$ with $n, j \neq 0$,*

then the operator $H = \left(\begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} + \sum_{i=1}^{m} \langle \cdot, g_i \rangle b_i \right) = \begin{pmatrix} A & -Q_1 \\ -Q_2 & -A^* \end{pmatrix}$ is a (discrete) Riesz-spectral operator.

Note that assumptions 2 and 3 imply that all the eigenvalues of both H and $\begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$ are simple. In Theorem 4.7, we did not have this restriction.

From the above it is clear that there is still work to be done about linear, bounded perturbations of discrete spectral operators.

It is also possible to remove assumption 4 of Theorem 4.7; i.e. all the eigenvalues of the Hamiltonian H should be simple. However, for this, we need a more general class of (discrete) Riesz-spectral operators. Applying Corollary XIX 2.8 of Dunford and Schwartz [6], it follows from assumptions 1 , 2, and 3 of Theorem 4.7 that all eigenvalues of the Hamiltonian H have finite multiplicity and all, except perhaps a finite number, are simple. Hence H has at most finitely many generalized eigenvectors.

For an operator which has finitely many generalized eigenvectors we can introduce a more general definition of Riesz-spectral operators. Namely, those spectral operators that possesses a Riesz basis of (generalized) eigenvectors, see Dunford and Schwartz [6]. In Zwart and Bontsema [30], a theory for this class of "general" Riesz-spectral operators is set up. They derive analogous formulas for this class, as we derived for our class. However, the proofs of these results are a lot more technical than in our case. Using the theory of Zwart and Bontsema [30], it can be shown that under assumptions 1, 2, and 3 of Theorem 4.7 the Hamiltonian H is a "general" Riesz-spectral operator. Since a "general" Riesz-spectral operator has at most finitely many generalized eigenvectors, we can decompose the Hamiltonian H into an infinite-dimensional part, which is Riesz-spectral in the usual sense, and a finite-dimensional part containing the generalized eigenvectors. Furthermore, the results are true for finite-dimensional systems. From these observations, it may be clear that if we prove the results for our class of Riesz-spectral operators, then they shall also hold for the class of "general Riesz-spectral operators. However, the actual proofs become very technical, and therefore we restrict ourselves in this paper to the case that the Hamiltonian H is a Riesz-spectral operator with simple eigenvalues. Summarizing we can state that all results to be derived in the latter sessions will go through in case the Hamiltonian H is a "general" Riesz-spectral operator.

5 General Form of the Solutions

In Mårtensson [18] and Potter [22] expressions for the solutions of the ARE are given in terms of the eigenvectors of the Hamiltonian H, provided the state space Z is finite-dimensional. In this section, we shall derive similar results in case H is a Riesz-spectral operator on an infinite-dimensional Hilbert space. We even shall give necessary and sufficient conditions such that every ARE has a linear bounded solution. For the proof of the main theorem of this section we need a few lemmas.

The first lemma gives three equivalent conditions in case X is a linear bounded solution of the ARE. Since this result is well-known we don't present a proof.

Lemma 5.1 *Suppose that Assumption 2.5 holds. For $\lambda \in \rho(A)$ the following three conditions are equivalent:*

1. *for all $z_1, z_2 \in \mathrm{D}(A)$*
$$\langle Az_1, Xz_2\rangle + \langle X^* z_1, Az_2\rangle + \langle Q_2 z_1, z_2\rangle - \langle Q_1 X^* z_1, Xz_2\rangle = 0; \tag{5.1}$$

2. *$Im(X\mid_{\mathrm{D}(A)}) \subset \mathrm{D}(A^*)$ and*
$$A^*X + XA + Q_2 - XQ_1X = 0 \text{ on } \mathrm{D}(A); and \tag{5.2}$$

3. $$A^*(\bar{\lambda}I - A^*)^{-1}X(\lambda I - A)^{-1} + (\bar{\lambda}I - A^*)^{-1}\{XA + Q_2 - XQ_1X\}(\lambda I - A)^{-1} = 0. \tag{5.3}$$

Note that equation (5.1) is the ARE as defined in Section 2. We need the next lemma in the proof of Lemma 5.3, Theorem 5.6 part 2, and Section 7.

Lemma 5.2 *Suppose that Assumptions 2.5 and 2.6 hold. If the index set $\mathbf{J} \subset \mathbf{Z}_0$ is such that $\{\eta_n, n \in \mathbf{J}\}$ is a Riesz basis for Z and the linear operator X is defined by*
$$X\eta_n = \zeta_n \text{ for } n \in \mathbf{J},$$
then

1. *$X \in \mathcal{L}(Z)$,*
2. *$\sigma_p(A - Q_1X) = \{\lambda_n, n \in \mathbf{J}\}$,*

and the eigenvector corresponding to λ_n is η_n.

Proof:
1. Since $\{\eta_n, n \in \mathbf{J}\}$ is a Riesz basis for Z, we have that every finite combination of the η_n's is linearly independent. Therefore X is well-defined on $\operatorname{span}_{n\in\mathbf{J}}\{\eta_n\}$. To show that $X \in \mathcal{L}(Z)$, it is sufficient to show that X is a bounded operator on the dense set $\operatorname{span}_{n\in\mathbf{J}}\{\eta_n\}$. Note that the index set $\mathbf{J} \subset \mathbf{Z}_0$ is isomorphic to $\mathbf{N}$. Hence without loss of generality, we may assume that $\mathbf{J} = \mathbf{N}$. For arbitrary N we

$$\begin{aligned}
\|X\sum_{n=1}^{N}\alpha_n\eta_n\|^2 &= \|\sum_{n=1}^{N}\alpha_n\zeta_n\|^2 \le \|\sum_{n=1}^{N}\alpha_n\zeta_n\|^2 + \|\sum_{n=1}^{N}\alpha_n\eta_n\|^2 \\
&= \|\sum_{n=1}^{N}\alpha_n\Phi_n\|^2 \le M\sum_{n=1}^{N}|\alpha_n|^2 \\
&\quad \text{since } \{\Phi_n\} \text{ is a Riesz basis for } Z\oplus Z \\
&\le \frac{M}{c}\|\sum_{n=1}^{N}\alpha_n\eta_n\|^2
\end{aligned}$$

since $\{\eta_n\}$ is a Riesz basis for Z. Using this once more, gives that $X \in \mathcal{L}(Z)$.

2. First we shall show that λ_n is an eigenvalue of $A-Q_1X$ with corresponding eigenvector η_n. By assumption $H\Phi_n = \lambda_n\Phi_n$, which implies that

$$A\eta_n - Q_1\zeta_n = \lambda_n\eta_n.$$

Using the fact that $X\eta_n = \zeta_n$ we get that

$$(A - Q_1X)\eta_n = \lambda_n\eta_n \qquad \text{for all } n \in \mathbf{J}. \tag{5.4}$$

Hence $\{\lambda_n, n \in \mathbf{J}\} \subset \sigma_p(A - Q_1X)$. Now we shall prove that

$$\sigma_p(A - Q_1X) \subset \{\lambda_n, n \in \mathbf{J}\}.$$

Suppose λ is an eigenvalue of $A - Q_1X$ with corresponding eigenvector z; i.e.

$$(A - Q_1X)z = \lambda z. \tag{5.5}$$

We assume now that $(A - Q_1X)$ is invertible; otherwise we can replace $A - Q_1X$ by $(\mu I - (A - Q_1X))$ with $\mu \in \rho(A - Q_1X)$. Since $X \in \mathcal{L}(Z)$, we have that $A-Q_1X$ generates a C_0-semigroup on Z and therefore $\rho(A-Q_1X) \neq \emptyset$. Now (5.5) becomes

$$z = (A - Q_1X)^{-1}\lambda z. \tag{5.6}$$

Let $\{\eta_n, \psi_n\}$ be an biorthogonal sequence. So we have for every $z \in Z$

$$z = \sum_{n\in\mathbf{J}} \langle z, \psi_n\rangle\eta_n.$$

Hence

$$\begin{aligned} (A - Q_1X)^{-1}\lambda z &= \sum_{n\in\mathbf{J}} \lambda\langle z, \psi_n\rangle(A - Q_1X)^{-1}\eta_n \\ &= \sum_{n\in\mathbf{J}} \tfrac{\lambda}{\lambda_n}\langle z, \psi_n\rangle\eta_n \qquad \text{by (5.4).} \end{aligned} \tag{5.7}$$

Note that since $(A - Q_1X)$ is invertible, $\lambda_n \neq 0$ for $n \in \mathbf{J}$. Using (5.6) and (5.7), we have that

$$\langle z, \psi_n\rangle = \langle \lambda(A - Q_1X)^{-1}z, \psi_n\rangle = \frac{\lambda}{\lambda_n}\langle z, \psi_n\rangle \qquad \text{for all } n \in \mathbf{J}.$$

Case 1: $\lambda \neq \lambda_n$ for all $n \in \mathbf{J}$. Therefore $\langle z, \psi_n\rangle = 0$ for all $n \in \mathbf{J}$. Since $\{\psi_n, n \in \mathbf{J}\}$ is a Riesz basis for Z (Lemma 2.2), we must have that $z = 0$, and therefore it is not an eigenvector.

Case 2: there exists an $n_0 \in \mathbf{J}$ such that $\lambda = \lambda_{n_0}$. This implies that for all $n \neq n_0 \in \mathbf{J}$, $\langle z, \psi_n \rangle = 0$. Thus $z = \alpha \eta_{n_0}$ and therefore it is not a new eigenvector. We conclude that

$$\sigma_p(A - Q_1 X) = \{\lambda_n, n \in \mathbf{J}\},$$

and the eigenvector corresponding to λ_n is η_n. ■

We remark that under the conditions of Lemma 5.2 $A - Q_1 X$ is a Riesz-spectral operator with $\mathrm{D}(A - Q_1 X) = \mathrm{D}(A)$.

Lemma 5.3 *Suppose that Assumptions 2.5 and 2.6 hold. If the index set* $\mathbf{J} \subset \mathbf{Z}_0$ *is such that* $\{\eta_n, n \in \mathbf{J}\}$ *is a Riesz basis for* Z *and the operator* X *is defined by*

$$X\eta_n = \zeta_n \text{ for } n \in \mathbf{J},$$

then

$$\overline{\operatorname{span}_{n \in \mathbf{J}}\{\eta_n\}} = \mathrm{D}(A),$$

where the closure is taken in $\mathrm{D}(A)$ *with the inner product*

$$\langle z_1, z_2 \rangle_{\mathrm{D}(A)} = \langle z_1, z_2 \rangle_Z + \langle A z_1, A z_2 \rangle_Z.$$

Proof: Since $\mathbf{J}$ is isomorphic to $\mathbf{N}$ we only have to prove this lemma for $\mathbf{J} = \mathbf{N}$. By Lemma 5.2, we know that the operator $A - Q_1 X$ is Riesz-spectral with eigenvalues λ_n and eigenvectors $\eta_n (n \in \mathbf{N})$, and $\mathrm{D}(A - Q_1 X) = \mathrm{D}(A)$. Let $\{\eta_n, \psi_n\}$ be the biorthogonal sequence. By Theorem 2.4 part 2 we have that

$$(A - Q_1 X)x = \sum_{n=1}^{\infty} \lambda_n \langle x, \psi_n \rangle \eta_n$$

for $x \in \mathrm{D}(A - Q_1 X)$, and

$$\mathrm{D}(A - Q_1 X) = \{x \in Z \mid \sum_{n=1}^{\infty} \mid \lambda_n \mid^2 \mid \langle x, \psi_n \rangle \mid^2 < \infty\}. \tag{5.8}$$

Choose an $x \in \mathrm{D}(A) = \mathrm{D}(A - Q_1 X)$, then it follows from (5.8) that

$$x = \sum_{n=1}^{\infty} \langle x, \psi_n \rangle \eta_n \qquad \text{with } \sum_{n=1}^{\infty} \mid \lambda_n \mid^2 \mid \langle x, \psi_n \rangle \mid^2 < \infty.$$

Thus we can choose $\varepsilon > 0$, N and N' such that

$$\| \sum_{n=N+1}^{\infty} \langle x, \psi_n \rangle \eta_n \|^2 < \frac{\varepsilon}{2}.$$

and

$$\| \sum_{n=N'+1}^{\infty} \lambda_n \langle x, \psi_n \rangle \eta_n \|^2 < \frac{\varepsilon}{2}.$$

If we define $m := \max(N, N')$, then we can deduce

$$\begin{aligned}
&\| \sum_{n=1}^{m} \langle x, \psi_n \rangle \eta_n - x \|^2_{\mathrm{D}(A)} \\
&\quad = \| \sum_{n=1}^{m} \langle x, \psi_n \rangle \eta_n - \sum_{n=1}^{\infty} \langle x, \psi_n \rangle \eta_n \|^2_{\mathrm{D}(A)} \\
&\quad = \| \sum_{n=m+1}^{\infty} \langle x, \psi_n \rangle \eta_n \|^2 + \| A \sum_{n=m+1}^{\infty} \langle x, \psi_n \rangle \eta_n \|^2 \\
&\quad \leq \| \sum_{n=m+1}^{\infty} \langle x, \psi_n \rangle \eta_n \|^2 + 3 \|(A - Q_1 X) \sum_{n=m+1}^{\infty} \langle x, \psi_n \rangle \eta_n \|^2 + \\
&\qquad 3 \| Q_1 X \|^2 \| \sum_{n=m+1}^{\infty} \langle x, \psi_n \rangle \eta_n \|^2 \qquad \text{since } Q_1 X \in \mathcal{L}(Z) \\
&\quad \leq 4 \max(1, \| Q_1 X \|^2) \Bigg[\| \sum_{n=m+1}^{\infty} \langle x, \psi \rangle \eta_n \|^2 + \\
&\qquad \| (A - Q_1 X) \sum_{n=m+1}^{\infty} \langle x, \psi_n \rangle \eta_n \|^2 \Bigg] \\
&\quad = 4 \max(1, \| Q_1 X \|^2) \Bigg[\| \sum_{n=m+1}^{\infty} \langle x, \psi \rangle \eta_n \|^2 + \\
&\qquad \| \sum_{n=m+1}^{\infty} \lambda_n \langle x, \psi_n \rangle \eta_n \|^2 \Bigg] \\
&\quad < 4 \max(1, \| Q_1 X \|^2) \left[\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \right] = 4 \max(1, \| Q_1 X \|^2) \varepsilon.
\end{aligned}$$

Hence for all $x \in \mathrm{D}(A)$ and all $\varepsilon > 0$, there exist m, $\alpha_n(\varepsilon)$ such that

$$\| \sum_{n=1}^{m} \alpha_n(\varepsilon) \eta_n - x \|^2_{\mathrm{D}(A)} < 4 \max(1, \| Q_1 X \|^2) \varepsilon.$$

In other words,

$$\overline{\underset{n \in \mathbf{J}}{\operatorname{span}} \{\eta_n\}} = \mathrm{D}(A).$$

■

The following lemma characterizes all invariant subspaces of the resolvent for Riesz-spectral operators in a simple manner.

Lemma 5.4 *Let T be a Riesz-spectral operator on the Hilbert space Z with the eigenvalues $\{\lambda_n, n \geq 1\}$ and the Riesz basis of eigenvectors $\{\phi_n, n \geq 1\}$. $\rho_\infty(T)$ denotes the (maximal) component of the resolvent set $\rho(T)$ of T which contains an interval $[r, \infty)$. Then we have*

1. *$\rho_\infty(T)$ is the whole resolvent set $\rho(T)$ and*
2. *if V is a closed subspace of Z, which is $(\lambda I - T)^{-1}$-invariant for a λ in $\rho_\infty(T)$, then V is $(\lambda I - T)^{-1}$-invariant for all λ in $\rho_\infty(T)$. Furthermore,*
$$V = \overline{\operatorname*{span}_{n\in\mathbf{J}}\{\phi_n\}},$$
where the subset $\mathbf{J}$ of $\mathbf{N}$ contains those $n \geq 1$ for which $\phi_n \in V$.

Proof: See Lemmas 2.5.6 and 2.5.8 of Curtain and Zwart [5]. ■

The following lemma gives an operator identity, which clarifies the structure of the (point) spectrum of the Hamiltonian H.

Lemma 5.5 *Suppose that Assumption 2.5 holds. Let the Hamiltonian H be given by equation (2.7), i.e.*
$$H = \begin{pmatrix} A & -Q_1 \\ -Q_2 & -A^* \end{pmatrix}$$
and define the operator $\tilde{H} : \mathrm{D}(\tilde{H}) \subset Z \oplus Z \longrightarrow Z \oplus Z$ by
$$\tilde{H} = \begin{pmatrix} A - Q_1 X & -Q_1 \\ 0 & -(A - Q_1 X^*)^* \end{pmatrix}.$$
If $X \in \mathcal{L}(Z)$ is a solution of the ARE, then the following relations hold between H and $\tilde{H}$

1. *on $\mathrm{D}(A) \oplus \mathrm{D}(A^*)$ the following identity holds*
$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}^{-1} \begin{pmatrix} A & -Q_1 \\ -Q_2 & -A^* \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} = \begin{pmatrix} A - Q_1 X & -Q_1 \\ 0 & -(A - Q_1 X^*)^* \end{pmatrix}; \tag{5.9}$$

2. $$\sigma_p(H) = \sigma_p(\tilde{H}) \text{ and } \sigma(H) = \sigma(\tilde{H});$$

3. $$\sigma_p(H) \subset \sigma_p(A - Q_1 X) \cup \sigma_p(-(A - Q_1 X^*)^*),$$
and
$$\sigma_p(A - Q_1 X) \subset \sigma_p(H);$$

4.

$$\sigma(H) \subset \sigma(A - Q_1X) \cup \sigma(-(A - Q_1X^*)^*);$$

and

5. if H is a Riesz-spectral operator, then

$$\sigma(H) = \sigma(A - Q_1X) \cup \sigma(-(A - Q_1X^*)^*).$$

Proof:
1. Recall from Lemma 5.1 that if $X \in \mathcal{L}(Z)$ is a solution of the ARE, then $Im\ (X\mid_{\mathrm{D}(A)}) \subset \mathrm{D}(A^*)$. Hence we have on $\mathrm{D}(A) \oplus \mathrm{D}(A^*)$

$$\begin{aligned} &\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}^{-1} \begin{pmatrix} A & -Q_1 \\ -Q_2 & -A^* \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \\ &\quad = \begin{pmatrix} A - Q_1X & -Q_1 \\ -XA + XQ_1X - Q_2 - A^*X & -(A - Q_1X^*)^* \end{pmatrix} \\ &\quad = \begin{pmatrix} A - Q_1X & -Q_1 \\ 0 & -(A - Q_1X^*)^* \end{pmatrix} \end{aligned}$$

since X is a solution of the ARE.

2. Since $\begin{pmatrix} I & 0 \\ -X & I \end{pmatrix}$ is a bijection, this follows immediately from part 1.

3. Let λ be an eigenvalue of $\tilde{H}$ with corresponding eigenvector $\begin{pmatrix} v \\ w \end{pmatrix}$, so

$$\begin{aligned} &(A - Q_1X)v - Q_1w = \lambda v, \\ &-(A - Q_1X^*)^*w = \lambda w. \end{aligned}$$

Now we have that either $\lambda \in \sigma_p(-(A - Q_1X^*)^*)$ or $\lambda \notin \sigma_p(-(A - Q_1X^*)^*)$. If $\lambda \notin \sigma_p(-(A - Q_1X^*)^*)$, then above equation implies that $w = 0$ and thus $\lambda \in \sigma_p(A - Q_1X)$. By using $\sigma_p(\tilde{H}) = \sigma_p(H)$, we can conclude that

$$\sigma_p(H) \subset \sigma_p(A - Q_1X) \cup \sigma_p(-(A - Q_1X^*)^*).$$

Assume now that λ is an eigenvalue of $A - Q_1X$ with eigenvector v. Then it is easy to see that λ is also an eigenvalue of $\tilde{H}$ with corresponding eigenvector $\begin{pmatrix} v \\ 0 \end{pmatrix}$. So

$$\sigma_p(A - Q_1X) \subset \sigma_p(\tilde{H}) = \sigma_p(H).$$

4. If $\lambda \in \rho(A - Q_1X) \cap \rho(-(A - Q_1X^*)^*)$, then an elementary calculation shows that

$$\begin{pmatrix} [\lambda I - (A - Q_1X)]^{-1} & -[\lambda I - (A - Q_1X)]^{-1} Q_1 [\lambda I + (A - Q_1X^*)^*]^{-1} \\ 0 & [\lambda I + (A - Q_1X^*)^*]^{-1} \end{pmatrix}$$

is a linear bounded operator, and it is the inverse of $(\lambda I - \tilde{H})$. Therefore, $\lambda \in \rho(\tilde{H})$. Combining this with the equality between $\rho(H)$ and $\rho(\tilde{H})$, we obtain that

$$\rho(A - Q_1X) \cap \rho(-(A - Q_1X^*)^*) \subset \rho(H).$$

Or equivalently,

$$\sigma(H) \subset \sigma(A - Q_1X) \cup \sigma(-(A - Q_1X^*)^*).$$

5. Since H is a Riesz-spectral operator, by part 1 and Definition 2.3, $\tilde{H}$ is a Riesz-spectral operator too. Furthermore, since $X \in \mathcal{L}(Z)$, we have that $A - Q_1X$ is the infinitesimal generator of a C_0-semigroup. Using these facts and the Hille-Yosida Theorem, we get that there exists a

$$\tilde{\lambda} \in \rho(\tilde{H}) \cap \rho_\infty(A - Q_1X). \tag{5.10}$$

Let us denote $(\tilde{\lambda} I - \tilde{H})^{-1}$ as $\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$, and so

$$\begin{aligned} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &= \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} \tilde{\lambda} I - (A - Q_1X) & -Q_1 \\ 0 & \tilde{\lambda} I + (A - Q_1X^*)^* \end{pmatrix} \\ &= \begin{pmatrix} P_{11}[\tilde{\lambda} I - (A - Q_1X)] & -P_{11}Q_1 + P_{12}[\tilde{\lambda} I + (A - Q_1X^*)^*] \\ P_{21}[\tilde{\lambda} I - (A - Q_1X)] & -P_{21}Q_1 + P_{22}[\tilde{\lambda} I + (A - Q_1X^*)^*] \end{pmatrix}. \end{aligned} \tag{5.11}$$

This gives that

$$0 = P_{21}[\tilde{\lambda} I - (A - Q_1X)].$$

It follows from equation (5.10) that we can postmultiply the above equation with $[\tilde{\lambda} I - (A - Q_1X)]^{-1}$, and hence

$$0 = P_{21}.$$

Thus we conclude that for $\tilde{\lambda} \in \rho(\tilde{H}) \cap \rho_\infty(A - Q_1X)$, $(\tilde{\lambda} I - \tilde{H})^{-1}$ is of the form $\begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}$. Define the closed linear subspace of $Z \oplus Z$ by $V = \begin{pmatrix} Z \\ 0 \end{pmatrix}$. Clearly, the above formula for $(\tilde{\lambda} - \tilde{H})^{-1}$ implies that $(\tilde{\lambda} I - \tilde{H})^{-1} V \subset V$. Applying Lemma 5.4, part 2 gives that $(\lambda I - \tilde{H})^{-1} V \subset V$

for all $\lambda \in \rho(\tilde{H})$. From the form of V, we conclude that $(\lambda I - \tilde{H})^{-1}$ is of the form $\begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}$ for every $\lambda \in \rho(\tilde{H})$; i.e. $P_{21} = 0$. Substituting $P_{21} = 0$ in equation (5.11), with $\tilde{\lambda}$ replaced by λ, we get that

$$P_{11} = [\lambda I - (A - Q_1 X)]^{-1} \text{ and } P_{22} = [\lambda I + (A - Q_1 X^*)^*]^{-1}$$

for all $\lambda \in \rho(\tilde{H})$. Using the equality between $\rho(H)$ and $\rho(\tilde{H})$, we conclude that

$$\rho(H) \subset \rho(A - Q_1 X) \text{ and } \rho(H) \subset \rho(-(A - Q_1 X^*)^*).$$

Equivalently,

$$\sigma(A - Q_1 X) \subset \sigma(H) \text{ and } \sigma(-(A - Q_1 X^*)^*) \subset \sigma(H).$$

Together with part 4 this gives that

$$\sigma(H) = \sigma(A - Q_1 X) \cup \sigma(-(A - Q_1 X^*)^*).$$

■

Personal communications with George Weiss indicates that the inclusions in parts 3 and 4 are strict in general.

Now we are in a position to state necessary and sufficient conditions such that the ARE has a linear bounded solution. The next theorem will not only show when we have existence of a solution of the ARE, but will also give a method for its construction. In Example 8.1, this will be illustrated.

Theorem 5.6 *Suppose that Assumptions 2.5 and 2.6 hold. Then we have the following results:*

1. *there exists an index set* $\mathbf{J} \subset \mathbf{Z}_0$ *such that* $\{\eta_n, n \in \mathbf{J}\}$ *(see Assumption 2.6) is a Riesz basis for* Z *if and only if there exists a linear bounded solution* X *of the ARE;*

2. *if the index set* $\mathbf{J} \subset \mathbf{Z}_0$ *is such that* $\{\eta_n, n \in \mathbf{J}\}$ *is a Riesz basis for* Z*, then the linear operator* X *defined by*

$$X\eta_n = \zeta_n \qquad \text{for } n \in \mathbf{J} \text{ (see Assumption 2.6)}$$

is an element of $\mathcal{L}(Z)$ *and it is a solution of the ARE; and*

3. *if* $X \in \mathcal{L}(Z)$ *is a solution of the ARE, then the index set* $\mathbf{J} \subset \mathbf{Z}_0$ *defined by*

$$\mathbf{J} = \begin{array}{l} \{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H) \\ \quad \text{and } \lambda_n \text{ is an eigenvalue of } A - Q_1 X\} \end{array} \tag{5.12}$$

is such that $\{\eta_n, n \in \mathbf{J}\}$ *is a Riesz basis for* Z*.*
$\mathbf{J}$ *is also equal to* $\{n \in \mathbf{Z}_0 \mid X\eta_n = \zeta_n\}$*.*

Proof: We shall prove parts 2 and 3, since part 1 has then been proven as well.

2. It follows from Lemma 5.2 part 1 that $X \in \mathcal{L}(Z)$. So it remains to show that X is a solution of the ARE.

For $\lambda \in \rho(A)$, we shall prove that X satisfies equation (5.3). Let λ_n be the eigenvalue of H corresponding to the eigenvector Φ_n, then we have that

$$\begin{aligned} A\eta_n - Q_1\zeta_n &= \lambda_n\eta_n, \\ -Q_2\eta_n - A^*\zeta_n &= \lambda_n\zeta_n. \end{aligned}$$

Using the equality $X\eta_n = \zeta_n$ and subtracting $(\overline{\lambda}I - A^*)^{-1}$ times the second equation from $(\overline{\lambda}I - A^*)^{-1}X$-times the first, gives

$$[(\overline{\lambda}I - A^*)^{-1}\{XA + A^*X + Q_2 - XQ_1X\}]\eta_n = 0 \qquad \text{for all } n \in \mathbf{J}.$$

Since $(\overline{\lambda}I - A^*)^{-1}A^* = A^*(\overline{\lambda}I - A^*)^{-1}$ on $\mathrm{D}(A^*)$, we have that

$$Q\eta_n = 0 \text{ for all } n \in \mathbf{J}, \tag{5.13}$$

where the operator Q is given by

$$Q = A^*(\overline{\lambda}I - A^*)^{-1}X + (\overline{\lambda}I - A^*)^{-1}\{XA + Q_2 - XQ_1X\}$$

with $\mathrm{D}(Q) = \mathrm{D}(A)$.

By Lemma 5.3, we have that $\overline{\operatorname{span}_{n\in\mathbf{J}}\{\eta_n\}} = \mathrm{D}(A)$. Furthermore, Q is a linear, bounded operator from $\mathrm{D}(A)$ to Z. Combining these results with equation (5.13) implies that $Q = 0$, and thus $Q(\lambda I - A)^{-1} = 0$. Substituting the expression for Q, gives

$$A^*(\overline{\lambda}I - A^*)^{-1}X(\lambda I - A)^{-1} + \\ (\overline{\lambda}I - A^*)^{-1}\{XA + Q_2 - XQ_1X\}(\lambda I - A)^{-1} = 0.$$

From Lemma 5.1 it follows that X satisfies ARE.

3. Let $X \in \mathcal{L}(Z)$ be a solution of the ARE. By Assumption 2.6 the Hamiltonian H is a Riesz-spectral operator with eigenvalues $\{\lambda_n, n \in \mathbf{Z}_0\}$ and eigenvectors $\{\Phi_n = \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix}, n \in \mathbf{Z}_0\}$. Let S denote the operator $\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$ on $Z \oplus Z$ and let $\tilde{H}$ denote the operator from Lemma 5.5 defined by

$$\tilde{H} = S^{-1}HS,$$

with $\mathrm{D}(\tilde{H}) = \mathrm{D}(H) = \mathrm{D}(A) \oplus \mathrm{D}(A^*)$, see (5.9). Since S is a bijection, we have that $\tilde{H}$ has also the eigenvalues $\{\lambda_n, n \in \mathbf{Z}_0\}$ and the corresponding eigenvectors are $\{S^{-1}\Phi_n = \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix}\begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix} = \begin{pmatrix} \eta_n \\ -X\eta_n + \zeta_n \end{pmatrix}, n \in \mathbf{Z}_0\}$. Clearly, by Definition 2.3, $\tilde{H}$ is a Riesz-spectral operator too.

From Lemma 5.5 we have that

$$\tilde{H} = \begin{pmatrix} A - Q_1 X & -Q_1 \\ 0 & -(A - Q_1 X^*)^* \end{pmatrix},$$

$$\sigma_p(\tilde{H}) = \sigma_p(A - Q_1 X) \cup \sigma_p(-(A - Q_1 X^*)^*),$$

and

$$\sigma_p(A - Q_1 X) \subset \sigma_p(\tilde{H}).$$

Since all the eigenvalues $\{\lambda_n, n \in \mathbf{Z}_0\}$ are simple, we can renumber them such that $\sigma_p(A - Q_1 X) = \{\lambda_n, n \in \mathbf{Z}^-\}$ and the remaining eigenvalues of $\tilde{H}$ as $\{\lambda_n, n \in \mathbf{Z}^+\}$, where $\mathbf{Z}^- = \{-1, -2, -3, \ldots\}$ and $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$.

Hence the index set $\mathbf{J} \subset \mathbf{Z}_0$, defined by (5.12), is now by definition equal to $\mathbf{Z}^-$.

First we shall prove that

$$\begin{aligned} \mathbf{Z}^- &= \mathbf{J} \\ &:= \{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H) \text{ is an eigenvalue of } A - Q_1 X\} \\ &= \{n \in \mathbf{Z}_0 \mid X\eta_n = \zeta_n\}. \end{aligned} \tag{5.14}$$

Let λ_i be an eigenvalue of $A - Q_1 X$ with eigenvector e_i. Thus there holds

$$(A - Q_1 X)e_i = \lambda_i e_i.$$

Premultiplying with X gives

$$(XA - XQ_1 X)e_i = \lambda_i X e_i.$$

Using the fact that X is a solution of the ARE, we have that

$$(A^* X + XA + Q_2 - XQ_1 X)e_i = 0.$$

Therefore, there holds

$$\begin{pmatrix} A & -Q_1 \\ -Q_2 & -A^* \end{pmatrix} \begin{pmatrix} e_i \\ Xe_i \end{pmatrix} = \lambda_i \begin{pmatrix} e_i \\ Xe_i \end{pmatrix}.$$

Hence $\begin{pmatrix} e_i \\ Xe_i \end{pmatrix}$ is an eigenvector of H, and so it must equal Φ_n for some $n \in \mathbf{Z}_0$. From the definition of η_n and ζ_n, we conclude that $\zeta_n = Xe_i = X\eta_n$. Hence $\mathbf{Z}^- \subset \{n \in \mathbf{Z}_0 \mid X\eta_n = \zeta_n\}$.

Now let us prove the converse inclusion. Assume that $X\eta_i = \zeta_i$ for $i \in \mathbf{Z}_0$. Substituting this in the derived formula for $S^{-1}\Phi_i$ gives, $S^{-1}\Phi_i = \begin{pmatrix} \eta_i \\ 0 \end{pmatrix}$. be Since

$$\tilde{H}\left(S^{-1}\Phi_i\right) = \lambda_i (S^{-1}\Phi_i),$$

this is equivalent to

$$\tilde{H}\begin{pmatrix}\eta_i\\0\end{pmatrix} = \lambda_i\begin{pmatrix}\eta_i\\0\end{pmatrix}.$$

Using the definition of $\tilde{H}$, this implies that

$$(A - Q_1X)\eta_i = \lambda_i\eta_i.$$

Thus λ_i is an eigenvalue of $(A - Q_1X)$ with corresponding eigenvector η_i, i.e. $\{n \in \mathbf{Z}_0 \mid X\eta_n = \zeta_n\} \subset \mathbf{Z}^-$. Hence we have proved equation (5.14).

It remains to prove that $\{\eta_n, n \in \mathbf{Z}^-\}$ is a Riesz basis for Z. Because $\tilde{H}$ is a Riesz-spectral operator, we have that $\{S^{-1}\Phi_n = \begin{pmatrix}\eta_n\\-X\eta_n+\zeta_n\end{pmatrix}, n \in \mathbf{Z}_0\}$ is a Riesz basis for $Z \oplus Z$. Therefore

$$\overline{\operatorname*{span}_{n\in\mathbf{Z}_0}\{\begin{pmatrix}\eta_n\\-X\eta_n+\zeta_n\end{pmatrix}\}} = Z \oplus Z.$$

First we shall show that $\overline{\operatorname*{span}_{n\in\mathbf{Z}^-}\{\eta_n\}} = Z$.

Define $V = \begin{pmatrix}Z\\0\end{pmatrix}$, i.e. a closed linear subspace of $Z \oplus Z$. By the proof of Lemma 5.5 part 5. we have that

$$(\lambda I - \tilde{H})^{-1}V \subset V \qquad \text{for a } \lambda \in \rho_\infty(\tilde{H}).$$

Now applying Lemma 5.4, gives us that

$$V = \begin{pmatrix}Z\\0\end{pmatrix} = \overline{\operatorname*{span}_{n\in\mathbf{K}}\begin{pmatrix}\eta_n\\-X\eta_n+\zeta_n\end{pmatrix}},$$

where $\mathbf{K}$ contains those $n \in \mathbf{Z}_0$ for which $\begin{pmatrix}\eta_n\\-X\eta_n+\zeta_n\end{pmatrix} \in \begin{pmatrix}Z\\0\end{pmatrix}$.
Equivalently, $\mathbf{K}$ contains those $n \in \mathbf{Z}_0$ for which $-X\eta_n + \zeta_n = 0$.
Using equation (5.14), it follows that $\mathbf{K} = \mathbf{Z}^-$. So we can conclude that

$$\begin{pmatrix}Z\\0\end{pmatrix} = \overline{\operatorname*{span}_{n\in\mathbf{Z}^-}\{\begin{pmatrix}\eta_n\\0\end{pmatrix}\}}.$$

Clearly this implies that

$$Z = \overline{\operatorname*{span}_{n\in\mathbf{Z}^-}\{\eta_n\}}.$$

Finally, Theorem 3.3 gives that $\{\eta_n, n \in \mathbf{Z}^-\}$ is a Riesz basis for Z. ■

The above theorem shows that to every linear bounded solution X of the ARE there corresponds exactly one index set $\mathbf{J} \subset \mathbf{Z}_0$ such that $\{\eta_n, n \in \mathbf{J}\}$ is a Riesz basis for Z.

It is also possible to prove the sufficiency of Theorem 5.6 part 1 without defining $\mathbf{J}$ explicitly as we did in equation (5.12), and without using the transformation S for the Hamiltonian H. For such a proof one only needs the theory of invariant subspaces of H and its resolvent operator.

6 Self-Adjoint and Nonnegative Solutions

In the theory of the ARE, one is interested in self-adjoint solutions and especially in nonnegative and stabilizing solutions. These solutions can be characterized in terms of the eigenvectors and eigenvalues of the Hamiltonian H too. We shall give this characterization for self-adjoint solutions in the following theorem and for nonnegative solutions in Theorem 6.2. In Section 7, we shall consider stabilizing solutions.

In the previous section, we gave a construction of linear bounded solutions of the ARE, in the case when Q_1 and Q_2 are self-adjoint operators. For nonnegative solutions we need the extra assumption that Q_1 and Q_2 are nonnegative.

Theorem 6.1 *Suppose that Assumptions 2.5 and 2.6 hold. Let the index set $\mathbf{J} \subset \mathbf{Z}_0$ be such that $\lambda_i \in \sigma_p(H)$ and $\lambda_i \neq -\overline{\lambda}_j$ for all $i, j \in \mathbf{J}$, where the bar denotes the complex conjugate. If $\{\eta_n, n \in \mathbf{J}\}$ is a Riesz basis for Z, then the linear operator X defined by*

$$X\eta_n = \zeta_n \qquad \text{for } n \in \mathbf{J}$$

is a self-adjoint solution of the ARE.

Proof: That X is a solution of the ARE follows from Theorem 5.6 part 2. So it remains to show that X is self-adjoint.

Let J denote the operator on $Z \oplus Z$ defined by

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

It is easy to see that

$$H^*J + JH = 0 \qquad \text{on } \mathrm{D}(H). \tag{6.1}$$

We shall show that $\langle X\eta_n, \eta_m\rangle - \langle \eta_n, X\eta_m\rangle = 0$ for all $n \in \mathbf{J}$. Since $\{\eta_n, n \in \mathbf{J}\}$ is a Riesz basis for Z and $X \in \mathcal{L}(Z)$, we may then conclude that X is self-adjoint.

$$\begin{aligned} &\langle X\eta_n, \eta_m\rangle - \langle \eta_n, X\eta_m\rangle \\ &\quad = \quad \langle \zeta_n, \eta_m\rangle - \langle \eta_n, \zeta_m\rangle \\ &\quad = \quad \langle \begin{pmatrix} \zeta_n \\ -\eta_n \end{pmatrix}, \begin{pmatrix} \eta_m \\ \zeta_m \end{pmatrix}\rangle = \langle J\Phi_n, \Phi_m\rangle \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\lambda_n + \overline{\lambda_m}} [\langle \lambda_n J\Phi_n, \Phi_m \rangle + \langle J\Phi_n, \lambda_m \Phi_m \rangle] \\ &= \frac{1}{\lambda_n + \overline{\lambda_m}} [\langle JH\Phi_n, \Phi_m \rangle + \langle J\Phi_n, H\Phi_m \rangle] \\ &= \frac{1}{\lambda_n + \overline{\lambda_m}} \langle [JH + H^* J]\Phi_n, \Phi_m \rangle = 0 \qquad \text{by (6.1).} \end{aligned}$$

■

Theorem 6.2 *Suppose that Assumptions 2.5 and 2.6 hold. Assume further that Q_1 and Q_2 are nonnegative. Denote by $\mathbf{J}_- := \{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H)$ and $Re(\lambda_n) < 0\}$. If $\{\eta_n, n \in \mathbf{J}_-\}$ is a Riesz basis for Z, then the linear operator X defined by*

$$X\eta_n = \zeta_n \qquad \textit{for } n \in \mathbf{J}_-$$

is a nonnegative self-adjoint solution of the ARE.

Proof: For the same reason as in Section 5, we can without loss of generality, assume that $\mathbf{J}_- = \mathbf{N}$.

It follows from Theorem 6.1 that X is a self-adjoint solution of the ARE, so we only have to prove that X is nonnegative. From Lemma 5.2 and its proof we have that $X \in \mathcal{L}(Z)$ and $A - Q_1X$ is a Riesz-spectral operator on the Hilbert space Z with eigenvalues $\{\lambda_n, n \in \mathbf{N}\}$ and corresponding eigenvectors $\{\eta_n, n \in \mathbf{N}\}$. Since $X \in \mathcal{L}(Z)$, we have that $A - Q_1X$ is an infinitesimal generator of the C_0-semigroup $S(t)$. Using the fact that $X \in \mathcal{L}(Z)$ is a self-adjoint solution of the ARE, we have by equation (2.8) that

$$\langle A\tilde{z}_1, X\tilde{z}_2 \rangle + \langle X\tilde{z}_1, A\tilde{z}_2 \rangle + \langle \tilde{z}_1, Q_2\tilde{z}_2 \rangle - \langle \tilde{z}_1, XQ_1X\tilde{z}_2 \rangle = 0 \tag{6.2}$$

for all $\tilde{z}_1, \tilde{z}_2 \in \mathrm{D}(A)$. We rewrite equation (6.2) as

$$\begin{aligned} &\langle (A - Q_1X)\tilde{z}_1, X\tilde{z}_2 \rangle + \\ &\qquad \langle X\tilde{z}_1, (A - Q_1X)\tilde{z}_2 \rangle = -\langle \tilde{z}_1, (Q_2 + XQ_1X)\tilde{z}_2 \rangle. \end{aligned} \tag{6.3}$$

Let $z \in \mathrm{D}(A)$, then $S(t)z \in \mathrm{D}(A)$ and

$$\begin{aligned} &\frac{d}{dt}\langle S(t)z, XS(t)z \rangle \\ &\quad = \langle \frac{d}{dt}S(t)z, XS(t)z \rangle + \langle S(t)z, \frac{d}{dt}XS(t)z \rangle \\ &\quad = \langle (A - Q_1X)S(t)z, XS(t)z \rangle + \langle S(t)z, X(A - Q_1X)S(t)z \rangle \\ &\quad = -\langle S(t)z, (Q_2 + XQ_1X)S(t)z \rangle \qquad \text{by equation (6.3)} \\ &\quad \leq 0 \qquad \text{since } Q_2 + XQ_1X \geq 0. \end{aligned} \tag{6.4}$$

Integrating (6.4) with respect to t yields

$$\int_0^\tau \frac{d}{dt}\langle S(t)z, XS(t)z\rangle dt \leq 0. \tag{6.5}$$

On the other hand, we have also that

$$\int_0^\tau \frac{d}{dt}\langle S(t)z, XS(t)z\rangle dt = \langle S(\tau)z, XS(\tau)z\rangle - \langle z, Xz\rangle. \tag{6.6}$$

Combining equations (6.5) and (6.6) gives

$$\langle S(\tau)z, XS(\tau)z\rangle - \langle z, Xz\rangle \leq 0 \qquad \text{for all } z \in \mathrm{D}(A). \tag{6.7}$$

Recall that $A - Q_1X$ has eigenvalues λ_n with $\mathrm{Re}(\lambda_n) < 0$. Since $\mathrm{Re}(\lambda_n)$ can converge to zero for $n \longrightarrow \infty$, we cannot conclude that the C_0-semigroup $S(t)$ is exponentially stable. Therefore, we cannot conclude directly that $\langle S(\tau)z, XS(\tau)z\rangle$ converges to zero for $\tau \longrightarrow \infty$ and thus nonnegativity of X. To prove this nonnegativity, we introduce the finite-dimensional subspaces $\tilde{Z}(n)$ of the Hilbert space Z. $\tilde{Z}(n) := \underset{i\in\{1,\ldots,n\}}{\mathrm{span}} \{\eta_i\}$, with inner product $\langle \tilde{z}_1, \tilde{z}_2\rangle_{\tilde{Z}(n)} := \langle \tilde{z}_1, \tilde{z}_2\rangle_Z$, and $\tilde{A}(n) := A - Q_1X \mid_{\tilde{Z}(n)}$. $\tilde{A}(n)$ is a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\mathrm{Re}(\lambda_i) < 0$ for $i = 1, \ldots, n$. So $\tilde{A}(n)$ is asymptotically stable.

Let $\tilde{z}_n \in \tilde{Z}(n)$. Using the fact that $S(t)|_{\tilde{Z}(n)} = e^{\tilde{A}(n)t}$, equation (6.7) gives

$$\langle e^{\tilde{A}(n)\tau}\tilde{z}_n, Xe^{\tilde{A}(n)\tau}\tilde{z}_n\rangle - \langle \tilde{z}_n, X\tilde{z}_n\rangle \leq 0. \tag{6.8}$$

It follows from the asymptotically stability of $\tilde{A}(n)$ that

$$\langle e^{\tilde{A}(n)\tau}\tilde{z}_n, Xe^{\tilde{A}(n)\tau}\tilde{z}_n\rangle \longrightarrow 0 \text{ for } \tau \longrightarrow \infty.$$

Therefore we have by equation (6.8) that $\langle \tilde{z}_n, X\tilde{z}_n\rangle \geq 0$ for all $\tilde{z}_n \in \tilde{Z}(n)$. Since $X \in \mathcal{L}(Z)$ and $\underset{n\in\mathbf{N}}{\cup} \tilde{Z}(n)$ is dense in Z, this inequality extends to all $z \in Z$, i.e.

$$\langle z, Xz\rangle \geq 0 \qquad \text{for all } z \in Z.$$

Equivalently, X is nonnegative. ■

Hence, to obtain a nonnegative self-adjoint solution of the ARE, we have to choose the index set $\mathbf{J} \subset \mathbf{Z}_0$ such that $\mathrm{Re}(\lambda_n) < 0$. In the next theorem, we shall answer the converse question. What kind of solution of the ARE do we need such that for the corresponding index set $\mathbf{J}$, defined in Theorem 5.6 part 3, holds: $\mathrm{Re}(\lambda_n) < 0$, $n \in \mathbf{J}\Gamma$

Theorem 6.3 *Suppose that Assumptions 2.5 and 2.6 hold. Assume further that Q_1 and Q_2 are nonnegative. Denote by $\mathbf{J}_0 := \{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H)$ and $Re(\lambda_n) \neq 0\}$, and let $\mathbf{J} \subset \mathbf{J}_0$. If $\{\eta_n, n \in \mathbf{J}\}$ is a Riesz basis for Z, and if the linear operator X defined by*

$$X\eta_n = \zeta_n \qquad \text{for } n \in \mathbf{J}$$

is self-adjoint and satisfies $\langle \eta_n, X\eta_n \rangle > 0$ for $n \in \mathbf{J}$, then $Re(\lambda_n) < 0$ for $n \in \mathbf{J}$.

Proof: From Theorem 5.6 part 2 we have that X is an element of $\mathcal{L}(Z)$ and it is a solution of the ARE. Since X is self-adjoint it follows that

$$\langle Az_1, Xz_2 \rangle + \langle Xz_1, Az_2 \rangle + \langle z_1, Q_2 z_2 \rangle - \langle z_1, XQ_1Xz_2 \rangle = 0$$

for all $z_1, z_2 \in \mathrm{D}(A)$.

Define the operator Q as $Q := A - Q_1X$ and M as $M := Q_2 + XQ_1X$. Using these definitions, it follows easily that

$$\langle Qz_1, Xz_2 \rangle + \langle Xz_1, Qz_2 \rangle = -\langle z_1, Mz_2 \rangle \tag{6.9}$$

for all $z_1, z_2 \in \mathrm{D}(A)$. By Lemma 5.2,

$$\sigma_p(Q) = \{\lambda_n, n \in \mathbf{J}\},$$

and the eigenvector corresponding to λ_n is η_n. Let $n \in \mathbf{J}$. Substituting $z_1 = z_2 = \eta_n$ in (6.9), we get

$$\begin{aligned} -\langle \eta_n, M\eta_n \rangle &= \langle Q\eta_n, X\eta_n \rangle + \langle X\eta_n, Q\eta_n \rangle \\ &= \lambda_n \langle \eta_n, X\eta_n \rangle + \overline{\lambda_n} \langle X\eta_n, \eta_n \rangle \\ &= 2\mathrm{Re}(\lambda_n) \langle \eta_n, X\eta_n \rangle. \end{aligned}$$

Since $\langle \eta_n, X\eta_n \rangle > 0$ and M is nonnegative, it follows that $\mathrm{Re}(\lambda_n) \leq 0$. From the definition of $\mathbf{J}$ we conclude that $\mathrm{Re}(\lambda_n) < 0$ for $n \in \mathbf{J}$. ■

7 Stabilizing Solutions

In this section, we shall study stabilizing solutions of the ARE. By a stabilizing solution X we mean that $A - Q_1X$ generates an exponentially stable C_0-semigroup.

First of all, we shall show that if there exists a stabilizing solution of the ARE, then it is unique and self-adjoint. Using the fact that for each linear, bounded solution of the ARE there corresponds exactly one index set $\mathbf{J} \subset \mathbf{Z}_0$ such that $\{\eta_n, n \in \mathbf{J}\}$ is a Riesz basis for Z (see Theorem 5.6), we shall give a method for constructing the stabilizing solution. Then, the next step shall be to find necessary and sufficient conditions for the existence of it.

Theorem 7.1 *Suppose that Assumptions 2.5 and 2.6 hold. If $X \in \mathcal{L}(Z)$ is a stabilizing solution of the ARE, then we have that*

1. *X is self-adjoint;*
2. *if Q_1 and Q_2 are nonnegative, then X is self-adjoint and nonnegative; and*
3. *X is unique.*

Proof:
1. Let $X \in \mathcal{L}(Z)$ be a stabilizing solution of the ARE; i.e. $A - Q_1X$ generates an exponentially stable C_0-semigroup on Z. From this it follows that

$$\sigma(A - Q_1X) \subset \mathbf{C}^-, \tag{7.1}$$

where $\mathbf{C}^-$ denotes the open left half plane.

From Theorem 5.6, we know that to the solution X there corresponds exactly one index set $\mathbf{J}_- \subset \mathbf{Z}_0$ such that $\{\eta_n, n \in \mathbf{J}_-\}$ is a Riesz basis for Z. Using part 3 of that theorem we get that

$$\begin{aligned}\mathbf{J}_- &:= \{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H) \text{ is an eigenvalue of } A - Q_1X\} \\ &= \{n \in \mathbf{Z}_0 \mid X\eta_n = \zeta_n\}.\end{aligned}$$

Now let $\lambda_i \in \sigma_p(H)$ be an eigenvalue of $A - Q_1X$. It follows immediately from equation (7.1) that $\text{Re}(\lambda_i) < 0$. Hence $\mathbf{J}_- \subset \{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H)$ and $\text{Re}(\lambda_n) < 0\}$. Applying Theorem 6.1 gives that X is self adjoint.

2. Similar as the proof of part 1, but now apply Theorem 6.2.

3. From part 1 we have that to the stabilizing solution X there corresponds exactly one index set $\mathbf{J}_- \subset \mathbf{Z}_0$ such that $\{\eta_n, n \in \mathbf{J}_-\}$ is a Riesz basis for Z and

$$\mathbf{J}_- := \{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H) \text{ is an eigenvalue of } A - Q_1X\}. \tag{7.2}$$

Now we shall prove that $\mathbf{J}_-$ defined by (7.2) is equal to the set $\{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H)$ and $\text{Re}(\lambda_n) < 0\}$. In part 1 we have already proven that $\mathbf{J}_- \subset \{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H)$ and $\text{Re}(\lambda_n) < 0\}$. We shall prove the converse inclusion by contradiction. Suppose that $\lambda_i \in \sigma_p(H)$ with $\text{Re}(\lambda_i) < 0$ is not an eigenvalue of $A - Q_1X$. By Lemma 5.5 part 3 and the fact that $X = X^*$, we have that

$$\sigma_p(H) \subset \sigma_p(A - Q_1X) \cup \sigma_p(-(A - Q_1X)^*).$$

So λ_i should be an eigenvalue of $-(A - Q_1X)^*$. Since $\sigma(-(A - Q_1X)^*) = \overline{-\sigma(A - Q_1X)}$, where the bar denotes the complex conjugate, we conclude from (7.1) that

$$\sigma(-(A - Q_1X)^*) \subset \mathbf{C}^+,$$

where $\mathbf{C}^+$ denotes the open right half plane.

Hence $\mathrm{Re}(\lambda_i) > 0$, which gives the contradiction. So we conclude that

$$\begin{aligned} \mathbf{J}_- &:= \{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H) \text{ is an eigenvalue of } A - Q_1X\} \\ &= \{n \in \mathbf{Z}_0 \mid X\eta_n = \zeta_n\}, \\ &= \{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H) \text{ and } \mathrm{Re}(\lambda_n) < 0\}, \end{aligned} \tag{7.3}$$

and $\{\eta_n, n \in \mathbf{J}_-\}$ is a Riesz basis for Z.

The last expression for $\mathbf{J}_-$ shows that this index set is the same for every solution X and thus unique. Using the bijection between a solution X of the ARE and the index set $\mathbf{J}_-$, gives the uniqueness of the stabilizing solution X. ■

In the proof of part 3 we derived a method for constructing the unique stabilizing solution.

Corollary 7.2 *Suppose that Assumptions 2.5 and 2.6 hold. Assume further that there exists a stabilizing solution X of the ARE. Then X is given by*

$$X\eta_n = \zeta_n \qquad \textit{for } n \in \mathbf{J}_-,$$

where $\mathbf{J}_- = \{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H)$ and $Re(\lambda_n) < 0\}$. Furthermore, $\{\eta_n, n \in \mathbf{J}_-\}$ is a Riesz basis for Z.

So far, we have proven that if there exists a stabilizing solution of the ARE, then it is self-adjoint, unique and we can construct it. However, we did not give conditions under which such a solution exists.

Much work has be done about the existence of a stabilizing solution of the ARE by using optimal control theory. The corresponding ARE for the optimal control problem is given by

$$\begin{aligned} &\langle Az_1, Xz_2\rangle + \langle X^*z_1, Az_2\rangle + \\ &\qquad \langle C^*Cz_1, z_2\rangle - \langle BR^{-1}B^*X^*z_1, Xz_2\rangle = 0, \end{aligned} \tag{7.4}$$

where B and C are in $\mathcal{L}(U, Z)$ and $\mathcal{L}(Z, Y)$, respectively, with U and Y are Hilbert spaces; see e.g. chapter 6 of Curtain and Zwart [5]. A full analysis of existence and uniqueness of the stabilizing solution for this ARE was first completed by Zabczyk [29]. Recently Louis and Wexler [16] have proven some interesting results.

Our aim is to give necessary and sufficient conditions for the existence and uniqueness of the stabilizing solution of the ARE, (7.4), by using only the Hamiltonian H and thus without using optimal control theory. Note that equation (7.4) corresponds with ARE defined in equation (2.8) by taking $Q_1 = BR^{-1}B^*$ and $Q_2 = C^*C$. It is obvious that these replacements appear in other expressions where Q_1 and Q_2 are involved, e.g. part 1, 2, ..., 4 of Assumption 2.5. However, before we can establish the existence and uniqueness of the stabilizing solution of the ARE, we need a few lemmas. The next lemma of Curtain and Rodman [4] will play an important role in the proof of the necessity part of Theorem 7.6.

Lemma 7.3 *Suppose that Assumption 2.5 holds with $Q_1 = BR^{-1}B^*$ and $Q_2 = C^*C$. If (A, B) is exponentially stabilizable, then the ARE has a unique maximal self-adjoint nonnegative solution in $\mathcal{L}(Z)$.*

The following result shows when the convergence of a sequence of closed operators on a Hilbert space Z implies convergence of their spectrum.

Lemma 7.4 *Let (T_n) be a sequence of closed operators on a Hilbert space Z and T another closed operator on Z. If*

1. *T_n converges uniformly to T for $n \longrightarrow \infty$, i.e. $\| T_n - T \| \longrightarrow 0$ for $n \longrightarrow \infty$,*
2. *$\rho(T) \neq \emptyset$, and*
3. *$\sigma(T)$ is totally disconnected,*

then

$$\lim_{n \longrightarrow \infty} \sigma(T_n) = \sigma(T).$$

Proof: See Theorem 11 of Newburgh [19]. Note that uniform convergence of a sequence of closed operators on a Hilbert space Z implies the convergence in the sense of Newburgh. ■

Finally, we give a relation between an exponentially stable C_0-semigroup and its point spectrum.

Lemma 7.5 *Suppose that A is the generator of a C_0-semigroup on the Hilbert space Z and that $B \in \mathcal{L}(U, Z)$, where U is finite-dimensional. Let (A, B) be exponentially stabilizable and let $F \in \mathcal{L}(Z, U)$. Then the C_0-semigroup generated by $A + BF$ is exponentially stable if and only if $\sigma_p(A + BF) \subset \mathbf{C}^-$.*

Proof: See Theorem A.6 of Logemann and Zwart [15]. ■

Now we are in a position to give necessary and sufficient conditions for the existence and uniqueness of a stabilizing solution of the ARE.

Theorem 7.6 *Suppose that Assumptions 2.5 and 2.6 hold, with $Q_1 = BR^{-1}B^*$ and $Q_2 = C^*C$. Assume further that dim $U < \infty$. Then (A, B) is exponentially stabilizable and $\sigma_p(H) \cap$im. as $= \emptyset$, if and only if there exists a unique stabilizing solution of the ARE in $\mathcal{L}(Z)$.*

Proof:
Necessity Assume that (A, B) is exponentially stabilizable and $\sigma_p(H) \cap$ im. as $= \emptyset$. Applying Lemma 7.3 gives that ARE has a unique maximal self-adjoint nonnegative solution in $\mathcal{L}(Z)$. We shall prove that this is a stabilizing solution. Then it follows from Theorem 7.1 that it is unique.

Let X_{max} denote the unique maximal self-adjoint nonnegative solution of the ARE in $\mathcal{L}(Z)$. We use the construction of X_{max} given in the proof of Lemma 7.3 by Curtain and Rodman [4]. They defined a nonincreasing sequence $\{X_n\}_{n=0}^{\infty}$ of self-adjoint nonnegative operators in $\mathcal{L}(Z)$ such that

1. X_n converges strongly to X_{max} for $n \longrightarrow \infty$ (notation: $X_n \xrightarrow{s} X_{max}$ for $n \longrightarrow \infty$) i.e. $\forall z \in Z \ \parallel X_n z - X_{max} z \parallel \longrightarrow 0$ for $n \longrightarrow \infty$;
2. the operator $A_n := A - BR^{-1}B^*X_n$ generates an exponentially stable C_0-semigroup $T_n(t)$; and
3. the operator $A_{max} := A - BR^{-1}B^*X_{max}$ generates a C_0-semigroup $T_{max}(t)$.

To prove that X_{max} is a stabilizing solution, we shall first prove that

$$\sigma(A_{max}) \subset \overline{\mathbf{C}^-},$$

where $\overline{\mathbf{C}^-}$ denotes the closed left half plane. From part 1 we have that

$$X_{max} - X_n \xrightarrow{s} 0 \text{ for } n \longrightarrow \infty.$$

Since $B \in \mathcal{L}(U, Z)$, this implies that

$$(X_{max} - X_n)B \xrightarrow{s} 0 \text{ for } n \longrightarrow \infty.$$

From the finite-dimensionality of U, it follows that dim(Range($(X_{max} - X_n)B$)) $< \infty$. So the operator $(X_{max} - X_n)B$ maps the finite-dimensional space U into another finite-dimensional space. This implies that strong convergence is equivalent to uniform convergence for the sequence of operators $((X_{max} - X_n)B)$ (see Kato [8]). Hence

$$\parallel (X_{max} - X_n)B \parallel \longrightarrow 0 \text{ for } n \longrightarrow \infty.$$

Since $A_n^* - A_{max}^* = (X_{max} - X_n)BR^{-1}B^*$, the above equation implies that

$$\parallel A_n^* - A_{max}^* \parallel \longrightarrow 0 \text{ for } n \longrightarrow \infty.$$

In Curtain and Zwart [5], it is shown that if A_{max} generates a C_0-semigroup, then A^*_{max} generates a C_0-semigroup as well. Thus $\rho(A^*_{max}) \neq \emptyset$. It follows from Lemma 5.5 part 5. that

$$\sigma(H) = \sigma(A_{max}) \cup -\sigma(A^*_{max}). \tag{7.5}$$

Since H is a Riesz-spectral operator, we know that $\sigma(H)$ is totally disconnected. Combining this with (7.5) gives that $\sigma(A^*_{max})$ is totally disconnected too.

From the above we see that we can apply Lemma 7.4 to the operators A^* and A^*_{max}. Thus we conclude that

$$\lim_{n \longrightarrow \infty} \sigma(A^*_n) = \sigma(A^*_{max}). \tag{7.6}$$

Now,

$$\sigma(A^*_n) = \overline{\sigma(A_n)} \text{ and } \sigma(A^*_{max}) = \overline{\sigma(A_{max})},$$

where the bar denotes the complex conjugate. Thus, by (7.6),

$$\lim_{n \longrightarrow \infty} \sigma(A_n) = \sigma(A_{max}). \tag{7.7}$$

Since A_n generates an exponentially stable C_0-semigroup, we have that $\sigma(A_n) \subset \mathbf{C}^-$. Now it follows from equation (7.7) that $\sigma(A_{max}) \subset \overline{\mathbf{C}^-}$ and thus also $\sigma_p(A_{max}) \subset \overline{\mathbf{C}^-}$, where $\overline{\mathbf{C}^-}$ is the closed left half plane. By Lemma 5.5 part 3 we have that $\sigma_p(A_{max}) \subset \sigma_p(H)$. Using the assumption that $\sigma_p(H) \cap$ im. as $= \emptyset$, we obtain that $\sigma_p(A_{max}) \subset \mathbf{C}^-$. From Lemma 7.5 we conclude that A_{max} generates an exponentially stable C_0-semigroup; i.e. X_{max} is a stabilizing solution of the ARE.

Sufficiency Let $X \in \mathcal{L}(Z)$ be a stabilizing solution of the ARE; i.e. $A - BR^{-1}B^*X$ generates an exponentially stable C_0-semigroup. By Theorem 7.1 part 1 we know that X is self-adjoint. Define the operator F as $F = -R^{-1}B^*X$. Then $F \in \mathcal{L}(Z, U)$ and thus by Definition 2.8 (A, B) is exponentially stabilizable.

By Lemma 5.5 part 3 we have that

$$\begin{aligned}
\sigma_p(H) &\subset \sigma_p(A - BR^{-1}B^*X) \cup \sigma_p(-(A - BR^{-1}B^*X)^*) \\
&\subset \sigma(A - BR^{-1}B^*X) \cup \sigma(-(A - BR^{-1}B^*X)^*) \\
&= \sigma(A - BR^{-1}B^*X) \cup -\overline{\sigma(A - BR^{-1}B^*X)},
\end{aligned}$$

where the bar denotes the complex conjugate. Since X is a stabilizing solution, we have that $\sigma(A - BR^{-1}B^*X) \subset \mathbf{C}^-$ and thus $-\overline{\sigma(A - BR^{-1}B^*X)} \subset \mathbf{C}^+$, where $\mathbf{C}^-$ and $\mathbf{C}^+$ denote the open left-half plane and the open right-half plane, respectively. This implies that $\sigma_p(H) \cap$ im. as $= \emptyset$. ∎

Hence under the Assumptions 2.5 and 2.6 with $Q_1 = BR^{-1}B^*$ and $Q_2 = C^*C$ and dim $U < \infty$, we have that if ARE has a stabilizing solution in $\mathcal{L}(Z)$, then it is the unique maximal self-adjoint nonnegative solution of the ARE.

Remark 7.7 The results of Theorem 7.6 remain valid if H is not Riesz-spectral.

Note that in the sufficiency part of the proof we only use that H is a Riesz-spectral operator for showing that X is self-adjoint. In the necessity part of the proof, we use the fact that H is Riesz-spectral to prove that there exists a stabilizing solution of the ARE in $\mathcal{L}(Z)$. If we remove this assumption, then we do not have that $\sigma(A^*_{max})$ is totally disconnected and therefore we cannot apply Lemma 7.4 anymore. However, we can prove again that the unique maximal self-adjoint nonnegative solution is a stabilizing solution of the ARE in $\mathcal{L}(Z)$. This proof is more technical than the proof of Theorem 7.6 and is therefore omitted. It can be found in the appendix of Kuiper and Zwart [14]. We can also prove the uniqueness of the stabilizing solution in the class of self-adjoint solutions. By showing that this is the maximal self-adjoint solution, we get that the stabilizing solution is unique.

We are also able to prove the uniqueness of the stabilizing solution in the class of self-adjoint solutions of the ARE in $\mathcal{L}(Z)$, in the case when Q_1 and Q_2 are arbitrary self-adjoint operators with Q_1 nonnegative, dimension of U is arbitrary and H is again not necessarily Riesz-spectral.

Remark 7.8 Recall that in the proof of Theorem 7.6, we have shown that the sequence of infinitesimal generators A^*_n converges uniformly to A^*_{max}. With some extra conditions we concluded convergence of the corresponding spectrum (see Lemma 7.4). This convergence cannot be shown if A^*_n converges strongly to A^*_{max}, because in general, one does not have upper semicontinuity of the spectrum under a perturbation of the operator in the strong sense. This is in contrast with the finite-dimensional case. Even if one knows that the C_0-semigroups corresponding to A^*_n converge strongly to the C_0-semigroup corresponding to A^*_{max}, one cannot conclude convergence of the spectrum of A^*_n. An example of this phenomena can be found in Zabczyk [28]. In our case the strong convergence of the corresponding C_0-semigroups can be proven by using the Uniform Boundedness Theorem, the Resolvent Equation and Chapter 3 of Pazy [21].

Recall that we have given, in Corollary 7.2, a method for constructing a stabilizing solution of the ARE, provided that it exists. In Theorem 7.6, we have given necessary and sufficient conditions for the existence of a stabilizing solution in case $Q_1 = BR^{-1}B^*$, $Q_2 = C^*C$ and dim $U < \infty$. Moreover, we can give conditions on the choice of the index set $\mathbf{J} \subset \mathbf{Z}_0$

such that the corresponding solution X of the ARE (see Theorem 5.6 part 2) is a stabilizing one.

Theorem 7.9 *Suppose that Assumptions 2.5 and 2.6 hold. Denote by* $\mathbf{J}_- := \{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H)$ *and* $Re(\lambda_n) < 0\}$*. If* $\{\eta_n, n \in \mathbf{J}_-\}$ *is a Riesz basis for* Z *and* $\sup_{n \in \mathbf{J}_-} Re(\lambda_n) < 0$*, then the operator* X *defined by*

$$X\eta_n = \zeta_n \qquad \textit{for } n \in \mathbf{J}_-$$

is the unique stabilizing solution of the ARE.

Proof: From Lemma 5.2 and its proof, we have that $X \in \mathcal{L}(Z)$ and $A - Q_1 X$ is a Riesz-spectral operator on the Hilbert space Z with eigenvalues $\{\lambda_n, n \in \mathbf{J}_-\}$ and corresponding eigenvectors $\{\eta_n, n \in \mathbf{J}_-\}$. Since $X \in \mathcal{L}(Z)$, we have that $A - Q_1 X$ is an infinitesimal generator of a C_0-semigroup $S(t)$.

By Theorem 2.3.5 of Curtain and Zwart [5], we have that the growth bound ω_0 of the C_0-semigroup $S(t)$ is given by

$$\omega_0 = \sup_{n \in \mathbf{J}_-} \operatorname{Re}(\lambda_n).$$

Hence, using the assumption $\sup_{n \in \mathbf{J}_-} \operatorname{Re}(\lambda_n) < 0$, we have that $A - Q_1 X$ is the infinitesimal generator of an exponentially stable C_0-semigroup $S(t)$; i.e. X is a stabilizing solution of the ARE.

The uniqueness follows immediately from Theorem 7.1. ■

8 An Example: The Heat Equation

In this section, we apply the derived results to the heat equation. This partial differential equation is also considered in Curtain and Pritchard [3] and Curtain and Zwart [5]. They have shown that it can be formulated as a state linear system on an infinite-dimensional Hilbert space Z. For this classical example, the stabilizing solution of the corresponding ARE is well known. However, for calculating this solution, it is assumed to be diagonal. This, however, is hard to see *a priori*. We shall recalculate the stabilizing solution of the ARE without any assumption. We shall even give a characterization of all solutions of the ARE in terms of the eigenvectors of the Hamiltonian H by using Theorem 5.6.

Example 8.1 Consider a metal bar of length one which can be heated along its length according to

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + u(x,t), \qquad z(x,0) = z_0(x),$$

$$\frac{\partial z}{\partial x}(0,t) = 0 = \frac{\partial z}{\partial x}(1,t).$$

$z(x,t)$ represents the temperature at position x at time t, $z_0(x)$ the initial temperature profile and $u(x,t)$ the addition of heat along the bar. We consider the cost functional

$$J(z_0;u) = \int_0^\infty \int_0^1 \mid z(x,t) \mid^2 + \mid u(x,t) \mid^2 \, dx dt.$$

First we reformulate the above partial differential equation as a state linear system and give some basic properties of it. The complete reformulation of this partial differential equation and the proofs of the basic properties can be found in Example 4.9 of Curtain and Pritchard [3] and in Example 6.2.8 of Curtain and Zwart [5].

The heat equation can be reformulated as an abstract differential equation of the form

$$\dot{z}(t) = Az(t) + Bu(t), \qquad t \geq 0, \quad z(0) = z_0, \ (z_0 \in Z)$$

$$y(t) = Cz(t),$$

on an infinite-dimensional state space Z with input u and output y, and corresponding cost functional

$$J(z_0;u) = \int_0^\infty \langle y(s), y(s)\rangle + \langle u(s), Ru(s)\rangle ds,$$

where

$$Z = L_2(0,1), \ U = L_2(0,1), \ Y = L_2(0,1),$$

$$A = \frac{d^2}{dx^2} \text{ with}$$

$$\mathrm{D}(A) = \{h \in L_2(0,1) \mid h, \tfrac{dh}{dx} \text{ are absolutely continuous,}$$

$$\tfrac{d^2h}{dx^2} \in L_2(0,1) \text{ and } \tfrac{dh}{dx}(0) = 0 = \tfrac{dh}{dx}(1)\},$$

$$B = I, \ C = I, \ \text{ and } R = I.$$

A has the eigenvalues $\nu_n = -(n+1)^2\pi^2$, $n = -1, -2, \ldots$, and the corresponding eigenvectors $\varphi_n = \sqrt{2}\cos((n+1)\pi x)$, $n = -2, -3, \ldots, \varphi_{-1} = 1$, form an orthonormal basis for $L_2(0,1)$. A is the infinitesimal generator of a C_0-semigroup $T(t)$, which is Riesz-spectral (see Example 2.3.7 of Curtain and Zwart [5]). Furthermore, the operator A is self-adjoint and has compact resolvent (see Example 2.40 of Curtain and Pritchard [3]).

The Hamiltonian H has the form ($Q_1 = BR^{-1}B^*$ and $Q_2 = C^*C$)

$$H = \begin{pmatrix} A & -I \\ -I & -A \end{pmatrix}, \tag{8.1}$$

and thus $X \in \mathcal{L}(Z)$ is a solution of the ARE if and only if

$$\langle Az_1, Xz_2 \rangle + \langle X^* z_1, Az_2 \rangle + \langle z_1, z_2 \rangle - \langle X^* z_1, Xz_2 \rangle = 0 \tag{8.2}$$

for all $z_1, z_2 \in \mathrm{D}(A)$.

Now we shall check that the Hamiltonian H is a Riesz-spectral operator by using Theorem 4.7. Since the Riesz-spectral operator A has compact resolvent, it follows from Definition 4.1 that it is discrete. So assumption 1 of Theorem 4.7 is satisfied.

Let $\mathbf{Z}^- := \{n \in \mathbf{Z} \mid n \leq 0\}$ and $\mathbf{Z}^+ := \{n \in \mathbf{Z} \mid n \geq 0\}$ and define

$$\begin{array}{lll} \mu_n & = \nu_n & = -(n+1)^2\pi^2 \text{ for } n \in \mathbf{Z}^-, \\ \text{and} & & \\ \mu_n & = -\nu_n & = (n+1)^2\pi^2 \text{ for } n \in \mathbf{Z}^+. \end{array} \tag{8.3}$$

Then, we have that $\{\mu_n, n \in \mathbf{Z}_0\}$ is an enumeration of $\sigma_p(A) \cup -\overline{\sigma_p(A)}$, where the bar denotes the complex conjugate. Hence only for the eigenvalue $\mu_{-1} = 0 \in \sigma_p(A)$ holds that $-\overline{\mu_{-1}} = 0 \in \sigma_p(A)$, and thus assumption 2 of Theorem 4.7 is satisfied.

Define d_n to be the distance from μ_n to $(\sigma_p(A) \cup -\overline{\sigma_p(A)}) - \{\mu_n\}$. We have to prove that

$$\sum_{n=-\infty,\, n\neq 0,}^{\infty} \frac{1}{d_n^2} < \infty, \tag{8.4}$$

where the sum is taken over those n for which $d_n \neq 0$. Clearly $d_{-1} = d_1 = 0$. Using equation (8.3) it follows that

$$\begin{aligned} \sum_{n=-\infty,\, n\neq 0,\, d_n \neq 0}^{\infty} \frac{1}{d_n^2} &= 2\sum_{n=2}^{\infty} \frac{1}{d_n^2} \\ &= 2\sum_{n=2}^{\infty} \frac{1}{\pi^4((n-1)^2 - (n-2)^2)^2} \\ &= 2\sum_{n=2}^{\infty} \frac{1}{\pi^4(2n-3)^2} < \infty. \end{aligned}$$

So assumption 3 of Theorem 4.7 is satisfied.

For checking assumption 4 of Theorem 4.7, we have to know $\sigma_p(H)$. Hence, we first calculate the eigenvalues and corresponding eigenvectors of H, i.e.

$$H \begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \lambda \begin{pmatrix} \eta \\ \zeta \end{pmatrix}.$$

Using formula (8.1) in the above equation, gives

$$\begin{aligned} \zeta &= -(\lambda I - A)\eta, \\ \eta &= -(\lambda I + A)\zeta. \end{aligned} \tag{8.5}$$

Substituting the second equation into the first one gives

$$((\lambda^2 - 1)I - A^2)\zeta = 0.$$

Recall that $\{\varphi_n, n \in \mathbf{Z}^-\}$ forms an orthonormal basis of $L_2(0,1)$. Hence, the above equation is equivalent with

$$\langle ((\lambda^2 - 1)I - A^2)\zeta, \varphi_n \rangle = 0 \qquad \text{for all } n \in \mathbf{Z}^-.$$

Substituting $A\varphi_n = \nu_n \varphi_n$, gives

$$\lambda^2 - 1 - \nu_n^2 = 0 \text{ or } \langle \zeta, \varphi_n \rangle = 0 \qquad \text{for all } n \in \mathbf{Z}^-.$$

So either $\zeta = 0$, or $\lambda = \pm\sqrt{\nu^2 + 1}$ for some n. If $\zeta = 0$, then (8.5) implies that $\eta = 0$ which is in contradiction with the fact that $\begin{pmatrix} \zeta \\ \eta \end{pmatrix}$ is an eigenvector. So we conclude that $\lambda = \pm\sqrt{\nu_n^2 + 1}$, and $\zeta = \varphi_n$ for some $n \in \mathbf{Z}^-$. Hence the Hamiltonian H has the eigenvalues

$$\begin{aligned} \lambda_n &= -\sqrt{\nu_n^2 + 1} = -\sqrt{(n+1)^4\pi^4 + 1}, \qquad n \in \mathbf{Z}^-, \\ \lambda_n &= \sqrt{\nu_{-n}^2 + 1} = \sqrt{(-n+1)^4\pi^4 + 1}, \qquad n \in \mathbf{Z}^+. \end{aligned} \tag{8.6}$$

From (8.5) it is easy to see that $\begin{pmatrix} (-\nu_n + \sqrt{\nu_n^2 + 1})\varphi_n \\ \varphi_n \end{pmatrix}$ is the eigenvector of the Hamiltonian H corresponding to the eigenvalue λ_n for $n \in \mathbf{Z}^-$. Define

$$\begin{aligned} a_n &:= -\nu_n + \sqrt{\nu_n^2 + 1} \\ &= (n+1)^2\pi^2 + \sqrt{(n+1)^4\pi^4 + 1} \qquad \text{for } n \in \mathbf{Z}^-. \end{aligned} \tag{8.7}$$

Then

$$\Phi_n := \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix} = \frac{1}{\sqrt{a_n^2 + 1}} \begin{pmatrix} a_n \varphi_n \\ \varphi_n \end{pmatrix}, \qquad n \in \mathbf{Z}^- \tag{8.8}$$

are the normalized eigenvectors of the Hamiltonian H corresponding to the eigenvalues λ_n, where $\varphi_n = \sqrt{2}\cos((n+1)\pi x)$, $n = -2, -3, \ldots$, and $\varphi_{-1} = 1$.

In a similar way, we can calculate the eigenvectors of the Hamiltonian H for $n \in \mathbf{Z}^+$. For this we define

$$\begin{aligned} a_n &:= -\nu_{-n} - \sqrt{\nu_{-n}^2 + 1} \\ &= (-n+1)^2\pi^2 - \sqrt{(-n+1)^4\pi^4 + 1} \qquad \text{for } n \in \mathbf{Z}^+. \end{aligned} \tag{8.9}$$

Then

$$\Phi_n := \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix} = \frac{1}{\sqrt{a_n^2 + 1}} \begin{pmatrix} a_n \varphi_{-n} \\ \varphi_{-n} \end{pmatrix}, \qquad n \in \mathbf{Z}^+ \tag{8.10}$$

are the normalized eigenvectors of the Hamiltonian H corresponding to the eigenvalues λ_n, where $\varphi_{-n} = \sqrt{2}\cos((-n+1)\pi x)$, $n = 2, 3, \ldots$, and $\varphi_{-1} = 1$.

We conclude from equation (8.6) that all the eigenvalues of the Hamiltonian H are simple; i.e. assumption 4 of Theorem 4.7 is satisfied. Hence the Hamiltonian $H = \begin{pmatrix} A & -I \\ -I & -A \end{pmatrix}$ is a (discrete) Riesz-spectral operator, with Riesz basis $\{\Phi_n,\, n \in \mathbf{Z}_0\}$ (see equations (8.8) and (8.10)). This implies that our state linear system, the heat equation, and the corresponding Hamiltonian H satisfy Assumptions 2.5 and 2.6.

Now we shall give a characterization for all solutions of the ARE. We begin by giving a characterization for the stabilizing solution. Denote

$$\mathbf{J}_- = \{n \in \mathbf{Z}_0 \mid \lambda_n \in \sigma_p(H) \text{ and } \mathrm{Re}(\lambda_n) < 0\}.$$

It follows from equation (8.6) that $\mathbf{J}_-$ is equal to $\mathbf{Z}^-$. We have to check that $\{\eta_n,\, n \in \mathbf{Z}^-\}$ forms a Riesz basis for Z. Using equation (8.7), we see that there exists a positive constant $\tilde{m}$ such that

$$\tilde{m} \le \frac{a_n}{\sqrt{a_n^2 + 1}} \qquad \text{for all } n \in \mathbf{Z}^-. \tag{8.11}$$

Clearly

$$\frac{a_n}{\sqrt{a_n^2 + 1}} \le 1 \qquad \text{for all } n \in \mathbf{Z}^-. \tag{8.12}$$

Since $\{\varphi_n,\, n \in \mathbf{Z}^-\}$ forms an orthonormal basis for $Z = L_2(0,1)$, we have that for arbitrary $N \in \mathbf{N}$ and arbitrary scalars α_n, $n = -N, \ldots, -1$, there holds

$$\begin{aligned} \| \sum_{n=-N}^{-1} \alpha_n \eta_n \|^2 &= \| \sum_{n=-N}^{-1} \frac{a_n}{\sqrt{a_n^2 + 1}} \alpha_n \varphi_n \|^2 \\ &= \sum_{n=-N}^{-1} \frac{a_n^2}{a_n^2 + 1} \mid \alpha_n \mid^2 . \end{aligned}$$

Combining this with equations (8.11) and (8.12) we have that

$$\tilde{m}^2 \sum_{n=-N}^{-1} \mid \alpha_n \mid^2 \leq \| \sum_{n=-N}^{-1} \alpha_n \eta_n \|^2 \leq \sum_{n=-N}^{-1} \mid \alpha_n \mid^2 . \tag{8.13}$$

Hence $\{\eta_n,\, n \in \mathbf{Z}^-\}$ forms a Riesz basis for Z. From equation (8.6), we know that $\sup_{n \in \mathbf{Z}^-} \operatorname{Re}(\lambda_n) = -1 < 0$. Hence, we may apply Theorem 7.9, which gives us that the stabilizing solution X_{stab} of the ARE is given by

$$X_{stab}\eta_n = \zeta_n \qquad \text{for } n \in \mathbf{Z}^-. \tag{8.14}$$

Substituting the formulas of equations (8.7) and (8.8) for η_n and ζ_n in equation (8.14) gives

$$X_{stab} \frac{a_n}{\sqrt{a_n^2+1}} \varphi_n = \frac{1}{\sqrt{a_n^2+1}} \varphi_n \qquad n \in \mathbf{Z}^-,$$

and so with (8.7),

$$X_{stab}\varphi_n = (-(n+1)^2\pi^2 + \sqrt{(n+1)^4\pi^4+1})\varphi_n.$$

In terms of the basis this becomes

$$X_{stab} = \sum_{n=-\infty}^{-1} (-(n+1)^2\pi^2 + \sqrt{(n+1)^4\pi^4+1})\langle \cdot, \varphi_n \rangle \varphi_n. \tag{8.15}$$

It follows from equation (8.15) that X_{stab} is a diagonal solution. Since $-(n+1)^2\pi^2 + \sqrt{(n+1)^4\pi^4+1}$ is positive for all n, X_{stab} is nonnegative. Note that this agrees with Theorem 7.1, from which we can conclude that X_{stab} is self-adjoint, nonnegative and unique.

Theorem 5.6 part 1 gives a bijection between the existence of a linear bounded solution X of the ARE and the existence of an index set $\mathbf{J} \subset \mathbf{Z}_0$ such that $\{\eta_n,\, n \in \mathbf{J}\}$ is a Riesz basis for Z. Hence, answering the question of existence of other linear bounded solutions X of the ARE is equivalent to the question of existence of other index sets $\mathbf{J} \subset \mathbf{Z}_0$ such that $\{\eta_n,\, n \in \mathbf{J}\}$ is a Riesz basis for Z.

Note that it follows from equation (8.7)–(8.10) that Φ_{-n} and Φ_n, $n \in \mathbf{Z}^+$, are both of the form $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \times$ (eigenvector of A), where c_1 and c_2 are constants. This similarity between Φ_n and Φ_{-n}, together with the fact that $\{\eta_n,\, n \in \mathbf{Z}^-\}$ forms a Riesz basis for Z, gives us the suggestion that $\{\eta_n,\, n \in \mathbf{Z}^+\}$ could form a Riesz basis for Z as well. By equations (8.9) and (8.10), we have that

$$\eta_n = \frac{a_n}{\sqrt{a_n^2+1}} \varphi_{-n} \qquad \text{for } n \in \mathbf{Z}^+.$$

The $a_n \longrightarrow 0$ for $n \longrightarrow \infty$. Hence $\|\eta_n\| \longrightarrow 0$ for $n \longrightarrow \infty$. Thus by Definition 2.1 part 2, we have that $\{\eta_n,\, n \in \mathbf{Z}^+\}$ does not form a Riesz basis for Z.

Similarly, if the set $\{\eta_n,\, n \in \mathbf{J} \subset \mathbf{Z}_0\}$ contains infinitely many η_n's with $n \in \mathbf{Z}^+$, we still have that $\|\eta_n\| \longrightarrow 0$ along the $n \in \mathbf{Z}^+$. Therefore, these η_n's do not form a Riesz basis for Z either. Perhaps we can find another index set $\mathbf{J} \subset \mathbf{Z}_0$ such that $\{\eta_n, n \in \mathbf{J}\}$ forms a Riesz basis for Z. Now, we take finitely many η_n's with $n \in \mathbf{Z}^+$, say $\eta_{i1}, \ldots, \eta_{ik}$. However, finitely many vectors $\eta_{i1}, \ldots, \eta_{ik}$ are never enough to form a Riesz basis for Z. Therefore we have to supply these vectors with infinitely many η_n's with $n \in \mathbf{Z}^-$. Since $\eta_{-n} = c_n \eta_n$, the set $\{\eta_n,\, n \in (i_1, \ldots, i_k)$ and $n \in \mathbf{Z}^-\}$ cannot form a Riesz basis for Z. Hence, we take

$$\begin{aligned}\{&\eta_{i1}, \ldots, \eta_{ik} \text{ (finitely many } \eta_n\text{'s with } n \in \mathbf{Z}^+\text{), and} \\ &\eta_n,\, n \in \mathbf{Z}^-,\ \text{except } \eta_{-i1}, \ldots, \eta_{-ik}\},\end{aligned} \tag{8.16}$$

i.e.

$$\mathbf{J} = \{n \in \mathbf{Z}_0 \mid n \in \{i_1, \ldots, i_k\} \text{ and } n \in \mathbf{Z}^- \setminus \{-i_1, \ldots, -i_k\}\}. \tag{8.17}$$

We cannot leave out any η_n's with $n \in \mathbf{Z}^- \setminus \{-i_1, \ldots, -i_k\}$, because then $\overline{\operatorname{span}_{n\in\mathbf{J}}\{\eta_n\}} \neq Z$. Note that $\eta_n = c_n \varphi_n$ and $\eta_{-n} = c_{-n}\varphi_n$ for $n \in \mathbf{Z}^-$. This, together with the fact that $\{\varphi_n,\, n \in \mathbf{Z}^-\}$ forms an orthonormal basis for Z, implies that

$$\overline{\operatorname{span}_{n\in\mathbf{J}}\{\eta_n\}} = Z,$$

where $\mathbf{J}$ is given by equation (8.17).

It follows from equation (8.9) that we can get similar bounds for $\frac{a_n}{\sqrt{a_n^2+1}}$, $n \in \mathbf{Z}^+$, as in (8.11) and (8.12). Thus there exists a positive constant m such that

$$m \leq \frac{a_n}{\sqrt{a_n^2+1}} \leq 1, \qquad \text{for all } n \in \mathbf{J}.$$

Using these equalities, we can, similarly as in equation (8.13), show that $\{\eta_n,\, n \in \mathbf{J}\}$ forms a Riesz basis for Z. Applying Theorem 5.6 part 2 gives that the corresponding linear bounded solution X of the ARE is given by

$$X\eta_n = \zeta_n \text{ for } n \in \mathbf{J}.$$

Using the formulas for η_n and ζ_n as given in equations (8.7)–(8.10), we rewrite the above equation as

$$X\varphi_n = b_n \varphi_n \qquad \text{for } n \in \mathbf{Z}^-,$$

where

$$\begin{array}{ll} b_n = -(n+1)^2\pi^2 - \sqrt{(n+1)^4\pi^4+1} & n \in \{-i_1, \ldots, -i_k\}, \\ b_n = -(n+1)^2\pi^2 + \sqrt{(n+1)^4\pi^4+1} & n \in \mathbf{Z}^- \setminus \{-i_1, \ldots, -i_k\}, \\ \varphi_{-1} = 1, & \\ \varphi_n = \sqrt{2}\cos((n+1)\pi x) & n = -2, -3, \ldots \end{array}$$

Note that these solutions are diagonal and thus self-adjoint, which corresponds with Theorem 6.1. These solutions, except the stabilizing one, are neither nonnegative nor nonpositive.

Varying the set $\{i_1, \ldots, i_k\}$, as defined in equation (8.16), over all finite subsets of $\mathbf{Z}^+$, we get all index sets $\mathbf{J} \subset \mathbf{Z}_0$ such that $\{\eta_n, n \in \mathbf{J}\}$ forms a Riesz basis for Z. Using the bijection between the existence of such an index set $\mathbf{J}$ and the existence of a linear bounded solution X of the ARE, we now have a characterization for all linear bounded solutions X of ARE. Applying Theorem 5.6 part 2 we have that X is given by

$$X\eta_n = \zeta_n \qquad \text{for } n \in \mathbf{J}.$$

Hence we have an explicit expression for all solutions of the ARE, among which there is a stabilizing one. In all we have infinitely many solutions. ∎

9 Conclusions

In this paper we studied the relation between linear bounded solutions X of the ARE and the eigenvectors $\{\Phi_n = \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix}, n \in \mathbf{Z}_0\}$ of the corresponding Hamiltonian. We considered the case that the Hamiltonian is a Riesz-spectral operator on an infinite-dimensional Hilbert space.

We presented some results about Riesz bases and gave sufficient conditions for a system such that the corresponding Hamiltonian is Riesz-spectral. We proved that to every linear bounded solution X of the ARE there corresponds exactly one index set $\mathbf{J} \subset \mathbf{Z}_0$ such that $\{\eta_n, n \in \mathbf{J}\}$ is a Riesz basis for the Hilbert space Z. A method for constructing all solutions has been given as well. Especially, we gave characterizations for self-adjoint, nonnegative and stabilizing solutions. These results are generalizations of the well-known results proved by Mårtensson [18] and Potter [22] for finite-dimensional systems. Finally, we applied the derived results to the heat equation.

As explained at the end of Section 4, we can introduce a more general definition of Riesz-spectral allowing for finitely generalized eigenvectors. Similar as in the finite-dimensional case, all results derived in this paper, except those in Section 4, go through in case the Hamiltonian is such a "general" Riesz-spectral operator. However, as in the finite-dimensional case, the proofs will become rather technical.

It would be interesting to extend the theory of this paper to the case that the Hamiltonian is not necessarily ("general") Riesz-spectral. We have already obtained some partial results [31], but more result need to be done. Furthermore, it is worth looking to the connections between J-spectral factorizations and the solutions of the ARE. In a forthcoming paper, Kuiper [11], we use the results of this article to get an ordering of all self-adjoint solutions of the ARE similar as in the finite-dimensional case (see Willems [26]).

We end with the remark that in Oostveen and Zwart [20] the discrete version of this paper in treated.

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C.R. KUIPER AND H.J. ZWART

University of Twente, Department of Applied Mathematics, P.O. Box 217, 7500 AE Enschede, The Netherlands

University of Twente, Department of Applied Mathematics, P.O. Box 217, 7500 AE Enschede, The Netherlands

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