UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL ATOMIC ENERGY AGENCY
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

SMR: 960/3

WORKSHOP ON NONLINEAR CONTROL AND CONTROL OF CHAOS

(17 - 28 June 1996)

"Controller/Observer Design for (Chaotic) Nonlinear Control Systems"

presented by:

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Linear control

$$\begin{cases} \dot{x} = Ax + Bu , & x(0) = x_0 \\ y = Cx \end{cases}$$

$x = x(t) \in \mathbb{R}^n$ state

$\neq$ parameters

$u = u(t) \in \mathbb{R}^m$ input / control

$y = y(t) \in \mathbb{R}^p$ output / measurement

A : (n,n)-matrix

B : (n,m)-matrix

C : (p,n)-matrix

$x_0 \in \mathbb{R}^n$ initial state

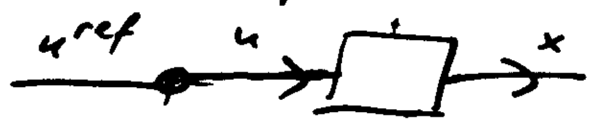
$u \in \mathcal{U}$: admissible controls

pcwise contin.

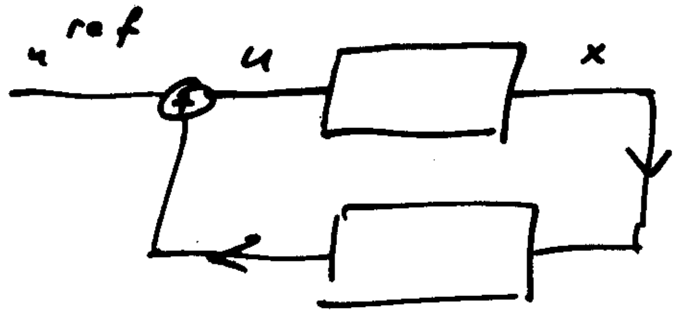
- linear , time-invariant , ...
- 'example'
- discrete-time

Use the control-possibilities as to influence the system in a desirable way.

* Different control objectives possible, e.g.
  - Stabilization
  - tracking
  - optimal control
* What information is used in the control
  - 'nothing' $u = u^{ref}(t)$ open-loop



  - state feedback



  - output feedback

closed loop

STATIC STATE FEEDBACK

$$u = Fx + Gv + u^{ref}$$

F : (m,n)-matrix , G : $(m \times m)$-matrix

v : new control signal

STATIC OUTPUT FEEDBACK

$$u = Ky + (Gv + u^{ref})$$

K: (m,p)-matrix

$$\dot{x} = (A+BF)\,x + BGv + Bu^{ref}$$

or

$$\dot{x} = (A+BKC)\,x + BGv + Bu^{ref}$$

Dynamic state / output feedback

$$\begin{cases} \dot{z} = Pz + Q\overset{y}{x} + Rv, & z \in \mathbb{R}^{\nu} \\ u = Sz + T\overset{y}{x} + Uv + u^{ref} \end{cases}$$

P, Q, R, S, T, U approp. dim.

Closed loop (with output feedback)

$$\begin{pmatrix} x \\ z \end{pmatrix}' = \begin{pmatrix} A+BTC & BS \\ QC & P \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} BU \\ R \end{pmatrix} v + \begin{pmatrix} B \\ 0 \end{pmatrix} u^{ref}$$

Controllability $\quad \dot{x} = Ax + Bu$

Given any $x_0, x_1 \in \mathbb{R}^n$, any $T > 0$

$\exists\, \bar{u} \in \mathcal{U} \quad \bar{u} : [0,T] \longrightarrow \mathbb{R}^m$ s.t.

$$x(T, 0, x_0, \bar{u}) = x_1$$

Thm: $\Sigma(A,B)$ controllable iff

$$\operatorname{rk} \underbrace{[B \,\vdots\, AB \,\vdots\, \cdots \,\vdots\, A^{n-1}B]}_{(n,nm)\text{-matrix}} = n$$

($m = 1$: invertibility condition)

Stabilization at $x = 0$

Is it possible to steer asymptotically ($\hat{=}$ exponentially!) any x_0 to 0 ?

(1) open loop, finite time : controllability

Closed loop

$u = Fx$

$\dot{x} = (A+BF)\,x$ asympt. stable

Sufficient condition : $\Sigma(A,B)$ controllable

STRONGER :

Σ controllable $\iff$ "pole-assignability"

Given any n symmetric (w.r.t. real axis) points in $\mathbb{C}$, say $\lambda_1, \dots, \lambda_n \Rightarrow$
$\exists F$ s.t. $\sigma(A+BF) = \{\lambda_1, \dots\, \lambda_n\}$
↑ real

What if Σ not controllable ?

$$R = \mathrm{Im}(B \,\vdots\, AB \,\vdots\, \dots \,\vdots\, A^{n-1}B)\ ,\quad R \subset \mathbb{R}^n$$

$$AR \subset R \qquad (\text{Cayley-Hamilton})$$

$$\Rightarrow \quad A \cong \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \qquad R = \mathrm{Im}\left\{\begin{pmatrix} * \\ 0 \end{pmatrix}\right\}$$

stabilizability $\iff \sigma(A_{22}) \subset \mathbb{C}^-$

Observability

$$\Sigma \begin{cases} \dot{x} = Ax \,(+Bu) \\ y = Cx \end{cases}, \quad x(0) = x_0$$

Σ observable if given $(u(t))$, $y(t)$, $t \in [0,T]$ can we uniquely determine x_0 (and thus $x(t)$, $t \geq 0$)

$$\mathcal{N} = \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

(np, n)-matrix

$p=1$ (n,n)-matrix

Thm Σ observable $\Leftrightarrow$ $\mathcal{N} = \{0\}$

Asymptotic reconstruction of $x(t)$

observer $\dot{\hat{x}} = A\hat{x} + K(\hat{y} - y)$ $\qquad \hat{y} = C\hat{x}$

$e \triangleq x - \hat{x}$ $\qquad K : (n,p)$-matrix

$\dot{e} = (A - KC)e$ $\qquad \hat{x}(0) = \hat{x}_0$ arbitrary

Thus, if $A-KC$ asympt. stable then $e \to 0 \iff \hat{x}(t) \to x(t)$ (exponentially)

Observability $\Longrightarrow \exists K$ s.t.

$A-KC$ asympt stable

Observability $\iff$ $\forall$ n symm. points in $\mathbb{C}$ $\lambda_1, \dots, \lambda_n$ $\exists K$ s.t.

$\sigma(A-KC) = \{\lambda_1, \dots, \lambda_n\}$

<u>Detectability</u>: If $\mathcal{N} \neq \{0\}$

$A\mathcal{N} \subset \mathcal{N} \longrightarrow$

$$A \cong \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \qquad \mathcal{N} = \mathrm{sp}\left\{\begin{pmatrix} * \\ 0 \end{pmatrix}\right\}$$

$\sigma(A_{22}) \subset \mathbb{C}^- \iff \exists K$ s.t.

$A-KC$ asympt stable

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$$\Sigma \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Desired to obtain dynamic output feedback controller that stabilizes Σ.

<u>Thm</u> Σ stabilizable & detectable

$\Longleftrightarrow$ $\exists$ dynamic output feedback that stabilizes Σ.

F s.t. $u = Fx$ $\rightsquigarrow$ $(A+BF)$ as. stable

$$\Downarrow$$

$$u = F\hat{x}$$

$$\dot{\hat{x}} = A\hat{x} + K(\hat{y} - y) + BF\hat{x}$$

$(A - KC)$ as. stable

$$e = x - \hat{x}$$

$$\begin{cases} \dot{x} = (A+BF)x - BFe \\ \dot{e} = 0 \quad (A-KC)e \end{cases} \quad \begin{pmatrix} A+BF & -BF \\ 0 & A-KC \end{pmatrix}$$

Separation principle

Tracking

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Tracking problem: Design a controller that asymptotically steers

$$y(t) \longrightarrow y_d(t)$$

↑ Given, 'smooth'

Assume $p = 1$ and $m = 1$

(y is 1 dimensional)
(u is 1 dimensional)

$$\begin{cases} \dot{x} = Ax + bu \\ y = cx \end{cases}$$

$y = cx$

$\dot{y} = cAx + cbu$

Either $cb = 0$ or $cb \neq 0$

STOP

$\ddot{y} = cA^2 b + \underbrace{cAb}u$

Either $cAb = 0$ or $cAb \neq 0$

stop

Let $\rho := \min_k cA^{k-1} b \neq 0$ relative degree

order

ρ is minimal number of derivatives of y s.t.

$y^{(\rho)}$ <u>explicitly</u> depends on u

$$y^{(\rho)} = cA^{\rho}x + cA^{\rho-1}b\,u$$

<u>Note</u> Either $\rho \leq n$ or $\rho = \infty$

output independent of u.

A <u>possible</u> tracking controller

$$u = -(cA^{\rho-1}b)^{-1} cA^{\rho}x + (cA^{\rho-1}b)^{-1} v$$

with

$$v = y_d^{(\rho)} - \alpha_{\rho-1} e^{(\rho-1)} - \cdots - \alpha_0 e$$

$$e \triangleq y - y_d$$

$$\Rightarrow e^{(\rho)} + \alpha_{\rho-1} e^{(\rho-1)} + \cdots + \alpha_0 e = 0$$

$$e_1 = e, \quad e_2 = \dot{e}, \quad \dots \quad e_p = e^{(p-1)}$$ (11

$$\dot{e}_1 = e_2$$

$$\vdots$$

$$\dot{e}_{p-1} = e_p$$

$$\dot{e}_p = -\alpha_0 e_1 \cdots -\alpha_{p-1} e_{p-1}$$

$$\begin{pmatrix} e_1 \\ \vdots \\ e_p \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix}}_{\bar{A}} \begin{pmatrix} e_1 \\ \vdots \\ e_p \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{\bar{b}} \begin{pmatrix} -\alpha_0 & \cdots & -\alpha_{p-1} \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_p \end{pmatrix}$$

$(\bar{A}, \bar{b})$ controllable $\Longrightarrow$

$\bar{F} = (-\alpha_0 \; \cdots \; -\alpha_{p-1})$ s.t. $\bar{A} + \bar{b}\bar{F}$ as. stable

Is this a '<u>good</u>' controller ?

Not always

Ex
$$\begin{cases} \dot{x}_1 = -x_2 + u \\ \dot{x}_2 = x_1 + u \end{cases}$$

$$y = x_1$$

$$\left.\begin{aligned} &\dot{y} = -x_2 + u \\ &y_d \end{aligned}\right\} \quad u = x_2 + \dot{y}_d - \alpha e$$

$$\Rightarrow \quad \dot{e} = -\alpha e \quad \rightsquigarrow \quad e \to 0$$

exponentially!

But $x_2(t) \rightsquigarrow$?

$$\left.\begin{aligned} &\\ &\dot{y}_d = 0,\ \alpha = 1 \end{aligned}\right\} \quad \dot{x}_2 = x_2$$

$$x_2 \longrightarrow \infty$$

Although the error-dynamics are (linear) exponentially stable, the unobservable part may become unstable.

This does **not** occur if

$\boxed{\rho = n}$ There is no other dynamical part or if

$\boxed{\rho < n}$ the system is minimum-phase

Let $u = -(cA^{\rho-1}b)^{-1} cA^{\rho} x \quad (v = 0)$

$$\dot{x} = Ax + Bu$$

Assume $y(0) = \dot{y}(0) = \cdots\ y^{(\rho-1)}(0) = 0$

Then $y(t) = 0$ for all t

Σ is minimum-phase if the resulting dynamics (of dimension $n-\rho$ are) are exponentially stable

Example 2nd order system

$$J\ddot{q} + F\dot{q} = u \quad (+\ T(t))$$

$$J > 0,\ F \geq 0$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & -F/J \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/J \end{pmatrix} u$$

$$y = q_1$$

Desired $y = q_{1d}(t)$ $\quad \rho = 2\ !\ (= \dim\ !)$

$$\Rightarrow u = J\ddot{q}_{1d} + Fq_2 - K_2\dot{e} - K_1 e$$

$$e = q_1 - q_{1d}$$

$$J\left(\ddot{e} + K_2/J\,\dot{e} + K_1/J\,e\right) = 0$$

ass. stable iff $K_1 > 0,\ K_2 > 0$

Alternative controller

$$u = \underline{J\ddot{q}_{1d} + F\dot{q}_d} - K_d\dot{e} - K_p e$$

$$\Rightarrow J\ddot{e} + (F + K_d)\dot{e} + K_p e = 0$$

feedforward ($u^{ref}(t)$!)

ass stable iff $K_p > 0,\ K_d > 0$

Including observer

$$J\ddot{q} + F\dot{q} = u$$

$$u = J\ddot{q}_d + F\dot{q}_d - K_d\dot{\hat{e}} - K_p\hat{e}$$

observer
$$\begin{cases} \dot{\hat{e}} = w + 2J^{-1}K_d(e-\hat{e}) - J^{-1}Fe \\ \dot{w} = 2J^{-1}K_p(e-\hat{e}) \end{cases}$$

Closed loop system

$$\begin{cases} J\ddot{e} + (F+K_d)\dot{e} + K_p e = K_d\dot{\tilde{e}} + K_p\tilde{e} \\ J\ddot{\tilde{e}} + K_d\dot{\tilde{e}} + K_p\tilde{e} = -K_d\dot{e} - K_p e \end{cases}$$

as. stable if $0 < K_p < J^{-1}K_d^2$

Many alternatives

Always observable

Nonlinear 2nd order control systems

Robustness / Adaptive Control