Hamiltonian restriction of Vlasov equations to rotating isopycnic and isentropic two-layer equations

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Abstract

A direct link between a Vlasov equation and the equations of motion of a rotating fluid with an effective pressure depending only on a pseudo-density is illustrated. In this direct link, the resulting fluid equations necessarily appear in flux conservative form when there are no topographical and rotational terms. In contrast, multilayer isopycnic and isentropic equations used in atmosphere and ocean dynamics, in the absence of topographical and rotational terms, can not be brought in a conservative flux form, and, hence, can not be derived directly from the Vlasov equations. Another route is explored, therefore, by deriving the Hamiltonian formulation of the two-layer isopycnic and isentropic equations as a restriction from a Hamiltonian formulation of two decoupled Vlasov equations. The work is motivated by our search for energy-preserving or even Hamiltonian (kinetic) numerical schemes.

Author Keywords: Vlasov equations, rotating two-layer equations, Hamiltonian formulation

1 Introduction

It is well-known that there is a link between the kinetic equation for a distribution function and the compressible Euler equations of motion (e.g., [1]). A similar link can be made between the Vlasov or Liouville equation, or collisionless kinetic equation, and a rotating fluid with an effective pressure $P = P(\sigma)$ which is a smooth, increasing function of a pseudo-density σ. (For the non-rotating shallow-water case with $P(\sigma) \propto \sigma^2$, this is essentially done in [2].) This direct link is illustrated here for a $P(\sigma)$–fluid, to clearly contrast it with the Hamiltonian route derived in the next section.

Consider, therefore, the rotating collisionless kinetic or Vlasov equation in two spatial dimensions (see [3] for the non-rotating case with $f = 0$)

$$\partial_t \mathsf{D} + \boldsymbol{\nabla} \cdot (\bar{\zeta}\, \mathsf{D}) - \boldsymbol{\nabla}_\zeta \cdot [(f\, \hat{\mathbf{z}} \times \bar{\zeta} + g\, \boldsymbol{\nabla} h_b)\, \mathsf{D}] = 0 \tag{1.1}$$

with the distribution function $\mathsf{D} = \mathsf{D}(\bar{x}, \bar{\zeta}, t)$ a function of horizontal coordinates $\bar{x} = (x, y)^T = (x_1, x_2)^T$ and transpose ^{T}, velocity coordinates $\bar{\zeta} = (\zeta_1, \zeta_2)^T$, and time t; partial derivative $\partial_t = \partial/\partial t$; a Coriolis parameter f; $\hat{\mathbf{z}}$ the vertical unit vector; potential $g\, h_b(\bar{x})$ with gravitational acceleration g; a spatial gradient operator $\boldsymbol{\nabla}$; and a velocity gradient operator $\boldsymbol{\nabla}_\zeta$.

Next, consider a function $\chi : \mathcal{R}^2 \to \mathcal{R}^+$ with the following symmetry properties

$$\chi(\cdot) \geq 0, \quad \int_{\mathcal{R}^2} \omega_i\, \chi(\bar{\omega}) \mathrm{d}\bar{\omega} = 0, \quad \int_{\mathcal{R}^2} \chi(\bar{\omega})\, \mathrm{d}\bar{\omega} = 1, \quad \int_{\mathcal{R}^2} \omega_i^2 \chi(\bar{\omega})\, \mathrm{d}\bar{\omega} = 1 \tag{1.2}$$

with $\bar{\omega} = (\omega_1, \omega_2)$, $\mathrm{d}\bar{\omega} = \mathrm{d}\omega_1\, \mathrm{d}\omega_2$, and $i = 1$ or $i = 2$. We restrict the distribution function D to be a function of the (Eulerian) fluid velocity $\mathbf{v} = \mathbf{v}(\bar{x}, t)$ and pseudo-density $\sigma = \sigma(\bar{x}, t)$ as follows

$$\mathsf{D} = K(\sigma)\, \chi\left((\bar{\xi} - \mathbf{v})/G(\sigma)\right). \tag{1.3}$$

In (1.3), the functions $K(\sigma)$ and $G(\sigma)$ are chosen such that

$$\sigma = K\,G^2 \qquad \text{and} \qquad P(\sigma) = K\,G^4. \tag{1.4}$$

Define the equations of motion for a $P(\sigma)$–fluid to be

$$\begin{aligned}
\partial_t \sigma + \partial_x(\sigma\, u) + \partial_y(\sigma\, v) &= 0\\
\partial_t(\sigma\, u) + \partial_x(\sigma\, u^2 + P) + \partial_y(\sigma\, u\, v) &= S_2\\
\partial_t(\sigma\, v) + \partial_x(\sigma\, u\, v) + \partial_y(\sigma\, v^2 + P) &= S_3
\end{aligned} \tag{1.5}$$

with pseudo-density σ and horizontal velocity $\mathbf{v} = (u, v)^T$ depending on $\bar{x}$ and time, partial derivatives $\partial_x = \partial/\partial x$ and so forth, and extra topographical and rotational "source" terms S. For example, system (1.5) with σ as the depth of a shallow layer of water over topography $h_b(\bar{x})$ and the choices $S_2 = f\,\sigma\,v - \sigma\,g\,\partial_x h_b$ and $S_3 = -f\,\sigma\,u - \sigma\,g\,\partial_y h_b$ and $P = g\,\sigma^2/2$ equals the rotating shallow water equations (see, e.g., [5]). The following proposition is the modest extension of a result in [2].

Proposition 1.1. *The two-dimensional equations of motion for a $P(\sigma)$–fluid, (1.5) with $S_2 = f\,\sigma\,v - \sigma\,g\,\partial_x h_b$ and $S_3 = -f\,\sigma\,u - \sigma\,g\,\partial_y h_b$, follow from the kinetic equation of motion (1.1) when the distribution function is constrained to (1.3) and (1.4) for a function $\chi(\cdot)$ with properties (1.2).*

Proof. From (1.2) and (1.3), one obtains

$$\sigma = K\,G^2 = \int \mathsf{D}\,\mathrm{d}\bar{\zeta}, \quad \mu_i \equiv \sigma\,u_i = K\,G^2\,u_i = \int \zeta_i\,\mathsf{D}\,\mathrm{d}\bar{\zeta} \quad \text{and} \quad K\,G^4 = \int (\zeta_i - u_i)^2\,\mathsf{D}\,\mathrm{d}\bar{\zeta}. \tag{1.6}$$

Multiply (1.1) by $(1, \bar{\zeta})^T$, and integrate over $\bar{\zeta}$ (which is the usual direct approach) to obtain

$$\int \left[\partial_t \mathsf{D} \begin{pmatrix} 1 \\ \bar{\zeta} \end{pmatrix} + \boldsymbol{\nabla}\cdot(\bar{\zeta}\,\mathsf{D}) \begin{pmatrix} 1 \\ \bar{\zeta} \end{pmatrix} - \boldsymbol{\nabla}_\zeta \cdot [(f\,\hat{\mathbf{z}}\times\bar{\zeta} + g\,\boldsymbol{\nabla} h_b)\,\mathsf{D}] \begin{pmatrix} 1 \\ \bar{\zeta} \end{pmatrix}\right] \mathrm{d}\bar{\zeta} = 0. \tag{1.7}$$

Evaluating the integrals in (1.7) while using definition (1.3), relations (1.6), the symmetry properties of χ, integration by parts (assuming $\mathsf{D} \to 0$ as $|\bar{\zeta}| \to \infty$) and (1.4), we find

$$\begin{aligned}
U &= \int \mathsf{D} \begin{pmatrix} 1 \\ \bar{\zeta} \end{pmatrix} \mathrm{d}\bar{\zeta} = \begin{pmatrix} \sigma \\ \sigma\,\mathbf{v} \end{pmatrix},\\
\boldsymbol{\nabla}\cdot F &= \int \boldsymbol{\nabla}\cdot(\bar{\zeta}\,\mathsf{D}) \begin{pmatrix} 1 \\ \bar{\zeta} \end{pmatrix} \mathrm{d}\bar{\zeta} = \begin{pmatrix} \boldsymbol{\nabla}\cdot(\sigma\,\mathbf{v}) \\ \partial_x(\sigma\,u^2 + P(\sigma)) + \partial_y(\sigma\,u\,v) \\ \partial_x(\sigma\,u\,v) + \partial_y(\sigma\,v^2 + P(\sigma)) \end{pmatrix},\\
S &= \int \boldsymbol{\nabla}_\zeta \cdot [(f\,\hat{\mathbf{z}}\times\bar{\zeta} + g\,\boldsymbol{\nabla} h_b)\,\mathsf{D}] \begin{pmatrix} 1 \\ \bar{\zeta} \end{pmatrix} \mathrm{d}\bar{\zeta} = \begin{pmatrix} 0 \\ f\,\sigma\,v - \sigma\,g\,\partial_x h_b \\ -f\,\sigma\,u - \sigma\,g\,\partial_y h_b \end{pmatrix}.
\end{aligned} \tag{1.8}$$

Combining (1.8) and (1.7) yields the proposition. Finally, we obtain K and G from (1.4). □

We can write (1.5) in the concise form

$$\partial_t U_i + \partial_{x_j} F_{ij} = S_i; \tag{1.9}$$

where $i = 1, 2, 3$; summation over repeated indices $j = 1, 2$ is understood; and U, F and S are identifiable from (1.8). Note that the above direct derivation from a Vlasov equation yields the fluid equations in flux form in the absence of topographical and rotational terms. In contrast, the multilayer isopycnic and isentropic equations used in atmosphere and ocean dynamics can not be brought in the flux form (1.9) even without the topographical and rotational source terms (see section 2). Hence, the direct approach of this section can not be used to derive these multilayer equations. Instead, the link between the Hamiltonian formulation of two Vlasov equations and the two-layer isopycnic and isentropic equations is used. It extends the work in [4] and [9], where both the direct and Hamiltonian approach can be used.

In conclusion, the Hamiltonian derivation presented next will be used as motivation to: (i) derive a spatial discretization of the two-layer equations by using a discretization of the Hamiltonian formulation of the Vlasov equations, and (ii) extend the Hamiltonian route to the general N-layer case and the continuously stratified rotating hydrostatic equations of motion. Preliminary numerical analysis shows that it is possible to derive a finite element or finite difference discretization preserving the anti-symmetry of the Vlasov and hence $P(\sigma)$-fluid equations. Mass and energy can thus be conserved in a spatial discretization.

2 Two-layer Isopycnic and Isentropic Equations

Hydrostatic primitive equations of motion for atmosphere and ocean dynamics can be simplified by dividing the fluid vertically in multiple isopycnic and isentropic layers, *i.e.* layers of constant density and entropy, respectively. In the oceanic case, this simplification to discrete vertical layers is exact (albeit for a special and restricted choice of piecewise constant initial conditions) when the water is considered incompressible and inviscid. In each isopycnic layer the density then has a constant but different value, the layer thickness is variable in the horizontal, and its horizontal velocity is constant across the layer but varies in the horizontal. In the atmospheric case, the simplification is also exact in the inviscid limit. In each isentropic layer the entropy is constant; the pseudo-density, defined by the pressure difference across the layer scaled by gravitational acceleration g, is variable in the horizontal; and the velocity only varies in the horizontal. Here, air is modeled as an ideal gas.

The multilayer continuity and momentum equations are ([5], [6])

$$\begin{aligned} \partial_t\sigma_\alpha + \partial_x(\sigma_\alpha\, u_\alpha) + \partial_y(\sigma_\alpha\, v_\alpha) &= 0 \\ \partial_t(\sigma_\alpha\, u_\alpha) + \partial_x(\sigma_\alpha\, u_\alpha^2) + \partial_y(\sigma_\alpha\, u_\alpha\, v_\alpha) &= -\sigma_\alpha\,\partial_x M_\alpha + S_{2,\alpha} \\ \partial_t(\sigma_\alpha\, v_\alpha) + \partial_x(\sigma_\alpha\, u_\alpha\, v_\alpha) + \partial_y(\sigma_\alpha\, v_\alpha^2) &= -\sigma_\alpha\,\partial_y M_\alpha + S_{3,\alpha} \end{aligned} \tag{2.1}$$

with the pseudo-density σ_α, horizontal velocity $\mathbf{v}_\alpha = (u_\alpha, v_\alpha)$, and momentum $\boldsymbol{\mu}_\alpha = \sigma_\alpha\,\mathbf{v}_\alpha$ in each layer α depending on the horizontal coordinates $\bar{x} = (x, y) = (x_1, x_2)$ and time t, a Montgomery potential M_α, extra terms S_α, and $\alpha = 1, \dots, N$ denoting the layer number. The first layer starts at the top and the N-th layer resides above the spatially variable topography at $z = z_N = h_b(\bar{x})$ with vertical coordinate z. The pseudo-density σ_α denotes the mass density in layer α

$$\sigma_\alpha = \begin{cases} \rho_\alpha\, h_\alpha & \text{isopycnic case} \\ (p_\alpha - p_{\alpha-1})/g & \text{isentropic case} \end{cases} \tag{2.2}$$

with h_α the thickness of the layer and ρ_α the constant density in the isopycnic case, and with at the bottom of layer α the pressure p_α in the isentropic case. At the top the atmospheric pressure $p_a = p_0 \approx 0$ and at the bottom topography $p_N = p_b$. Only the two-layer case with $N = 2$ is considered although the N-layer extension is similar. The Montgomery potential

$$M_1 = \begin{cases} g\,(\sigma_1/\rho_1 + \sigma_2/\rho_2) & \text{isopycnic case} \\ c_p\,\theta_2\,(p_2/p_r)^\kappa + c_p\,(\theta_1 - \theta_2)\,(p_1/p_r)^\kappa & \text{isentropic case} \end{cases} \tag{2.3}$$

$$M_2 = \begin{cases} g\,(\sigma_1/\rho_2 + \sigma_2/\rho_2) & \text{isopycnic case} \\ c_p\,\theta_2\,(p_2/p_r)^\kappa & \text{isentropic case} \end{cases}, \tag{2.4}$$

where θ_α is the potential temperature of layer α, $\kappa = R/c_p$, the gas constant $R = c_p - c_v$ with c_p and c_v the specific heat at constant pressure and volume. When only topography is included and no other source terms, we can derive

$$S_{i+1,1} = -\,f\,\sigma_1\,(\hat{z}\times\mathbf{v}_1)_i - \sigma_1\,\partial_{x_i}\,g\,h_b \quad \text{and} \quad S_{i+1,2} = -f\,\sigma_2\,(\hat{z}\times\mathbf{v}_2)_i - \sigma_2\,g\,\partial_{x_i}h_b, \tag{2.5}$$

where $h_b + h_1 + h_2$ is the top of the first isentropic layer, $\partial_{xi} = \partial/\partial x_i$ and with $i = 1, 2$.

It is important to note that, in contrast to the one-layer cases and the $P(\sigma)$-fluid described in the previous section, the multilayer system (2.1) without source terms $S_{2,\alpha}, S_{3,\alpha}$ *can not* be brought in conservative form (1.9) since $\sigma_\alpha \boldsymbol{\nabla} M_\alpha \neq \boldsymbol{\nabla} \tilde{P_\alpha}$ for some effective pressure $\tilde{P_\alpha}$.

The energy density of the two-layer isopycnic equations emerges by integration in the vertical or z-direction over the two layers each with a different but constant density ρ_1 and ρ_2 going down

$$
\begin{aligned}
H_{\text{isopycnic}} &= \int_{h_b}^{h_b+h_2} \left(\frac{1}{2}\,\rho_2\,|\mathbf{v}_2|^2 + \rho_2\,g\,z\right) \mathrm{d}z + \int_{h_b+h_2}^{h_b+h_2+h_1} \left(\frac{1}{2}\,\rho_1\,|\mathbf{v}_1|^2 + \rho_1\,g\,z\right) \mathrm{d}z \\
&= \frac{1}{2}\,\sigma_1|\mathbf{v}_1|^2 + \frac{1}{2}\,g\,\sigma_1^2/\rho_1 + g\,\sigma_1\,\sigma_2/\rho_2 + \frac{1}{2}\,\sigma_2|\mathbf{v}_2|^2 + \frac{1}{2}\,g\,\sigma_2^2/\rho_2 + g\,(\sigma_1+\sigma_2)\,h_b. \qquad (2.6)
\end{aligned}
$$

The energy density of the two-layer isentropic equations emerges by integration in the vertical direction over the two isentropic layers each with a different but constant entropy or potential temperature s_1 or θ_1 and s_2 or θ_2 going down. The potential temperature θ is related to the temperature T as follows $T = \theta\,(p/p_r)^\kappa$ ([5]) with a constant reference pressure p_r. We also use the ideal gas law $p = \rho\,R\,T$, the internal energy $U(s,\rho) = c_v\,T$ for this ideal gas, and hydrostatic balance $\partial p/\partial z = -\rho\,g$. The resulting energy density is ([5], [6])

$$
\begin{aligned}
H_{\text{isentropic}} &= \int_{h_b}^{h_b+h_2+h_1} \left(\frac{1}{2}\,\rho\,|\mathbf{v}|^2 + \rho\,g\,z + \rho\,U(s,\rho)\right) \mathrm{d}z \\
&= \int_{h_b}^{h_b+h_2+h_1} \left(\frac{1}{2}\,(-\partial_z p/g)\,|\mathbf{v}|^2 - \partial_z(p\,z) - \frac{c_p\,\theta}{g}\,(p/p_r)^\kappa\,\partial_z p\right) \mathrm{d}z \\
&= \frac{1}{2}\,\sigma_1|\mathbf{v}_1|^2 + \frac{c_p\,\theta_1}{g}\,\frac{p_r}{\kappa+1}\,\left((p_1/p_r)^{\kappa+1} - (p_a/p_r)^{\kappa+1}\right) + \\
&\quad \frac{1}{2}\,\sigma_2|\mathbf{v}_2|^2 + \frac{c_p\,\theta_2}{g}\,\frac{p_r}{\kappa+1}\,\left((p_2/p_r)^{\kappa+1} - (p_1/p_r)^{\kappa+1}\right) + g\,(\sigma_1+\sigma_2)\,h_b \qquad (2.7)
\end{aligned}
$$

with pseudo-densities $\sigma_1 = (p_1 - p_a)/g$ and $\sigma_2 = (p_2 - p_1)/g$. The variable pressure at the topography $z = h_b$ is p_2, on the interface at $z = h_b + h_2$ it is p_1, with a small or zero but constant pressure p_a on the top surface at $z = h_b + h_1 + h_2$ so we can neglect the terms $p_a\,(h_1 + h_2)$.

2.1 Two-layer kinetic theory

Consider a Hamiltonian formulation of the collisionless kinetic or Vlasov equations in two spatial dimensions in the layers denoted by $\alpha = 1 \dots N$ from top to bottom

$$
\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\}. \qquad (2.8)
$$

Here, the generalized Poisson bracket is

$$
\{\mathcal{F}, \mathcal{G}\} = \sum_{\alpha=1}^{N} \int\!\!\int \mathsf{D}_\alpha \left[\partial_{x_i}\left(\frac{\delta\mathcal{F}}{\delta\mathsf{D}_\alpha}\right)\partial_{\zeta_i'}\left(\frac{\delta\mathcal{G}}{\delta\mathsf{D}_\alpha}\right) - \partial_{x_i}\left(\frac{\delta\mathcal{G}}{\delta\mathsf{D}_\alpha}\right)\partial_{\zeta_i'}\left(\frac{\delta\mathcal{F}}{\delta\mathsf{D}_\alpha}\right)\right] \mathrm{d}\bar{x}\,\mathrm{d}\bar{\zeta}' \qquad (2.9)
$$

for arbitrary functionals $\mathcal{F} = \mathcal{F}$ and $\mathcal{G}$ of D_α, coordinates $\bar{x} = (x_1, x_2)$, modified velocity coordinates $\bar{\zeta}' = (\zeta_1', \zeta_2')^T = (\zeta_1 - f\,y/2, \zeta_2 + f\,x/2)^T$ and $i = 1, 2$ in two dimensions. The Hamiltonian is

$$
\mathcal{H}[\mathsf{D}_\alpha] = \sum_{\alpha=1}^{N} \int\!\!\int \mathsf{D}_\alpha \left(\frac{1}{2}\,|\bar{\zeta}|^2 + \phi_\alpha(\bar{x})\right) \mathrm{d}\bar{x}\,\mathrm{d}\bar{\zeta}' = \sum_{\alpha=1}^{N} \int\!\!\int \mathsf{D}_\alpha \left(\frac{1}{2}\,|\bar{\zeta}|^2 + \phi_\alpha(\bar{x})\right) \mathrm{d}\bar{x}\,\mathrm{d}\bar{\zeta} \qquad (2.10)
$$

with potentials $\phi_\alpha(\bar{x})$. Note that the distribution function $\mathsf{D}_\alpha = \mathsf{D}_\alpha(\bar{x}, \bar{\zeta}', t)$ depends on the spatial coordinates $\bar{x}$ and modified velocity coordinates $\bar{\zeta}'$, and time t. The bracket (2.9) is Hamiltonian because it is skew-symmetric $\{\mathcal{F}, \mathcal{G}\} = -\{\mathcal{G}, \mathcal{F}\}$, and satisfies Jacobi's identity $\{\mathcal{F}, \{\mathcal{G}, \mathcal{K}\}\} +$

$\{\mathcal{G},\{\mathcal{K},\mathcal{F}\}\}+\{\mathcal{K},\{\mathcal{F},\mathcal{G}\}\}=0$ (see [3] for the case $\alpha=1$ and $f=0$; for general background see, e.g., [5] and [7]). Since the layers are decoupled in the bracket, the one-layer Hamiltonian structure carries over to the multilayer case.

The equations of motion emerging from the Hamiltonian formulation (2.8)–(2.10) are

$$\partial_t \mathsf{D}_\alpha + \boldsymbol{\nabla}\cdot(\bar{\zeta}\,\mathsf{D}_\alpha) - \boldsymbol{\nabla}_{\zeta'}\cdot[(f\,\hat{\mathbf{z}}\times\bar{\zeta}/2+\boldsymbol{\nabla}\phi_\alpha)\,\mathsf{D}_\alpha]=0 \tag{2.11}$$

with spatial gradient operator $\boldsymbol{\nabla}$ and velocity gradient operator $\boldsymbol{\nabla}_{\zeta'}$. Note that these equations, the bracket and the Hamiltonian are all linear in D_α, and are brought in conservation form. Space-velocity integrals of any (smooth) function of D_α and the Hamiltonian $\mathcal{H}$ are conserved quantities (in time), which follows either by direct manipulation from (2.11), or by using the Hamiltonian formulation, i.e., $\mathrm{d}\mathcal{H}/\mathrm{d}t=\{\mathcal{H},\mathcal{H}\}=0$ by skew-symmetry of the bracket.

We constrain or restrict the distribution function D_α to functions of the form

$$\mathsf{D}_\alpha = K_\alpha(\sigma_\alpha)\,\chi\left(|\bar{\xi}'-\mathbf{v}'_\alpha|/G_\alpha(\sigma_\beta)\right) = K_\alpha(\sigma_\alpha)\,\chi\left(|\bar{\xi}-\mathbf{v}_\alpha|/G_\alpha(\sigma_\beta)\right) \tag{2.12}$$

with modified velocity $\mathbf{v}'_\alpha=\mathbf{v}'_\alpha(\bar{x},t)=\mathbf{v}_\alpha+f\,\hat{\mathbf{z}}\times\bar{x}/2$ and Eulerian fluid velocity $\mathbf{v}_\alpha=\mathbf{v}_\alpha(\bar{x},t)$ and $\sigma_\alpha=\sigma_\alpha(\bar{x},t)$ for all $\beta=1,\ldots,N$. Define the pseudo-density

$$\sigma_\alpha(\bar{x},t)=\int_{\mathcal{R}^2} K_\alpha(\sigma_\alpha)\,\chi\left(\frac{|\bar{\xi}'-\mathbf{v}'_\alpha|}{G_\alpha(\sigma_\beta)}\right)\mathrm{d}\bar{\xi}=\int_{\mathcal{R}^2} K_\alpha(\sigma_\alpha)\,\chi\left(\frac{|\bar{\xi}-\mathbf{v}_\alpha|}{G_\alpha(\sigma_\beta)}\right)\mathrm{d}\bar{\xi}. \tag{2.13}$$

From properties (1.2) and definition (2.13) we find that $\sigma_\alpha=K_\alpha\,G_\alpha^2$. The functions K_α and G_α are determined to yield the relevant shallow-layer equations. For the two-layer case, note that $K_\alpha=K_\alpha(\sigma_1,\sigma_2)$ and $G_\alpha=G_\alpha(\sigma_1,\sigma_2)$.

Proposition 2.1. *There exist functions K_α and G_α defining D_α in (2.12) using the function $\chi(\cdot)$ with properties (1.2) such that the Hamiltonian formulation (2.8), (2.17)–(2.21) of the two-layer isopycnic and isentropic equations follows from the Hamiltonian formulation (2.8)–(2.10) of the two-layer kinetic equations when the distribution functions are constrained as in (2.12)–(2.13).*

Proof. From (1.2), (2.12) and (2.13), one obtains with $\alpha=1,\ldots,N$ and $N=2$

$$\sigma_\alpha=K_\alpha\,G_\alpha^2=\int \mathsf{D}_\alpha\,\mathrm{d}\bar{\zeta}' \quad\text{and}\quad \mu'_{i,\alpha}\equiv\sigma_\alpha\,u'_{i,\alpha}=K_\alpha\,G_\alpha^2\,u'_{i,\alpha}=\int \zeta_i\,\mathsf{D}_\alpha\,\mathrm{d}\bar{\zeta}'. \tag{2.14}$$

The functional derivatives between the variables $\{\sigma_\alpha,\mathbf{v}'_\alpha\}$ and D_α are related by using (2.14)

$$\begin{aligned}
\delta\mathcal{F} &= \int\left(\frac{\delta\mathcal{F}}{\delta\sigma_\alpha(\bar{x},t)}\,\delta\sigma_\alpha(\bar{x},t)+\frac{\delta\mathcal{F}}{\delta\mu'_{i,\alpha}(\bar{x},t)}\,\delta\mu'_{i,\alpha}(\bar{x},t)\right)\mathrm{d}\bar{x}\\
&= \int\int\left(\frac{\delta\mathcal{F}}{\delta\sigma_\alpha(\bar{x},t)}+\frac{\delta\mathcal{F}}{\delta\mu'_{i,\alpha}(\bar{x},t)}\,\zeta'_i\right)\delta\mathsf{D}_\alpha(\bar{x},\bar{\zeta}',t)\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{\zeta}'\\
&= \int\int\frac{\delta\mathcal{F}}{\delta\mathsf{D}_\alpha(\bar{x},\bar{\zeta}',t)}\,\delta\mathsf{D}_\alpha(\bar{x},\bar{\zeta}',t)\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{\zeta}'.
\end{aligned} \tag{2.15}$$

The result from (2.15) is

$$\frac{\delta\mathcal{F}}{\delta\mathsf{D}_\alpha}=\frac{\delta\mathcal{F}}{\delta\sigma_\alpha}+\frac{\delta\mathcal{F}}{\delta\mu'_{i,\alpha}}\,\zeta'_i. \tag{2.16}$$

Substituting (2.16) into the bracket (2.9), while using properties (1.2), constraints (2.12), and integrating over $\bar{\zeta}$ yields the bracket

$$\begin{aligned}
\{\mathcal{F},\mathcal{G}\}=\sum_{\alpha=1}^{N}\int \sigma_\alpha &\left(\frac{\delta\mathcal{G}}{\delta\mu'_{k,\alpha}}\,\partial_{x_k}\frac{\delta\mathcal{F}}{\delta\sigma_\alpha}-\frac{\delta\mathcal{F}}{\delta\mu'_{k,\alpha}}\,\partial_{x_k}\frac{\delta\mathcal{G}}{\delta\sigma_\alpha}\right)+\\
\mu'_{j,\alpha}&\left(\frac{\delta\mathcal{G}}{\delta\mu'_{k,\alpha}}\,\partial_{x_k}\frac{\delta\mathcal{F}}{\delta\mu'_{j,\alpha}}-\frac{\delta\mathcal{F}}{\delta\mu'_{k,\alpha}}\,\partial_{x_k}\frac{\delta\mathcal{G}}{\delta\mu'_{j,\alpha}}\right)\mathrm{d}\bar{x},
\end{aligned} \tag{2.17}$$

where we used the relations (2.14). The bracket (2.17) is a summation over all layers of the single-layer bracket (4.4.7) in [8] for the rotating case.

The Hamiltonian (2.10) for the kinetic equations transforms under the constraint (2.12) to

$$\mathcal{H}[\sigma_1,\sigma_2,\mathbf{v}_1,\mathbf{v}_2] = \sum_{\alpha=1}^{N} \int \frac{1}{2} K_\alpha\, G_\alpha^2\, |\mathbf{v}_\alpha|^2 + \frac{1}{2} K_\alpha\, G_\alpha^4 + K_\alpha\, G_\alpha^2\, \phi_\alpha \mathrm{d}\bar{x}. \tag{2.18}$$

Because $K_\alpha\, G_\alpha^2 = \sigma_\alpha$, the Hamiltonian for the isopycnic case with density (2.6) emerges by choosing

$$K_1\, G_1^4 + K_2\, G_2^4 = \sigma_1\, G_1^2 + \sigma_2\, G_2^2 = g\,\sigma_1^2/\rho_1 + 2\, g\sigma_1\,\sigma_2/\rho_2 + g\,\sigma_2^2/\rho_2. \tag{2.19}$$

Using $K_\alpha\, G_\alpha^2 = \sigma_\alpha$ and making the partition

$$\sigma_1\, G_1^2 = g\,\sigma_1^2/\rho_1 + g\sigma_1\,\sigma_2/\rho_2 \quad \text{and} \quad \sigma_2\, G_2^2 = g\,\sigma_2^2/\rho_2 + g\sigma_1\,\sigma_2/\rho_2, \tag{2.20}$$

we can obtain K_α and G_α.

The Hamiltonian for the isentropic case with density (2.7) emerges by choosing

$$\begin{aligned} &\frac{1}{2} K_1\, G_1^4 + \frac{1}{2} K_2\, G_2^4 = \frac{1}{2}\sigma_1\, G_1^2 + \frac{1}{2}\sigma_2\, G_2^2 \\ &= \frac{c_p\,\theta_1}{g}\,\frac{p_r}{\kappa+1}\left((p_1/p_r)^{\kappa+1} - (p_a/p_r)^{\kappa+1}\right) + \frac{c_p\,\theta_2}{g}\,\frac{p_r}{\kappa+1}\left((p_2/p_r)^{\kappa+1} - (p_1/p_r)^{\kappa+1}\right). \end{aligned} \tag{2.21}$$

Using $K_\alpha\, G_\alpha^2 = \sigma_\alpha$ and making the partition (note that $p_1 = p_a + g\,\sigma_1$ and $p_2 = p_a + g\,(\sigma_1+\sigma_2)$)

$$\begin{aligned} \sigma_1\, G_1^2 &= \frac{2\, c_p\,\theta_1}{g}\,\frac{p_r}{\kappa+1}\left((p_1/p_r)^{\kappa+1} - (p_a/p_r)^{\kappa+1}\right) \qquad \text{and} \\ \sigma_2\, G_2^2 &= \frac{2\, c_p\,\theta_2}{g}\,\frac{p_r}{\kappa+1}\left((p_2/p_r)^{\kappa+1} - (p_1/p_r)^{\kappa+1}\right). \end{aligned} \tag{2.22}$$

one can obtain K_α and G_α. We take $\phi_\alpha = g\, h_b(\bar{x})$. □

It may be verified that the equations of motion (2.1) emerge for the isopycnic and isentropic case from $d\mathcal{F}/dt = \{\mathcal{F},\mathcal{H}\}$, the bracket (2.17) and Hamiltonian $\mathcal{H} = \int H \mathrm{d}\bar{x}$ with the respective energy densities $H = H_{\text{isopycnic}}$ and $H = H_{\text{isentropic}}$ defined in (2.6) and (2.7).

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