Balanced Outcome-Set Expected Utility*

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Abstract

We introduce and analyze the Balanced Outcome-Set Expected Utility (BOSEU) model for decision-making under uncertainty. A prior and a *balanced outcome-set utility* characterize a BOSEU decision-maker. Her prior is defined over those events she perceives as unambiguous and hence willing to quantify with a precise probability. Using her prior, the decision-maker transforms each act into an outcome-set act that assigns a *set* of outcomes to each state. Her balanced outcome-set utility assigns a utility to each outcome-set relative to a "balance" point. She implicitly ranks each act according to the balance point required to equate the balanced outcome-set expected utility of the associated outcome-set act with zero. As a consequence her risk preferences need only exhibit betweenness allowing for behavior that can accommodate Allais-type paradoxes.

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Key words: uncertainty, ambiguity, betweenness.

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1 Introduction

In the tradition of the voluminous literature initiated by Ellsberg (1961), consider a decision-maker (hereafer, DM) operating in a setting of purely subjective uncertainty who perceives there to be ambiguity arising as a result of her possessing only partial information about the underlying stochastic process that determines the resolution of the uncertainty she faces. In particular, this means she is not comfortable quantifying with a precise probability the uncertainty she associates with each and every event. However, as is the case in Gul and Pesendorfer's (2014) (hereafter, GP) expected uncertain utility (EUU) model, our DM has *rich coherent beliefs* by which we mean there exists a rich collection of events she deems *measurable* over which she can define her *prior*, a unique probability that represents her beliefs over those events.

An object of choice for our DM is an uncertain prospect or *act*, formally identified as a mapping from the state space that describes the uncertainty she faces to a space of outcomes that we assume contains, an outcome she ranks best, and another outcome she ranks worst. Using her prior, each measurable act (that is, one that is adapted to the set of events deemed measurable by the DM) can be mapped to a corresponding probability distribution or *lottery* over outcomes. Since the richness of her beliefs guarantees for each lottery the existence of an act for which it is the corresponding lottery, this allows us to identify the DM's risk-preferences with the relation over lotteries induced from the restriction of her preferences to measurable acts.

In the EUU model the events deemed by the DM to be measurable correspond precisely to those for which the event and its complement jointly satisfy a version of Savage's (1954) postulate **P2**. Letting $\succsim$ denote the DM's preferences over acts and writing $f_E g$ for the act that agrees with the act f on the event E and with the act g on its complement, the DM deems an event E measurable or (employing GP's terminology) *ideal*, if, for any four acts f, g, h, and h':

$$[f_E h \succsim g_E h \;\&\; h_E f \succsim h_E g] \implies [f_E h' \succsim g_E h' \;\&\; h'_E f \succsim h'_E g]\ .$$

Since the restriction of $\succsim$ to ideal acts satisfies **P2** by definition, unsurprisingly this entails the DM's risk-preferences must satisfy (mixture) independence and hence conform to expected utility. However, as the Allais paradoxes clearly illustrate, **P2** is not only challenged when probabilities are unknown, certainly if it comes to descriptive modeling (see, for example, Tversky and Shafir (1992) for empirical findings), but also on normative grounds (see, for example, Heukelom (2015) for an extensive historical account). Hence our goal is to characterize a class of DMs who have rich coherent beliefs but for which the corresponding risk preferences need not conform to expected utility theory.

One approach taken by Grant et al. (2022) that allows for risk-preferences that can accommodate Allais style violations of expected utility, is to assume the set of measurable events are exogeneously specified and then axiomatize a family of preference relations in which the evaluation of an arbitrary act is characterized by a mapping to an equivalent measurable act along with a generalized notion of the certainty equivalent of a lottery.[1]

The approach taken here, however, is to consider an alternative property an event must satisfy in order for it to be deemed measurable by the DM and then explore its implications for the corresponding risk-preferences. The one we propose is simple and as it corresponds to Grant et al.'s (2000) (weak) decomposability property (in its strict form), we refer to an event R as *decomposable*, if for any pair of acts f and g:

$$[f_R g \succ g \text{ and } g_R f \succ g] \implies f \succ g .$$

Grant et al. (2000) contend that by interpreting a statement like *"f would be preferred to g if the event R were known to obtain"* as *only* entailing $f_R g \succ g$, decomposability may be interpreted as encapsulating the following reasoning.

> *If the DM would prefer f to g knowing R obtains, and she would prefer f to g knowing R does not obtain, then she should prefer f to g even though she currently does not know whether R will or will not obtain.*

That is, similar to GP's notion of an ideal event, decomposability provides a way to operationalize Savage's (1954) *extralogical Sure-Thing Principle* (STP).

With measurable events classified as those that satisfy decomposability, as Grant et al. (2000) establish, the corresponding risk-preferences need only exhibit the betweenness property of Chew (1983) and Dekel (1986) and thus can accommodate Allais style violations of expected utility. In our framework, it turns out, the motivation for decomposability critically depends on the measurability of the event, as we explain later on.

Our second point of departure concerns the handling of non-measurable acts. To model her valuation of such acts, we adapt GP's construction that assigns to each act a measurable *envelope*. We keep their notion of a *measurable split* of the state space induced by the preimage of the act. Each element of this split

[1] Grant et al. (2022) also consider endogenizing the set of events the DM views as measurable by utilizing Epstein and Zhang's (2001) preference-based definition for classifying measurable events. Even then, however, in their representation theorem (Theorem 5, p14) the structure of the set of meaurable events needed for the evaluation of arbitrary acts is *assumed* as part of the hypothesis of the theorem and not derived from the postulates they propose.

corresponds to an outcome-set (a finite subset of outcomes), for which, given her limited information about the stochastic process determining the resolution of uncertainty, renders her incapable of attributing any fraction of the probability her prior assigns to that element of the split to any strict subset of the corresponding outcome-set. The essential difference is that whereas GP define an envelope as a mapping from states to intervals of outcomes, defined in terms of minimum and maximum outcomes of the corresponding outcome-set, we retain the entire outcome-set. That is, the envelope of the act maps to an outcome-set precisely those states in the element of the measurable split induced by the act's preimage that corresponds to that subset of outcomes. This in turn means the envelope of an act can be characterized as the minimal (with respect to set-inclusion) measurable mapping from states to outcome-sets that contains the act. Indeed from a perceptual perspective, we contend it makes sense to view the DM *as incapable of distinguishing among acts that have a common envelope*.[2] The axioms we adopt guarantee the existence and uniqueness of envelopes.

The interpretation of envelopes in terms of belief and plausibility functions, as described in GP, becomes even more straightforward: the belief in a particular outcome-set obtaining is the total probability assigned to that outcome-set and all its subsets in the envelope, while its plausibility is the total probability of all subsets containing at least one element of that outcome-set. Moreover, the prior and the envelope of an act, induces the outcome-set lottery in which for each outcome-set, the probability assigned to that outcome-set is given by the probability the prior assigns to the set of states that the envelope maps to that particular outcome-set.

Imposing that the DM is indifferent among acts inducing the same outcome-set lottery, we introduce the *Balanced Outcome-Set Expected Utility (BOSEU) preference model* corresponding to the family of preferences that admit an *implicit* probability equivalent representation characterized by a pair $\langle \mu, U \rangle$, where μ is the DM's prior defined over those events she deems measurable and a *balanced outcome-set utility*, $U(Y, p)$, that specifies the utility of an outcome-set Y in a lottery that has a *probability* equivalent value of p, by which we mean a lottery the DM views as equally valuable as a binary gamble that yields the best outcome with probability p and the worst outcome with the complementary probability $1-p$. It exhibits a natural (outcome-set) monotonicity with respect to its first argument.

Thus we obtain a clean separation of the ambiguity she perceives to be present

[2] In this regard, the measurable split induced by an act's inverse image is reminiscent of Ghirardato's (2001, p249) second scenario of an underspecified state space as one possible way to interpret his model in which preferences are defined over outcome-set acts.

given her knowledge about the random process governing the resolution of the uncertainty she faces from her attitude toward risk (that is, measurable uncertainty). The former is characterized by those events that lie outside the domain of her prior while the restriction of her balanced outcome-set utility to singleton outcome-sets encodes the latter. Finally, her attitude toward (general) uncertainty involving both risk and ambiguity is embodied in her (unrestricted) balanced outcome-set utility.

This interpretation clarifies why we do not impose decomposability for non-measurable events. The principle relies on a sharp separation between outcomes of an act on an event and its complement, but this gets blurred when events are non-measurable, since outcomes on an event may contribute to the outcome-set outside that event. We refer to GP (Section 5) for an illustration by way of the Ellsberg paradox of this effect. Hence, their motivation not to impose **P2** for non-measurable events in essence also applies to restricting decomposability to measurable events only.[3]

We develop the formal definition of BOSEU preferences in section 2 with its axiomatic characterization appearing in section 3. We provide three examples in section 4, combining Chew's (1983) and Dekel's (1986) betweenness property with EEU, Eichberger and Pasichnichenko's (2021) Quasi-Arithmetic Mean Utility, and Gul and Pesendorfer's (2015) Hurwicz Expected Utility. We conclude in section 5. Proofs appear in the appendix.

2 The Model

Our setting is one in which the purely subjective uncertainty the DM faces is described by a state space Ω. The objects of choice are acts that for each state of nature $\omega \in \Omega$, deliver an outcome x from a set X. Each act f is simple, that is, its image $f(\Omega)$ is a finite subset of X.

We denote the set of all acts by F. We identify any outcome $x \in X$ with the (constant) act f in which $f(\omega) = x$ for all ω. And with further (albeit fairly standard) abuse of notation, X will also refer to the set of constant acts.

For any pair of events $E, B \subseteq \Omega$, $B \backslash E$ shall denote the set of elements that are in B but not in E. For any pair of acts f and g in F and any event $E \subset \Omega$, we write $f_E g$ for the act that agrees with f on E and with g on $\Omega \backslash E$.

The DM is characterized by her preferences over acts, a binary relation $\succsim$ on

[3] Decomposability accommodates the Ellsberg preferences, but complications arise in updating when new information renders originally decomposable events no longer decomposable. The questions this raises on dynamic consistency will be treated in a separate paper.

F, with asymmetric and symmetric parts denoted by $\succ$ and $\sim$, respectively. We assume X contains a best outcome $\bar{x}$, a worst outcome $\underline{x}$, and a "middle/moderate" outcome $\widetilde{x}$. That is, $\bar{x} \succ \widetilde{x} \succ \underline{x}$, and $\bar{x} \succsim x \succsim \underline{x}$ for all x in X.

We begin our description of the Balanced Outcome-Set Expected Utility preference model by first noting the DM possesses *rich coherent beliefs*. This entails the existence of a sufficiently rich collection of (*risky*) events, constituting a σ-algebra of subsets of Ω, over which can be defined a countably-additive and convex-ranged probability measure μ (her 'prior') with which the DM can *quantify precisely* the uncertainty she associates with each risky event. We denote the domain of μ by $\mathbf{R}$.

Countable-additivity requires the probability of the union of a countable collection of disjoint measurable events from $\mathbf{R}$ equals the infinite sum of the probabilities of these events. For μ to be convex-ranged requires for any event R in $\mathbf{R}$ and any r in $(0,1)$ there exists a subset $B \subset R$ that is in $\mathbf{R}$ and for which $\mu(B) = r\mu(R)$. Let $F_\mu \subset F$ denote the set of acts that are measurable with respect to μ.

From this point on, an *outcome-set* will refer to any non-empty finite subset of X with generic elements denoted by Y, Z, Y' et cetera. As we alluded to in the introduction, there is a natural way to use this prior to identify with each act its *outcome-set* envelope. Let $\mathbf{F}_\mu$ be the set of measurable functions $\mathbf{f}\colon \Omega \to \{Y \subset X \colon Y \neq \emptyset, |Y| < \infty\}$. We refer to elements of $\mathbf{F}_\mu$ as outcome-set acts.

Definition 1 (Envelope of an act) *The outcome-set act* $\mathbf{f} \in \mathbf{F}_\mu$ *is the* envelope *of* f *if*

(*i*) $f(\omega) \in \mathbf{f}(\omega)$ *for all* $\omega \in \Omega$, *and*

(*ii*) *for any outcome-set act* $\mathbf{g} \in \mathbf{F}_\mu$:

$$f(\omega) \in \mathbf{g}(\omega) \quad \textit{for all } \omega \in \Omega \implies \mu\left(\{\omega \in \Omega \colon \mathbf{f}(\omega) \subseteq \mathbf{g}(\omega)\}\right) = 1\,.$$

To construct the envelope of an act, it is useful first to define the *inner measure* of μ, denoted by μ_*, that is derived from the prior by assigning to each event $E \subset \Omega$ the weight $\mu_*(E) \in [0,1]$ that is the solution to:

$$\sup_{R \in \mathbf{R}, R \subseteq E} \mu(R)\ .$$

Since μ is countably additive, the supremum is attained. We shall refer to the measurable event $[E]_* \in \mathbf{R}$, as the *inner-sleeve* of E, if $[E]_* \subseteq E$ and $\mu([E]_*) = \mu_*(E)$.[4]

[4] Notice that the inner-sleeve is unique up to a set of μ-measure 0.

Following GP, we associate with each act a measurable partition of the state space generated by the act's preimage as follows.

Definition 2 (Measurable split) *The* measurable split *(of the state space) associated with the act $f : \Omega \to X$ and denoted by $\{R_f^Y \in \mathbf{R} : Y \subseteq f(\Omega), Y \neq \emptyset\}$, is inductively defined as follows:*

1. *For each element $x \in f(\Omega)$, set $R_f^{\{x\}} := [f^{-1}(x)]_*$.*
2. *For each $Y \subseteq f(\Omega)$ such that $|Y| > 1$, set*

$$R_f^Y := \left[f^{-1}(Y)\right]_* \setminus \left(\bigcup_{Z \subset Y, Z \neq \emptyset} R_f^Z \right).$$

*We refer to R_f^Y as the f-*marginal inner-sleeve *of the outcome-set Y.*

To see how the envelope of an act can be constructed using the measurable split generated by its inverse image, first consider a binary act $x_A y$. The measurable split is the three element partition of the state-space

$$\{\underset{[A]_*}{\underset{||}{R_{x_A y}^{\{x\}}}},\ \underset{[\Omega\setminus A]_*}{\underset{||}{R_{x_A y}^{\{y\}}}},\ \underset{\Omega\setminus\left([A]_* \cup [\Omega\setminus A]_*\right)}{\underset{||}{R_{x_A y}^{\{x,y\}}}}\}.$$

The first (respectively, second) element corresponds to the largest measurable subset in which the binary act $x_A y$ yields the outcome x (respectively, y). For the third element, all the DM can discern is that the outcome will be either x or y. However, she is unable to attribute any fraction of the probability her prior assigns to this element of the split to either x or y obtaining alone. Thus, it readily follows from Definition 1 that the envelope of $x_A y$ is the outcome-set act $\mathbf{f} \in \mathbf{F}_\mu$ for which,

$$\mathbf{f}(\omega) = \begin{cases} \{x\} & \text{if } \omega \in [A]_* \\ \{y\} & \text{if } \omega \in [\Omega \setminus A]_* \\ \{x, y\} & \text{otherwise} \end{cases}$$

This method readily extends to an arbitrary act f in F. Using the measurable split $\{R_f^Y \in \mathbf{R} : Y \subseteq f(\Omega), Y \neq \emptyset\}$, its (essentially unique) envelope is the outcome-set act $\mathbf{f} \in \mathbf{F}_\mu$ constructed by setting $\mathbf{f}(\omega) := Y$ whenever $\omega \in R_f^Y$.

An *outcome-set lottery* is a finite ranged function $L : \{Y \subset X : Y \neq \emptyset, |Y| < \infty\} \to [0, 1]$ satisfying $\sum_{Y \subset X, |Y| < \infty} L(Y) = 1$. We associate with the act f the outcome-set lottery $\mu \circ \mathbf{f}^{-1}$.

Recalling the approach of Dempster (1967) and Shafer (1976), we shall interpret the outcome-set lottery $\mu \circ \mathbf{f}^{-1}$ as encoding how the DM weights that part of the evidence supporting the belief that the act f leads to an outcome in a given set of outcomes obtaining that is not well-specified enough to allow her to distribute any of it across any of the elements of that set or any of the other strict subsets of that set of outcomes. In other words, for each outcome-set $Y \subseteq f(\Omega)$ we shall interpret $\mu \circ \mathbf{f}^{-1}(Y) \; \left(= \mu\left(R_f^Y\right)\right)$ as the weight assigned by the DM to evidence that directly supports the act f leading to an outcome in Y obtaining that cannot be further refined in terms of any of the strict subsets of Y.

Analogous to Grant's (1995 p163) rendition of Machina and Schmeidler's (1992) concept of probabilistic sophistication, we require that no relevant preference information is lost by this association.

Definition 3 (Coherent Beliefs) *The prior μ is a* coherent belief *for the preference relation $\succsim$, if for each pair of acts f and $\widehat{f}$, with respective envelopes $\mathbf{f}$ and $\widehat{\mathbf{f}}$, $f \sim \widehat{f}$ whenever $\mu \circ \mathbf{f}^{-1} = \mu \circ \widehat{\mathbf{f}}^{-1}$.*

One more element is needed, a *balanced outcome-set utility* that specifies the utility of each outcome-set in a lottery of a given value, its *balance-probability.* Restricted to singletons, we impose the standard properties of utility in betweenness models, but to determine a useful concept of monotonicity for sets turns out to be a more delicate issue. It involves the choice of a dominance relation between sets of outcomes. To impose a complete dominance relation, as for singletons, would overly restrict the model class. However, it is natural to require the DM prefer one outcome-set over another if the worst element of the former is strictly preferred by the DM to the best element of the latter.

We assume a balanced outcome-set utility $U \colon \{Y \subset X : Y \neq \{\emptyset\}, |Y| < \infty\} \times [0,1] \to \mathbb{R}$ is continuous in the second argument and exhibits *outcome-set monotonicity,* in the sense that for any p in $(0,1)$ and any pair of outcome-sets Y and Z, $U(Y, p) > U(Z, p)$, whenever $y \succ z$ for all $(y,z) \in Y \times Z$.

We now have assembled all the parts required for our formal definition.

Definition 4 (BOSEU Preferences) *A preference relation $\succsim$ conforms to the* Balanced Outcome-Set Expected Utility Preference Model *if there exists a prior μ and a balanced outcome-set utility U, such that $\succsim$ admits a probability equivalent representation $V \colon F \to [0,1]$, in which $V(f)$ is the unique solution to*

$$\sum_{Y \subseteq f(\Omega)} U(Y, V(f))\, \mu\left(\mathbf{f}^{-1}(Y)\right) = 0\,, \tag{1}$$

where $\mathbf{f}$ is the envelope of f.

A sufficient condition for the solution of (1) to be unique is that $U(Y, p)$ is strictly decreasing in p. Utility is determined by $\succsim$ up to a scalar $\lambda_p > 0$ for each p, which we typically choose so that $U(\bar{x}, p) - U(\underline{x}, p) \equiv 1$ and $pU(\bar{x}, p) + (1 - p)U(\underline{x}, p) \equiv 0$. This amounts to setting

$$U(\{\bar{x}\}, p) := 1 - p \text{ and } U(\{\underline{x}\}, p) := -p\,.$$

Notice that this is p lower than Dekel's normalization rules, to let zero utility correspond to equally good as the balance-utility level p.

3 Characterization

We begin our characterization of the class of BOSEU decision-makers by first specifying what property an event must satisfy that allows us to infer the DM deems it to be 'risky' and thus 'measurable'. As we noted above in the introduction, in GP's EUU theory, this requires both it and its complement satisfy a version of Savage's (1954) postulate **P2**. We instead propose the following property of *decomposability.*

Definition 5 (Decomposable Events) *An event $R \subseteq \Omega$ is* decomposable *if for every pair of acts f and g in F, $f_R g \succ g$ and $g_R f \succ g \implies f \succ g$.*

Our axioms will ensure that decomposability is equivalent to the criterion with $\succ$ replaced by $\prec$, and entails the non-strict variants with $\precsim$ and $\succsim$ as well.[5] We shall refer to $\mathbf{R}$ as the set of decomposable events. Notice that by construction the set $\mathbf{R}$ (like the set of ideal events) is closed under complements. Also it readily follows that if $\succsim$ is complete and transitive then any ideal event is also decomposable.[6] The converse, however, need not hold.

We refer to an act as *decomposable* if it is measurable with respect to $\mathbf{R}$, that is, an act g is decomposable if $g^{-1}(\{x\}) \in \mathbf{R}$ for all $x \in X$. Hence, loosely speaking, a decomposable act coincides with its own envelope. Let $G \subset F$ denote the set of decomposable acts.

A subclass of decomposable events are those for which modifying any act on that event leaves it in the same indifference set. These are known as (Savage-)null events.

[5] Where $\precsim$ is the relation derived from $\succsim$ by setting $f \precsim f'$ if $f' \succsim f$, and $\prec$ is the asymmetric part of $\precsim$.

[6] If the event E is ideal, then it follows from GP's Lemma B0 (p25) that it is also left ideal, that is, $g_E f \succsim f$ implies $g \succsim f_E g$ or equivalently, $\neg(g \succsim f_E g)$ implies $\neg(g_E f \succsim f)$. Thus, it follows from the completeness of $\succsim$ that $f_E g \succ g$ implies $f \succ g_E f$. Hence if, in addition, we have $g_E f \succ g$, then $f \succ g$ follows from the transitivity of $\succ$, which in turn follows from the completeness and transitivity of $\succsim$.

Definition 6 (Null events) *An event $N \subseteq \Omega$ is* null *if $f \sim g_N f$ for all $f, g \in F$. Let* $\mathbf{N}$ *denote the set of null events.*

Set $\mathbf{R}^+ := \mathbf{R} \backslash \mathbf{N}$, the class of non-null decomposable events. And for each non-null decomposable event $R \in \mathbf{R}^+$ and each act $f \in F$, set $f(R)^+ := \{y \in f(R) \colon f^{-1}(y) \cap R \notin \mathbf{N}\}$. That is, $f(R)^+$ contains each element in the image of $f(R)$ for which its preimage has a non-null intersection with R.

Analogous to the role played by ideal events in GP, we suppose a BOSEU DM uses elements of $\mathbf{R}^+$ to quantify the uncertainty of any event. So it seems natural to view an event as *maximally ambiguous* if it and its complement contain no element of $\mathbf{R}^+$. Adopting the terminology of GP, we will refer to such an event (as well as its complement) as *diffuse*.

Definition 7 (Diffuse Events) *An event $D \subseteq \Omega$ is* diffuse *if, for every non-null decomposable event $R \in \mathbf{R}^+$, $D \cap R \notin \mathbf{N}$ and $(\Omega \backslash D) \cap R \notin \mathbf{N}$ (that is, D has a non-null intersection with both R and its complement). We denote by $\mathbf{D}$ the set of diffuse events.*

We say an act h is *diffuse* if its inverse image generates a diffuse partition of Ω, by which we mean $h^{-1}(x) \in \mathbf{D}$ for all $x \in h(\Omega)^+$. Let $H \subset F$ denote the set of diffuse acts.[7]

With these preliminaries in hand we can now state the axioms. Our first is the standard ordering axiom.

Axiom 1 (Ordering) *The binary relation $\succsim$ is complete and transitive.*

We next require the collection of events the DM deems unambiguous to be closed under conjunctions. That is, we require for any pair of decomposable events R and $\widehat{R}$ their intersection is also decomposable.

Axiom 2 (Decomposable Closure under Intersection) *For any pair of decomposable events $R, \widehat{R} \in \mathbf{R}^+$ and any pair of acts $f, f' \in \mathcal{F}$,*

$$\text{if} \;\; f_{R \cap \widehat{R}} f' \succ f' \;\; \text{and} \;\; f'_{R \cap \widehat{R}} f \succ f' \;\; \text{then} \; f \succ f' .$$

We readily acknowledge this is not without loss of generality. As a number of researchers have demonstrated, various hedging opportunities against ambiguity may be based on events one would expect a DM to deem unambiguous but which are not closed under intersection.[8] However, just as was the case in GP's model of

[7] It will turn out that the envelope of any diffuse act $h \in H$ will be the constant function $\mathbf{h}(\omega) = h(\Omega)^+$ for all $\omega \in \Omega$.

[8] See for example, Zhang (2002), Epstein and Zhang (2001), Kopylov (2007), and Nehring (2009).

EUU, as our theory builds on the induced order over the (meaurable) envelopes the DM associates with acts, we require the set of events the DM deems measurable, constitute a σ-algebra thereby guaranteeing the (measurable) envelope the DM associates with each act is well-defined. Axiom 2 in conjunction with the continuity property imposed by Axiom 5 (see below), plays a key role in establishing that the collection of events $\mathbf{R}$ is a σ-algebra.

Recall from above, a diffuse event is one in which the DM cannot find any non-null decomposable event that can "fit" inside that event that would help her quantify the uncertainty she associates with that event. Correspondingly, for a given diffuse act $h \in H$, the DM is unable to estimate the *relative* likelihood of *any strict subset* of the objects in $h(\Omega)^+$ obtaining, either unconditionally or conditionally on any decomposable event R obtaining. Thus the next axiom requires the DM is incapable of expressing a strict preference between any pair of diffuse acts h and $\widehat{h}$ for which $h(\Omega)^+ = \widehat{h}(\Omega)^+$. This indifference extends to any pair of acts that only differ on a decomposable event R, with one having the diffuse act h determine the outcome should R obtain while the other has $\widehat{h}$. Furthermore, her evaluation of a diffuse act h conditional on a decomposable event R obtaining cannot exceed (respectively, be no worse) than her conditional evaluation of any outcome in $h(\Omega)^+$.

Axiom 3 (Decomposable Eventwise Monotonicity) *For all diffuse and constrant acts h in $H \cup X$, all acts f in F, and all non-null decomposable events R in $\mathbf{R}^+$:*

(i) $h_R f \sim \widehat{h}_R f$ *for any diffuse act* $\widehat{h}$ *in* H, *in which* $\widehat{h}(\Omega)^+ = h(\Omega)^+$;

(ii) $x_R f \succsim (\succ) h_R f$ *for any* x *such that* $x \succsim (\succ) y$ *for all* $y \in h(\Omega)^+$; *and,*

(iii) $h_R f \succsim (\succ) z_R f$ *for any* z *such that* $y \succsim (\succ) z$ *for all* $y \in h(\Omega)^+$.

This axiom can be interpreted as *recognizable* monotonicity: if the DM can deduce from the envelopes of f and g that f dominates g on Ω, which is the case if and only if f's envelope dominates that of g (according to the strong set-order), then she should prefer f to g.

We do not impose, however, GP's Axiom 2 of statewise monotonicity. In our interpretation, this would be the axiom of *plausible* monotonicity, requiring that f should be preferred to f' whenever the DM cannot exclude, on the basis of their envelopes, that f may dominate f' on Ω. For diffuse acts h and h', with $h(\Omega)^+ = Y$ and $h'(\Omega)^+ = Y'$, this is already the case when both the worst outcome and best outcome in Y are each strictly preferred to the respectively, worst and best in Y'. Moreover, the utility of a set Y would then be completely determined by its worst and best element, that is, we would arrive at interval-utilities. For instance, for the case where X is a subset of the real line this would require the set $Y = \{0.1, 1, 2, 3, 10.1\}$ be assigned a strictly higher utility than the set $Y' = \{0, 10\}$, not as a matter of taste, but as axiomatic principle, for which we see no ground.

We follow with an axiom that serves the role played by Savage's comparative probability axiom **P4** in GP's characterization of EUU. However, since the decomposability property does not entail the full separability implied by the definition of an ideal event, it is not enough for us to modify **P4** by substituting decomposable events for ideal events. So instead we adopt the weak comparative probability axiom**P4*** from Epstein and LeBreton (1993) in indifference form and extend it to apply to diffuse acts as well.

Axiom 4 (Comparative Conditional Probability) *For any pair of disjoint decomposable events R and $\widehat{R}$ in $\mathbf{R}$, any outcomes $x^* \succ x$ in X, and any f in F,*

$$(x^*_R x)_{R\cup\widehat{R}} f \sim (x^*_{\widehat{R}} x)_{R\cup\widehat{R}} f \implies (h^*_R h)_{R\cup\widehat{R}} f \sim (h^*_{\widehat{R}} h)_{R\cup\widehat{R}} f \,,$$

for all h^* and h in $H \cup X$.

To interpret this axiom, notice that the first indifference allows us to infer that, conditional on the act f determining the outcome should neither R nor $\widehat{R}$ obtain, the DM views R "as equally as likely" as $\widehat{R}$. The axiom then requires these revealed conditional equal likelihoods should still obtain no matter what pair of *outcome-sets* serve as the "stakes" for bets made on R against $\widehat{R}$.

We finish with two continuity axioms. The first helps ensure the collection of events the DM deems unambiguous is closed under countable unions.

Axiom 5 (Cumulative Decomposable-Event Continuity) *Let $f^n = f_{R^n} f^*$ with $\{R^n\} \subset \mathbf{R}$ and $R^n \subset R^{n+1}$, and set $R^\infty := \cup_n R^n$. If $f' \succsim f^n \succsim f''$ for all n, then $f' \succsim f_{R^\infty} f^* \succsim f''$.*

Our second continuity axiom plays the same role as Savage's (1954) postulate **P6**, namely, requiring the set of decomposable events is sufficiently rich so that the derived comparative likelihood relation over decomposable events is both fine and tight.

Axiom 6 (Small Decomposable-Event Continuity) *For any pair of acts $f, f' \in \mathcal{F}$, if $f \succ f'$, then for each outcome $x \in X$ there exists a finite decomposable partition $\{R_1, \ldots, R_N\}$ of Ω, with $R_n \in \mathbf{R}$, such that $\underline{x}_{R_n} f \succ f'$ and $f \succ \bar{x}_{R_n} f'$ for all $n = 1, \ldots, N$.*

Our main representation result follows.

Theorem 1 *Fix a binary relation $\succsim$. Then the following are equivalent.*

1. *The relation $\succsim$ satisfies Axioms 1 – 6.*
2. *The relation $\succsim$ conforms to the BOSEU Preference Model.*

The proof appears in the appendix but here we provide an outline of how the axioms enable us to obtain a BOSEU representation.

Step 1: *Deriving the prior*
We first establish that the set of decomposable events, $\mathbf{R}$, constitutes a σ-algebra of events. Next, we show the restriction of the DM's preferences to binary bets on decomposable events involving only the best and worst outcomes admits a standard expected utility representation, which is characterized by a countably-additive and convex-ranged probability μ defined on $\mathbf{R}$. Since μ is convex-ranged, this means that for any $p \in [0,1]$, there exists a decomposable event R_p, for which $\mu(R_P) = p$.

Step 2: *Variants of decomposability*
To prepare for the construction of the utility, we formulate a technical result on the variants of the decomposability criterion with $\succ$ replaced by $\prec$, $\precsim$ and $\succsim$. This establishes conditional independence of acts in an indifference class.

Step 3: *Constructing the Balanced Outcome-Set Utility*
The value of the outcome-set utility $U(Y,p)$ is determined as follows. Set $U(\{\bar{x}\},p) := 1-p$ and $U(\{\underline{x}\},p) := -p$. For a singleton $Y = \{y\}$, determine R and R_p in $\mathbf{R}$ such that $\mu(R_p) = p$ and (i) in case $y \succsim \bar{x}_{R_p}\underline{x}$, such that $y_R\underline{x} \sim \bar{x}_{R_p}\underline{x}$, otherwise (ii) such that $y_R\bar{x} \sim \bar{x}_{R_p}\underline{x}$. Define $q := \mu(R)$. The only candidate for a solution to the BOSEU equation turns out to be, respectively

$$\text{(i)} : U(\{y\},p) = p\frac{(1-q)}{q} \quad \text{and} \quad \text{(ii)} : U(\{y\},p) = -(1-p)\frac{(1-q)}{q}\,.$$

Do the same for non-singletons Y, with y replaced by a diffuse act h^Y in H that has associated with it an envelope that maps every state to Y.

Step 4: *Verification*
We show that the prior μ from step 1 and the balanced outcome-set utility $U(\cdot,\cdot)$ from step 3 indeed determine a properly constituted BOSEU representation of $\succsim$.

4 Examples of BOSEU Preferences

We present three examples of BOSEU preferences. The first is a betweenness-extension of GP's EUU model, covering a disappointment aversion variant as a special case, while the second is a subjective version of a preference model over outcome-set lotteries (or equivalently, *belief functions*) introduced by Eichberger and Pasichnichenko (2021). The third example is inspired by Gul and Pesendorfer's (2015) model of Hurwicz expected utility (HEU).

In what follows, we take X to be a closed interval $[\underline{x}\,, \bar{x}]$ of real numbers and for each outcome-set Y, we set $\underline{y} := \min_{z\in Y} z$ and $\bar{y} := \max_{z\in Y} z$.

4.1 Balanced Interval Expected Utility (BIEU)

A balanced interval utility is a function

$$u : (\underline{y}, \bar{y}; c) \mapsto \mathbb{R} \text{ on the domain } \{\underline{y}, \bar{y}, c \in X \mid \underline{y} \leqslant \bar{y}\}$$

that is continuous and monotonic in the sense that, for every $c \in X$,

$$u(\underline{y}, \bar{y}; c) > u(\underline{y}', \bar{y}'; c) \text{ whenever } \underline{y} > \underline{y}' \text{ and } \bar{y} > \bar{y}'\,.$$

Furthermore, u is strictly decreasing in its third argument and normalized at $u(c, c; c) = 0$. The corresponding outcome-set utility is defined by setting

$$U(Y, p) := u(\underline{y}, \bar{y}; c) \text{ with } c \text{ such that } pu(\bar{x}, \bar{x}; c) + (1-p)u(\underline{x}, \underline{x}; c) = 0\,.$$

Hence preferences from this subclass of BOSEU admit a Balanced Interval Expected Utility (BIEU) *certainty equivalent* representation, where $ce(f)$ in X, the certainty equivalent of the act f, is the unique solution to

$$\sum_{Y\subseteq f(\Omega)} u(\underline{y}, \bar{y}\,; ce(f))\, \mu\left(\mathbf{f}^{-1}(Y)\right) = 0\,.$$

As we explained in the introduction, BIEU is a subclass of BOSEU that satisfies *plausible monotonicity.* Standard definitions of aversion to ambiguity from the literature correspond to $u(\underline{y}, \bar{y}, c) \leqslant (u(\underline{y}, \underline{y}; c) + u(\bar{y}, \bar{y}; c))/2$ for all c.

EUU is the special case with cardinally equivalent interval utility functions $u(\cdot, \cdot; c)$ for different c, hence of the form

$$u(\underline{y}, \bar{y}; c) = v(\underline{y}, \bar{y}) - v(c, c)$$

with v an interval utility function as in EUU. Then $v(c, c) = p$, and $U(Y, p) = v(\underline{y}, \bar{y}) - p$.

The best-known example of using the extra flexibility of betweenness is Gul's

(1991) risk preference model of *disappointment aversion*,

$$u(\underline{y}, \bar{y}; c) = \begin{cases} \dfrac{v(\underline{y}, \bar{y}) - v(c, c)}{1 + \beta} & \text{if positive} \\ v(\underline{y}, \bar{y}) - v(c, c) & \text{otherwise} \end{cases}$$

Gul highlights a key feature of disappointment-averse preferences is that they admit a representation that is only one parameter richer than expected utility. This is the *risk* disappointment-aversion β that features in the expression above for the balanced outcome-set utility. Risk preferences that conform to expected utility correspond to $\beta = 0$ while risk aversion requires a concave v and $\beta \geqslant 0$. The property of disappointment aversion which Gul shows is both necessary and sufficient to generate Allais style choice patterns, requires $\beta > 0$.

4.2 Quasi-Arithmetic Mean Uncertain Utility (QAMUU)

We take the concept of a *quasi-arithmetic mean* as starting point for an example in which all elements of an outcome set Y matter. The function $v : X \to [0\,,1]$ now denotes a (first order) Bernoulli utility that is continuous, monotonic in the sense that $(v(x) - v(x'))\,(x - x') > 0$ whenever $x \neq x'$, and normalized by setting $v(\underline{x}) := 0$ and $v(\bar{x}) := 1$. Let ϕ denote a (second-order) utility $\phi\colon [0\,,1] \to \mathbb{R}$ that is continuous, monotonic in the sense that $(\phi(v) - \phi(v'))\,(v - v') > 0$ whenever $v \neq v'$, and normalized by setting $\phi(0) := 0$ and $\phi(1) := 1$. Define M_ϕ as the ϕ-induced quasi-arithmetic mean,

$$M_\phi := \phi^{-1}\left(\frac{1}{|Y|}\sum_{x \in Y} \phi\left(v(x)\right)\right) .$$

The balanced outcome-set utility in the QAMUU class is defined by setting

$$U\left(Y, p\right) = M_\phi(Y) - p\,,$$

which conforms to expected utility on the envelopes of acts. Eichberger and Pasichnichenko argue this embodies a *principle of insufficient reason* for evaluating ambiguous sets of possible outcomes. Ambiguity aversion corresponds to concave ϕ. Expanding to a betweenness form, we can take

$$U\left(Y, p\right) = M_{\phi_p}(Y) - p$$

with the arithmetic means now depending on p, in such a way that the resulting outcome-set utility satisfies the required properties, and is strictly decreasing in

p in order to guarantee that the BOSEU equation has a unique solution for $V(f)$ for all acts.

4.3 Disappointment Averse-Implicit Hurwicz Expected Utility (DA-IHEU)

For the third example we return to interval utilities, and combine ambiguity and disappointment aversion in a Hurwicz-style model. Decomposability allows for levels of ambiguity aversion to depend on p.[9] Let v be a utility function as in the previous example. Let $\alpha\colon [0,1] \to [0\,,1]$ parametrize ambiguity aversion, and $\beta \in \mathbb{R}$ the parameter of disappointment aversion. Define

$$U(Y,p) := \begin{cases} \dfrac{\alpha(p)\, v\left(\underline{y}\right) + (1-\alpha(p))\, v(\bar{y}) - p}{1+\beta} & \text{if positive} \\ \alpha(p)\, v\left(\underline{y}\right) + (1-\alpha(p))\, v(\bar{y}) - p & \text{otherwise} \end{cases}$$

By imposing α is differentiable in p, with derivative $\alpha'(p) > -1$ for all p, it is guaranteed that $V(f)$ is uniquely defined by the BOSEU equation. The standard definition of ambiguity aversion is $\alpha(p) \leqslant 1/2$.

In order for the third example to conform to EUU requires α to be constant and $\beta = 0$. As this is precisely Gul and Pesendorfer's (2015) model of Hurwicz expected utility (HEU), example three provides a generalization of that model that allows for non-constant absolute ambiguity aversion as well as disappointment aversion.

5 Concluding Comments

Like GP's EUU model, the BOSEU model affords the outside observer the ability to infer those events the DM deems measurable, *solely* from her preferences. And similar to GP, these may be viewed as those events for which Savage's (1954) *sure-thing principle* applies.

Unlike Gul and Pesendorfer, however, we have not "operationalized" this principle by imposing Savage's postulate **P2**. Instead, following Grant et al. (2000), we have interpreted statements like *"f would be preferred to g if the event R were known to obtain"* as only entailing $f_R g \succ g$.

As a consequence and following as an immediate corollary to Theorem 1, for a DM who exhibits rich coherent beliefs, her risk preferences (over outcome lotteries), which can naturally be identified with the restriction of her preferences to

[9] We thank the co-editor for suggesting we consider such a possibility in one of our examples.

measurable acts, need only satisfy the betweenness property of Chew (1983) and Dekel (1986), rather than (full) independence. Moreover, the third example from section 4, provides us with a parsimoniously parameterized model that not only can accommodate Ellsberg style choice patterns and exhibit more nuanced comparative static wealth effects over ambiguous acts than Hurwicz Expected Utility, but it can also produce Allais style choice patterns for risky (that is, measurable) acts.

Preferences that exhibit betweenness properties have been relatively understudied in the context of choice under risk and, to our knowledge, are almost completely absent in the ambiguity literature. However, in light of the above discussion we contend the BOSEU family of preferences provides us with a normatively attractive approach for modelling choice under uncertainty when decision makers possess only partial information about the stochastic process resolving the uncertainty they face.

Appendix

Proof of Theorem 1

To set the stage for the proof, we first describe how every act can be expressed in terms of constants acts or (and) diffuse acts on its measurable split, as explained in GP. Without loss of generality we take $f(\Omega) = f(\Omega)^+$.

Lemma 1 *Fix a act $f \in F$ with measurable split $\{R_f^Y \in \mathbf{R} : Y \subseteq f(\Omega), Y \neq \emptyset\}$. For each non-null element R_f^Y in the measurable split of f, if $|Y| > 1$, there exists a diffuse act $h^Y \in H$ such that $h^Y_{R_f^Y} f = f$, and if $Y = \{y\}$, $f = y_{R_f^Y} f$.*

Proof. Given a non-decomposable $f \in F$, choose a non-null R_f^Y with $Y = \{y_1, \cdots, y_n\}$, $n > 1$ (the case $n = 1$ is obvious). There exists a sequence of disjoint events $\{B_n\}$ such that $B_i = f^{-1}(y_i) \cap R_f^Y$ for all i. Lemma A2 of GP show that there exists a diffuse partition $\{D_1, \cdots, D_n\} \in \mathbf{D}$ of Ω. Now define

$$D_1^* = (D_1 \cap (\Omega \backslash R_f^Y)) \cup B_1,$$

$$\cdots\cdots$$

$$D_n^* = (D_n \cap (\Omega \backslash R_f^Y)) \cup B_n.$$

We next show that $\{D_1^*, \cdots, D_n^*\}$ is a diffuse partition of Ω. Assume by way of contradiction that D_i^* is not a diffuse event for some i. Then there is $R \in \mathbf{R}^+$ such that $R \in D_i^*$. Since $\mathbf{R}$ is a σ-algebra, $(\Omega \backslash (R_f^Y)) \backslash R \in \mathbf{R}$. Moreover, $(\Omega \backslash (R_f^Y)) \backslash R \in B_i$, which contradicts B_i containing no decomposable event. Thus, D_i^* is a diffuse event. It is easy to check that D_i^*s are all disjoint and their union is Ω. Therefore, $\{D_1^*, \cdots, D_n^*\}$ is also a diffuse partition of Ω. Set $h^Y = (D_1^* : y_1, \cdots, D_n^* : y_n)$. Then $h^Y_{R_f^Y} f = f$. ■

Sufficiency of the axioms

Step 1: Deriving the Prior

First we show that the set of decomposable events constitutes a σ-algebra: that is, 1) it contains both the universal set Ω and the empty set, $\emptyset$; 2) it is closed under complements; 3) it is closed under intersection; and, 4) it is closed under countable unions.

Lemma 2 *The set of decomposable events* $\mathbf{R}$ *is a* σ*-algebra.*

Proof. 1) From the definition of a decomposable event, it is immediate that $\emptyset \in \mathbf{R}$ and $\Omega \in \mathbf{R}$. 2) If $R \in \mathbf{R}$, then also by definition we have $\Omega \backslash R \in \mathbf{R}$. 3) Axiom 2 ensures $\mathbf{R}$ is closed under intersection.

4) Finally, let $\{R^n\}$ be a set of (increasing) decomposable events with $R^n \subset R^{n+1}$. Assume by way of contradiction that $R^\infty := \bigcup R^n$ is not a decomposable event. That is, there exist $f, g \in \mathcal{F}$ such that $f_{R^\infty} g \succ g$ and $g_{R^\infty} f \succ g$ but $g \succsim f$. But since each R^n is decomposable, this means for every n either $g \succsim f_{R^n} g$ or $g \succsim g_{R^n} f$ (or both). Hence, we can find an infinite subsequence $\{\widehat{R}^n\}$ of $\{R^n\}$ with $\bigcup \widehat{R}^n = R^\infty$, and for which

- (i) either $g \succsim f_{\widehat{R}^n} g$ $(\succsim \underline{x})$ for all n,
- (ii) or $g \succsim g_{\widehat{R}^n} f$ $(\succsim \underline{x})$ for all n.

If (i) (respectively, (ii)) holds, axiom 5 implies $g \succsim f_{R^\infty} g$ (respectively, $g \succsim g_{R^\infty} f$) contradicting $f_{R^\infty} g \succ g$ (respectively, $g_{R^\infty} f \succ g$). Thus we have established there must exist at least one n for which $f_{R^n} g \succ g$ and $g_{R^n} f \succ g$ which, since R^n is decomposable, in turn implies $f \succ g$. A contradiction. Thus, $\bigcup R^n$ is a decomposable event. Therefore $\mathbf{R}$ is a σ-algebra. ■

The following auxiliary result is fundamental and used both here and in subsequent steps.

Lemma 3 *For all* $x^* \succ x$*, all* $f, f' \in F$ *and* $R \in \mathbf{R}^+$*, if* $x^*_R f \succsim f' \succsim x_R f$ *then there is a* $R' \in \mathbf{R}$ *for which* $(x^*_{R'} x)_R f \sim f'$*.*

Proof. If either $x^*_R f \sim f'$ or $f' \sim x_R f$, set $R' := \Omega$ or set $R' := \emptyset$. We now only need to find a decomposable event R' when $x^*_R f \succ f' \succ x_R f$.

Since $x^*_R f \succ f'$, Axiom 6 implies there is a decomposable event $R_1 \subset R$ such that $(x_{R_1} x^*)_R f \succ f'$. Applying Axiom 6 again implies there is a decomposable event $R_2 \subset R \backslash R_1$ such that $(x_{R_1 \cup R_2} x^*)_R f \succ f'$. By repeating the argument, this yields a series $\{R_n\} \subset \mathbf{R}^+$ of disjoint subsets of R that satisfies

$$(x_{\bigcup_{n=1}^k R_n} x^*)_R f \succ f' \text{ for all } k. \tag{2}$$

Let A denote the collection of all such series, and define $B := \{\bigcup_{n=1}^{\infty} R_n : \{R_n\} \in A\}$. Notice that $B \subset \mathbf{R}$, and that $(x_{\widehat{R}} x^*)_R f \succsim f'$ for each $\widehat{R} \in B$, by Axiom 5. We show that we can take $R' = R \backslash M$ with M a maximal element of B, if it exists, and invoke Zorn's lemma to establish that otherwise we can take M as the upperbound outside B of a chain in B.

Lemma (Zorn) *Let $\mathcal{P}$ be a partially ordered set in which each chain C has an upperbound. Then $\mathcal{P}$ has at least one maximal element.*

This applies to B as a partially ordered set, by set-inclusion. First assume B has a maximal element M. We can exclude that $(x_M x^*)_R f \succ f'$, since otherwise the procedure above would determine $\tilde{R} \in \mathbf{R}^+$ for which still $((x_{M \cup \tilde{R}} x^*)_E f \succ f'$, implying that also $M \cup \tilde{R} \in B$, as limit of the series $(M \cup \tilde{R}, \emptyset, ...) \in A$, contradicting that M is maximal of B with $M \subset M \cup \tilde{R}$. So $(x^*_{R \backslash M} x)_f \sim f'$.

Next suppose B has no maximal element. By Zorn's lemma, B must contain a chain C with upperbound $\bigcup_{\tilde{R} \in C} \tilde{R} \notin B$. Now we can take M as this upperbound, we can again exclude that $(x_{M \cup \tilde{R}} x^*)_R f \succ f'$.

To conclude, in both cases, we can take R' as the decomposable event $R \backslash M$ such that $(x_M x^*)_R f \sim f'$. ■

Next we establish Machina and Schmeidler's (1992) axiom **P4*** holds on the restriction of $\succsim$ to G (the set of decomposable acts). To do so, we first show Epstein and LeBreton's (1993) axiom **P4**^{c} holds.

Lemma 4 *For any decomposable events $R, \widehat{R}, T \in \mathbf{R}$, $R \cup \widehat{R} \subseteq T$, any four outcomes x^*, x, y^* and y in X, and any act $f \in F$: if $x^*_T f \succ x_T f$ and $y^*_T f \succ y_T f$ then $(x^*_R x)_T f \succsim (x^*_{\widehat{R}} x)_T f \implies (y^*_R y)_T f \succsim (y^*_{\widehat{R}} y)_T f$.*

Proof. By Axiom 3, $x^*_T f \succ x_T f$ and $y^*_T f \succ y_T f$ implies $x^* \succ x$ and $y^* \succ y$ respectively. Hence from $(x^*_R x)_T f \succsim (x^*_{\widehat{R}} x)_T f \succ x_T f$, by applying Lemma 3 we can find a decomposable event $R' \subseteq R$ such that $(x^*_{R'} x)_T f \sim (x^*_{\widehat{R}} x)_T f$. Thus by Axiom 4 it follows $(y^*_{R'} y)_T f \sim (y^*_{\widehat{R}} y)_T f$. And since $R' \subseteq R$ it follows from Axiom 3 that $(y^*_R y)_T f \succsim (y^*_{R'} y)_T f$. The desired implication then follows from the transitivity of $\succsim$. ■

Our structural assumption on X that $\bar{x} \succ \widetilde{x} \succ \underline{x}$ means we can apply Epstein and LeBreton's (1993) analysis to show Machina and Schmeidler's (1992) axiom **P4*** holds on the restriction of $\succsim$ to G.

Now we consider the restriction of $\succsim$ to binary bets on decomposable events involving the best $\bar{x}$ and worst $\underline{x}$ outcomes, denoted as $\mathcal{F}^{\{\underline{x},\bar{x}\}}$, and establish it admits an SEU representation. Our structural assumption on X that $\bar{x} \succ \underline{x}$ implies (Savage's) **P5**. Axiom 1 is **P1**. Axiom 3 directly implies **P3**. **P4** is redundant, and Axiom 6 is **P6**. Finally, in a setting with exactly two outcomes $\bar{x} \succ \underline{x}$, **P4*** (as derived above from Axiom 4) is equivalent to **P2**, cf. (Machina and Schmeidler, 1992, p764).

Thus, there is a finitely additive, convex-ranged μ on Σ and a function $v : \{\underline{x}, \bar{x}\} \to \mathbb{R}$ such that $V : \mathcal{F}^{\{\underline{x},\bar{x}\}} \to \mathbb{R}$, defined by $V(\bar{x}_R\underline{x}) = \mu(R)v(\bar{x}) + (1-\mu(R))v(\underline{x})$, represents $\succsim$ on $\mathcal{F}^{\{\underline{x},\bar{x}\}}$. Without loss of generality we can set $v(\underline{x}) := 0$ and $v(\bar{x}) := 1$. Hence

$$V(\bar{x}_R\underline{x}) = \mu(R). \tag{3}$$

Axiom 5 implies that $V(\bar{x}_{R^1\cup\cdots\cup R^n}\underline{x})$ converges to $V(\bar{x}_{\bigcup_{n=1}^{\infty} R^n}\underline{x})$ for any disjoint sequence of decomposable events $\{R_n\}$, which yields

$$\lim_{n\to\infty} \sum_{n=1}^{n} \mu(R_i) = \mu(\bigcup_{n=1}^{\infty} R^n)\,,$$

that is, μ is countably additive.

Step 2: Variants of decomposability

To prepare for the construction and validation of the BOSEU representation, we address some variants of decomposability conditions.

Lemma 5 *Let $\succsim$ be a relation that satisfies Axiom 1 and Axiom 3. For any decomposable event $E \in \mathbf{R}$ and any pair of acts $f, g \in F$:*

1. $f \succ f_E g$ and $f \succ g_E f$ implies $f \succ g$;

and furthermore, if $\succsim$ also satisfies Axioms 2 and 6 then

2. $f \succsim f_E g$ and $f \succsim g_E f$ implies $f \succsim g$; and,

3. $f_E g \succsim g$ and $g_E f \succsim g$ implies $f \succsim g$.

Proof. Fix $E \in \mathbf{R}$. We show that statement 1 holds by the same techniques in Grant et al. (2000). Assume by way of contradiction that there exist two acts $f, g \in F$, such that $f \succ f_E g$, $f \succ g_E f$ and $g \succsim f$. We consider two cases *(a)* Suppose $f_E g \succsim g_E f$. Then we have

$$g \succsim f \succ f_E g \succsim g_E f.$$

Set $\widehat{f} := f_E g$ and $\widehat{g} := g_E f$. Notice $\widehat{f}_E \widehat{g} = f$ and $\widehat{g}_E \widehat{f} = g$. Thus,

$$\widehat{g}_E \widehat{f} \succsim \widehat{f}_E \widehat{g} \succ \widehat{f} \succsim \widehat{g}.$$

Since E is decomposable, $\widehat{g}_E \widehat{f} \succsim \widehat{f}_E \widehat{g} \succ \widehat{f}$ implies that $\widehat{g} \succ \widehat{f}$, which contradicts $\widehat{f} \succsim \widehat{g}$.

(b) Now suppose, $g_E f \succ f_E g$. Then we have

$$g \succsim f \succ g_E f \succ f_E g.$$

So again, set $\widehat{f} := f_E g$ and $\widehat{g} := g_E f$, and again notice that $\widehat{f}_E \widehat{g} = f$ and $\widehat{g}_E \widehat{f} = g$. Thus,

$$\widehat{g}_E \widehat{f} \succsim \widehat{f}_E \widehat{g} \succ \widehat{g} \succ \widehat{f}.$$

Since E is decomposable, $\widehat{g}_E \widehat{f} \succsim \widehat{f}_E \widehat{g} \succ \widehat{g}$ implies that $\widehat{f} \succ \widehat{g}$, which contradicts $\widehat{g} \succ \widehat{f}$.

Therefore, we have established that statement 1 holds.

For statement 2, we first show that for all (arbitrary) acts $f \in F$ and decomposable acts $g \in G$.

$$f \sim f_E g \sim g_E f \ \text{ implies } \ f \sim g \tag{4}$$

Assume by way of contradiction that $f \sim f_E g \sim g_E f$ and $[f \succ g \text{ or } g \succ f]$

(a) $f \sim f_E g \sim g_E f$ and $f \succ g$.

(i) In case $E \subset g^{-1}(\bar{x})$ or $\Omega \backslash E \subset g^{-1}(\bar{x})$. If $E \subset g^{-1}(\bar{x})$, then

$$f \sim f_E g \sim \bar{x}_E f \succ g = \bar{x}_E g \ \text{ that is, } \ f_E g \succ \bar{x}_E g$$

and axiom 3 is violated. Similarly, if $\Omega \backslash E \subset g^{-1}(\bar{x})$, then

$$f \sim f_E \bar{x} \sim g_E f \succ g = g_E \bar{x} \ \text{ that is, } \ g_E f \succ g_E \bar{x}\,.$$

Again, axiom 3 is violated. The same argument applies when $g(E)$ or $g(\Omega \backslash E)$ only has outcomes indifferent to $\bar{x}$.

(ii) Otherwise, axiom 6 implies there is a partition $\{R_i\}$ such that $f \succ \bar{x}_{R_i} g$ for all R_i. Since E is non-null, there exist R_j such that $R_i \cap E$ is non-null. Applying Axiom 6again, there is a partition $\{R_i'\}$ such that $f \succ \bar{x}_{R_i'}(\bar{x}_{R_j} g)$ for all R_i'. Then there is R_j' such that $R_j' \cap E^c$ is non-null. Let $R = R_j \cup R_j'$ and so $f \succ \bar{x}_R g$ with both $R \cap E$ and $R \cap E^c$ non-null. Together with Axiom 3, $f_E(\bar{x}_R g) \succ f$ and $(\bar{x}_R g)_E f \succ f$, and so $\bar{x}_R g \succ f$ and we reach a contraction.

(b) $f \sim f_E g \sim g_E f$ and $g \succ f$. This case can be proved in the same way as what we have done in part (a) by applying Axiom 6 on $\underline{x}$.

Next we show statement 2 holds. Assume for contradiction that there are f, g such any of the following holds.

(i) $f \sim f_E g \sim g_E f$ and $g \succ f$.

(ii) $f \succ f_E g, f \sim g_E f$ and $g \succ f$.

(iii) $f \sim f_E g, f \succ g_E f$ and $g \succ f$.

We only need to show case (i) since the other two are in favor of the direction to get contradiction. Applying Lemma 3, there exist decomposable events R_g^1 and R_g^2 such that

$$f \sim f_E(\underline{x}_{R_g^1}\bar{x}) \sim (\underline{x}_{R_g^2}\bar{x})_E f$$

and so $f \sim \underline{x}_{R_g^1 \cup R_g^2}\bar{x}$ by Eq. 4. Similarly, there there exist decomposable events R_f^1 and R_f^2 such that

$$f \sim g_E(\underline{x}_{R_f^1}\bar{x}) \sim (\underline{x}_{R_f^2}\bar{x})_E g \tag{5}$$

Now we let $f^* = \underline{x}_{R_g^1 \cup R_g^2}\bar{x}$ and $g^* = \underline{x}_{R_f^1 \cup R_f^2}\bar{x}$, the above preferences are reduced to

$$f \sim f^* \sim f_E f^* \sim f_E^* f \sim g_E g^* \sim g_E^* g \sim f_E g \sim g_E f$$

Notice that $f_E^* f = (f_E^* g^*)_E(g_E f)$ and $g_E g^* = (g_E f)_E(f_E^* g^*)$ and $f_E^* g^* \in G$ and so $f_E^* g^* \sim f^*$ by expression 4. Similarly, $g_E^* f^* \sim f^*$, so $f^* \sim g^*$. From (3) it follows that $\mu(R_f^1) = \mu(R_g^1)$ and $\mu(R_f^2) = \mu(R_g^2)$. Expression 5 becomes

$$f \sim g_E(\underline{x}_{R_g^1}\bar{x}) \sim (\underline{x}_{R_g^2}\bar{x})_E g \sim \underline{x}_{R_g^1 \cup R_g^2}\bar{x}$$

and so $g \sim \underline{x}_{R_g^1 \cup R_g^2}\bar{x} \sim f$, which gives us the contradiction.

Statement 3 can be proved in the same way as statement 2. ■

Lemma 5 implies the conditional independence of acts on an indifference class.

Proposition 1 *For any event $R \in \mathbf{R}$ and any four acts $f, f', f'', f^* \in F$:*

$$f_R f' \sim f_R f'' \sim f^*_R f' \implies f^*_R f'' \sim f^*_R f' \sim f_R f' \sim f_R f''\,.$$

Proof. By the non-strict criterion in Lemma 5, $f_R f' \succsim f_R f''$ and $f_R f' \succsim f^*_R f'$ implies $f_R f' \succsim f^*_R f''$. Similarly, $f_R f' \precsim f_R f''$ and $f_R f' \precsim f^*_R f'$ implies $f_R f' \precsim f^*_R f''$. Thus, $f^*_R f' \sim f_R f'$ implies $f^*_R f'' \sim f^*_R f' \sim f_R f' \sim f_R f''$. ■

Step 3: Constructing the Balanced Outcome-Set Utility

The preliminary results above (particularly, lemmas 3 and 5) enable us to define the balanced outcome-set utility $U(\cdot\,,\cdot)$, as follows. Set $U(\{\bar{x}\},p) := 1-p$ and $U(\{\underline{x}\},p) := -p$, for all $p \in [0,1]$.

Fix an outcome set Y and $p \in [0,1]$. We employ a short-hand notation $\bar{x}_p\underline{x}$ for an act of the form $\bar{x}_R\underline{x}$ with $\mu(R) = p$. When Y is not a singleton, choose a diffuse act $h^Y \in H$, for which its envelope $\mathbf{h}^Y$ is the constant function $\mathbf{h}^Y(\omega) = Y$ for all $\omega \in \Omega$. When $Y = \{y\}$, set $h^Y := y$.

(*a*) For the case $h^Y \succ \bar{x}_p\underline{x}$, determine a decomposable event R such that $h^Y_R\underline{x} \sim \bar{x}_p\underline{x}$. Such an R exists, since the balance probability of $h^Y_{R'}$ with $R' \in \mathbf{R}$ only depends on $\mu(R')$, again by Axiom 4, and we take any $R \in \mathbf{R}$ with $\mu(R) = q$, for q the maximum probability of R' such that $h^Y_{R'} \succsim \bar{x}_p\underline{x}$. Notice that q does not depend on the choice of h^Y, by Axiom 3(i). In order for $\succsim$ to admit a BOSEU representation requires

$$q\,U(Y,p) + (1-q)\,U(\{\underline{x}\},p) \;=$$
$$p\,U(\{\bar{x}\},p) + (1-p)\,U(\{\underline{x}\},p) = 0.$$

Solving for $U(Y,p)$ yields

$$U(Y,p) := \frac{1-q}{q} \times p\,.$$

(*b*) Otherwise, determine a decomposable event R such that $h^Y_R\bar{x} \sim \bar{x}_p\underline{x}$, and set $q := \mu(R)$. Again, q does not depend on the choice of R and h^Y. In order for $\succsim$ to admit a BOSEU representation requires

$$q\,U(Y,p) + (1-q)\,U(\{\bar{x}\},p) \;= 0,$$

which yields

$$U(Y,p) := -\frac{1-q}{q} \times (1-p).$$

By construction this function satisfies the two properties required for a balanced outcome-set utility.

Step 4: Establishing the BOSEU Representation.

It remains to verify that μ and U, as specified above, constitute a proper BOSEU representation that represents the given ordering $\succsim$ on F.

Fix an arbitrary act f in F that has associated with it the measurable split $\{R_f^Y : Y \subseteq f(\Omega)^+\}$, and the diffuse acts h_f^Y for which $(h_f^Y)_{R_f^Y} f = f$.

There exists a $p \in [0,1]$, such that $f \sim \bar{x}_p \underline{x}$ (again by Lemma 3, taking $E = \Omega$). For each $Y \subseteq f(\Omega)^+$, we can find a decomposable subevent $\bar{R}_f^Y \subseteq R_f^Y$ for which $[\bar{x}_{\bar{R}_f^Y} \underline{x}]_{R_f^Y} f \sim f$. Also, there is $A \subset \Omega \backslash R_f^Y$ in $\mathbf{R}$ such that $f \sim f_{R_f^Y}[\bar{x}_A \underline{x}]$, and Proposition 1 guarantees that also $[\bar{x}_{\bar{R}_f^Y} \underline{x}]_{R_f^Y}[\bar{x}_A \underline{x}] \sim f$.

To analyse these expressions, notice that the value of an act of the form $(h_R^Y \underline{x})_E \bar{x}$ for measurable events $R \subset E$, only depends on $\mu(R)$ and $\mu(E)$, again by Axiom 4. Since we can split R and E in k subsets R_i, E_i with equal probabilities, respectively $\mu(R)/k$ and $\mu(E)/k$, we have $(h_{R_i}^Y \underline{x})_{E_i} \bar{x} \sim (\bar{x}_{A_i} \underline{x})_{E_i} \bar{x}$ for events $A_i \subset E_i$, with also $\mu(A_i)$ independent of i. From Proposition 1 it follows now that $(h_R^Y \underline{x})_E \bar{x} \sim (\bar{x}_A \underline{x})_E \bar{x}$. This means that this indifference in fact only prescribes the ratio $\mu(R)/\mu(A)$, for any $E \in \mathbf{R}$.

More generally, we know from Axiom 4 that the ratio $\mu(\bar{R}_f^Y)/\mu(R_f^Y)$ must be the same in all acts of the form $h_{R_p^Y}^Y f'$ that are on the same indifference curve as f, being all indifferent to $h_{R_p^Y}^Y[\bar{x}_A \underline{x}]$ for some $A \in \mathbf{R}$. Since we defined $U(Y,p)$ from the rule $h_{R_p^Y}^Y \underline{x} \sim \bar{x}_{R_p} \underline{x}$ (when $h^Y \succsim f$) and $h_{R_p^Y}^Y \bar{x} \sim \bar{x}_{R_p} \underline{x}$ (when $h^Y \prec f$), we can determine this ratio as

$$\frac{\mu(\bar{R}_f^Y)}{\mu(R_f^Y)} = \begin{cases} \dfrac{p}{q} = U(Y,p) + p \text{ if } h^Y \succ f \\ \\ \dfrac{q-1+p}{\mu(R_p^Y)} = U(Y,p) + p \text{ otherwise.} \end{cases}$$

So the contribution of each outcome set Y to the BOSEU equation is

$$\mu(R_f^Y) \left[\frac{\mu(\bar{R}_f^Y)}{\mu(R_f^Y)} (1-p) + \left(1 - \frac{\mu(\bar{R}_f^Y)}{\mu(R_f^Y)} \right) (-p) \right] = \mu(R_f^Y) U(Y,p),$$

as desired.

Necessity of the axioms

Axiom 1 follows from the fact that for any $f \in F$, there exists a unique $p \in [0,1]$ such that (1) holds true for $V(f) = p$.

Let $\mathbf{R}$ denote the domain of μ, which is a σ-algebra. To show that the events in $\mathbf{R}$ are decomposable, consider an arbitrary $R \in \mathbf{R}$, and fix a pair of acts $f, g \in F$, with $f \sim \underline{x}_R \bar{x} \in X$ and $\mu(R) = p$. If $g_R f \succ f$, then

$$\sum_{Y \in \mathcal{X}^{(\Omega)}} U(Y\,, p)\, \mu \left(\mathbf{g}^{-1}\,(Y) \cap R\right) > \sum_{Y \in \mathcal{X}^{f(\Omega)}} U(Y\,, p)\, \mu \left(\mathbf{f}^{-1}\,(Y) \cap R\right)\,. \qquad (6)$$

And $f_R g \succ f$ implies that

$$\sum_{Y \in \mathcal{X}^{g(\Omega)}} U(Y\,, p)\, \mu \left(\mathbf{g}^{-1}\,(Y) \cap \Omega \backslash R\right) > \sum_{Y \in \mathcal{X}^{f(\Omega)}} U(Y\,, p)\, \mu \left(\mathbf{f}^{-1}\,(Y) \cap \Omega \backslash R\right)\,. \quad (7)$$

Adding inequalities 6 and 7, we get

$$\sum_{Y \in \mathcal{X}^{g(\Omega)}} U(Y\,, p)\, \mu \left(\mathbf{g}^{-1}\,(Y)\right) > 0\,.$$

Hence $V(g) > p$, that is $g \succ f$. So $\mathbf{R}$ only contains decomposable events.

Next we show that any event outside $\mathbf{R}$ is not decomposable. . Consider an event $E \subset \Omega$ but $E \notin \mathbf{R}_\mu$. Let $[E]_*$ (respectively, $[\Omega \backslash E]_*$) denote the inner-sleeve of E (respectively, $\Omega \setminus E$), and define $\tilde{E} := \Omega \setminus (E_* \cup [\Omega \setminus E]_*)$. Since $E \notin \mathbf{R}$, $r := \mu(\tilde{E}) > 0$. By Lemma A2 (p22) and the proof of B11 (p31) in GP, we can partition $E \backslash [E]_*$ (respectively, $(\Omega \backslash E) \backslash [\Omega \backslash E]_*$) into two non-null events B^{11} and B^{12} (respectively, B^{21} and B^{22}). Notice by construction none of the four events B^{11}, B^{12}, B^{21} and B^{22} contain any non-null measurable event.

To show E is not decomposable, observe that $U(\{\bar{x}\}, 0) > U(\{\underline{x}\}, 0) = 0$ and $U(\{\bar{x}\}, 0) \geqslant U(\{\underline{x}, \bar{x}\}, 0) \geqslant U(\{\underline{x}\}, 0) = 0$. Hence at least one of following two inequalities (i) $U(\{\bar{x}, \underline{x}\}, 0) > 0$ and (ii) $U(\{\bar{x}\}, 0) > U(\{\bar{x}, \underline{x}\}, 0)$ must hold.

Consider first, the case $U(\{\bar{x}, \underline{x}\}, 0) > U(\{\underline{x}\}, 0)$ and the act $f = \bar{x}_{B^{11} \cup B^{21}} \underline{x}$. Since $\underline{x}_E f = \bar{x}_{B^{21}} \underline{x}$ and $f_E \underline{x} = \bar{x}_{B^{11}} \underline{x}$ it follows that $\mathbf{f} = \{\underline{x}\}_E \mathbf{f} = \mathbf{f}_E \{\underline{x}\}$, (that is, the envelope of each of those three acts are all the same). Since by construction, the measure μ is a coherent belief for the preferences generated by the (implicitly defined) BOSEU functional, this means $f \sim \underline{x}_E f \sim f_E \underline{x}$. However, since

$$[1 - \mu([E]_*) - \mu([\Omega\backslash E]_*)]\, u(\{\bar{x}, \underline{x}\}, 0) + [\mu([E]_*) + \mu([\Omega\backslash E]_*)]\, u(\{\underline{x}\}, 0)$$
$$> [1 - \mu([E]_*) - \mu([\Omega\backslash E]_*)]\, u(\{\underline{x}\}, 0) + [\mu([E]_*) + \mu([\Omega\backslash E]_*)]\, u(\{\underline{x}\}, 0)\, (= 0\,),$$

it follows that $f \succ \underline{x}$, say $f \sim \bar{x}_{R_p}\underline{x}$ for some R_p with $\mu(R_p) = p > 0$. To arrive at a violation of the decomposability criterion, choose a measurable event $R \subset \tilde{E}$ with $0 < \mu(R) < p$, so that $f \succ f' := \underline{x}_R f \succ \underline{x}$. Since the envelopes of $f'_E\underline{x}$ and $\underline{x}_E f'$ are the same as those of respectively $f_E\underline{x}$ and $\underline{x}_E f$, we have $f'_E\underline{x} \succ f'$ and $\underline{x}_E f' \succ f'$, yet $f' \succ \underline{x}$. So E is not decomposable.

So now consider the case $U(\{\bar{x}\}, 0) > U(\{\bar{x}, \underline{x}\}, 0)$ and the pair of acts $f = \bar{x}_{B^{11}\cup B^{21}}\underline{x}$ and $f' = \underline{x}_{[E]_*\cup[\Omega\backslash E]_*}\bar{x}$. Since $f_E f' = \underline{x}_{B^{11}\cup B^{21}\cup B^{22}}\bar{x}$ and $f'_E f = \underline{x}_{B^{11}\cup B^{12}\cup B^{22}}\bar{x}$, $\mathbf{f} = \mathbf{f_E f'} = \mathbf{f'_E f}$, that is, all three acts come from the same indifference set and hence $f \sim f_E f'$ and $f \sim f'_E f$. However, since

$$[1 - \mu([E]_*) - \mu([\Omega\backslash E]_*)]\, u(\{\bar{x}\}, 0) + [\mu([E]_*) + \mu([\Omega\backslash E]_*)]\, u(\{\underline{x}\}, 0)$$
$$> [1 - \mu([E]_*) - \mu([\Omega\backslash E]_*)]\, u(\{\bar{x}, \underline{x}\}, 0) + [\mu([E]_*) + \mu([\Omega\backslash E]_*)]\, u(\{\underline{x}\}, 0)\, (= 0\,),$$

it follows $f' \succ f$. A violation of the decomposability criterion for E can be established as above. So also in this case E is not decomposable, and hence **R** is the set of all decomposable events.

Axiom 2 now follows directly from the assumption that **R** is a σ-algebra. The necessity of the rest of the axioms follows straightforwardly from the BOSEU representation combined with the fact that for any pair of acts f and g with respective envelopes **f** and **g**, and any decomposable event R in **R**, the envelope of $g_R f$ is $\mathbf{g}_R\mathbf{f}$. In particular, the envelope of $h_R f$ has outcome-set $h(\Omega)$ on R, and equals **f** outside R. The necessity of Axiom 4 is now obvious. Axioms 3 and 5 follow directly from the corresponding properties of U. Axiom 6 follows from the fact that μ is convex-ranged.

■

References

Chew, S. H. (1983). A generalization of the quasilinear mean with applications to meaurement of income inequality and decision theory resolving the Allais paradox. *Econometrica 51*(4), 1065–1092.

Dekel, E. (1986). An axiomatic characterization of preferences under uncertainty: weakening the independence axiom. *Journal of Economic Theory 40*, 304–318.

Dempster, A. P. (1967). Upper and Lower Probabilities Induced by a Multivalued Mapping. *Annals of Mathematical Statistics 38*, 325–339.

Eichberger, J. and I. Pasichnichenko (2021, December). Decision making with partial information. *Journal of Economic Theory 198*.

Ellsberg, D. (1961). Risk, ambiguity, and the Savage axioms. *Quarterly Journal of Economics 75*, 643–669.

Epstein, L. and M. LeBreton (1993). Dynamically Consistent Beliefs Must Be Bayesian. *Journal of Economic Theory 61*, 1–22.

Epstein, L. G. and J. Zhang (2001). Subjective probabilities on subjectively unambiguous events. *Econometrica 69*(2), 265–306.

Ghirardato, P. (2001). Coping with ignorance: unforseen contingencies and non-additive uncertainty. *Economic Theory 17*, 247–276.

Grant, S. (1995). Subjective Probability without Monontonicity: or How Machina's Mom may also be probabilistically sophisticated. *Econometrica 63*(1), 159–189.

Grant, S., A. Kajii, and B. Polak (2000). Decomposable Choice under Uncertainty. *Journal of Economic Theory 92*, 167–197.

Grant, S., P. Rich, and J. Stecher (2022). Bayes and Hurwicz without Bernoulli. *Journal of Economic Theory 199*(105027).

Gul, F. (1991). A Theory of Disappointment Aversion. *Econometrica 59*(3), 667–686.

Gul, F. and W. Pesendorfer (2014). Expected uncertain utility. *Econometrica 82*(1), 1–39.

Gul, F. and W. Pesendorfer (2015). Hurwicz expected utility and subjective sources. *Journal of Economic Theory 159*, 465–488.

Heukelom, F. (2015). A history of the Allais Paradox. *The British Journal for the History of Science 48*(1), 147–169.

Kopylov, I. (2007). Subjective probabilities on "small" domains. *Journal of Economic Theory 133*(1), 236–265.

Machina, M. J. and D. Schmeidler (1992). A More Robust Definition of Subjective Probability. *Econometrica 60*(4), 745–780.

Nehring, K. (2009). Imprecise probabilistic beliefs as a context for decision-making under ambiguity. *Journal of Economic Theory 144*(3), 1054–1091.

Savage, L. J. (1954). *The Foundations of Statistics.* John Wiley & Sons.

Shafer, G. (1976). *A Mathematical Theory of Evidence.* Princeton, New Jersey: Princeton University Press.

Tversky, A. and E. Shafir (1992). The disjunction effect in choice under uncertainty. *Psychological science 3*(5), 305–310.

Zhang, J. (2002). Subjective Ambiguity, Expected Utility, and Choquet Epected Utility. *Economic Theory 20*, 159–181.